CHAPTER 1

Circuit Analysis using Laplace Transforms

Let us consider the circuit shown in Fig. 1. The voltage source is a generated by an electrical sensor that is measuring something in the physical world. The source is connected to a read-out display that is represented by the 8 Ω load resistor, $R_3$. That display shows the voltage over the load resistor. The circuit components between the source and the load resistor model the circuit connecting the sensor to the read-out display. If we know there is a given input voltage, $v_{in}(t)$, generated by the sensor, then the question is what is the voltage over the read-out display, $v_{out}(t)$, and how does that voltage compare to the input voltage?

The practical importance of this problem should be apparent. There are many applications where electrical sensors are embedded in the physical world, and one wants the voltage displayed on the readout sensor to represent what the sensor measures. The system shown in Fig. 1 is commonly used to monitor environmental processes such as a lake system. In this case a number of different sensors measure physical variables of interest to both scientists and local officials responsible for the management of the lake as a shared environmental resource. The circuitry shown in Fig. 1 represents a common part of a sensor network. One may, for instance, be interested in detecting a change in some important environmental variable such as nutrient level or water level. In the event of an extreme rain event, these variables may change abruptly, and resource managers may be very interested in detecting the change in environmental variable as quickly as possible. The question is whether the “circuitry” between the sensor and read-out distorts $v_{out}$ so it does not represent a good measure of, $v_{in}$, what the sensor actually saw. The circuitry may, for instance, introduce “artifacts” into the logged record that did not occur in the actual physical variable. The detection of the variable change may be delayed because of the device (capacitor/inductor) choices made when initially deploying the system. What we want to do is determine the relationship between $v_{out}$ and $v_{in}$ to more clearly identify what parts of the observed data record represent “circuit artifacts” and which represent actual information about the event that one needs to act on.

This type of problem is distinct from the earlier ones we considered in DC or AC circuit analysis. In DC circuit analysis, we assumed that the input voltage was constant and that all of the circuitry was purely resistive. in AC circuit analysis we assumed the input was a sinusoid and were only interested in the steady-state output sinusoid. In this case, the input voltage $v_{in}(t)$ is a general time-varying waveform and the circuit components are no longer purely resistive.

We start the analysis by first finding a set of ordinary differential equations characterizing how the voltages/currents of the circuit’s energy storing devices are related to each other. For the circuit in Fig. 1 there are 4 energy storing devices. The capacitors $C_1$ and $C_2$ whose important variable are the nodal voltages $v_1(t)$
and \( v_2 \), respectively. The inductors \( L_2 \) and \( L_3 \) are the other two energy storing devices with states \( i_2(t) \) and \( i_3(t) \), respectively. We refer to these variables as states because knowledge of these variables at an initial time \( t_0 \) and knowledge of the future input completely determines the future evolution of these variables. We usually represent this evolution in terms of ordinary differential equations (ODE).

The ODE’s characterizing these four state variables \( (v_1, v_2, i_2, \text{ and } i_3) \) are determined by applying KVL and KCL. Applying KVL around the mesh currents \( i_1(t) \), \( i_2(t) \), and \( i_3(t) \) shown in Fig. 1 yields

\[
\begin{align*}
\text{(1) } v_{\text{in}}(t) &= R_1 i_1(t) + v_1(t) \\
\text{(2) } v_1(t) &= R_2 i_1(t) + L_2 \frac{di_2}{dt} + v_2 \\
\text{(3) } v_2(t) &= L_3 \frac{di_3}{dt} + R_3 i_3(t) \\
\text{and we take introduce an output equation} \\
\text{(4) } v_{\text{out}}(t) &= R_3 i_3(t)
\end{align*}
\]

Note that the preceding equations only contain derivatives for two of the state variables \( (i_2(t) \text{ and } i_3(t)) \). We still need to obtain state equations for the other states for the capacitor voltages. These differential equations are obtained from the constitutive relationships for these capacitors,

\[
\begin{align*}
\text{(5) } i_1(t) - i_2(t) &= C_1 \frac{dv_1}{dt} \\
\text{and} \\
\text{(6) } i_2(t) - i_3(t) &= C_2 \frac{dv_2}{dt}
\end{align*}
\]

We now have equations containing all four derivatives and rewrite these equations to obtain 4 coupled first order ordinary differential equations. Rewrite equation (5) to get an expression for \( i_1(t) \) and substitute this
back into equation (1) and equation (2). We then adjoin (3) and (6) to the revised equations (1-2) to get the following set of 4 equations.

\[ v_{in}(t) = R_1 \left( C_1 \frac{dv_1}{dt} + i_2(t) \right) + v_1(t) \]

\[ v_1(t) = R_2 i_2 + L_2 \frac{di_2}{dt} + v_2 \]

\[ v_2(t) = L_3 \frac{di_3}{dt} + R_3 i_3(t) \]

\[ 0 = C_2 \frac{dv_2}{dt} - i_2(t) + i_3(t) \]

Note that this set of equations can be rewritten as a set of four coupled linear ordinary differential equations

\[ \frac{dv_1}{dt} = \frac{v_{in}(t) - i_2(t)}{R_1 C_1} - \frac{v_1(t)}{R_1 C_1} \]

\[ \frac{dv_2}{dt} = \frac{1}{C_2} i_2(t) - \frac{1}{C_2} i_3(t) \]

\[ \frac{di_2}{dt} = \frac{1}{L_2} v_1 - \frac{1}{L_2} v_2 - \frac{R_2}{L_2} i_2 \]

\[ \frac{di_3}{dt} = \frac{1}{L_3} v_2(t) - \frac{R_3}{L_3} i_3(t) \]

We can rewrite this in matrix-vector form

\[
\begin{bmatrix}
  v_1 \\
  v_2 \\
  i_2 \\
  i_3
\end{bmatrix}
= 
\begin{bmatrix}
  -\frac{1}{R_1 C_1} & 0 & -\frac{1}{C_1} & 0 \\
  0 & 0 & \frac{1}{C_1} & \frac{1}{C_2} \\
  \frac{1}{L_2} & -\frac{1}{L_2} & -\frac{R_2}{L_2} & 0 \\
  0 & \frac{1}{L_3} & 0 & -\frac{R_3}{L_3}
\end{bmatrix}
\begin{bmatrix}
  v_1(t) \\
  v_2(t) \\
  i_2(t) \\
  i_3(t)
\end{bmatrix}
+ 
\begin{bmatrix}
  \frac{1}{R_1 C_1} \\
  0 \\
  0 \\
  0
\end{bmatrix}
\begin{bmatrix}
  v_{in}(t)
\end{bmatrix}
\]

We can write this as a matrix differential equation and adjoin the output equation (4) to get

\[
\frac{d}{dt} x = Ax + B v_{in}(t)
\]

\[ v_{out}(t) = C x \]

with

\[
x = \begin{bmatrix}
  v_1(t) \\
  v_2(t) \\
  i_2(t) \\
  i_3(t)
\end{bmatrix}, \quad A = \begin{bmatrix}
  -\frac{1}{R_1 C_1} & 0 & -\frac{1}{C_1} & 0 \\
  0 & 0 & \frac{1}{C_1} & \frac{1}{C_2} \\
  \frac{1}{L_2} & -\frac{1}{L_2} & -\frac{R_2}{L_2} & 0 \\
  0 & \frac{1}{L_3} & 0 & -\frac{R_3}{L_3}
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
  \frac{1}{R_1 C_1} \\
  0 \\
  0 \\
  0
\end{bmatrix}, \quad C = \begin{bmatrix}
  0 & 0 & R_3
\end{bmatrix}
\]

So what have we done? We’ve shown that the voltages across the two capacitors, \( v_1 \) and \( v_2 \), and the currents, \( i_2 \) and \( i_3 \), through the two inductors satisfy a set of first-order ordinary differential equations and we’ve shown that the output voltage is a linear combination of these four variables. This means that if we can find a way to
easily solve this system of ordinary differential equations, then we can find the response, $v_{\text{out}}(t)$, of the circuit to any arbitrary input voltage $v_{\text{in}}(t)$. We are no longer confined to DC voltages or sinusoidal AC voltages. In the following we will introduce two approaches for solving this system of ordinary differential equations. The first approach is precisely what we studied above, namely, the use of Laplace transform methods. The second approach is to numerically integrate the differential equations. This second approach constitutes a digital simulation of the circuit. In the analysis of the preceding circuit we will use both methods as part of a systematic recursive framework that is frequently used in designing large-scale and complex circuits.

1. Solving the Problem using Laplace Transform Methods

From our earlier work, we know that we can solve linear constant coefficient differential equations using Laplace transform methods. We can do this for the circuit in Fig. 1 using the differential equations we derived in the preceding section. In particular, if we define the time-varying vector $x(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ i_2(t) \\ i_3(t) \end{bmatrix}$. We can then take the Laplace transform of this vector as $X(s) = \mathcal{L}[x] = \begin{bmatrix} V_1(s) \\ V_2(s) \\ I_2(s) \\ I_3(s) \end{bmatrix}$.

