Lecture Week 7: Lyapunov Stability of Linear Time-varying Systems

This week's lecture has two objectives. The first part of the week extends Lyapunov stability concepts to linear time-varying (LTV) systems. For such systems, one can no longer use the eigenvalue conditions or the matrix Lyapunov equation to certify stability. Lyapunov stability for time-varying systems has to be refined to hold "uniformly" with respect to the initial time. The second objective of this week's lecture is to extend stability concepts to forced systems. This leads to the notion of input-output stability that we define with respect to the $\mathcal{L}_p$ norms of the signal spaces. We examine the relationship between $\mathcal{L}_p$ stability and Lyapunov stability and use this relationship to set up our next topic addressing the controllability and observability of linear dynamical systems.

Mathematical Preliminaries: Consider a C^1 function $v : \mathbb{R} \to \mathbb{R}$ and let there be a function $f : \mathbb{R} \to \mathbb{R}$. The function v is said to satisfy a *differential inequality* if such that

$$\dot{v}(t) \leq f(v(t))$$

for all t. We can use this relationship to *bound* the $v(t)$ for all time. In particular, let $x : \mathbb{R} \to \mathbb{R}$ be a function that satisfies the initial value problem,

$$\dot{x}(t) = f(x(t)), \quad x(0) = v(0)$$

Then a result known as the *comparison principle* allows us to assert that $v(t) \leq x(t)$ for all $t \geq 0$.

1. Uniform Stability Concepts

We need to *refine* the earlier Lyapunov stability concept when it is applied to time-varying systems. To help illustrate the need for this refinement, consider the LTV system

$$\dot{x}(t) = (6t\sin(t) - 2t)x(t)$$

One may view this as a linear system, $\dot{x}(t) = a(t)x(t)$ whose time-varying coefficient $a(t)$ is subject to a damping force and a sinusoidal perturbation that both grow over time. The solution may be obtained by first separating the variables, x and t

$$\frac{dx}{x} = (6t\sin(t) - 2t)dt$$

and then integrating both sides of the equation from t_0 (initial time) to $t > t_0$ assuming $x(t_0) = x_0$.

$$\begin{aligned} x(t) &= x(t_0)\exp\left\{\int_{t_0}^{t}(6s\sin(s) - 2s)ds\right\} \\ &= x(t_0)\exp\left\{6\sin(t) - 6t\cos(t) - t^2 - 6\sin(t_0) + 6t_0\cos(t_0) + t_0^2\right\} \end{aligned}$$

For a fixed initial time, t_0, one sees that eventually the quadratic term, t^2, will denominate the exponential function's behavior, thereby implying that the exponential function in the above equation is bounded above by a function of t_0, say $c(t_0)$. It therefore follows that

$$|x(t)| \leq c(t_0)|x(t_0)|, \quad \text{for all } t \geq t_0$$

If we then consider any $\epsilon > 0$ and select the initial neighborhood $\delta(\epsilon) = \dfrac{\epsilon}{c(t_0)}$ then clearly $|x(t)|$ remains with a ϵ-neighborhood of the equilibrium and so we can conclude the equilibrium at 0 is stable in the sense of Lyapunov.

The issue we face here, however, is that δ is not just a function of ϵ, it is also a function of the initial time t_0. Our worry is that as $t_0 \to \infty$ that $\delta(\epsilon, t_0)$ could approach a finite limit that is, in fact, zero. If we think of t_0 as the system's current "age". Then this says as the system ages (i.e. t_0 gets larger) our ability to keep $|x(t)| < \epsilon$ is harder and harder as we have to start in a smaller and smaller distance, δ, from the origin.

This is precisely the case with this particular system. In particular, consider a sequence of initial times $\{t_{0n}\}_{n=0}^{\infty}$ where $t_{0n} = 2n\pi$ for $n = 0, 1, 2, \ldots, \infty$. Let us evaluate $x(t)$ exactly π time units after t_{0n} to see that

$$\begin{aligned} x(t_{0n} + \pi) &= x(t_{0n})\exp\left\{6(2n+1)\pi - (2n+1)^2\pi^2 + 6(2n\pi) + (2n)^2\pi^2\right\} \\ &= x(t_{0n})\exp\left\{24n\pi + 6\pi - 4n\pi^2 - \pi^2\right\} \\ &= x(t_{0n})\exp\left\{(4n+1)(6-\pi)\pi\right\} \end{aligned}$$

and for any $x(t_0) \neq 0$ we then can see that the ratio

$$\frac{x(t_{0n} + \pi)}{x(t_{0n})} = e^{(6-\pi)\pi(4n+1)} = 7942.2e^{35.918n} \to \infty$$

as $n \to \infty$. In other words, as the system ages in the sense that t_{0n} goes to infinity, we see that ratio of x at $t_{0n} + \pi$ and t_{0n} become unbounded. This means that the origin becomes "less" stable as $t_0 \to \infty$ since $|x(t)|$ gets larger and larger as $t_0 \to \infty$ assuming $x(t_0)$ starts in the same δ-sized neighborhood of the origin.

These issues also become relevant when we consider asymptotic stability of the equilibrium. To be asymptotically stable, we require the origin to be stable and that $x(t) \to 0$ as $t \to \infty$. Formally, this asymptotic behavior may be seen as requiring for any $\epsilon > 0$ there exists a time $T > 0$ and initial neighborhood, $N_\delta(0)$, such that starting $x_0 \in N_\delta(0)$ implies $|x(t)| < \epsilon$ for all $t \geq T$. T represents the time it takes for the system state to reach the desired ϵ-neighborhood. In general this convergence time is a function of ϵ as well. But if the system is time-varying then we can also expect T to be a function of the initial time t_0. Our worry is that as $t_0 \to \infty$ (i.e. as the system ages) we have $T(\epsilon, t_0) \to \infty$. In other words, as the system ages its convergence time gets slower and slower.

We will use the following LTV system

$$\dot{x}(t) = -\frac{x(t)}{1+t}$$

to illustrate this other convergence issue. Again we separate the variables, x and t, and integrate from t_0 to t to obtain

$$x(t) = x(t_0)\frac{1+t_0}{1+t}$$

The origin is Lyapunov stable since for any t_0 we have $|x(t)| \leq |x(t_0)|$ for $t \geq t_0$. So for any $\epsilon > 0$, we can choose δ so it is independent of t_0. Note that the origin, however, is also *asymptotically stable*. So for all ϵ, we can find $T > 0$ such that $|x(t)| < \epsilon$ for all $t \geq t_0 + T$. In particular, given ϵ we can bound $|x(t)|$ as

$$|x(t)| \leq |x(t_0)|\frac{1+t_0}{1+t_0+T} < \epsilon$$

which can be rearranged to isolate T and get

$$T > t_0\left(\frac{|x(t_0)|}{\epsilon} - 1\right) - 1$$

Note that this is a lower bound on the time, T, it takes to reach the target ϵ-neighborhood. So $1/T$ may be taken as the "rate" at which the state converges to the origin. But the lower bound on T in the above equation is also a function of t_0 and we can readily see that $\frac{1}{T} \to 0$ as $t_0 \to \infty$. In other words, as the system ages (i.e. t_0 gets larger), the system's convergence rate, $1/T$, gets slower and slower. In terms of behavior, this system would appear to "stall out" on its approach to the origin.

