Solutions of State Equations

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week 5

Linear Homogeneous Dynamical System

- Consider a linear homogeneous (LH) system whose state trajectory $x : \mathbb{R} \rightarrow \mathbb{R}^n$ satisfies the IVP

$$\dot{x}(t) = \mathbf{A}(t)x(t), \qquad x(t_0) = x_0$$

  for $t \geq t_0$.
- All solutions of the LH system form an n-dimensional linear space.
- Let V denote the set of all solutions to LF over $[t_0, T]$. If $\phi_1, \phi_2 \in V$, then

$$\begin{aligned} \frac{d}{dt}(\alpha_1\phi_1(t) + \alpha_2\phi_2(t)) &= \alpha_q\mathbf{A}(t)\phi_1(t) + \alpha_2\mathbf{A}(t)\phi_2(t) \\ &= \mathbf{A}(t)\left[\alpha_1\phi_1(t) + \alpha_2\phi_2(t)\right] \end{aligned}$$

  So V is *closed* with respect to addition/dilation and must be a linear space.

Solutions of LH Systems

- To show V is n-dimensional, we first need to find a basis.
- Choose n linearly independent vectors, $\{x_{i0}\}_{i=1}^{n}$ that span $\mathbb{R}^n$. Let $\{\phi_i\}_{i=1}^{n}$ denote n solutions to the LH using initial conditions $\phi_i(t_0) = x_{i0}$.
- Assume solutions are not linearly independent, then $\sum_{i=1}^{n} \alpha_i \phi_i(t) = 0$ where not α_i are zero.
- This holds at t_0, so $\sum_{i=1}^{n} \alpha_i x_{i0} = 0$, which contradicts the linear independence assumption so $\mathrm{span}\{\phi_1, \ldots, \phi_n\} \subset V$.
- Let $\phi \in V$ such that $x(t_0) = x_0$ and $x_0 = \sum_{i=1}^{n} \alpha_i x_{i0}$. So

  $$\phi(t) = \sum_{i=1}^{n} \alpha_i \phi_i(t) \text{ and } V \subset \mathrm{span}\{\phi_1, \ldots, \phi_n\}$$

Solutions of LH Systems

- Find all solutions to $\dot{x} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} x$.
- We know the linear space, V, of solutions has dimension 2. So we only need to find 2 linearly independent solutions.
- We can readily verify that these functions satisfy the LHS.

$$\phi_1(t) = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}, \quad \phi_2(t) = \begin{bmatrix} \frac{1}{2}e^{t} \\ e^{-t} \end{bmatrix}$$

- Since these two solutions are clearly linearly independent

$$x(t) = \alpha_1 \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} \frac{1}{2}e^{t} \\ e^{-t} \end{bmatrix}$$

Solutions to LH Systems

- A set of n linearly independent solutions of $\dot{x} = \mathbf{A}(t)x$ is called a set of *fundamental solutions*. The matrix

$$\mathbf{\Psi}(t) = \left[\begin{array}{cccc} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \end{array} \right]$$

  whose columns are fundamental solutions is called a *fundamental matrix*.

- Any fundamental matrix satisfies the matrix differential equation

$$\dot{\mathbf{\Psi}}(t) = \mathbf{A}(t)\mathbf{\Psi}(t)$$

- A solution $\mathbf{\Psi}$ of $\dot{\mathbf{\Psi}}(t) = \mathbf{A}(t)\mathbf{\Psi}(t)$ is a fundamental matrix iff $\mathbf{\Psi}(t)$ is nonsingular for all t.

Solutions of LH Systems

- Let $\mathbf{\Psi}$ be a fundamental matrix and let ϕ be any solution to the LHS. There are coefficients (no all zero) such that

$$\phi(t) = \sum_{i=1}^{n} \alpha_i \phi_i(t) = \mathbf{\Psi}(t)\overline{\alpha}$$

- For any time t, the LAE $\phi(t) = \mathbf{\Psi}(t)\overline{\alpha}$ only has a unique solution if $\mathbf{\Psi}(t)$ is nonsingular for all t.
- Conversely, assuming $\mathbf{\Psi}$ satisfies the matrix differential equation and is nonsingular for all t. This means $\det(\mathbf{\Psi}(t)) \neq 0$ for all t and so the columns of $\mathbf{\Psi}$ are linearly independent. This means $\mathbf{\Psi}$ is a fundamental matrix.