With this notational convention, the preceding system of ordinary differential equations

$$
\begin{align*}
\dot{x} &= Ax + Bv_{\text{in}}(t) \\
y &= Cx
\end{align*}
$$

with the single output equation can be transformed to obtain

$$
\begin{align*}
sX(s) - x(0^-) &= AX(s) + BV_{\text{in}}(s) \\
Y(s) &= CX(s)
\end{align*}
$$

where $I$ is a 4 by 4 identity matrix (since $x$ is a 4-vector). Note that the term in the parentheses on the left side of the equation is a 4 by 4 matrix and provided the inverse $(sI - A)^{-1}$ exists, then we can multiply both sides of the equation to get

$$
X(s) = (sI - A)^{-1}x(0^-) + (sI - A)^{-1}V_{\text{in}}(s)
$$

Since $V_{\text{out}}(s) = CX(s)$, we can then readily conclude that the Laplace transform of the output voltage $v_{\text{out}}(t)$ is

$$
V_{\text{out}}(s) = CX(s) = C(sI - A)^{-1}x(0^-) + C(sI - A)^{-1}BV_{\text{in}}(s)
$$

where $V_{\text{in}}(s)$ is the Laplace transform of the input voltage $v_{\text{in}}(t)$ and $x(0^-)$ is a vector representing the initial voltage/current on each of the energy storing elements at $t = 0$. In equation (8) the first term on the
right represents the system response due to the initial voltage/currents in the energy storing elements. We sometimes call this the circuit’s natural response. The second term represents the forced response due solely to the applied input voltage $v_{in}(t)$.

Can we find the inverse $(sI - A)^{-1}$? This matrix is sometimes called the resolvent of $A$. If the matrix is of low order then we can compute it symbolically using something like Cramer’s rule. For instance if $A$ is the 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its resolvent is

\[
(sI - A)^{-1} = \left( s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^{-1} = \begin{bmatrix} s - a & b \\ c & s - d \end{bmatrix}^{-1} = \begin{bmatrix} s - d & -b \\ -c & s - a \end{bmatrix}.
\]

We can compute inverses for larger matrices relying essentially on the same Gaussian elimination methods that we talk about earlier. Because these methods are “algorithmic” in nature (i.e. there is a well-defined sequence of steps to be taken to achieve the desired goal), one can write a program that automates the computation of this inverse, the only difference being that rather than treating the $s$ as a number, we treat it as a “symbolic” variable that we just carry along in the computation. This is sometimes called computer algebra and tools such as Mathematica, Maple, and SINGULAR are all computer programs that do this. MATLAB even has a symbolic engine built into it, so in principle we can do the Laplace transform analysis of our 4 element circuit using the computer, rather than doing a “hand” calculation.

The following script does this computation for the circuit in question. In particular, it declares $s$ as a symbolic variable and then computes the resolvent $(sI - A)^{-1}$ using the inv command. From that we can immediately compute $V_{out}(s)$ using the equation (8).

```matlab
% declare 's' as a symbolic variable
syms s

% Laplace transform of step input
step_input = 1/s;

% Laplace transform of inv(sI-A)
resolvent = simplify(inv(s*eye(4) - A));

% natural response, forced response, and total response
natural_response = C*resolvent*x0;
forced_response = C*resolvent*B*step_input;
```
total_response = simplify(forced_response+natural_response)

The output generated by this bit of script is

\[
\text{total_response} = \frac{100}{s(10s^4 + 21s^3 + 222s^2 + 312s + 201)}
\]

So we know that

\[
V_{\text{out}}(s) = \frac{10}{s^5 + 2.1s^4 + 22.2s^3 + 31.2s^2 + 20.1s}
\]

Recall that what we are interested in doing is characterizing the output voltage \(v_{\text{out}}(t)\) as a function of time. In particular, we want to know if there is an abrupt step change in the \(v_{\text{in}}(t)\) (the voltage directly measuring the environmental variable of interest) then 1) how quickly will the logged output voltage \(v_{\text{out}}(t)\) take before we can detect this step change on the input and 2) does the circuit we've built introduce any additional “artifacts” into our logged measurement that were not present in the input voltage \(v_{\text{in}}(t)\). The main tool we use to find \(v_{\text{out}}(t)\) is the partial fraction expansion (PFE) we discussed before that is used to compute the inverse transform of \(V_{\text{out}}(s)\).

We start the PFE approach by finding the zeros of the denominator polynomial and then using those zeros to factor the denominator into first (and maybe second order) factors. Using \texttt{roots} on the denominator polynomials we find five simple poles (4 complex and 1 real) \(p_{1,2} = -0.2966 \pm 4.4976j, p_{3,4} = -0.7534 \pm 0.649j,\) and \(p_5 = 0\). So the factored Laplace transform is

\[
V_{\text{out}}(s) = \frac{10}{s(s^2 + 0.5932s + 20.3164)(s^2 + 0.15068s + 0.9888)}
\]

The expansion into partial fractions yields,

\[
V_{\text{out}}(s) = \frac{K_1}{s + 0.2966 - 4.4976j} + \frac{K_1^*}{s + 0.2966 + 4.4976j}
+ \frac{K_2}{s + 0.7534 - 0.649j} + \frac{K_2^*}{s + 0.7534 + 0.649j} + \frac{K_3}{s}
\]

The residues, \(K_1, K_2,\) and \(K_3\) are then computed in the usual manner. For the pole at \(p_1 = -0.2966 + 4.4976j\)

\[
K_1 = \lim_{s \to p_1} (s - p_1)V_{\text{out}}(s) = \lim_{s \to p_1} \frac{10}{s(s + 0.2966 + 4.4976j)(s^2 + 0.15068s + 0.9888)}
= 0.0122 + 0.0017j = 0.0123 \angle 8.067^\circ
\]
For the pole at $p_3 = -0.7534 + 0.649j$ we get

$$K_2 = \lim_{s \to p_3} (s - p_3)V_{out}(s) = \lim_{s \to p_3} \frac{10}{s(s^2 + 0.5932s + 20.3264)(s + 0.7534 + 0.649j)} = -0.2610 + 0.2852j = 0.3865 \angle 132.45^\circ$$

For the pole at $p_5 = 0$ we get

$$K_3 = \lim_{s \to 0} sV_{out}(s) = \frac{10}{20.1} = 0.4975 \angle 0^\circ$$

The procedure outlined above can be easily automated using the computer. So as one might expect there is a MATLAB command that computes the residues for a Laplace transform that can be represented as a proper rational function. This function is residue and is used in the following script to find the check that the residues we computed above are correct.

```matlab
%extract numerator/denomator polynomial
% in total_response
[num,den] = numden(total_response);

%extract coefficients of num/den polynomial
% be sure to convert to double
num = double(coeffs(num,'All'));
den = double(coeffs(den,'All'));

%compute residues of rational function num/den
[r,p,k] = residue(num,den)
```

In this script, we first extract the numerator and polynomial polynomials using the command numden. The results are symbolic and need to be converted to a floating point array before it can be used by MATLAB to find the residues. The command coeffs extracts the coefficients of the polynomials and we need to use double to convert them from a symbolic type to a double floating point type. The MATLAB command residue automates what you did by hand in finding the residues of the partial fraction expansion. This command returns three floating point arrays, $r$ is an array of residues, $p$ is an array of poles, and $k$ is nonempty if the rational function is not strictly proper. In our case, the output generated by this script is

$$r = 0.0122 + 0.0017i$$
\[ V_{\text{out}}(s) = \frac{0.0122 + 0.0017j}{s + 0.2966 - 4.4976j} + \frac{0.0122 - 0.0017j}{s + 0.2966 + 4.4976j} \]
\[ + \frac{-0.2610 + 0.2852j}{s + 0.7534 - 0.6494j} + \frac{-0.2610 - 0.2852j}{s + 0.7534 + 0.6494j} \]
\[ + \frac{0.4975}{s} \]
\[ = \frac{K_1}{s + \alpha_1 - j\beta_1} + \frac{K_1^*}{s + \alpha_1 + j\beta_1} + \frac{K_2}{s + \alpha_2 - j\beta_2} + \frac{K_2^*}{s + \alpha_2 + j\beta_2} + \frac{K_3}{s} \]

These transforms are in our table we can immediately see the time domain response is

\[ v_{\text{out}}(t) = 2|K_1|e^{\alpha_1 t}\cos(\beta_1 t + \theta_1) \]
\[ + 2|K_2|e^{\alpha_2 t}\cos(\beta_2 t + \theta_2) \]
\[ + K_3 \]

for \( t \geq 0 \) in which
In this table the pole is represented as $\alpha_i + j\beta_i$ and the residue for the $i$th pole is $K_i = |K_i|\angle\theta_i$. Note that this is exactly what we computed by hand before, thereby verifying our earlier hand computation of the residues.

The preceding expression for $v_{\text{out}}(t)$ is better represented by combining the complex-valued first order factors to get

$$v_{\text{out}}(t) = K_3 u(t) + \sum_{i=1}^{2} 2|K_i|e^{\alpha_i t} \cos(\beta_i t + \theta_i)u(t)$$

(9) $$= 0.49u(t) + 0.024e^{-0.29t}\cos(4.49t + 8.0^\circ)u(t) + 0.77e^{-0.75t}\cos(0.649t + 132^\circ)u(t)$$