The preceding concerns motivate a refinement of our earlier Lyapunov stability concept. We say the equilibrium at the origin is *uniformly stable* if for all $\epsilon > 0$ there exists $\delta > 0$

that is independent of t_0 such that

$$|x(t_0)| \leq \delta, \quad \Rightarrow \quad |x(t)| < \epsilon \quad \text{for all } t \geq t_0.$$

The equilibrium is *uniformly asymptotically stable* if it is uniformly stable and there exists δ independent of t_0 such that for all $\epsilon > 0$ there exists $T > 0$ that is also independent of t_0 such that $|x(t)| < \epsilon$ for all $t \geq t_0 + T(\epsilon)$ and all $|x(t_0)| \leq \delta(\epsilon)$. Finally, we say the origin is *globally uniformly asymptotically stable* (GUAS) if it is uniformly stable and δ can be chosen so that $\delta(\epsilon) \to \infty$ as $\epsilon \to \infty$ and there exist constants T and δ, both independent of t_0, such that for any $\epsilon > 0$ we have $|x(t)| < \epsilon$ for all $t \geq t_0 + T(\epsilon)$ when $|x(t_0)| < \delta(\epsilon)$.

Example: Consider the LTV system

$$\dot{x}(t) = \begin{bmatrix} -1 & t \\ 0 & -1 \end{bmatrix} x(t)$$

- Use the definition of asymptotic stability to show the origin is asymptotically stable.
- Use the definition of uniform asymptotic stability to show the origin is not UAS.

We take the initial time to be t_0 and let $x_{10} = x_1(t_0)$ and $x_{20} = x_2(t_0)$. Note that $x_2(t) = e^{-(t-t_0)}x_{20}$ for all $t \geq t_0$. This means that the first ODE is

$$\dot{x}_1(t) \;=\; -x_1(t) + te^{-(t-t_0)}u(t-t_0)$$

where u is a unit step function The solution for this ODE is

$$\begin{aligned} x_1(t) &= e^{-(t-t_0)}x_{10} + \int_{t_0}^{t} \tau e^{-(\tau-t_0)} e^{-(t-\tau)} x_{20} d\tau \\ &= e^{-(t-t_0)}x_{10} + x_{20}e^{-(t-t_0)} \int_{t_0}^{t} \tau d\tau \\ &= e^{-(t-t_0)} \left(x_{10} + \frac{x_{20}}{2}(t^2 - t_0^2) \right) \end{aligned}$$

Note that for $t \geq t_0$ we get

$$\begin{aligned} |x(t)|^2 &= x_1^2(t) + x_2^2(t) \\ &= e^{-2(t-t_0)} \left(x_{20}^2 + x_{10}^2 + x_{10}x_{20}(t^2 - t_0^2) + \frac{x_{20}^2}{4}(t^2 - t_0^2)^2 \right) \\ &\leq e^{-2(t-t_0)} \left(|x_0|^2 + |x_0|^2(t^2 - t_0^2) + \frac{|x_0|^2}{4}(t^2 - t_0^2)^2 \right) \\ &= |x_0|^2 e^{-2(t-t_0)} \left(1 + (t^2 - t_0^2) + \frac{1}{4}(t^2 - t_0^2)^2 \right) \end{aligned}$$

To assess stability, let $\epsilon > 0$ and $t_0 = 0$, then we get

$$|x(t)|^2 \leq |x_0|^2(1+K)$$

where $K = \max_t e^{-2t}(t^2 + t^4/4)$. We choose $\delta < \dfrac{\epsilon}{\sqrt{1+K}}$ to show the origin is stable. We can also see that

$$|x(t)|^2 \leq |x_0|^2(e^{-2t} + t^2e^{-2t} + t^4e^{-2t}/4) \to 0$$

as $t \to \infty$ which establishes the origin is asymptotically stable.

To assess uniform asymptotic stability, we need to keep t_0. In particular, we see this means

$$|x(t)|^2 \leq |x_0|^2 e^{-2(t-t_0)} \left(1 + (t^2 - t_0^2) + \frac{1}{4}(t^2 - t_0^2)\right)$$

The problem is that

$$t^2 - t_0^2 = (t - t_0)^2 + 2t_0(t_0 - t)$$

which means δ is also a function of t_0, not just $t - t_0$. As a result we cannot conclude the origin is uniformly stable and hence cannot be UAS.

2. Lyapunov Stability for Linear Time-varying Systems

As before, there is a Lyapunov theorem (direct method) for time-varying systems that certifies the uniform asymptotic stability of the equilibrium. This theorem is stated below without proof since it uses techniques that are not of direct interest to this course. We will use this theorem in establishing uniform stability results for LTV systems.

theorem 1. (Direct Method for Time-Varying Systems) *Let $x = 0$ be an equilibrium for $\dot{x}(t) = f(t, x)$, and let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be C^1 in both arguments. If there exist continuous positive definite functions $\underline{W}$, $\overline{W}$, and W all mapping $\mathbb{R}^n$ onto $\mathbb{R}$ such that*

$$\underline{W}(x) \leq V(t, x) \leq \overline{W}(x)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f(t, x) \leq -W(x)$$

for all $t \geq 0$ and $x \in D$, then the origin is uniformly asymptotically stable.

Let us compare Theorem 1 to our earlier direct method for time invariant systems. We first see that the Lyapunov function is now a function of time, t, and state, x, whereas the time-invariant V was only a function of state. This means that to establish similar conditions we

need to "bound" the time variation in V. So the condition for $V(x) > 0$ now becomes one where $V(t,x)$ is "sandwiched" between two positive definite functions $\underline{W}(x)$ and $\overline{W}(x)$ which are *independent of* t. In a similar spirit, we now require

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x)$$

to be bounded above by a negative definite $-W(x)$ which is also *independent of* t. So the conditions in this theorem are essentially the same as those in the direct method for time-invariant systems. Uniform asymptotic stability of the equilibrium is certified when $V(t,x)$ is sandwiched between two positive definite functions of state and $\dot{V}(t,x)$ is bounded above by a negative definite function of state. As before a function $V(t,x)$ that satisfies these conditions is called a Lyapunov function for UAS.

Let us now apply theorem 1 to an LTV system. So consider the LTV system

$$\dot{x}(t) = \mathbf{A}(t)x(t), \quad x(t_0) = x_0$$

which has an equilibrium at the origin. We let $\mathbf{A}(t)$ be a continuous function of t and suppose there is a C^1 symmetric positive definite matrix-valued function $\mathbf{P} : \mathbb{R} \to \mathbb{R}^{n\times n}$ with positive constants c_1 and c_2 such that

$$0 < c_1\mathbf{I} \leq \mathbf{P}(t) \leq c_2\mathbf{I}, \quad \text{for all } t \geq t_0 \tag{1}$$

and such that $\mathbf{P}(t)$ satisfies the matrix differential equation

$$-\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{P}(t) + \mathbf{Q}(t) \tag{2}$$

where $\mathbf{Q}(t)$ is a continuous symmetric positive definite matrix valued function and positive constant c_3 such that

$$\mathbf{Q}(t) \geq c_3\mathbf{I} > 0, \quad \text{for all } t \geq t_0 \tag{3}$$

We want to show that $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ taking values

$$V(t,x) = x^T\mathbf{P}(t)x$$

is a Lyapunov function for the LTV system.