State Transition Matrix

- The fundamental matrices for an LH problem are not unique.
- We would like a "unique" way to characterize the LH problem's solution. This is done through the idea of a *transition matrix*, $\mathbf{\Phi}$.
- In particular, we denote the state transition matrix over $[t_0, t]$ as $\mathbf{\Phi}(t; t_0)$ and define it as the fundamental matrix whose ith column is a solution to the LHS with initial condition $x(t_0) = e_i$.
- Note that $\mathbf{\Phi}(t; t_0) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)$ where $\mathbf{\Psi}(t)$ is *any* fundamental matrix of the LHS.
- If $\mathbf{T}$ is a nonsingular matrix and we let $\mathbf{\Psi}_1 = \mathbf{\Psi}_2$, then $\mathbf{\Psi}_2$ is aso a fundmental matrix.
- We also have

$$
\begin{aligned}
\mathbf{\Phi}(t; t_0) &= \mathbf{\Psi}_1(t)\mathbf{\Psi}_1^{-1}(t_0) = \mathbf{\Psi}_2\mathbf{T}\mathbf{T}^{-1}\mathbf{\Psi}_2(t_0) \\
&= \mathbf{\Psi}_2(t)\mathbf{\Psi}_2^{-1}(t_0)
\end{aligned}
$$

Transition Matrices

- $\mathbf{\Phi}(t; t_0)$ is unique solution to the matrix differential equation

$$\frac{\partial}{\partial t}\mathbf{\Phi}(t; t_0) = \mathbf{A}(t)\mathbf{\Phi}(t; t_0), \quad \mathbf{\Phi}(t_0; t_0) = \mathbf{I}$$

Proof:

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{\Phi}(t; t_0) &= \dot{\mathbf{\Psi}}(t)\mathbf{\Psi}^{-1}(t_0) \\ &= \mathbf{A}(t)\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0) = \mathbf{A}(t)\mathbf{\Phi}(t; t_0) \end{aligned}$$

- For all $t, \tau, \sigma \in \mathbb{R}$ we have

$$\mathbf{\Phi}(t; \tau) = \mathbf{\Phi}(t; \sigma)\mathbf{\Phi}(\sigma; \tau)$$

Proof: Note that

$$\begin{aligned} \mathbf{\Phi}(t; \tau) &= \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(\tau) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(\sigma)\mathbf{\Psi}(\sigma)\mathbf{\Psi}^{-1}(\tau) \\ &= = \mathbf{\Phi}(t; \sigma) = \mathbf{\Phi}(\sigma; \tau) \end{aligned}$$

Transition Matrices

- $\mathbf{\Phi}(t;t_0)$ is nonsingular for all t, t_0 and $[\mathbf{\Phi}(t;t_0)]^{-1} = \mathbf{\Phi}(t_0;t)$.

  Since $\det(\mathbf{\Psi}(t)) \neq 0$ for any t we have

  $$\det(\mathbf{\Phi}(t;t_0)) = \det(\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)) = \det\mathbf{\Psi}(t) \times \det\mathbf{\Psi}^{-1}(t_0) \neq 0$$

  Also note that

  $$[\mathbf{\Phi}(t;t_0)]^{-1} = \left[\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\right]^{-1} = \mathbf{\Psi}(t_0)\mathbf{\Phi}^{-1}(t)$$

- The unique solution $x(t;t_0,x_0)$ to LHS is

  $$x(t;t_0,x_0) = \mathbf{\Phi}(t;t_0)x_0$$

Solutions of Inhomogeneous Systems

- Consider solutions to *inhomogeneous problem*

$$\dot{x}(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t), \quad x(t_0) = x_0$$

  where u is known input.

- We claim the solution is

$$x(t; t_0, x_0) = \mathbf{\Phi}(t; t_0)x_0 + \int_{t_0}^{t} \mathbf{\Phi}(t; \tau)\mathbf{B}(\tau)u(\tau)d\tau$$

  where $\mathbf{\Phi}$ is the state transition matrix for the LH system. This is readily verified by substituting back into ODE.

