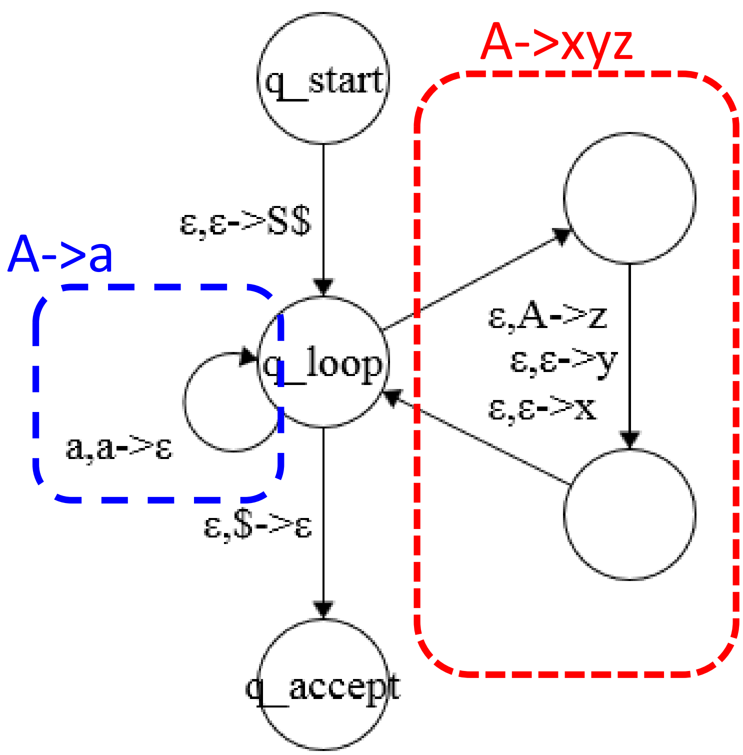
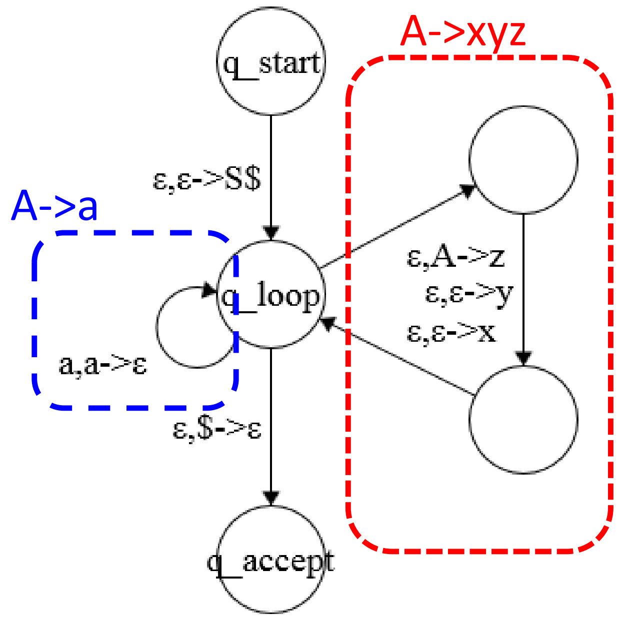
(pp. 117-124) **PDAs and CFGs** (Sec. 2.2)

- A language is context free iff all strings in L can be generated by some context free grammar
- Theorem 2.20: **L is Context Free iff a PDA accepts it**
  - I.e. if L is context free than some PDA accepts it
  - AND if a PDA accepts L, then it is context free
- Outline of proof: must prove in both directions
  - If language A is CF, then we construct a PDA P
    - Use stack to keep the right hand of the intermediate string that includes the leftmost variable on top
    - Create transition rules from grammar rules
    - Use nondeterministic choice of rules to match terminals
  - If a PDA P recognizes L, then L is CF
    - Proof by constructing a CFG that matches