Certifying V as a Lyapunov function simply means checking the conditions in Theorem 1. Clearly

$$c_1|x|^2 \leq V(t,x) = x^T\mathbf{P}(t)x \leq c_2|x|^2$$

based on our assumptions on $\mathbf{P}(t)$ in equation (1). So we can take $\underline{W}(x) = c_1|x|^2$ and $\overline{W}(x) = c_2|x|^2$, both of which are clearly positive definite. We now check the other condition by computing the directional derivative of $V(t,x)$,

$$\begin{aligned}
\dot{V}(t,x) &= x^T\dot{\mathbf{P}}(t)x + x^T\mathbf{P}(t)\dot{x} + \dot{x}^T\mathbf{P}(t)x \\
&= x^T\left\{\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{P}(t)\right\}x \\
&= -x^T\mathbf{Q}(t)x \leq -c_3|x|^2
\end{aligned}$$

where the last line comes from our assumption on $\mathbf{Q}(t)$ in equation (3). So we can take $W(x) = c_3|x|^2$ which is also clearly positive definite. As both conditions of Theorem 1 are satisfied we can conclude that the equilibrium of the LTV system is UAS provided the conditions in equations (1), (2), and (3) are all satisfied.

The Lyapunov conditions in equations (1-2) are only sufficient UAS of an LTV system. Unlike the LTI case, we cannot use eigenvalues as a necessary and sufficient condition for UAS because these eigenvalues are changing over time. The following theorem provides an alternative "eigenvalue" condition for GUAS of the LTV system,

theorem 2. *The origin of $\dot{x}(t) = \mathbf{A}(t)x(t)$ with initial condition $x(t_0) = x_0$ is uniformly asymptotically stable (UAS) if and only if there are positive constants, k and λ, such that the system's state transition matrix satisfies*

$$\|\mathbf{\Phi}(t;t_0)\| \leq ke^{-\lambda(t-t_0)}$$

for all $t \geq t_0 > 0$.

Proof: Since $\mathbf{\Phi}$ is the system's transition matrix, we have for $t \geq t_0$ that

$$\begin{aligned}
|x(t)| &\leq |\mathbf{\Phi}(t;t_0)x(t_0)| \\
&\leq \|\mathbf{\Phi}(t;t_0)\| \ |x(t_0)| \\
&\leq k|x(t_0)|e^{-\lambda(t-t_0)}
\end{aligned}$$

Since this is true for any $x(t_0)$, we can see that $|x(t)| \to 0$ as $t \to \infty$ at a rate, λ, which is independent of t_0. So the condition is sufficient for *global* UAS.

Conversely, assume that the origin is UAS, then there must exist a class $\mathcal{KL}$ function, $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$, such that

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0), \quad \text{for all } t \geq t_0 \text{ and all } x(t_0) \in \mathbb{R}^n$$

A class $\mathcal{KL}$ function, $\beta(r,s)$, is a continuous function that is continuous and increasing in r with $\beta(0,s) = 0$ and that is also asymptotically decreasing to zero in s. It is what we sometimes refer to as a *comparison function*.

Now note that the induced matrix norm of $\mathbf{\Phi}$ has the property

$$\|\mathbf{\Phi}(t;t_0)\| = \max_{|x|=1} |\mathbf{\Phi}(t;t_0)x| \leq \max_{|x|=1} \beta(|x|, t - t_0) = \beta(1, t - t_0)$$

Since $\beta(1,s) \to 0$ as $s \to \infty$, there exists $T > 0$ such that $\beta(1,T) \leq \frac{1}{e}$. For any $t \geq t_0$, let N be the smallest positive integer such that $t \leq t_0 + NT$. Divide the interval $[t_0, t_0 + (N-1)T]$ into $(N-1)$ equal subintervals of width T. Using the transition matrix' group property we can write

$$\mathbf{\Phi}(t;t_0) = \mathbf{\Phi}(t, t_0 + (N-1)T)\ \mathbf{\Phi}(t_0 + (N-1)T, t_0 + (N-2)T)\ \cdots\ \mathbf{\Phi}(t_0 + T, t_0)$$

and so

$$\begin{aligned}
\|\mathbf{\Phi}(t;t_0)\| &\leq \|\mathbf{\Phi}(t, t_0 + (N-1)T)\| \prod_{k=1}^{n-1} \|\mathbf{\Phi}(t_0 + kT, t_0 + (k-1)T)\| \\
&\leq \beta(1,0) \prod_{k=1}^{N-1} \frac{1}{e} = e\beta(1,0)e^{-N} \\
&\leq e\beta(1,0)e^{-(t-t_0)/T} = ke^{-\lambda(t-t_0)}
\end{aligned}$$

where $k = e\beta(1,0)$ and $\lambda = 1/T$. $\diamondsuit$

Example: Consider the LTV system

$$\dot{x}(T) = \begin{bmatrix} -1 & \alpha(t) \\ -\alpha(t) & -1 \end{bmatrix} x$$

where $\alpha(t)$ is continuous for all $t \geq 0$. Is the origin uniformly asymptotically stable?

We can solve this by checking a somewhat easier condition suggested by the preceding theorem. In particular, let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ and note that

$$\dot{V}(x) = x_1(-x_1 + \alpha(t)x_2) + x_2(-\alpha(t)x_1 - x_2) = -x_1^2 - x_2^2 = -2V$$

In other words, we know that $V(x(t))$ will satisfy the differential equation

$$\dot{V}(t) = -2V(t)$$

which has the solution $V(t) = V(0)e^{-2t}$ for $t \geq 0$. Since $V(x) = \frac{1}{2}|x|^2$ we know this implies the system is uniformly exponentially stable and so by the preceding theorem it must also be UAS.

Example: Consider the linear homogeneous system

$$\dot{x}(t) = \begin{bmatrix} -2 & t \\ 0 & -2 \end{bmatrix} x(t)$$

Determine if the origin is uniformly asymptotically stable. We computed the transition matrix for a related system in week 5. Using those methods here, we find this system's state transition matrix is

$$\mathbf{\Phi}(t;\tau) = \begin{bmatrix} e^{-2(t-\tau)} & \frac{t^2-\tau^2}{2}e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix} = \begin{bmatrix} 1 & \frac{t^2-\tau^2}{2} \\ 0 & 1 \end{bmatrix} e^{-2(t-\tau)}$$

The matrix norm is clearly dominated by the $e^{-2(t-\tau)}$ term and so we know this system is UAS according to the preceding theorem.

This theorem means that for linear systems, UAS is equivalent to uniform exponential stability (i.e. asymptotic stability where $|x(t)| < ke^{-\lambda t}$). This condition is not as useful as the eigenvalue condition we had for LTI systems because it needs knowledge of the transition matrix that can only be obtained by solving the state equations. In other words, the preceding theorem is of limited value as a "test" for UAS.