Solutions to Linear Inhomogeneous Problem

- Since Transition matrix plays such a prominent role in solution to inhomogeneous system, we will discuss several approaches for finding $\mathbf{\Phi}(t; t_0)$.
- Consider

$$\dot{x} = \begin{bmatrix} 2 & t \\ 0 & 2 \end{bmatrix} x$$

- Note that $\dot{x}_2 = 2x_2$ implies $x_2(t) = e^{2t}x_{20}$.
- Note that $\dot{x}_1 = 2x_2 + te^{2t}x_{20}$ is the first ODE and we can use our prior formula to get

$$\begin{aligned} x_1(t) &= e^{2t}x_{10} + \int_0^t \tau e^{2t} x_{20} d\tau \\ &= e^{2t}x_{10} + x_{20}\frac{t^2}{2}e^{2t} \end{aligned}$$

Transition Matrices

- So selection a pair of linearly independent x_0, say $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ to get fundamental matrix

$$\mathbf{\Psi}(t) = \begin{bmatrix} e^{2t} & \frac{t^2}{2} e^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

- The inverse of the fundamental matrix is

$$\mathbf{\Psi}^{-1}(t) = \frac{1}{e^{4t}} \begin{bmatrix} e^{2t} & -\frac{t^2}{2} e^{2t} \\ 0 & e^{2t} \end{bmatrix} = \begin{bmatrix} e^{-2t} & -\frac{t^2}{2} e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

- So LHS state transition matrix is

$$\mathbf{\Phi}(t;\tau) = \begin{bmatrix} e^{2(t-\tau)} & \frac{t^2-\tau^2}{2} e^{2(t-\tau)} \\ 0 & e^{2(t-\tau)} \end{bmatrix}$$

Matrix Exponential

- For homogeneous LTI systems the transition matrix can be written as the matrix exponential function $e^{\mathbf{A}t}$.

$$\begin{aligned}\mathbf{\Phi}(t;t_0) &= e^{\mathbf{A}(t-t_0)} \\ &= \mathbf{I} + \mathbf{A}(t-t_0) + \frac{1}{2!}\mathbf{A}^2(t-t_0)^2 + \cdots + \frac{1}{m!}\mathbf{A}^m(t-t_0)^m + \cdot\end{aligned}$$

- Given the ODE $\dot{x} = \mathbf{A}x$, the solution can also be written as

$$x(t) = x_0 + \int_{t_0}^{t} \mathbf{A}x(\tau)d\tau$$

- We are going to prove that the above series formula is indeed the *unique* solution to the ODE.
- Our proof of uniqueness and the solution is based on a successive approximations that form a contraction mapping over a completed normed linear signal space.

Matrix Exponential

- Consider a linear transformation $\mathbf{G} : \mathcal{L}_\infty \to \mathcal{L}_\infty$. This transformation is a *contraction mapping* if for any $x, y \in \mathcal{L}_\infty$ there exists $0 \leq \gamma < 1$ such that

$$\|\mathbf{G}[x] - \mathbf{G}[y]\}\,|_{\mathcal{L}_\infty} \leq \gamma\,|x - y\|_{\mathcal{L}_\infty}$$

- We focus on $\mathcal{L}_\infty$ because it is a complete normed linear space and we know that every *Cauchy sequence* is convergent to an $\mathcal{L}_\infty$ function. This fact is used to prove the *contraction mapping principle*
- Let X be a Banach space, let $S \subset X$ and $\mathbf{G} : S \to X$ be a *contraction mapping*, then there exists a unique element $x^* \in X$ such that $x^* = \mathbf{G}[x^*]$.

Matrix Exponential

- Recall that our LHS has solution

$$x(t) = x_0 + \int_0^t \mathbf{A}x(s)ds$$

  We view the RHS of equation as a *linear transformation* on the function x.