The preceding table was computed directly from the output of the `residue` command and then used to construct the response of each mode using the following script.

```matlab
K1 = abs(r(1));
theta1 = angle(r(1));
alfa1 = real(p(1));
beta1 = imag(p(1));
yx(1,:) = 2*K1*exp(alfa1*time).*cos(beta1*time+theta1);

K2 = abs(r(3));
theta2 = angle(r(3));
alfa2 = real(p(3));
beta2 = imag(p(3));
yx(2,:) = 2*K2*exp(alfa2*time).*cos(beta2*time+theta2);

K3 = abs(r(5));
yx(3,:) = K3.*ones(size(time));
```

figure(2);
subplot(2,1,1);
plot(data(:,1),data(:,7),'linewidth',2);
title('input and output voltages');
axis([0 50 0 1]);
subplot(3,1,2);
plot(time,yx(1,:)+yx(2,:)+yx(3,:),'linewidth',2);
```
Equation (9) shows that the time-domain behavior of \( v_{out}(t) \) consists of three distinct terms. The first term \( 0.4975u(t) \) is a step function and represents the long-term steady-state response of the circuit. The second term

\[
0.024e^{-0.29t} \cos(4.49t + 8^\circ)u(t)
\]

is a small exponentially decaying oscillation with a frequency of 4.49 rad/sec. The third term

\[
0.77e^{-7.5t} \cos(0.649t + 132^\circ)
\]

is a larger exponentially decaying oscillation with a frequency much longer than that of the second term. These three terms are often referred to as *modes* of the circuit. We can think of the first term as the “ideal” response that the circuit should exhibit and we can think of the other modes as *circuit artifacts* that distort the “ideal” response. The last two modes represent “fundamental” behaviors of the circuit in the sense that for any other input, \( v_{in}(t) \), these modes would still be present as they are dependent on the specific devices (capacitors and inductors) found in the circuit rather than the exogenous applied input \( v_{in}(t) \).

Fig. 2 plots the total response (assuming zero initial condition) for the \( v_{out}(t) \) in equation (9) computed from the partial fraction expansion. The top plot shows the total response \( v_{out}(t) \). What we see here is that it takes about 10 seconds for \( v_{out}(t) \) to settle to its steady state value. This means that if we were trying to observe a step event in the lake systems we were monitoring, it would be at least 10 seconds before we could detect the event’s occurrence.

The lower plot shows the response of each of the modes identified through the PFE analysis of the circuit. From this we see that the high frequency oscillation in the top plot is not something that occurred in \( v_{in}(t) \), it is an artifact introduced by the circuit that we can safely ignore. The other thing we notice is that the second mode governs the rate at which we detect the step change. From an analytical standpoint it should be possible to relate the poles of this mode with specific combinations of circuit components and this suggests we can use that knowledge to “speed” up the response of that mode so we can detect the step change more quickly.

The preceding discussion demonstrates one of the great advantages of the Laplace transform analysis of circuits. The analysis not only tells us the circuit’s total response, it provides decomposition of that response into modes that can then be used to explain important features in that response and how we might change those features. The next question, of course, is that we may not be sure if our PFE analysis was correct. After
all it is an involved calculation that may be subject to human error. To double check our analysis there are a couple of things we can do; both of which are briefly discussed in the next section.

2. Verification of Analysis Results

This section discusses methods that might be used to verify the results of the preceding section’s Laplace transform analysis. We can use this verification to 1) identify possible errors we made in the Laplace transform analysis or 2) to identify possible deviations that a real-life implementation of the circuit from the idealized linear circuit analysis.

One easy way of approaching the verification problem is to use the initial and final value theorems to compute the initial and final values from the Laplace transform and then check this against the initial/final values predicted by the computed time-domain function. For the example given above, we know that

\[ V_{out}(s) = \frac{10}{s^5 + 2.1s^4 + 22.2s^3 + 21.2s^2 + 20.1s} \]
The initial value theorem says
\[
\lim_{t \to 0} v_{\text{out}}(t) = \lim_{s \to \infty} sV_{\text{out}}(s)
\]
\[
= \lim_{s \to \infty} \frac{10}{s^4 + 2.1s^3 + 22.2s^2 + 21.2s + 20.1}
\]
\[
= \frac{10}{\infty} = 0
\]

The final value theorem says
\[
\lim_{t \to \infty} v_{\text{out}}(t) = \lim_{s \to 0} sV_{\text{out}}(s)
\]
\[
= \lim_{s \to 0} \frac{10}{s^4 + 2.1s^3 + 22.2s^2 + 21.2s + 20.1}
\]
\[
= \frac{10}{20.1} = 0.4975
\]

Let us now look at the initial and final value predicted from the time-domain \(v_{\text{out}}\) computed using the Laplace transform analysis. The initial value is
\[
\lim_{t \to 0} v_{\text{out}}(t) = \left[0.49 + 0.024e^{-0.29t} \cos(4.49t + 8.0^\circ) + 0.77e^{-0.75t} \cos(0.649t + 132^\circ)\right] u(t)
\]
\[
= 0.4975 + 0.024 \cos(8^\circ) + 0.77 \cos(132^\circ)
\]
\[
= 0.4975 + 0.0244 - 0.5220 = 0
\]

and the final value is
\[
\lim_{t \to \infty} v_{\text{out}}(t) = \left[0.49 + 0.024e^{-0.29t} \cos(4.49t + 8.0^\circ) + 0.77e^{-0.75t} \cos(0.649t + 132^\circ)\right] u(t)
\]
\[
= 0.4975
\]

Both of these values match the what was predicted by the Laplace transform analysis and so we have some confidence that the answer we computed was correct.

There is, however, another issue we need to consider. Namely that the Laplace transform analysis assumes the circuit we are building is \textit{linear}. The real implementation of the circuit may have nonlinear characteristics. For example the mathematical model of a resistor’s current/voltage characteristic is
\[v(t) = i(t)R\]

which suggests that the larger the current going through the resistor the greater the voltage. In reality, the resistor is a physical device with a power rating. If that power rating is exceeded the “linear” characteristic begins to break down and we have a nonlinear relationship between voltage and current. The Laplace transform analysis cannot predict what happens if we replace the ideal linear resistor with the more realistic nonlinear device. While we might use the Laplace transform analysis to initially check to see how the circuit we build will respond, in the end you need a more accurate verification that shows how the physical implementation of the linear model will behave once built. In practice this is done through digital simulation models.

The easiest way of building a digital simulation of the circuit is to take the first order ODE’s for the circuit and then numerically integrate them using the computer. As a simple example, let us start from a differential
equation of the form
\[
\frac{dx}{dt}(t) = ax(t) + bu(t)
\]
where \(a\) and \(b\) are real constant parameters, \(u(t)\) is an applied input signal, and \(x\) is the time-domain function.

We can approximate the derivative of \(x\) as
\[
\frac{dx(t)}{dt} \approx \frac{x(t+h) - x(t)}{h}
\]
where \(h > 0\) is a small time increment. If we insert this back into the differential equation we get
\[
\frac{x(t+h) - x(t)}{h} = ax(t) + bu(t)
\]
which we rewrite as
\[
x(t+h) = x(t) + (ax(t) + bu(t)) h
\]
This is a recursive equation that takes the value of the state at time \(t\) and predicts what it will be at time \(t+h\) by simply extrapolating by the derivative of \(x\). Note that \(\frac{dx(t)}{dt} = ax(t) + bu(t)\), so this is essentially the same as writing
\[
x(t+h) \approx x(t) + \left[\frac{dx(t)}{dt}\right] h
\]
This is sometimes called the forward Euler method for integration and provided \(h\) is sufficiently small then it can provide the basis for an algorithm that recursively solves the above equation to compute \(x\) as a function of time. Of course to start the recursion we need to know an initial value for \(x\) say at time \(t = 0\).

In our example, we showed we could write the circuit equations as
\[
\frac{dx(t)}{dt} = Ax(t) + Bv(t)
\]
So the preceding discussion suggests we could compute a trajectory for the voltages and currents. In particular, we can implement this in MATLAB. I used the following script.