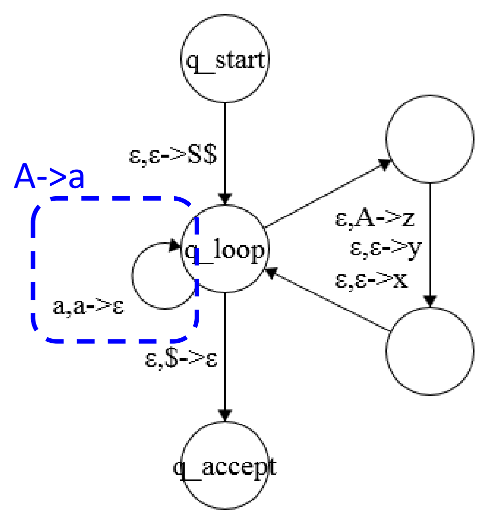
- (p117) 1st part: **if G=(V,∑,R,S) is CFG for L, then some PDA P=(Q,∑,Γ,δ,q_{start},F) accepts it**
  - **Proof: Build the PDA from the Grammar**
  - Overview of proof by construction
  - Assume G=(V,∑,R,**S**)
    - Γ_ε = V U ∑ U {$, ε}
      - Alphabet + non-terminals+special characters $, ε
    - 3 common states: q_{start}, q_{loop}, q_{accept}
      - F = {q_{accept}}
      - Additional states added for each grammar rule
        - Extra states for rules like A->xyz
        - Self-loop on q_{loop} for rules like A->a



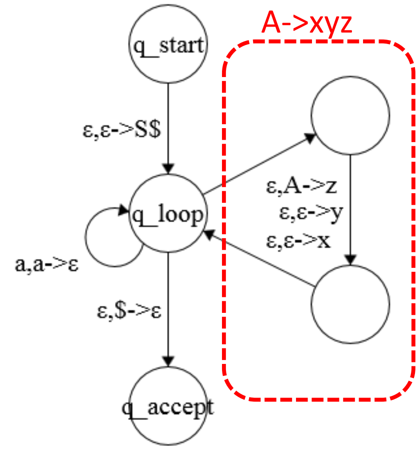


- **Overview of PDA P in operation (see Fig. 2.26)**:
  - Start with pushing "**S**$" onto stack (with **S** on top)
    - Used $ to mark bottom of stack
  - Repeatedly loop around state q_{loop}
    - **If stack top is $**, enter accept state
    - **If stack top is non-terminal A**, select an edge (non-deterministically) based on a rule for A
      - Pop the variable
      - Push the RHS (in reverse order)
    - **If stack top is terminal "a",** next symbol on input must be "a" to be accepted. Pop a.

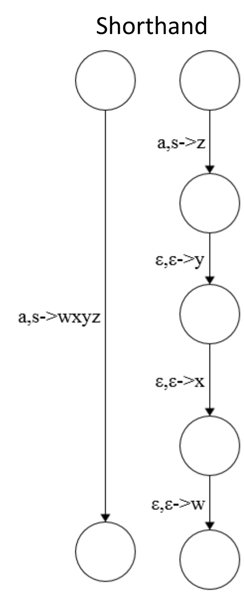
- Formal Construction of P from Grammar
  - Remember PDA transition rule specifies pair (q, x)
    - q is next state
    - x is character to push on stack
  - $Q = \{q_{start}, q_{loop}, q_{accept}\} \cup E$
    - q_{loop} is special state where all grammar rules start & end
    - E = all states generated by grammar rules as discussed below
  - $F = \{ q_{accept}\}$
  - Add startup transitions to push S$ on start
    - $\delta(q_{start}, \varepsilon, \varepsilon) = \{(q_1, \$)\}$, q_1 a new state in E
    - $\delta(q_1, \varepsilon, \varepsilon) = \{(q_{loop}, S)\}$
    - Note **shorthand** "single edge" ε, ε->S$
  - For each terminal a in ∑, add the following self-loop
    - $\delta(q_{loop}, a, a) = \{(q_{loop}, \varepsilon)\}$ (We match the a and pop from stack)
  - To detect acceptance, add rule
    - $\delta(q_{loop}, \varepsilon, \$) = \{(q_{accept}, \varepsilon)\}$