- So consider the linear transformation $\mathbf{G} : \mathcal{L}^\infty \to \mathcal{L}_\infty$ that takes values

$$\mathbf{G}[x](t) = x_0 + \in_0^T \mathbf{A}x(s)ds$$

  Clearly if x^* is a fixed point of this transformation (i.e. $x^* = \mathbf{G}[x^*]$) then $x^* \in \mathcal{L}_\infty$ is a solution to the LHS.

Matrix Exponential

- Consider a sequence of function $\{x_k\}_{k=1}^{\infty}$ generated by $\mathbf{G}$, (i.e., $x_{k+1} = \mathbf{G}[x_k]$).
- Assume $x_k : [0, T] \to \mathbb{R}^n$ is defined over $[0, T]$. For any $t \in [0, T]$ we have

$$
\begin{aligned}
|\mathbf{G}[x](t) - \mathbf{G}[y](t)| &= \left| \int_0^t (\mathbf{A}x(s) - \mathbf{A}y(s)) ds \right| \\
&\leq \int_0^t |\mathbf{A}(x(s) - y(s))| \, ds \\
&\leq \|\mathbf{A}\| \int_0^t |x(s) - y(s)) ds
\end{aligned}
$$

where $\|\mathbf{A}\|$ is matrix norm of $\mathbf{A}$.

Transition Matrix

- Since $t < T$ we can bound this as

$$|\mathbf{G}[x](t) - \mathbf{G}[y](t)| \leq \|\mathbf{A}\} \int_0^T |x(s) - y(s)| ds = \|\mathbf{A}\| \|x - y\|_{\mathcal{L}_\infty}$$

- This holds for all $t \in [0, T]$ and so we have

$$\|\mathbf{G}[x] - \mathbf{G}[y]\|_{\mathcal{L}_\infty} \leq T \|\mathbf{A}\| \|x - y\|_{\mathcal{L}_\infty}$$

- If we let $T < \frac{1}{\|\mathbf{A}\|}$, then $\mathbf{G}$ is a contraction mapping and the contraction mapping principle implies there exists a unique solution to the LTI ystem over the interval existence $[0, T]$.
- Note that this method can be applied to any ODE, $\dot{x} = f(x)$ to show it has unique solution provided f is Lipschitz.
- for linear systems, the interval existence can be extended to infinity by simply applying the same idea over and over again.

Matrix Exponential

We now show how the series solution to the matrix exponential is obtained.

$$
\begin{aligned}
x_1(t) &= x_0 \\
x_2(t) &= x_0 + \int_0^t \mathbf{A} x_1(s) ds = x_0 + \mathbf{A} \int_0^t x_0 ds \\
&= x_0 \left(\mathbf{I} + \mathbf{A} t \right) \\
x_3(t) &= x_0 + \int_0^t \mathbf{A} x_2(s) ds = x_0 + \mathbf{A} \int_0^t \left(\mathbf{I} + \mathbf{A} s \right) x_0 ds \\
&= x_0 \left(\mathbf{I} + \mathbf{A} t + \frac{1}{2} (\mathbf{A} t)^2 \right) \\
&\vdots \\
x_m(t) &= x_0 \left(\mathbf{I} + \mathbf{A} t + \cdots + \frac{1}{m!} (\mathbf{A} t)^m \right) \\
&\vdots
\end{aligned}
$$

Matrix Exponential

- Consider the LTI inhomogenous system

$$\begin{aligned} \dot{x} &= \mathbf{A}x + \mathbf{B}u, \quad x(0) = x_0 \\ y(t) &= \mathbf{C}x(t) \end{aligned}$$

- Using our matrix exponential and equation for y we get

$$y(t) = \mathbf{C}e^{\mathbf{A}t}x_0 + \int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

- The RHS two terms are the zero input and zero-state response of the system. The impulse response function is readily seen to be

$$g(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}u(t)$$

where u is unit step function.

Matrix Exponential

- The matrix exponential is an infinite power series

$$f(\mathbf{A}) = e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \beta_k \mathbf{A}^k t^k$$

  We want to find finite series representations for $e^{\mathbf{A}t}$ so we can actually get a closed form expression for the solution.