```matlab
R1 = 1; R2 = .01; R3 = 1;
C1 = 1; C2 = 1;
L2 = .1; L3 = 1;
% define ODE matrices
A = [-1/(R1*C1) 0 -1/C1 0; 0 0 1/C1 -1/C2; 1/L2 -1/L2 -R2/L2 0; 0 1/L3 0 -R3/L3];
B = [1/(R1 *C1); 0; 0; 0];
C = [0 0 0 R3];
tstart = 0;
h = 1.e-2;
tduration = 50;
time = tstart:h:tstart+tduration;
x = zeros(4,1);
v = ones(size(time));
%vin = cos(0.1*time);
v(t<10) = 0;
data = [];
```
for t = 1:length(time)
    xdot = A*x + B*vin(t);
    x = x + xdot*h;
    vout = C*x;
    data = [data ; time(t) x' vin(t) vout];
end

figure(1);
clf(1);
subplot(3,1,1);
plot(data(:,1),data(:,2),data(:,1),data(:,3),'linewidth',2);
title('capacitor voltages');
legend('v1(t)','v2(t)');

subplot(3,1,2);
plot(data(:,1),data(:,4),data(:,1),data(:,5),'linewidth',2);
title('inductor currents');
legend('i2(t)','i3(t)');

subplot(3,1,3);
plot(data(:,1),data(:,6),data(:,1),data(:,7),'linewidth',2);
title('input and output voltages');
legend('vin(t)','vout(t)');

Figure 3. (a) step response (b) AC response

Figure 3(a) shows the capacitor voltages, $v_1(t)$ and $v_2(t)$, inductor currents, $i_2(t)$ and $i_3(t)$, and the input/output voltage waveforms, $v_{in}(t)$ and $v_{out}(t)$ computed by this script for time between 0 and 50 seconds with step size $h = 0.01$. The input waveform, $v_{in}(t)$, is a step input, that is zero for $t < 10$ and is 1 V for $t > 10$. What we see in the other plots is the same basic behavior. That prior to the step input, the voltages, currents, and outputs are all zero. After the input, these waveforms begin to rise until they reach a steady state value. In other words, the behavior of the circuit, both in terms of its internal variables (we often call these states) and the output have two phases. There is a transient interval over which these values are changing and as time gets large, the system settles down to a constant value. In our earlier work with DC analysis, we were only concerned with the steady state behavior. But clearly the transient behavior may be of interest also. In this example, we see the initial voltages/currents are highly oscillatory immediately after the step occurs. For obvious reasons, one may be concerned with what happens during this transient region.

We can also look and see what might happen with a sinusoidal input, by simply redefining the input as
\[ \text{vin} = \cos(0.1 \times \text{time}); \]

This input voltage is a unit amplitude sinusoid with a frequency of 0.3 rad/sec and zero phase angle. The simulation plots are shown in Fig. 3(b), which again shows a highly oscillatory transient interval, after which the response settles down to a sinusoid whose frequency is equal to that of the input voltage. This is, of course, the steady-state AC circuit analysis that we studied in the preceding part of these lectures.

If we go back to the specific results we obtained from the Laplace transform analysis we get the result shown in Fig. 4. Comparing the response in this figure to the top plot in Fig. 2 obtained from the Laplace transform analysis we see that they are essentially identical. We can, therefore, take this as an independent verification of our earlier Laplace transform result.

![Figure 4. \( v_{out}(t) \) obtained using digital simulation](image)

We now have two ways to analyze a linear circuit: the Laplace transform method or the simulation method. We see that the Laplace transform method provides considerable insight into the behavior of a circuit by identifying the “modes” that govern that circuit’s behavior. The limitation is that it requires the circuit models to be linear and real-life implementations have inherent nonlinearities in all devices. The simulation method is easy to program and there are many commercial tools that rely on this approach, but it does little more than compute a response without providing guidance regarding the relationship between the system’s response and the devices comprising that circuit. This suggests that in practice we need to use both methods for analysis and design. This is, in fact the case.

Fig. 5 shows the flowchart that is typically followed in the design of complex systems (circuits or any other engineering systems). We usually start with a set of design requirements and from those requirements we synthesize linear circuit that is intended to meet those requirements. The initial design is usually done with \( s \)-domain methodologies like the Laplace transform method we discussed above. After that has been done, however, we do not take the circuit as a final design. Instead we go ahead and build a digital simulation of that circuit that includes as much of the real-world characteristics as can be incorporated into the linear model. This might include saturation nonlinearities in device characteristics or other behaviors that were initially neglected in the design. The digital simulation is used to verify that these additional real-world aspects of
the implementation are indeed negligible. The outcome from the simulation is used to verify whether or not we expect the physical implementation of our physical design will satisfy the design requirement. If the requirement is satisfied, then we are done. But usually this is not the case. The design will fail the verification step in some way and the outcome of that failure will be used as a basis for either 1) adjusting the original linear design or 2) adjusting the design requirements. Once these changes have been done, we then repeat the verification procedure as shown in the figure. In other words, we see that the design of a real-life circuit that meets the design requirements is a recursive process. In this recursion, the linear models provide the basis for the initial design whereas the simulation is used to verify how this design functions in the real world.

3. Circuit Elements in the $s$-Domain

Rather than using the nodal analysis to write out the differential equations as we did above, one can start directly in the $s$-domain. We start by considering the constitutive relations for each circuit element.

From Ohm’s law we know that

$$v(t) = Ri(t)$$

But if we take the Laplace transform of both sides we get

$$V(s) = RI(s) \Rightarrow I(s) = GV(s)$$

where $G = \frac{1}{R}$.

Now consider the current-voltage equation for an inductor,

$$v(t) = L \frac{di(t)}{dt}$$
We take the Laplace transform of both sides to get

\[ V(s) = L(sI(s) - i(0^-)) = sLI(s) - Li(0^-) \]

We can invert this relation to solve for \( I(s) \)

\[ I(s) = \frac{V(s)}{sL} + \frac{i(0^-)}{s} \]

Now consider the current-voltage equation for a capacitor,

\[ i(t) = C \frac{dv(t)}{dt} \]

Applying the transform yields,

\[ I(s) = C(sV(s) - v(0^-)) = sCV(s) - Cv(0^-) \]

As before if we put this in terms of \( V(s) \) we get

\[ V(s) = \left( \frac{1}{sC} \right) I(s) + \frac{v(0^-)}{s} \]

The \( s \)-domain current/voltage relationships for each passive device (resistor, inductor, capacitor) are illustrated in Fig. 6. Essentially, they allow one to “redraw” and “relabel” the impedance diagram drawn in the time-domain into an \( s \)-domain impedance diagram. Note that in the \( s \)-domain, each circuit component with a zero initial condition can be written as

\[ V(s) = Z(s)I(s), \quad I(s) = Y(s)V(s) \]

where \( Z(s) \) is the \( s \)-domain impedance of the device in units of ohms. For the resistor, \( Z(s) = R \), for the inductor \( Z(s) = sL \) and for capacitor \( Z(s) = \frac{1}{sC} \). We can also choose to write the relationship in terms of admittances; in which case the resistor’s admittance becomes \( Y(s) = \frac{1}{R} \), the inductor’s admittance becomes \( Y(s) = \frac{1}{sL} \) and the capacitor’s admittance becomes \( Y(s) = sC \). These impedance/admittance values are used to relabel the time-domain impedance diagram. If there are also initial currents through the inductor or initial voltages over the capacitor, the circuit element would need to be augmented with the voltage or current source, respectively, to account for that initial condition. Once this is done we have an \( s \)-domain impedance/admittance diagram for the circuit and from that one can readily use KCL/KVL or the circuit reduction methods discussed before to identify a set of algebraic equations that can be solved for the Laplace transforms of the current/voltages in the original circuit. The benefit of this transformation is that solving for that set of Laplace transforms is an algebraic problem and is much easier to do than the numerical integration we presented above. Moreover, since the circuit is linear, we know the resulting transforms will be rational functions and this means we can readily invert the transforms using partial fraction methods to obtain a complete closed-form time-domain solution of the circuit.
**Time Domain**

\[ i(t) \rightarrow \text{R} \rightarrow v(t) \rightarrow - \]

\[
v(t) = Ri(t)
\]

\[ i(t) \rightarrow \text{L} \rightarrow v(t) \rightarrow - \]

\[
v(t) = L\frac{di(t)}{dt}
\]

\[ i(t) = \frac{1}{L} \int_{0^-}^{t} v(\tau)d\tau + i(0^-) \]

**s (Frequency) Domain**

\[ I(s) \rightarrow \text{R} \rightarrow V(s) \rightarrow - \]

\[
V(s) = RI(s)
\]

\[ I(s) \rightarrow \text{sL} \rightarrow V(s) \rightarrow - \]

\[
V(s) = sLI(s) - Li(0^-)
\]

\[
I(s) = \frac{V(s)}{sL} + \frac{i(0^-)}{s}
\]

\[ i(t) \rightarrow \text{C} \rightarrow v(t) \rightarrow - \]

\[
v(t) = C\frac{dv(t)}{dt}
\]

\[ i(t) = \frac{1}{C} \int_{0^-}^{t} i(\tau)d\tau + v(0^-) \]