- (p.11(0 For kth rule $S\to u_1u_2\ldots u_L$, u_i from ∑ U V
  - $\delta(q_{loop}, \varepsilon, S)$ includes $(q_{k,1}, u_L)$, q_1 a new state
  - Add L-1 transitions to push $u_1u_2\ldots u_{L-1}$ onto stack, with u_1 on top as follows
    - $\delta(q_{k,1}, \varepsilon, \varepsilon) = \{(q_{k,2}, u_{L-1})\}$, $q_{k,2}$ a new state in E
    - $\delta(q_{k,2}, \varepsilon, \varepsilon) = \{(q_{k,3}, u_{L-2})\}$, $q_{k,3}$ a new state in E
    - …
    - $\delta(q_{k,L-2}, \varepsilon, \varepsilon) = \{(q_{k,L-1}, u_2)\}$, q_l a new state in E
    - $\delta(q_{k,L-1}, \varepsilon, \varepsilon) = \{(q_{loop}, u_1)\}$



- (p. 119, Fig. 2.23) Book uses shorthand a,s->w (w a string) on edge for sequence of steps:
  - $a,s\to w_n$
  - $\varepsilon,\varepsilon\to w_{n-1}$
  - …
  - $\varepsilon,\varepsilon\to w_1$



- (p. 120) Final machine looks like Fig.2.24
- (p.120) Example problem Fig.2.25
- (p. 155) See also problems 2.5, 2.7, 2.9 esp. 2.11, and create PDAs from CFGs 2.13, 2.14. 2.46

- (p. 121) **Now prove if PDA accepts L, L must be CF**
- **Again by construction of a CFG from PDA**
  - Modify P slightly
    - Ensure a single accept state q_{accept}
    - From any prior accept state, add set of transitions that ensure stack is empty before final accept state
    - Ensure each transition *either* pushes or pops *but not both or neither*
      - If a transition does both (δ(q,a,x)->{(q',y)}),
        - add new intermediate state
        - Transition from original state does pop: a,x->ε
        - Transition from new state does push: ε,ε->y
      - If a transition does neither (δ(q,ε,ε)->{(q',ε)},
        - add new state and use any terminal x
        - Transition from original state pushes x: a,ε->x
        - Transition from new state pops that x: ε,x->ε

- Construct G = (V,∑,R,S)
  - ∑ the same
  - V = {A_{pq} | p, q in Q} – 1 symbol for each pair of states
  - S = $A_{q0,qaccept}$
  - Construct grammar rules R as follows
    - For each p,q,r,s in Q, u in Γ, and a,b in ∑
      - If δ(p, a, ε) contains (r,u) (we are pushing u)
      - And δ(s, b, u) contains (q, ε) (we are popping u)
      - Then add grammar rule A_{pq} -> aA_{rs}b
    - For each p,q,r in Q
      - Then add grammar rule A_{pq} -> $A_{pr}A_{rs}$
    - For each p in Q
      - Then add grammar rule A_{pp} -> ε
- See p.122 Figs. 2.28 for notional pictures of stack height

- (p. 123) Claim 2.30. If variable A_{pq} generates string x, then x can bring P from state p with an empty stack to state q with empty stack
  - Proof by induction on # of steps in derivation of x
  - Basis step: it took 1 step
    - Only grammar rules with no RHS variables are A_{pp}->ε
    - i.e. ε must take P from p to p without pushing anything onto empty stack
  - Induction Hypothesis: assume true for derivations of length at most k, k≥1.
  - Induction step: prove true for derivations of length k+1
    - Suppose A_{pq}=>x with k+1 steps
    - Two possibilities
      - First case: A_{pq} -> aA_{rs}b for some a,b, r,s
        - A_{rs} must have generated y where x = ayb
          - But this must have happened in k steps, so P can go from r to s on empty stack
          - Because A_{pq} -> aA_{rs}b is a rule in G
            - δ(p, a, ε) contains (r, u) for some u
              - i.e. it pushes u
            - and δ(s, b, u) contains (q, ε)
              - i.r. it pops u

- Thus if P starts at p with empty stack
  - After reading a it goes to state r with u on stack
  - Then reading y brings P to s and leaves u on stack
- Second case: $A_{pq} \Rightarrow A_{pr}A_{rq}$
  - Assume x=yz where
    - $A_{pr} \Rightarrow y$ in at most k steps
    - $A_{rs} \Rightarrow z$ in at most k steps
- Then induction hypothesis says y can bring P from p to r, and z can bring P from r to q, with empty stacks on both ends

- (p.123) Claim 2.31: If we can bring P from p to q with empty stacks on both sides then A_{pq} generates x
- (p. 124) Corollary 2.32. Every regular language is context free.