There are special cases, however, where we can use the transition matrix to assess whether an LTV system is UAS. This occurs when the $\mathbf{A}(t)$ matrix commutes with itself. In these cases there are useful methods for determining the transition matrix. In particular let assume that

$$\mathbf{A}(t)\mathbf{A}(\tau) = \mathbf{A}(\tau)\mathbf{A}(t)$$

for all t, τ. This would imply that

$$\mathbf{A}(t)\int_{\tau}^{t}\mathbf{A}(s)ds = \left[\int_{\tau}^{t}\mathbf{A}(s)ds\right]\mathbf{A}(t)$$

If we then consider the LTV system $\dot{x} = \mathbf{A}(t)x$ with $x(t_0) = x_0$, note that

$$x(t) = \left[\mathbf{I} + \left(\int_{t_0}^{t} \mathbf{A}(s)ds\right) + \frac{1}{2!}\left(\int_{t_0}^{t} \mathbf{A}(s)ds\right)^2 + \cdots\right] x_0$$

Now compute to show that

$$\begin{aligned}
\frac{d}{dt}\left(\int_{t_0}^{t} \mathbf{A}(s)ds\right) &= \mathbf{A}(t) \\
\frac{d}{dt}\left(\int_{t_0}^{t} \mathbf{A}(s)ds\right)^2 &= \mathbf{A}(t)\int_{t_0}^{t} \mathbf{A}(s)ds + \int_{t_0}^{t} \mathbf{A}(s)ds\mathbf{A}(t) \\
&= 2\mathbf{A}(t)\int_{t_0}^{t} \mathbf{A}(s)ds
\end{aligned}$$

and by induction one can then deduce that

$$\frac{d}{dt}\left(\int_{t_0}^{t} \mathbf{A}(s)ds\right)^k = k\mathbf{A}(t)\left(\int_{t_0}^{t} \mathbf{A}(s)ds\right)^{k-1}$$

So if we use this last fact in our expression for $\dot{x}$ we get

$$\begin{aligned}
\dot{x} &= \left[\mathbf{A}(t) + \frac{2}{2!}\mathbf{A}(t)\left(\int_{t_0}^{t} \mathbf{A}(s)ds\right) + \frac{3}{3!}\mathbf{A}(t)\left(\int_{t_0}^{t} \mathbf{A}(s)ds\right)^2 + \cdots\right] x_0 \\
&= \mathbf{A}(t)\left[\mathbf{I} + \left(\int_{t_0}^{t} \mathbf{A}(s)ds\right) + \ +\frac{1}{2!}\left(\int_{t_0}^{t} \mathbf{A}(s)ds\right)^2 + \cdots\right] x_0 \\
&= \mathbf{A}(t)\exp\left[\int_{\tau}^{t} \mathbf{A}(s)ds\right] x_0 = \mathbf{A}(t)x(t)
\end{aligned}$$

This means, therefore that when $\mathbf{A}(t)\mathbf{A}(\tau) = \mathbf{A}(\tau)\mathbf{A}(t)$ then

$$\Phi(t,\tau) = \exp\left[\int_{\tau}^{t} \mathbf{A}(s)ds\right] = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_{\tau}^{t} \mathbf{A}(s)ds\right)^k$$

Let us see how we can use this to evaluate whether the origin of an LTV system is UAS. We consider the special case where $\mathbf{A}(t) = \alpha(t)\mathbf{M}$. It is trivial to see that $\mathbf{A}(t)\mathbf{A}(\tau) = \mathbf{A}(\tau)\mathbf{A}(t)$. So we can immediately conclude that this system's transition matrix is

$$\Phi(t,\tau) = \exp\left[\int_{\tau}^{t} \mathbf{A}(s)ds\right] = \exp\left[\int_{\tau}^{t} \alpha(s)ds\mathbf{M}\right]$$

we can then calculate $\exp^{\lambda\mathbf{M}}$ using any method we want, then simply replace λ by $\int_{\tau}^{t} \alpha(s)ds$.

Let us consider the case

$$\mathbf{A}(t) = \begin{bmatrix} t & t \\ 0 & 0 \end{bmatrix} = t \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Note that

$$\int_\tau^t s ds = \frac{1}{2}(t^2 - \tau^2)$$

so we know that

$$\Phi(t,\tau) = \exp\left[\frac{1}{2}(t^2 - \tau^2)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right]$$

and we have $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

We find $e^{\lambda \mathbf{M}}$ using the Laplace transform methods. So

$$(s\mathbf{I} - \mathbf{M})^{-1} = \begin{bmatrix} s-1 & -1 \\ 0 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s(s-1)} \\ 0 & \frac{1}{s} \end{bmatrix}$$

Inverting this we get

$$e^{\mathbf{M}\lambda} = \begin{bmatrix} e^\lambda & e^\lambda - 1 \\ 0 & 1 \end{bmatrix}$$

and so we can see that

$$\Phi(t,\tau) = \begin{bmatrix} e^{\frac{1}{2}(t^2-\tau^2)} & e^{\frac{1}{2}(t^2-\tau 62)} - 1 \\ 0 & 1 \end{bmatrix}$$

Note by our earlier theorem a necessary and sufficient condition for the origin to be UAS in an LTV system is that the matrix norm of its transition matrix be bounded by an exponentially decaying function of time. This is clearly not the case for the $\Phi(t,\tau)$ we computed above. So we can conclude that this system is unstable.

Note that if we had tried to use the eigenvalue condition, we would have gotten the wrong answer. In particular the characteristic polynomial of $\mathbf{A}(t) = \begin{bmatrix} t & t \\ 0 & 0 \end{bmatrix}$ is

$$p(s) = \det \begin{bmatrix} s-t & -t \\ 0 & s \end{bmatrix} = s(s-t)$$

for fixed t, none of the eigenvalues have positive real parts, which would lead us to think the trajectories remain bounded. But as shown when we computed the transition matrix, we found this was not true. This simply emphasizes the point that eigenvalue analysis of the $\mathbf{A}(t)$ matrix cannot be used to deduce whether or not the origin is stable, let alone uniformly stable.

Nonetheless, we can use it as a condition that implies the existence of a Lyapunov function for the system and thereby obtain a *converse theorem* for LTV systems in which the existence of a Lyapunov function is necessary and sufficient for the uniform exponential stability of the LTV system. This result is stated and proven in the following theorem.

theorem 3. *Let $x = 0$ be the uniformly exponentially stable equilibrium of $\dot{x}(t) = \mathbf{A}(t)x(t)$ where $\mathbf{A}(t)$ is continuous and bounded. Let $\mathbf{Q}(t)$ be continuous bounded symmetric positive definite matrix function of time. Then there is a continuously differentiable bounded positive definite symmetric matrix function, $\mathbf{P}(t)$, that satisfies the matrix differential equation*

$$-\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{P}(t) + \mathbf{Q}(t)$$

and so $V(t, x) = x^T\mathbf{P}(t)x$ is a Lyapunov function for this system.