\[
v(t) = \frac{1}{C} \int_{0^-}^{t} i(\tau)d\tau + v(0^-)
\]

\[
V(s) = \frac{1}{sC} I(s) + \frac{v(0^-)}{s}
\]

\[
I(s) = sCV(s) - Cv(0^-)
\]

**Figure 6.** s-domain circuits
4. Simple Examples

Natural Response of RC Circuit: Let us first consider the RC circuit on the left hand side of Fig. 7. This is an RC circuit with an initial voltage over the capacitor of $V_0$ and the switch then closes at $t = 0$ allowing current, $i(t)$, from the capacitor to flow through the resistor. The resulting output voltage $v(t)$ represents the “natural” response of the RC circuit without forcing.

In the time domain, we get the following differential equation model,

$$0 = C \frac{dv(t)}{dt} + \frac{v(t)}{R}, \quad v(0^-) = V_0$$

We can start by taking the Laplace transform to get

$$sCV(s) - CV_0 + \frac{V(s)}{R} = 0$$

Solving for $V(s)$ gives

$$V(s) = \frac{RCV_0}{RCs + 1} = \frac{V_0}{s + 1/RC}$$

Taking the inverse transform from the table gives

$$v(t) = V_0 e^{-t/RC} u(t)$$

We can get this same Laplace transform relation by redrawing the time-domain impedance diagram using the transformations in Fig. 6. If model the initial voltage as a source, then we get the middle figure. Applying KVL around the loop gives

$$\frac{V_0}{s} = \left( \frac{1}{sC} + R \right) I(s)$$

which gives

$$I(s) = \frac{CV_0}{RCs + 1} = \frac{V_0/R}{s + (1/RC)}$$

Taking the inverse transform gives

$$i(t) = \frac{V_0}{R} e^{-t/(RC)} u(t)$$
and since \( v(t) = Ri(t) \) we can conclude
\[
v(t) = V_0 e^{-t/(RC)} u(t)
\]
which is the same thing we got above.

We can also model the initial voltage on the capacitor using a current source. KCL at the top node gives
\[
CV_0 = sCV(s) + \frac{V(s)}{R}
\]
and solving \( V(s) \) gives
\[
V(s) = \frac{V_0}{s + (1/RC)}
\]
which is identical to what we got with the time-domain analysis.

**Step Response of Parallel RLC Circuit:** Let us now consider the parallel RLC circuit shown in Fig. 8.
In this case the switch is closed prior to time \( t = 0 \) so there is no initial voltage over the capacitor (i.e. \( v_C(0^-) = 0 \)) and no initial current through the inductor. Immediately after the switch is open the current flows through the parallel elements.

We can apply KCL at the top node to get
\[
I_{dc} u(t) = \frac{v(t)}{R} + C \frac{dv(t)}{dt} + \frac{1}{L} \int_0^t v(t) dt
\]
We can solve this by taking the Laplace transform
\[
\frac{I_{dc}}{s} = \frac{V(s)}{R} + sCV(s) + \frac{V(s)}{sL}
\]
and solving for \( V(s) \) gives
\[
V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}
\]
We get this same equation if we apply KCL at the top node of the \( s \)-domain diagram on the right side of Fig. 8.

![Figure 8. Parallel RLC](image-url)
and substituting the numerical values for $I_{dc}$, $R$, $L$, and $C$ we get

$$I_L(s) = \frac{384 \times 10^5}{s(s^2 + 64,000s + 16 \times 10^8)}$$

We factor the terms in the quadratic part of the dominator to get

$$I_L(s) = \frac{384 \times 10^5}{s(s + 32,000 - j24,000)(s + 32,000 + j24,000)}$$

We can test the $s$-domain representation for $I_L$ by using the initial value theorem to make sure that $i_L(\infty) = 24mA$. In particular, we see

$$\lim_{s \to 0} sI_L(s) = \frac{384 \times 10^5}{16 \times 10^8} = 24 \text{ mA}$$

So this checks out.

We now proceed with a partial fraction expansion to find the inverse Laplace transform.

$$I_L(s) = \frac{K_1}{s} + \frac{K_2}{s + 32000 - j24000} + \frac{K_2^*}{s + 32000 + j24000}$$

and evaluating the coefficients gives

$$K_1 = \frac{384 \times 10^5}{16 \times 10^8} = 24 \times 10^{-3}$$

$$K_2 = \frac{384 \times 10^5}{(-32000 + j24000)(j48000)} = 20 \times 10^{-3} \angle 126.87^\circ$$

Substituting numerical values for $K_1$ and $K_2$ and inverse transforming gives,

$$i_L(t) = \left[24 + 40e^{-32,000t}\cos(24,000t + 126.87^\circ)\right] u(t), \text{ mA}$$

Another example of using the Laplace transform arises from replacing the DC current source in Fig. 8 with a sinusoidal current source of the form

$$i_g(t) = I_m \cos \omega t$$

where $I_m = 24 \text{ mA}$ and $\omega = 40,000 \text{ rad/s}$. As before the initial stored energy in the inductor and capacitor is zero.

The $s$-domain expression for the source current is

$$I_g(s) = \frac{sI_m}{s^2 + \omega^2}$$

The voltage across the parallel elements is

$$V(s) = \frac{(I_g(s)/C)s}{s^2 + (1/RC)s + (1/LC)}$$

$$= \frac{(I_m/C)s^2}{(s^2 + \omega^2)(s^2 + (1/RC)s + (1/LC))}$$

The current through the inductor is then

$$I_L(s) = \frac{V(s)}{sL} = \frac{(I_m/LC)s}{(s^2 + \omega^2)(s^2 + (1/RC)s + (1/LC))}$$
Substituting numerical values for the parameters gives

\[ I_L(s) = \frac{384 \times 10^5 s}{(s^2 + 16 \times 10^8)(s^2 + 64,000s + 16^8)} \]

which we factor as

\[ I_L(s) = \frac{384 \times 10^5 s}{(s - j\omega)(s + j\omega)(s + \alpha - j\beta)(s + \alpha + j\beta)} \]

where \( \omega = 40,000 \), \( \alpha = 32,000 \), and \( \beta = 24,000 \).

WE expand this into a sum of partial fractions

\[ I_L(s) = \frac{K_1}{s - j40,000} + \frac{K_1^*}{s + j40,000} + \frac{K_2}{s + 32000 - j24000} + \frac{K_2^*}{s + 32000 + j24000} \]

Evaluating the numerical coefficients gives

\[ K_1 = \frac{384 \times 10^5 (j40,000)}{(j80,000)(32,000 + j16,000)(32,000 + j65,000)} = 7.5 \times 10^{-3} \angle -90^\circ \]

\[ K_2 = \frac{384 \times 10^5 (-32,000 + j24,000)}{(-32,000 - j16,000)(-32,000 + j64,000)(j48,000)} = 12.5 \times 10^{-3} \angle 90^\circ \]

Inverting this then gives

\[ i_L(t) = (15 \cos(40,000t - 90^\circ) + 25e^{-32000t} \cos(24000t + 90^\circ)) u(t) \]

The steady state current is \( 15 \sin(40000t) \) mA which can be verified using the phasor analysis we studied earlier.

**Step Response of Multi-Mesh Circuit:** We now apply the Laplace transform method to determine the step response of a circuit with two meshes in it. The circuit is shown in Fig. 9.

Here we want to find the branch currents \( i_1(t) \) and \( i_2(t) \) when the 336 V DC voltage source is suddenly applied across the circuit. We assume the initial current though both inductors is zero. There are two mesh
current equations that in the $s$-domain are
\[
\frac{336}{s} = (42 + 8.4s)I_1(s) - 42I_2(s)
\]
\[
0 = -42I_1(s) + (90 + 10s)I_2(s)
\]

We can write this in Matrix vector form as
\[
\begin{bmatrix}
\frac{336}{s} \\
0
\end{bmatrix}
= 
\begin{bmatrix}
(42 + 8.4s) & -42 \\
-42 & (90 + 10s)
\end{bmatrix}
\begin{bmatrix}
I_1(s) \\
I_2(s)
\end{bmatrix}
\]

Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, one can easily verify that
\[
A^{-1} = \begin{bmatrix}
d & -b \\ -c & a
\end{bmatrix} \frac{1}{ad - bc}
\]

So we can readily show that
\[
\begin{bmatrix}
I_1(s) \\
I_2(s)
\end{bmatrix}
= 
\frac{1}{(42 + 8.4s)(90 + 10s) - 42^2}
\begin{bmatrix}
(90 + 10s) & 42 \\
-42 & (42 + 8.4s)
\end{bmatrix}
\begin{bmatrix}
\frac{336}{s} \\
0
\end{bmatrix}
\]

Expanding this out into partial fractions gives
\[
I_1(s) = \frac{15}{s} - \frac{14}{s + 2} - \frac{1}{s + 12}
\]
\[
I_2(s) = \frac{7}{s} - \frac{8.4}{s + 2} + \frac{1.4}{s + 12}
\]

and taking the inverse transform gives
\[
i_1(t) = (15 - 14e^{-2t} - e^{-12t})u(t)
\]
\[
i_2(t) = (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t)
\]

Let us try what we did above for the sensor network circuit we considered in Fig. 1. We again assume the circuit is initially at rest so we don’t have to consider the current/voltage source extensions. If we simply redraw the circuit in Fig. 1 using the transformations in Fig. 6 we get the circuit diagram in Fig. 10. Rather than writing out the differential equations for the energy storage elements, we directly do a mesh analysis by applying KVL around the current loops in Fig. 1. What is different here, however, is that these mesh equation are written out in terms of the $s$-domain device impedances. For the first loop we get
\[
V_{in}(s) = R_1I_1(s) + \frac{1}{sC_1}(I_1(s) - I_2(s))
\]

the mesh equation for $I_2(s)$ is
\[
0 = \frac{1}{sC_1}(I_2(s) - I_1(s)) + I_2(s)(R_2 + sL_2) + \frac{1}{sC_2}(I_2(s) - I_3(s))
\]
and finally for the $I_3(s)$ loop we get

$$0 = \frac{1}{sC_2} (I_3(s) - I_2(s)) + sL_3 I_3(s) + R_3 I_3(s)$$

We collect terms to get

$$V_{in}(s) = \left( R_1 + \frac{1}{sC_1} \right) I_1(s) - \frac{1}{sC_1} I_2(s)$$

$$0 = -\frac{1}{sC_1} I_1(s) + \left( \frac{1}{sC_1} + R_2 + sL_2 + \frac{1}{sC_2} \right) I_2(s) - \frac{1}{sC_2} I_3(s)$$

$$0 = -\frac{1}{sC_2} I_2(s) + \left( \frac{1}{sC_2} + sL_3 + R_3 \right) I_3(s)$$

As before we find it more convenient to write the preceding three equations in matrix vector form. Assuming

$$V_{in}(s) = \frac{1}{s}$$

(a unit step) we get

$$\begin{bmatrix} \frac{1}{s} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + \frac{1}{sC_1} & -\frac{1}{sC_1} & 0 \\ -\frac{1}{sC_1} & \left( \frac{1}{sC_1} + R_2 + sL_2 \right) & -\frac{1}{sC_2} \\ 0 & -\frac{1}{sC_2} & \left( \frac{1}{sC_2} + R_3 + sL_3 \right) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{bmatrix}$$
Note that this is in the form of a linear algebraic equation \( b = AI \) where

\[
b = \begin{bmatrix}
\frac{1}{s} \\
0 \\
0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
R_1 + \frac{1}{sC_1} & -\frac{1}{sC_1} & 0 \\
-\frac{1}{sC_1} & \left(\frac{1}{sC_1} + \frac{1}{sC_2} + R_2 + sL_2\right) & -\frac{1}{sC_2} \\
0 & -\frac{1}{sC_2} & \left(\frac{1}{sC_2} + R_3 + sL_3\right)
\end{bmatrix}
\]

\[
I = \begin{bmatrix}
I_1(s) \\
I_2(s) \\
I_3(s)
\end{bmatrix}
\]

So we can compute \( I \) symbolically using MATLAB with the following script,