- We can use the division algorithm to rewrite $e^{\mathbf{A}t}$ as

$$e^{\mathbf{A}t} = p(\mathbf{A})q(\mathbf{A}) + r(\mathbf{A})$$

  where $p(\mathbf{A}) = \det(s\mathbf{I} - \mathbf{A})$ and $q(s)$ and $r(s)$ are polynomials with $r(s)$ having a degree less than $n-1$.

- $p(\mathbf{A}) = 0$ by Cayley-Hamilton so we can write

$$e^{\mathbf{A}t} = r(\mathbf{A}) = \sum_{k=0}^{n-1} \alpha_k(t)\mathbf{A}^k$$

- . We find $\alpha_k(t)$ as follows. Assume **A** has n distinct eigenvalues, $\{\lambda_i\}_{i=1}^n$.
- Since $p(\lambda_i) = 0$ we have

$$f(\lambda_i) = e^{\lambda_i t} = p(\lambda_i)q(\lambda_i) + r(\lambda_i) = r(\lambda_i) = \sum_{k=0}^{n-1} \alpha_k(t)\lambda_i^k$$

- We have n distinct λ_i, so this gives n linear equations that we can solve for $\alpha_k(t)$.
- Consider $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ with $p(s) = s^2 + 3s + 2$.
- The eigenvalues of **A** are $\lambda_1 = -2$ and $\lambda_2 = -2$. so our finite order representation is

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A}$$

Matrix Exponential

- The LAE formed from the eigenvalues are

$$\begin{aligned} e^{-t} &= \alpha_0(t) - \alpha_1(t) \\ e^{-2t} &= \alpha_0(t) - 2\alpha_1(t) \end{aligned}$$

- We can solve symbolically for α_0 and α_1

$$\begin{aligned} \alpha_0(t) &= 2e^{-t} - e^{-2t} \\ \alpha_1(t) &= e^{-t} - e^{-2t} \end{aligned}$$

- So our expression for matrix exponential is

$$\begin{aligned} e^{\mathbf{A}t} &= (2e^{-t} - e^{-2t})\mathbf{I} + (e^{-t} - e^{-2t})\mathbf{A} \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \end{aligned}$$

- See lecture notes for cases where eigenvalues not distinct or not real.

Matrix Exponential

- COmputing $e^{\mathbf{A}t}$ using a truncated series. This is exact if $\mathbf{A}^k = 0$ for some k. But in general truncating the series is not advised because the series converges very slowly.
- Most people use Laplace transforms to find $e^{\mathbf{A}t}$. Note that matrix exponential satisfies

$$\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t}, \quad e^{\mathbf{A}0} = \mathbf{I}$$

- Take bilateral Laplace transform since $e^{\mathbf{A}t}$ is defined for all t.

$$s\widehat{\mathbf{\Phi}}(s) - e^{\mathbf{A}0} = \mathbf{A}\widehat{\mathbf{\Phi}}(s)$$

- Solving for $\widehat{\mathbf{\Phi}}(s)$ gives

$$\widehat{\mathbf{\Phi}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$$

- This gives concrete representation for Laplace transforms of $e^{\mathbf{A}t}$ which we can then invert using the residue method.

Matrix Exponential

- $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ with

$$\widehat{\mathbf{\Phi}}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{s+3}{(s+2)(s+1)} & \frac{1}{(s+2)(s+1)} \\ \frac{-2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \end{bmatrix}$$

- We now expand as PFE

$$\widehat{\mathbf{\Phi}}(s) \quad = \quad \begin{bmatrix} \frac{K_{10}}{s+2} + \frac{K_{20}}{s+1} & \frac{K_{11}}{s+2} + \frac{K_{21}}{s+1} \\ \frac{K_{12}}{s+2} + \frac{K_{22}}{s+1} & \frac{K_{13}}{s+2} + \frac{K_{23}}{s+1} \end{bmatrix}$$

- Evaluate residues — a bit tedious to get

$$\begin{bmatrix} \frac{-1}{s+2} + \frac{2}{s+1} & \frac{-1}{s+2} + \frac{1}{s+1} \\ \frac{2}{s+2} + \frac{-2}{s+1} & \frac{2}{s+2} + \frac{-1}{s+1} \end{bmatrix} \Rightarrow \begin{bmatrix} -e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

Matrix Exponential

- In general if you use PFE of $(s\mathbf{I} - \mathbf{A})^{-1}$ to find $e^{\mathbf{A}t}$ we get

$$e^{\mathbf{A}t} = \sum_{i=1}^{\sigma} \sum_{k=0}^{m_i - 1} \mathbf{R}_{ik} t^k e^{\lambda_i t}$$

- The residues R_{ik} are called *modes* of the system.
- if $\mathrm{Re}(\lambda_i) < 0$, then the mode asymptotically goes to zero and we say this mode is asymptotically stable.
- This eigenvalue condition is necessary and sufficient for the asymptotic stability of the LTI system's origin.