Proof: This theorem is proven by making an intelligent guess for $\mathbf{P}(t)$. So let

$$\mathbf{P}(t) = \int_t^\infty \mathbf{\Phi}^T(\tau, t)\mathbf{Q}(\tau)\mathbf{\Phi}(\tau, t)d\tau$$

and let $\phi(\tau; t, x) = \mathbf{\Phi}(\tau, t)x$ denote the state trajectory with initial condition at time t being x. With this notational convention, we can write

$$x^T\mathbf{P}(t)x = \int_t^\infty \phi^T(\tau; t, x)\mathbf{Q}(\tau)\phi(\tau; t, x)d\tau$$

So by the preceding theorem we know that if the equilibrium is uniformly exponentially stable (UAS), then there exist $k > 0$ and $\lambda > 0$ such that

$$\|\mathbf{\Phi}(\tau, t)\| \leq ke^{-\lambda(\tau - t)}$$

Since we assumed $\mathbf{Q}(t)$ is bounded and positive definite, there are constants c_3 and c_4 such that $c_3\mathbf{I} \leq \mathbf{Q}(t) \leq c_4\mathbf{I}$ and we can then say

$$\begin{aligned} x^T\mathbf{P}(t)x &= \int_t^\infty \phi^T(\tau;t,x)\mathbf{Q}(\tau)\phi(\tau;t,x)d\tau \\ &= \int_t^\infty x^T\mathbf{\Phi}^T(\tau,t)\mathbf{Q}(\tau)\mathbf{\Phi}(\tau,t)xd\tau \\ &\leq \int_t^\infty c_4 \left\|\mathbf{\Phi}(\tau,t)\right\|^2 |x|^2 d\tau \\ &\leq \int_t^\infty k^2 e^{-2\lambda(\tau-t)} d\tau c_4 |x|^2 \\ &= \frac{k^2 c_4}{2\lambda}|x|^2 \end{aligned}$$

So we take $\overline{W}(x) = \dfrac{k^2c_4}{2\lambda}|x|^2$.

On the other hand since there exists $L > 0$ such that $\|\mathbf{A}(t)\| \leq L$ for all time, we can bound the solution from below by

$$|\phi(\tau;t,x)|^2 \geq |x|^2 e^{-2L(\tau-t)}$$

and so

$$\begin{aligned} x^T\mathbf{P}(t)x &\geq \int_t^\infty c_3|\phi(\tau;t,x)|^2 d\tau \\ &\geq \int_t^\infty e^{-2L(\tau-t)} d\tau c_3|x|^2 \\ &= \frac{c_3}{2L}|x|^2 \end{aligned}$$

So we take $\underline{W}(x) = \dfrac{c_3}{2L}|x|^2$ and what we've established is that

$$\underline{W}(x) = \frac{c_3}{2L}|x|^2 \leq x^T\mathbf{P}(t)x \leq \frac{k^2c_4}{2\lambda}|x|^2 = \overline{W}(x)$$

which establishes the first condition we need for a Lyapunov function.

We now show that the derivative property holds. We use the fact that

$$\frac{\partial}{\partial t}\mathbf{\Phi}(\tau,t) = -\mathbf{\Phi}(\tau,t)\mathbf{A}(t)$$

to show that

$$\begin{aligned}
\dot{\mathbf{P}}(t) &= \int_t^\infty \mathbf{\Phi}^T(\tau,t)\mathbf{Q}(\tau)\frac{\partial}{\partial t}\mathbf{\Phi}(\tau,t)d\tau \\
&\quad + \int_t^\infty \left[\frac{\partial}{\partial t}\mathbf{\Phi}^T(\tau,t)\right]\mathbf{Q}(\tau)\mathbf{\Phi}(\tau,t)d\tau - \mathbf{Q}(t) \\
&= -\int_t^\infty \mathbf{\Phi}^T(\tau,t)\mathbf{Q}(\tau)\mathbf{\Phi}(\tau,t)\mathbf{A}(t) \\
&\quad -\mathbf{A}^T(t)\int_t^\infty \mathbf{\Phi}^T(\tau,t)\mathbf{Q}(\tau)\mathbf{\Phi}(\tau,t)d\tau - \mathbf{Q}(t) \\
&= -\mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}^T(t)\mathbf{P}(t) - \mathbf{Q}(t)
\end{aligned}$$

which establishes that $V(t,x) = x^T\mathbf{P}(t)x$ is a Lyapunov function. $\diamondsuit$

3. $\mathcal{L}_p$ STABILITY:

Lyapunov stability is a property of the equilibrium for a system that is not being forced by an unknown exogenous input. For systems with exogenous inputs, the state equation takes the form,

$$\dot{x}(t) = f(x(t), w(t))$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is now a function of the state $x : \mathbb{R} \to \mathbb{R}^n$ and an applied input signal $w : \mathbb{R} \to \mathbb{R}^m$. One can think of w as a *disturbance*. The disturbance is a signal that we don't know and that fluctuates about a bias or trend line. If $0 = f(0, w(t))$ for any w, then the origin is an equilibrium point for the forced system and one can examine the Lyapunov stability of that equilibrium. In general, however, the disturbance is *non-vanishing* in the sense that $f(0,w) \neq 0$ and this means that the origin will not be an equilibrium point and so the Lyapunov stability concept cannot be used.

For systems with non-vanishing perturbations, one often uses a *stability concept* that focuses on the input/output behavior of the system. In particular, we say a forced system is *input/output stable* if all bounded inputs to the system result in a bounded response. "Bounded", in this case, means that signals have finite norms and so the system is viewed as a linear transformation between two normed linear signal spaces. A stable system is then a linear transformation, $\mathbf{G} : \mathcal{L}_{\text{in}} \to \mathcal{L}_{\text{out}}$, that takes any signal, $w \in \mathcal{L}_{\text{in}}$, such that $\|w\|_{\mathcal{L}_{\text{in}}} < \infty$ onto an output signal $\mathbf{G}[w] \in \mathcal{L}_{\text{out}}$ such that $\|\mathbf{G}[w]\|_{\mathcal{L}_{\text{out}}}$ is also finite. It is

customary to consider signals that are linear transformations between two $\mathcal{L}_p$ spaces[1] because these spaces are Banach spaces. Such systems are said to be $\mathcal{L}_p$ stable. The purpose of this section is to formalize the $\mathcal{L}_p$ stability concept and discuss the special case when $p = 2$.

$\mathcal{L}_p$-stability is defined for systems, $\mathbf{G} : \mathcal{L}_{\text{pe}} \to \mathcal{L}_{\text{pe}}$, that are linear transformations between two *extended* $\mathcal{L}_p$ spaces. In particular, $\mathcal{L}_{\text{pe}}$ is the space of all functions, w, such that the truncation of w for any finite T

$$w_T(t) = \begin{cases} w(t) & \text{for } t \leq T \\ 0 & \text{otherwise} \end{cases}$$

is in $\mathcal{L}_p$. We say this space is "extended" because it contains all signals in $\mathcal{L}_p$ as well as unbounded signals whose truncations are bounded. For such systems we say $\mathbf{G}$ is $\mathcal{L}_p$ *stable* if and only if there exists a class $\mathcal{K}$ function[2], $\alpha : [0, \infty) \to [0, \infty)$, and a non-negative constant, β, such that

$$\|\mathbf{G}[w]_T\|_{\mathcal{L}_p} \leq \alpha(\|w_T\|_{\mathcal{L}_p}) + \beta$$

for all $w \in \mathcal{L}_{\text{pe}}$ and $T \geq 0$. The constant β is called a *bias*.