```matlab
syms s
Ax = [R1+1/(s*C1) -1/(s*C1) 0; -1/(s*C1) (1/(s*C1)+1/(s*C2)+R2+s*L2) -1/(s*C2); 0 -1/(s*C2) (1/(s*C2)+s*L3+R3)];
bx = [1/s;0;0];
Voutx = simplify(inv(Ax)*bx)
```

The output from this script gives the following Laplace transforms for each loop current,

\[
I_1(s) = \frac{s^4 + 1.1s^3 + 21.1s^2 + 20.1s + 10}{s(s^4 + 2.1s^3 + 22.2s^2 + 31.2s + 20.1)}
\]

\[
I_2(s) = \frac{10(s^2 + s + 1)}{s(s^4 + 2.1s^3 + 22.2s^2 + 31.2s + 20.1)}
\]

\[
I_3(s) = \frac{10}{s(s^4 + 2.1s^3 + 22.2s^2 + 31.2s + 20.1)}
\]

Since \( V_{out}(s) = R_3I_3(s) \) and \( R_3 = 1 \) ohm, we can readily see that

\[
V_{out}(s) = \frac{10}{s(s^4 + 2.1s^3 + 22.2s^2 + 31.2s + 20.1)}
\]

which is exactly the same answer we obtained by first constructing the fourth order system of ODE’s and then using PFE’s to compute the inverse Laplace transform for \( V_{out}(s) \). Note that we only needed the three mesh currents in this approach, which is significantly less complicated than the earlier approach we first derived the equations in the time domain, rather than the \( s \)-domain.

**Example using \( s \)-domain diagram:** The circuit shown in Fig. 11 has the switch in position \( a \) for a “long time” for \( t < 0 \). At \( t = 0 \) it switches to position \( b \). What is the output voltage, \( v_o(t) \) for \( t \geq 0 \).
For $t < 0$, the circuit may be drawn as shown in Fig. 12(a) where we treat the inductor as a short and the capacitor as an open circuit. Note that the current entering the parallel combination $R_2 \parallel R_3 \parallel R_4$ is $i_g = 5$ mA. We can therefore see that the initial capacitor voltage is

$$v_o(0^-) = (R_2 \parallel R_3 \parallel R_4)(i_g) = (100 \parallel 500 \parallel 2000)(0.005) = (80)(0.005) = 0.4 \text{ V}$$

Since we know the voltage over the inductor resistor ($R_3$) series combination, we can see the initial inductor current is

$$i_L(0^-) = \frac{v_o(0^-)}{R_3} = \frac{0.4}{500} = 0.8 \text{ mA}$$

This gives the initial cap and inductor conditions when the switch is closed at time $t = 0$.

For $t > 0$ the circuit configuration changes to that shown in Fig. 12(b). We convert this into the $s$-domain impedance diagram using the initial inductor current, $i_L(0^-)$, and capacitor voltage $v_o(0^-)$ computed above.

We apply KCL at the top node of the $s$-domain diagram to get

$$0 = \frac{V_o(s) - L i_L(0^-)}{R_3 + sL} + \frac{V_o(s)}{R_4} + \frac{V_o(s) - (v_o(0^-)/s)}{1/sC}$$

$$= \frac{V_o + 0.8 \times 10^{-3}}{500 + s} + \frac{V_o}{2000} + \frac{V_o - (0.4/s)}{10^6/s}$$
Gathering terms for \( V_o \) yields,
\[
V_o \left( \frac{1}{500 + s} + \frac{1}{2000} + \frac{s}{10^6} \right) = 0.4 \times 10^{-6} - \frac{0.8 \times 10^{-3}}{500 + s}
\]
and solving for \( V_o \) gives
\[
V_o(s) = \frac{0.4(s - 1500)}{s^2 + 1000s + 125 \times 10^4}
\]
Since the denominator polynomial is quadratic in \( s \), we can use the quadratic formula to identify the poles of \( V_o(s) \),
\[
p_{1,2} = -500 \pm \frac{1}{2} \sqrt{10^6 - 4(125 \times 10^4)} = -500 \pm j1000
\]
We now write out \( V_o(s) \) in its partial fraction form,
\[
V_o(s) = \frac{K_1}{s + 500 - j1000} + \frac{K_1^*}{s + 500 + j1000}
\]
and evaluate the residue \( K_1 \)
\[
K_1 = \lim_{s \to p_1} \frac{0.4(s - 1500)}{s + 500 + j1000} = 0.2 + j0.4 = 0.4472 \angle 63.4349^\circ
\]
From this we can immediately write down that
\[
v_o(t) = 0.894e^{-500t} \cos(1000t + 64.43^\circ)u(t)
\]
Use of Thevenin Equivalent: Consider the \( s \)-domain circuit shown in Fig. 13. We assume the circuit is initially at rest and that at \( t = 0 \) a 480V constant voltage is applied across the RLC combination. What we want to do is determine the capacitor current, \( i_c(t) \) and capacitor voltage, \( v_c(t) \), waveforms. We will do this by constructing a Thevenin equivalent circuit for the circuit shown in the yellow box.

![Figure 13. Example 13.17](image)

The Thevenin equivalent circuit is found by first redrawing the impedance diagram in the \( s \)-domain and then determining the open circuit voltage when the capacitor is not connected. This is obtained by a voltage divider across the 20 ohm resistor and the 0.002s inductor,
\[
V_{oh}(s) = \frac{480}{s} \frac{0.002s}{20 + 0.002s} = \frac{480}{(20/0.002) + s} = \frac{480}{s + 10^4}
\]
The Thevenin equivalent impedance is obtained by first computing the short circuit current, \( I_{sc} \). Note that if we short terminals \( a \) and \( b \), then the impedance seen by the voltage source is
\[
Z(s) = 20 + \frac{(60)(0.002s)}{60 + 0.002s} = \frac{80(s + 7500)}{s + 30000}
\]
The total current drawn out of the source is

\[ I(s) = \frac{480}{sZ(s)} = \frac{6(s + 30000)}{s(s + 7500)} \]

and the short circuit current is obtained by a current divider,

\[ I_{sc}(s) = I(s) \frac{0.002s}{60 + 0.002s} = \frac{6(s + 30000)}{s(s + 7500)} \frac{s}{s + 30000} = \frac{6}{s + 7500} \]

The Thevenin equivalent impedance is then

\[ Z_{th}(s) = \frac{V_{th}(s)}{I_{sc}(s)} = \frac{480}{s + 10000} \frac{s + 7500}{6} = \frac{80(s + 7500)}{s + 10000} \]

which is shown in Fig. 13.

The \( I_c(s) \) through the capacitor is then obtained as The capacitor’s impedance is \( Z_c(s) = \frac{2 \times 10^5}{s} \). The voltage, \( V_c(s) \) across the capacitor is obtained by a simple voltage divider between \( Z_{th} \) and \( Z_c \).

\[ V_c(s) = V_{th} \frac{Z_c}{Z_c + Z_{th}} = \frac{480}{s + 10^4} \frac{2 \times 10^5}{s} \frac{1}{s + 10^4} \frac{s + 7500}{s + 10^4} + \frac{2 \times 10^5}{s} = \frac{12 \times 10^5}{(s + 5000)^2} \]

The current through the capacitor is then

\[ I_c(s) = sCV_c(s) = \frac{s}{2 \times 10^5} \frac{12 \times 10^5}{(s + 5000)^2} = \frac{6s}{(s + 5000)^2} \]

We now compute the inverse transform for both the current and the voltage. The current’s inverse transform is obtained from a partial fraction expansion,

\[ I_c(s) = \frac{6s}{(s + 5000)^2} = \frac{-30000}{(s + 5000)^2} + \frac{6}{s + 5000} \]

which gives the inverse transform,

\[ i_c(t) = (-30,000te^{-5000t} + 6e^{-5000t})u(t) \]

and the voltage is simply

\[ v_c(t) = 12 \times 10^5te^{-5000t}u(t) \]

**Example with Mutual Inductance:** The next example looks at the transient response of a circuit with a mutual inductance. Fig. 14 shows the circuit. The switch shown in Fig. 14 switches the circuit topology at time \( t = 0 \). We want to determine the current \( i_2(t) \).

![Figure 14. Mutual Inductance Example](image-url)
This circuit has a pair of coupled inductors $L_1$ and $L_2$ whose mutual inductance is $M$. We can represent this as a transformer as shown in Fig. 15(a) whose primary and secondary voltages are given as

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$
$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

Because KCL doesn’t apply over the coupled coils, we replace it with an equivalent ($T$-equivalent see appendix C) model for which KCL does hold. The $T$-model is shown in Fig. 15(b) with inductances $L_a$, $L_b$, and $L_c$. For this model the primary and secondary voltages satisfy

$$v_1 = L_a \frac{di_1}{dt} + L_c \left( \frac{di_1}{dt} + \frac{di_2}{dt} \right)$$
$$v_2 = L_b \frac{di_2}{dt} + L_c \left( \frac{di_1}{dt} + \frac{di_2}{dt} \right)$$

For the two circuits to have equivalent voltage/current characteristics we require $L_c = M$, $L_a = L_1 - M$ and $L_b = L_2 - M$.

![Figure 15. coupled inductors and their $T$-equivalent circuit model](image)

We use this transform to redraw our circuit Fig. 14(a). In particular, for $t < 0$, we need the initial inductor currents. Since we are at DC for $t < 0$, both inductors act as short circuits. This means that the current in the secondary is $i_2(0^-) = 0$. The current in the primary only sees the series resistance of $9 + 3$ ohms. So the initial primary current