Matrix Exponential

- You can also use similarity transformations to compute $e^{\mathbf{A}t}$.
- This is particularly useful when $\mathbf{A}$ is diagonalizable through a matrix $\mathbf{V}$ whose columns are the eigenvectors of $\mathbf{A}$.

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$$

- The transition matrix of $\mathbf{\Lambda}$ is

$$e^{\mathbf{\Lambda}t} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$$

and the transition matrix is then

$$\mathbf{\Phi}(t) = \mathbf{V}\mathrm{diag}(e^{\lambda_i t})\mathbf{V}^{-1}$$

Matrix Exponential

- In our example for $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$,so we have $\mathbf{V} = \begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix}$.
- So we get

$$\begin{aligned} \mathbf{\Phi}(t) &= \mathbf{V} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \mathbf{V}^{-1} \\ &= \begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-t} - e^{-t} \end{bmatrix} \end{aligned}$$

Discrete Time Transition Matrix

- Consider discrete-time state equations

$$\begin{aligned} x(k+1) &= \mathbf{A}(k)x(k) + \mathbf{B}(k)u(k) \\ y(k) &= \mathbf{C}(k)x(k) + \mathbf{D}(k)u(k) \end{aligned}$$

- Consider the homogeneous problem

$$x(k+1) = \mathbf{A}(k)x(k)$$

and note that

$$\begin{aligned} x(k+2) &= \mathbf{A}(k+1)x(k+1) \\ &= \mathbf{A}(k+1)\mathbf{A}(k)x(k) \\ &\vdots \\ x(k+n) &= \mathbf{A}(n-1)\mathbf{A}(n-2)\cdots\mathbf{A}(k+1)\mathbf{A}(k)x(k) \\ &= \prod_{j=k}^{n-1} \mathbf{A}(j)x(k) \end{aligned}$$

Discrete-time Transition Matrix

- This suggests the transition matrix is

$$\mathbf{\Phi}(n; k) = \prod_{j=k}^{n-1} \mathbf{A}(j), \quad n > k$$

and that $\mathbf{\Phi}(k; k) = \mathbf{I}$.

- So the solution to the homogeneous problem is

$$x(n) = \mathbf{\Phi}(n; k_0) x_{k_0} = \prod_{j=k_0}^{n-1} \mathbf{A}(j) x(k_0)$$

for all $n > k_0$.

Discrete-time Transition Matrix

- Computing closed form expression for $\mathbf{\Phi}(n; k_0)$ is difficult, We usually have to resort to inductive arguments.
- We can use $(z\mathbf{I} - \mathbf{A})^{-1}$ to find time-invariant transition matrices. here we need $|\lambda_k| < 1$ for $\mathbf{A}^k \to 0$.
- Some useful properties of discrete-time transition matrices are similar to those of continuous time. One notable exception is the group property. For discrete-time we only have a semi-group property

$$\mathbf{\Phi}(k; \ell) = \mathbf{\Phi}(k; m)\mathbf{\Phi}(m; \ell), \quad k \geq m \geq \ell$$

- The solution of the inhomogeneous problem for the time-invariant case will be

$$\begin{aligned} y(k) &= \mathbf{C}\mathbf{A}^{(k-k_0)}x(k_0) + \sum_{j=k_0}^{k-1} \mathbf{C}\mathbf{A}^{k-(j+1)}\mathbf{B}u(j) + \mathbf{D}u(k), \quad k > k_0 \\ y(k_0) &= \mathbf{C}x(k_0) + \mathbf{D}u(k_0) \end{aligned}$$