Lyapunov analysis is often just concerned with declaring whether or not the equilibrium is stable. But for $\mathcal{L}_p$ stability, one talks about how well the "disturbance", w, is attenuated at the system's output and this degree of attenuation is characterized through the concept of the system's *gain*. In particular, we say that $\mathbf{G} : \mathcal{L}_{\text{pe}} \to \mathcal{L}_{\text{pe}}$ is *finite-gain* $\mathcal{L}_p$ *stable* if there exist $\gamma > 0$ such that

$$\|\mathbf{G}[w]_T\|_{\mathcal{L}_p} \leq \gamma \|w_T\|_{\mathcal{L}_p} + \beta$$

This constant γ is called a *gain*.

Note that this gain depends on the w that was chosen. Moreover, if there is any other $\gamma_1 > \gamma$, then γ_1 is also a gain. We are therefore interested in defining the gain as a property of the *system* and in a way that is unique. This is done by taking the infimum over all

[1]The $\mathcal{L}_p$ norm of a integrable function $x : \mathbb{R} \to \mathbb{R}$ is defined as $\|x\|_{\mathcal{L}_p} := \left(\int_{-\infty}^{\infty} |x(s)|^p ds \right)^{1/p}$ where p is a positive integer. We've already introduced examples of this norm when $p = 1, 2$, or ∞. The $\mathcal{L}_p$ space consists of all functions with finite $\mathcal{L}_p$ norms.

[2]A function $\alpha : [0, a) \to [0, \infty)$ is class $\mathcal{K}$ if and only if it is a continuous and increasing with $\alpha(0) = 0$.

inputs, w, of those scalars that can be gains. We call this the $\mathcal{L}_p$ *induced gain* of the system that is formally defined as

$$\|\mathbf{G}\|_{\mathcal{L}_p-\text{ind}} := \inf\left\{\gamma \,:\, \|(\mathbf{G}[w])_T\|_{\mathcal{L}_p} \leq \gamma\|w_T\|_{\mathcal{L}_p} + \beta, \quad \text{for all } w \in \mathcal{L}_p \text{ and } T \geq 0\right\}$$

Note that if the bias, β, is zero then the above gain is identical to the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ defined we discussed in earlier lectures.

The prior lectures provided an explicit formula for the $\mathcal{L}_\infty$ induced gain of an LTI system with a known impulse response function. But the $\mathcal{L}_2$-induced gain we derived for LTI systems required finding the maximum of the system's gain-magnitude function, $|\mathbf{G}(j\omega)|$. Finding this maximum can be very difficult to do if the system has many sharp resonant peaks; which is usually the case for mechanical systems with a large number of vibrational modes. So the rest of this section presents an alternative way of determining the $\mathcal{L}_2$-induced gain that is computationally efficient. The following theorem will play an important role in this approach.

theorem 4. *Suppose* $\mathbf{G} \stackrel{s}{=} \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & 0 \end{array}\right]$ *is a system realization with* $\mathbf{A}$ *being Hurwitz, then* $\|\mathbf{G}\|_{\mathcal{L}_2-\text{ind}} < \gamma$ *if and only if the matrix*

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^T \\ -\mathbf{C}^T\mathbf{C} & -\mathbf{A}^T \end{bmatrix}$$

has no eigenvalues on the $j\omega$*-axis.*

Proof: Let $\mathbf{\Phi}(s) = \gamma^2\mathbf{I} - \mathbf{G}^*(s)\mathbf{G}(s)$. It should be apparent that because $\|\mathbf{G}\|_{\mathcal{L}_2-\text{ind}} = \max_\omega |\mathbf{G}(j\omega)|$ then $\|\mathbf{G}\|_{\mathcal{L}_2-\text{ind}} < \gamma$ if and only if $\mathbf{\Phi}(j\omega) > 0$ for all $\omega \in \mathbb{R}$. This means that $\mathbf{\Phi}(s)$ has no zeros on the imaginary axis, or rather than $\mathbf{\Phi}^{-1}(s)$ has no poles on the imaginary axis.

One can readily verify that a state space realization for $\mathbf{\Phi}^{-1}(s)$ is

$$\mathbf{\Phi}^{-1}(s) \stackrel{s}{=} \left[\begin{array}{c|c} \mathbf{H} & \begin{bmatrix} \frac{1}{\gamma^2}\mathbf{B} \\ \mathbf{0} \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{0} & \frac{1}{\gamma^2}\mathbf{B}^T \end{bmatrix} & \frac{1}{\gamma^2} \end{array}\right]$$

If $\mathbf{H}$ has no imaginary eigenvalues, then clearly $\mathbf{\Phi}^{-1}(s)$ has no imaginary poles.

So let $j\omega_0$ be an eigenvalue of $\mathbf{H}$. This means there is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$ such that

$$\begin{aligned} 0 &= (j\omega_0\mathbf{I} - \mathbf{H})x \\ &= \begin{bmatrix} j\omega_0\mathbf{I} - \mathbf{A} & -\frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^T \\ \mathbf{C}^T\mathbf{C} & j\omega_0\mathbf{I} + \mathbf{A}^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

which means

$$(j\omega_0\mathbf{I} - \mathbf{A})x_1 = \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^T x_2 \tag{4}$$

$$(j\omega_0\mathbf{I} + \mathbf{A}^T)x_2 = -\mathbf{C}^T\mathbf{C}x_1 \tag{5}$$

The mode associated with this eigenvalue will be $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{j\omega_0 t}$. For this mode not to appear in the system's output, we would require for all t that

$$0 = \begin{bmatrix} 0 & \frac{1}{\gamma^2}\mathbf{B}^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{j\omega_0 t} = \frac{1}{\gamma^2}\mathbf{B}^T x_2 e^{m\omega_0 t}$$

This can only occur in if $\mathbf{B}^T x_2 = 0$, which when we insert this into equations (4-5) gives

$$\begin{aligned} (j\omega_0\mathbf{I} - \mathbf{A})x_1 &= 0 \\ (j\omega_0\mathbf{I} + \mathbf{A}^T)x_2 &= -\mathbf{C}^T\mathbf{C}x_1 \end{aligned}$$

The first equation implies $x_1 = 0$ since $\mathbf{A}$ is Hurwitz, and inserting this into the second equation implies $x_2 = 0$. So we've shown that if the $j\omega_0$-mode is activated and it does not appear at the output, then $x = 0$, which cannot happen. So in this case $j\omega_0$ cannot be an eigenvalue of $\mathbf{H}$. A similar case can be made if we require that the applied input not excite those x_2 components in the $j\omega_0$-mode. $\diamondsuit$

The preceding theorem is useful because it uses an eigenvalue test to certify whether $\|\mathbf{G}\|_{\mathcal{L}_2-\text{ind}} < \gamma$. We refer to this test as an algorithmic *oracle* since for a given state space realization it can easily declare whether or not a given γ is an upper bound on the induced gain. In particular, we use this oracle as the basis of a binary or bisection search to efficiently search from minimum γ. This bisection algorithm is stated below.