$$i_1(0^-) = \frac{60}{9 + 3} = 5 \text{ A}$$

Now that we know the initial currents, we first transform the time-domain impedance diagram seen after $t > 0$ to the circuit shown in Fig. 14(b) using the $T$-equivalent model of the coupled inductors. We then use the transformations in Fig. 6 to transform Fig 14(b) into the $s$-domain as shown in Fig. 14(c) where we added the 5 A current source to model the initial currents. This source was added to the vertical leg of the "T" to account for the initial value of the current tin the 2 H inductor, $(L_1)$, of $i_1(0^-) + i_2(0^-) = 5$ amps. The
branch with the inductor $L_1 - M$ has no source because $L_1 - M = 0$. The branch with the inductor $L_2 - M$ has no source because $i_2(0^-) = 0$.

We now use the mesh analysis on the $s$-domain circuit in Fig. 14(c). The two $s$-domain mesh equations are

\[
(3 + 2s)I_1(s) + 2sI_2(s) = 10
\]
\[
2sI_1(s) + (12 + 8s)I_2(s) = 10
\]

and solving for $I_2$ yields,

\[
I_2(s) = \frac{2.5}{(s + 1)(s + 3)}
\]

Expanding this as a sum of partial fractions generates

\[
I_2(s) = \frac{1.25}{s + 1} - \frac{1.25}{s + 3}
\]

and taking the inverse transform yields,

\[
i_2(t) = (1.25e^{-t} - 1.25e^{-3t})u(t)
\]

The plot for this response is shown in Fig. 16 where we see that after the switch is moved to position $b$, the current $i_2$ increases from zero to a peak value of 481 mA in 550 ms before decaying down to zero. Knowing the size of this peak transient is sometimes important in sizing system components.

![Figure 16. Example Response](image)

**Use of Superposition:** Figure 17 shows a circuit that has two sources. We will use the principle of superposition to determine this circuit’s transient response where we want to find the voltage $V_2(s)$. The left side of Fig. 17 shows the time-domain impedance diagram with an initial capacitor voltage $\gamma$ V and an initial inductor current of $\rho$ A. The $s$-domain impedance diagram is shown on the right side of Fig. 17 where we opted to shows the initial voltages and currents as current sources because we’ll be using a nodal voltage method.

To find $V_2(s)$ by superposition, we calculate the component of $V_2$ resulting from each of the sources acting alone, and then sum up the components. We begin with $V_g(s)$ acting alone. Opening each of the three current sources deactivates them. Fig. 18(a) shows the resulting circuit for $V_g$ alone. The two equations that describe
this circuit are
\[
\left( \frac{1}{R_1} + \frac{1}{sL} + sC \right) V_1^{(a)}(s) - sC V_2^{(a)}(s) = \frac{V_g(s)}{R_1} \\
-sC V_1^{(a)}(s) + \left( \frac{1}{R_2} + sC \right) V_2^{(a)} = 0
\]

For convenience, we represent the admittances in the above equation as
\[
Y_{11}(s) = \frac{1}{R_1} + \frac{1}{sL} + sC \\
Y_{12}(s) = -sC \\
Y_{22}(s) = \frac{1}{R_2} + sC
\]

Substituting back into the equation yields
\[
Y_{11}(s)V_1^{(a)}(s) + Y_{12}(s)V_2^{(a)}(s) = \frac{V_g(s)}{R_1} \\
Y_{12}(s)V_1^{(a)}(s) + Y_{22}(s)V_2^{(a)}(s) = 0
\]

We can solve these equations using any method you want to see that
\[
V_2^{(a)}(s) = \frac{-Y_{12}(s)/R_1}{Y_{11}(s)Y_{22}(s) - Y_{12}^2(s)} V_g(s)
\]

We now look at the current source $I_g(s)$ acting alone. The resulting circuit is obtained by shorting the voltage source and opening the other current sources. The simplified circuit is shown in Fig. 18(b). In this case, the two nodal equations describing the circuit are
\[
Y_{11}(s)V_1^{(b)}(s) + Y_{12}(s)V_2^{(b)}(s) = 0 \\
Y_{12}(s)V_1^{(b)}(s) + Y_{22}(s)V_2^{(b)}(s) = I_g(s)
\]

which we solve for $V_2^{(b)}(s)$ to obtain
\[
V_2^{(b)}(s) = \frac{Y_{11}(s)}{Y_{11}(s)Y_{22}(s) - Y_{12}^2(s)} I_g(s)
\]
To find the component of $V_2(s)$ resulting from the initial energy stored in the inductor, we solve the circuit shown in Fig. 18(c). In this case the nodal equations are

\[
Y_{11}(s)V_1^{(c)}(s) + Y_{12}(s)V_2^{(c)}(s) = -\frac{\rho}{s}
\]

\[
Y_{12}(s)V_1^{(c)}(s) + Y_{22}(s)V_2^{(c)}(s) = 0
\]

which we solve to get

\[
V_2^{(c)}(s) = \frac{Y_{12}(s)/s}{Y_{11}(s)Y_{22}(s) - Y_{12}^2(s)}\rho
\]

Finally we determine the component of $V_2$ resulting from the initial energy stored in the capacitor. The nodal equations for this circuit in Fig. 18(d) are

\[
Y_{11}(s)V_1^{(d)}(s) + Y_{12}(s)V_2^{(d)}(s) = \gamma C
\]

\[
Y_{12}(s)V_1^{(d)}(s) + Y_{22}(s)V_2^{(d)}(s) = -\gamma C
\]

which gives

\[
V_2^{(d)}(s) = -\frac{(Y_{11}(s) + Y_{12}(s))C}{Y_{11}(s)Y_{22}(s) - Y_{12}^2(s)}\gamma
\]

So the total expression for $V_2(s)$ is obtained by summing these four components together.

\[
V_2(s) = V_2^{(a)}(s) + V_2^{(b)}(s) + V_2^{(c)}(s) + V_2^{(d)}(s)
\]

\[
= -\frac{(Y_{12}/R_1)}{Y_{11}Y_{22} - Y_{12}^2}V_g + \frac{Y_{11}}{Y_{11}Y_{22} - Y_{12}^2}I_g
\]

\[
+ \frac{Y_{12}/s}{Y_{11}Y_{22} - Y_{12}^2}\rho + \frac{-C(Y_{11} + Y_{12})}{Y_{11}Y_{22} - Y_{12}^2}\gamma
\]
Dependent Source: Consider the circuit shown in Fig. 19(a) where switch moves from position 𝑎 to position 𝑏 at time 𝑡 = 0. Find $i_L(0^-)$, $V_o(s)$, and $v_o(t)$ for $t \geq 0$.

We start by finding the initial conditions on all energy storing elements. Because the capacitor is an open circuit at steady state, we expect $v_C(0^-) = 0$. The inductor, however, will have an initial current since the switch is initially in position 𝑎. In particular, the current through the inductor just prior to $t = 0$ will be

$$i_L(0^-) = \frac{v_g}{R_1} = \frac{24}{3} = 8 \text{ A}$$

Note the direction of the current is upwards.

After $t = 0$, we can redraw the circuit as shown in Fig. 19(b). We can view this as a circuit connected to the “yellow” box. We are going to find the Thevenin equivalent of that yellow box. In particular, Let us consider the box in isolation. Since there is no stored energy in the capacitor, the circuit in the box has an $s$-domain diagram as shown in Fig. 20(a). The current $I_\phi(s)$ can be obtained from a current divider rule

$$I_\phi(s) = \frac{\frac{1}{C_1} I_T(s)}{\frac{1}{C_1} + R_3} = \frac{1}{R_3 C s + 1} I_T(s)$$
The current/voltage relationship (i.e. \( V_T(s) \) versus \( I_T(s) \)) is

\[
V_T(s) = 25I_\phi(s) + I_T(s) \left( \frac{R_3 \left( \frac{1}{C_s} \right)}{R_3 + \frac{1}{C_s}} \right)
\]

\[
= 25I_\phi(s) + \left( \frac{25}{R_3C_s + 1} + \frac{R_3}{R_3C_s + 1} \right) I_T(s)
\]

\[
= \left( \frac{25}{R_3C_s + 1} \right) I_T(s) = Z(s)I_T(s)
\]

So the \( s \)-domain representation of the box is simply an impedance,

\[
Z(s) = \frac{25 + R_3}{R_3C_s + 1}
\]

We now use this in the \( s \)-domain representation of the original circuit for \( t > 0 \). This reduced circuit is shown in Fig. 20(b). Applying KCL at the top node gives

\[
\frac{8}{s} = \frac{V_o(s)}{sL} + \frac{V_o(s)}{R_2} + \frac{V_o(s)}{Z(s)}
\]

\[
= \left( \frac{1}{sL} + \frac{1}{R_2} + \frac{1}{Z(s)} \right) V_o(s)
\]

Solving for the Laplace transform of the output voltage is therefore

\[
V_o(s) = \frac{8}{s} \left( \frac{1}{sL} + \frac{1}{R_2} + \frac{1}{Z(s)} \right)^{-1}
\]

We now plug in the values we have for the variables to get

\[
V_o(s) = \frac{8}{s} \left( \frac{1}{5.025s} + \frac{1}{2s + 1} \right)^{-1}
\]

\[
= \frac{180}{s^2 + 5s + 4} = \frac{180}{(s + 1)(s + 4)}
\]

So the circuit has two simple poles at \( s = -1 \) and \( s = -4 \).

We now invert the Laplace transform to get the output voltage in the time domain. Rewriting \( V_o(s) \) as a partial fraction expansion gives

\[
V_o(s) = K_1 \frac{1}{s + 1} + K_2 \frac{1}{s + 4}
\]

The residue for the pole at \(-1\) is

\[
K_1 = \lim_{s \to -1} \frac{180}{s + 4} = \frac{180}{3} = 60
\]

The residue for the pole at \(-4\) is

\[