(1) Select an upper and lower bound, γ_u and γ_ℓ, respectively, such that

$$\gamma_\ell \leq \|\mathbf{G}\|_{\mathcal{L}_2-\text{ind}} \leq \gamma_u$$

(2) If $\dfrac{\gamma_u - \gamma_\ell}{\gamma_\ell} < \epsilon$ where ϵ is a specified error tolerance, then STOP and declare $\dfrac{\gamma_u + \gamma_\ell}{2}$ as the induced gain. Otherwise continue

(3) Set $\gamma = \dfrac{\gamma_u + \gamma_\ell}{2}$

(4) Form the Hamiltonian matrix $\mathbf{H}$ and compute its eigenvalues

- if no eigenvalues are purely imaginary, then set $\gamma_\ell = \gamma$ and go to step 2.
- If any eigenvalue is imaginary then set $\gamma_u = \gamma$ and go to step 2.

This algorithm works in a very simple manner. It assumes we already know that the induced gain is bounded between the two initial guesses, γ_ℓ and γ_u. The value of γ that we check is chosen halfway between γ_ℓ and γ_u, thereby dividing the region of uncertainty in half. Let γ_0 denote this initial guess. The eigenvalue test tells us whether the initial guess is or is not an upper bound on the induced gain. If it is an upper bound, then we know $\gamma_\ell < \|\mathbf{G}\|_{\mathcal{L}_2-\text{ind}} < \gamma_0$ and we can reset the upper bound to $\gamma_u = \gamma_0$. If the eigenvalue test tells us γ_0 is not an upper bound on the induced gain, then we know $\gamma_0 < \|\mathbf{G}\|_{\mathcal{L}_2-\text{ind}} < \gamma_u$ and we can reset the lower bound to $\gamma_\ell = \gamma_0$. With this reset, the interval of uncertainty around the induced gain, $[\gamma_\ell, \gamma_u]$ is cut in half. We then repeat this game with the smaller uncertainty interval. What this algorithm guarantees is that we will determine the induced gain to an accuracy of $\dfrac{\gamma_u - \gamma_\ell}{2^n}$ after n recursions. So for a specified tolerance level, ϵ, we can actually determine how many recursions are needed to complete the search.

Example: Consider the LTI system

$$\mathbf{G} \stackrel{s}{=} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -\omega_n^2 & -0.1 & 1 \\ \hline 1 & 0 & 0 \end{array}\right]$$

Determine the system's induced $\mathcal{L}_2$ gain.

We know the transfer function for this system is $\mathbf{G}(s) = \frac{1}{s^2+0.1s+\omega_n^2}$ and so

$$|\mathbf{G}(j\omega)|^2 = \frac{1}{(\omega_n^2 - \omega^2)^2 + 0.01\omega^2}$$

Computing the first derivative and setting it equal to zero yields,

$$2(\omega_n^2 - \omega) = 0.01 = 0$$

The solution, ω_0, is the peak in the gain magnitude function and satisfies the quadratic equation

$$\omega_0^2 + \omega_n^2 - \frac{0.01}{2}$$

which has a positive solution positive for $\omega_n^2 > 0.01/2$. For $\omega_n^2 \leq 0.01/2$, the function is monotone decreasing which means the peak occurs for $\omega_0 = 0$. We therefore see that

$$\|\mathbf{G}\|_{\mathcal{L}_2-\text{ind}} = |\mathbf{G}(j\omega_0)|^2 = \begin{cases} \frac{1}{(0.01/2)^2+(0.01)(0.01/2+\omega_n^2} \approx \frac{100}{\omega_n^2} & \text{for } \omega_n > 0.01/2 \\ \frac{1}{\omega_n^2} & \text{for } \omega_n^2 \leq 0.01/2 \end{cases}$$

Example: Consider a system with the following state space realization

$$\mathbf{G} \stackrel{s}{=} \left[\begin{array}{cccccccc|c} -0.016 & 16.19 & 0 & 0 & 0 & 0 & 0 & 0 & 1.30 \\ -16.19 & -0.162 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 \\ 0 & 0 & -0.01058 & 10.58 & 0 & 0 & 0 & 0 & 1.1 \\ 0 & 0 & -10.58 & -0.0106 & 0 & 0 & 0 & 0 & -0.09 \\ 0 & 0 & 0 & 0 & -0.004 & 3.94 & 0 & 0 & -0.22 \\ 0 & 0 & 0 & 0 & -3.94 & -0.004 & 0 & 0 & -0.0019 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.00056 & 0.568 & 0.074 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.568 & -0.0006 & 10^{-4} \\ \hline 2\times 10^{-5} & -0.0025 & -0.0075 & -0.095 & -0.00032 & 0.0366 & 0.00016 & -0.127 & 0 \end{array}\right]$$

Plot the system's frequency response function at 100 equally spaced points between 0.1 to 100 rad/sec. Use that plot to estimate the $\mathcal{L}_2$ induced gain of the system. Then use the preceding bisection algorithm to estimate the induced gain.

The frequency response plotted using MATLAB is shown in Fig. 1. This plots shows points at 100 sample points between 0.1 and 100 rad/sec (blue asterisks) as well as a more finely sampled plot with $10,000$ sample points (shown in solid red line). The peak found by the coarse sampling was 1.0822 (0.3671 dB). So one would estimate the induced gain to be 1.0822. The peak found by the more finely sampled plot was 4.8912 (13.7883 dB). If we had used the preceding bisection algorithm, we would have found the actual peak to be 8.2664 (18.3463 dB). The induced gain estimated from the gain-magnitude plot depended greatly on how finely we sampled the plot. The problem here was that we have no way of relating the sampling interval to the error in our estimate of the induced gain. The bisection algorithm, on the other hand, does provide a relationship between the number of recursions and the accuracy of the estimate. In particular, one can guarantee that with an initial guess between 0 and 10 one would only need 10 recursions to get within 0.01 of the actual gain.

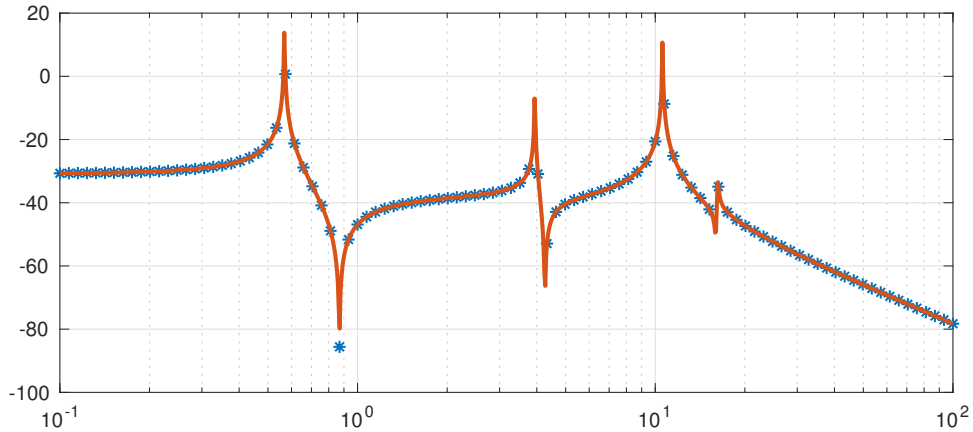

FIGURE 1. Sampled Gain magnitude being used to estimate L2 induced gain

4. $\mathcal{L}_p$ STABILITY AND LYAPUNOV STABILITY

Since Lyapunov stability is such an important concept, it is useful to characterize the relationship between $\mathcal{L}_p$ stability and Lyapunov stability. This is done in the following theorem

theorem 5. *Consider the input-output system $\dot{x}(t) = f(x(t), w(t))$ and $y(t) = h(x(t), w(t))$ where the origin is an exponentially stable equilibrium of $\dot{x}(t) = f(x(t), 0)$. Assume there exist positive constants L, r, r_w, η_1, and η_2 such that*

$$\begin{aligned} |f(x,w) - f(x,0)| &\leq L|w| \\ |h(x,w)| &\leq \eta_1|x| + \eta_2|w| \end{aligned}$$

for all $|x| < r$ and $|w| < r_w$. If there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ and non-negative constants c_1, c_2, c_3, and c_4 such that

$$\begin{aligned} c_1|x|^2 &\leq V(x) \leq c_2|x|^2 \\ \dot{V}(x,0) &\leq -c_3|x|^2 \\ \left|\frac{\partial V}{\partial x}\right| &\leq c_4|x| \end{aligned}$$

then the system is finite gain $\mathcal{L}_p$-stable.

Proof: Consider $\dot{V}$ along trajectories of the forced system

$$\dot{V} = \frac{\partial V}{\partial x} f(x,0) + \frac{\partial V}{\partial x}[f(x,w) - f(x,0)]$$

with the given bounds and the Lipschitz constant, L, for f we get

$$\dot{V} \leq -c_3|x|^2 + c_4 L|x||w|$$

Take $W(t) = \sqrt{V(x(t))}$ and note that

$$\dot{W} = \frac{\dot{V}}{2\sqrt{V}}$$

We obtain the following differential inequality

$$\dot{W} \leq -\frac{1}{2}\frac{c_3}{c_2}W + \frac{c_4 L}{2\sqrt{c_1}}|w(t)|$$

By the *comparison principle* we can therefore conclude that

$$W(t) \leq e^{-\frac{c_3}{2c_2}t}W(0) + \frac{c_4 L}{2\sqrt{c_1}}\int_0^t e^{-(t-s)\frac{c_3}{2c_2}}|w(s)|ds$$

which implies that

$$|x(t)| \leq \frac{c_2}{c_1}|x_0|e^{-\frac{c_3}{2c_2}t} + \frac{c_4 L}{2c_1}\int_0^t e^{-(t-s)\frac{c_3}{2c_2}}|w(s)|ds$$

and so we can conclude that if $|x_0| \leq \frac{r}{2}\sqrt{\frac{c_1}{c_2}}$ and $\|w\|_{\mathcal{L}_\infty} \leq \frac{c_1 c_3 r}{2c_2 c_4 L}$, then $|x(t)| \leq r$ for all time.

This means that the bound on h holds for all time and so

$$|y(t)| \leq k_1 e^{-at} + k_2\int_0^t e^{-a(t-s)}|w(s)|ds + k_3|w(t)|$$

where $k_1 = \sqrt{\frac{c_1}{c_2}}|x_0|\eta_1$, $k_2 = \frac{c_4 L\eta_1}{2c_1}$, $k_3 = \eta_2$, and $a = \frac{c_3}{2c_2}$. Assign to each of these terms in the above equation a signal y_1, y_2, and y_3. If $w \in \mathcal{L}_{\text{pe}}$ with $\|w\|_{\mathcal{L}_\infty}$ sufficiently small then for any $T > 0$

$$\|y_{2T}\|_{\mathcal{L}_p} \leq \frac{k_2}{a}\|w_T\|_{\mathcal{L}_p}, \quad \|y_{3T}\|_{\mathcal{L}_p} \leq k_3\|w_T\|_{\mathcal{L}_p}, \quad \|y_{1T}\|_{\mathcal{L}_p} \leq k_1\rho$$

where $\rho = \begin{cases} 1 & \text{if } p = \infty \\ (1/ap)^{1/p} & \text{otherwise} \end{cases}$. So that since

$$\|y_T\|_{\mathcal{L}_p} \leq \|y_{1T}\|_{\mathcal{L}_p} + \|y_{2T}\|_{\mathcal{L}_p} + \|y_{3T}\|_{\mathcal{L}_p}$$

we can use the above bounds to conclude

$$\begin{aligned}\|y_T\|_{\mathcal{L}_p} &\leq k_1\rho + \frac{k_2}{a}\|w_T\|_{\mathcal{L}_p} + k_3\|w_T\|_{\mathcal{L}_p} \\ &= \left(\frac{k_2}{a} + k_3\right)\|w_T\|_{\mathcal{L}_p} + k_1\rho = \gamma\|w_T\|_{\mathcal{L}_p} + \beta\end{aligned}$$

which identify the finite gain and bias for this system. $\diamondsuit$

For linear time-invariant systems, since the existence of a Lyapunov function is necessary and sufficient for asymptotic stability of the state, then it is much easier to see that the system will be $\mathcal{L}_p$ stable. The converse, however is not true. Let us consider,

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}w(t) \\ y(t) &= \mathbf{C}x(t)\end{aligned}$$

If the origin is asymptotically stable, then it is exponentially stable and so there exist $K > 0$ and $\gamma > 0$ such that

$$e^{\mathbf{A}t}x_0 \leq Ke^{-\gamma t}$$

for $t \geq 0$. Since the impulse response is $\mathbf{C}e^{\mathbf{A}t}\mathbf{B}u(t)$, it is easy to see that this must be integrable and so if the state-based system is exponentially stable, it must also be $\mathcal{L}_p$ stable.

The converse may not be true as is seen in the following example. Let us consider the state-based system,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This system is clearly unstable since the $\mathbf{A}$ matrix has one eigenvalue with a positive real part. This system however is $\mathcal{L}_p$ stable since its transfer function is $\frac{1}{s+1}$ which only has a single pole on the left hand side of the complex plane. The reason why this system is $\mathcal{L}_p$ stable but not asymptotically stable is because the unstable mode of the state-based system is not observed at the system's output. Moreover, it is not influenced by the input either. State-based models that have this property are said to be *uncontrollable* and *unobservable*.

Informally, we define these concepts as follows

- Controllability and Reachability mean that there exists a *finite* duration input that can drive the system state to any state in $\mathbf{R}^n$.

- Observability and Constructibility mean that we can deduce the system's initial state, $x(0)$, given a finite duration set of inputs and outputs.

The main thing to notice here is that controllability and observability are *finite* time concepts whereas stability characterizes the system's behavior over an *infinite* time interval. The following week's lectures provide a detailed discussion on the controllability and observability of linear dynamical systems.