K_2 = \lim_{s \to -4} \frac{180}{s + 1} = \frac{180}{-3} = -60
\]

So the partial fraction expansion is

\[
V_o(s) = \frac{60}{s + 1} - \frac{60}{s + 4}
\]
from which we can immediately deduce that

\[ v_o(t) = (60e^{-t} - 60e^{-4t})u(t) \]

**Example 1:** An example to help with homework/exam. Consider the circuit shown in Fig. 21. The switch is closed for a long time before it is opened at \( t = 0 \).

- Construct the \( s \)-domain circuit for \( t \geq 0 \).
- Find \( I_o(s) \) and \( i_o(t) \).

\[ R_1 = 240 \text{ ohms}, \ R_2 = 80 \text{ ohms}, \ R_3 = 32 \text{ ohms} \]
\[ C = 25 \mu \text{F}, \ L = 200 \text{ mH}, \ v_g = 40 \text{ V} \]

For \( t < 0 \), we know \( C \) is an open circuit and \( L \) is a short circuit. So the circuit before \( t < 0 \) is shown in Fig. 22. The initial inductor current is obtained by first noting that the impedance seen by \( v_g \) is

\[ Z_{eq} = R_3 + R_2 \parallel (R_2 + R_1) + 2R_3 \]
\[ = 3R_3 + \frac{R_2(R_2 + R_1)}{2R_2 + R_1} \]
\[ = 3(32) + \frac{160 + 240}{160 + 240} = 160 \text{ ohms} \]

So

\[ i_L(0^-) = \frac{v_g}{Z_{eq}} = \frac{40}{160} = 0.25 \text{ A} \]
The initial capacitor voltage is given by

\[ v_c(0^-) = i_1(R_2) - R_3i_L(0^-) + v_g \]

\( i_1 \) is obtained by a current divider

\[ i_1 = -\frac{R_2}{2R_2 + R_1}i_L(0^-) = -\frac{80}{400} \left(\frac{1}{4}\right) = -0.05 \text{ A} \]

So

\[ v_c(0^-) = (-0.05)80 - (32)/4 + 40 = 28 \text{ V} \]

When \( t = 0 \) the switch opens and the circuit changes its configuration to that shown in Fig. 23(a). Note that the upper branch in this figure is a parallel combination \( R_1 \) and \( 2R_2 \)

\[ R_1 || 2R_2 = \frac{(240)(160)}{400} = 96 \text{ ohms} \]

So we can redraw this as shown in Fig. 23(b) where all of the variables have been replaced with their appropriate values. The initial capacitor voltage of 28 V is shown as is the initial inductor current \( i_L(0^-) = 1/4 \text{ A} \). We now redraw this last circuit in the \( s \)-domain using these initial conditions to add additional sources as shown in Fig. 23(c).

Applying KVL around the \( s \)-domain circuit in Fig. 23(c) gives

\[ \frac{28}{s} + 0.05 = \left(160 + \frac{4 \times 10^4}{s} + 0.2s\right)I_o(s) \]

which implies that

\[ I_o(s) = \frac{0.25(s + 450)}{s^2 + 800s + 200000} \]
4. SIMPLE EXAMPLES

The partial fraction expansion of $I_o(s)$ is therefore

$$I_o(s) = \frac{K_1}{s + 400 - j200} + \frac{K_1^*}{s + 400 + j200}$$

We now evaluate the residue

$$K_1 = \lim_{s \to -400+j200} \frac{0.25(s + 560)}{s + 400 + j200}$$

$$= 0.125 - 0.1j$$

$$= 0.16\angle -38.6598^\circ$$

and so we can readily determine that

$$i_o(t) = 0.32e^{-400t}\cos(200t - 38.66^\circ)u(t)$$

**Example 2 for Exam:** Let us now consider the circuit shown in Fig. 24(a) where $R = 10$ ohms and $C = 100$ mF. The current source is $i_g(t) = 9u(t)$. We assume that prior to the energizing of the circuit at $t = 0$ that there is no energy stored in the circuit. We need to find $I_a(s)$, $i_a(t)$, $I_b(s)$, and $i_b(t)$. We then use these currents to find the voltages $V_a(s)$, $v_a(t)$, $V_b(s)$, $v_b(t)$, $V_c(s)$, and $v_c(t)$. Finally we assume that the capacitors breakdown if the voltage exceeds 1000 V. We are to determine how long after the source turns on will a capacitor fail.

We can readily redraw our circuit in the $s$-domain to obtain the diagram in Fig. 24(b). Note that $1/CS = 10/s$ and $R = 10$ so we replaced these values in the $s$-domain diagram. We draw 3 mesh currents, but note the mesh containing the current source has the mesh current $9/s$. So there are only two mesh currents to determine $I_1(s)$ and $I_2(s)$ as marked in Fig. 24(b).
The mesh equations are readily seen to be

\[0 = \frac{10}{s} I_1 + \frac{10}{s} (I_1 - I_2) + 10 \left( I_1 - \frac{9}{s} \right)\]

\[0 = \frac{10}{s} \left( I_2 - \frac{9}{s} \right) + \frac{10}{s} (I_2 - I_1) + 10I_2\]

We collect terms to get

\[9 = (s + 2)I_1 - I_2\]

\[\frac{9}{s} = -I_1 + (s + 2)I_2\]

and rewrite in matrix-vector form to get

\[
\begin{bmatrix}
9 \\
\frac{9}{s}
\end{bmatrix} =
\begin{bmatrix}
s + 2 & -1 \\
-1 & s + 2
\end{bmatrix}
\begin{bmatrix}
I_1(s) \\
I_2(s)
\end{bmatrix}
\]

This last equation can be solved to get

\[
\begin{bmatrix}
I_1(s) \\
I_2(s)
\end{bmatrix} =
\begin{bmatrix}
\frac{s + 2}{(s + 2)^2 - 1} & 1 \\
\frac{1}{(s+1)(s+3)} & \frac{1}{s+2}
\end{bmatrix}
\begin{bmatrix}
9 \\
\frac{9}{s}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{s+2}{(s+1)(s+3)} & \frac{1}{(s+1)(s+3)} \\
\frac{1}{s+2} & \frac{18}{s(s+3)}
\end{bmatrix}
\begin{bmatrix}
9 \\
\frac{9}{s}
\end{bmatrix}
\]

From the circuit diagram we know

\[I_a(s) = I_1(s) = \frac{9(s+1)}{s(s+3)} = \frac{K_1}{s} + \frac{K_2}{s+3}\]
where the poles at 0 and −3 were used to write out the partial fraction expansion. We then evaluate the residues to get

\[
K_1 = \lim_{{s \to 0}} \frac{9(s + 1)}{s + 3} = \frac{9}{3} = 3
\]

\[
K_2 = \lim_{{s \to -3}} \frac{9(s + 1)}{s} = \frac{9(-2)}{-3} = 6
\]

So

\[
I_a(s) = \frac{3}{s} + \frac{6}{s + 3}
\]

whose inverse transform gives us the time-domain signal

\[
i_a(t) = (3 + 6e^{-3t})u(t)
\]

From the circuit diagram we can also see that

\[
I_b = \frac{9}{s} - I_1 = \frac{9}{s} - \frac{9(s + 1)}{s(s + 3)}
\]

\[
= \frac{9(s + 3) - 9(s + 1)}{s(s + 3)} = \frac{18}{s(s + 3)}
\]

\[
= \frac{K_1}{s} + \frac{K_2}{s + 3}
\]

where we used the poles at 0 and −3 to write out the PFE. Evaluating the residues gives us

\[
K_1 = \lim_{{s \to 0}} \frac{18}{s + 3} = \frac{18}{3} = 6
\]

\[
K_2 = \lim_{{s \to -3}} \frac{18}{s} = \frac{18}{-3} = -6
\]

so that

\[
I_b(s) = \frac{6}{s} - \frac{6}{s + 3} \quad \Rightarrow \quad i_b(t) = (6 - 6e^{-3t})u(t)
\]

We now look to evaluate the voltages. I’ll only do this in detail for \(v_a(t)\). Again from the circuit diagram we can see that

\[
V_a(s) = \frac{10}{s}I_b(s) = \frac{10}{s} \left( \frac{3}{s} + \frac{6}{s + 3} \right)
\]

\[
= \frac{30}{s^2} + \frac{60}{s(s + 3)}
\]

there is a pole at 0 with multiplicity of 2 and a simple pole at −3. This means the PFE of \(V_a(s)\) is

\[
V_a(s) = \frac{K_{11}}{s^2} + \frac{K_{12}}{s} + \frac{K_2}{s + 3}
\]
We now evaluate the residues to get

\[ K_{11} = \lim_{s \to 0} s^2 V_a(s) = \lim_{s \to 0} \left( 30 + \frac{60s}{s+3} \right) = 30 \]

\[ K_{12} = \lim_{s \to 0} \frac{d}{ds} s^2 V_a(s) \]

\[ = \lim_{s \to 0} \frac{d}{ds} \left( 30 + \frac{60s}{s+3} \right) \]

\[ = \lim_{s \to 0} \frac{(s+3)(60) - 60s}{(s+3)^2} = \frac{180}{9} = 20 \]

\[ K_2 = \lim_{s \to -3} (s + 3)V_a(s) \]

\[ = \lim_{s \to -3} \left( \frac{30(d + 3) + 60s}{s^2} \right) = \frac{60(-3)}{(-3)^2} = \frac{60}{9} = -20 \]

So we can see that

\[ V_a(s) = \frac{30}{s^2} + \frac{20}{s} - \frac{20}{s+3} \Rightarrow v_a(t) = [30t + 20 - 20e^{-3t}] u(t) \]

Similar analysis for the other voltages shows

\[ V_b(s) = \frac{10}{s} (I_2 - I_1) = \frac{30}{s^2} - \frac{40}{s} + \frac{40}{s+3} \]

\[ V_c(s) = \frac{10}{s} \left( \frac{9}{s} - I_2 \right) = \frac{30}{s^2} + \frac{20}{s} - \frac{20}{s+3} \]

so that

\[ v_b(t) = [30t - 40 + 40e^{-3t}] u(t) \]

\[ v_c(t) = [30t + 20 - 20e^{-3t}] u(t) \]

The last thing to do is see when the capacitor breaks down because the voltage over the capacitor has exceeded 1000 volts. This means we need to find the time \( t \) when

\[ 1000 = 30t + 20 - 20e^{-3t} \]

\[ 1000 = 30t - 40 + 40e^{-3t} \]

Note that for \( t \) large enough, that the exponential terms are negligible. We can therefore find our times as

\[ 30t + 20 = 1000 \text{ and } 30t - 40 = 1000 \]

which gives the times

\[ t = \frac{980}{30} = 32.67 \text{ sec} \]

\[ t = \frac{1040}{30} = 34.67 \text{ sec} \]

The first one occurs first and so the circuit will break down in 32.67 seconds.