Problem 3.1: Show that extremals for the catenary problem satisfy,

\[ y(x) = c \cosh \left( \frac{x + d}{c} \right) + \lambda \tag{1} \]

for \( c > 0 \).

Problem 3.2: Consider the problem of finding local extremals for

\[ J(y) = \int_{-1}^{0} (y'(x))^3 \, dx \]

on \( \mathcal{D} = \{ y \in C^1[-1,0] : y(-1) = 0, y(0) = 2/3 \} \) subject to the integral constraint,

\[ G(y) = \int_{-1}^{0} xy'(x) \, dx = -\frac{4}{15} \]

Show that \( y^*(x) = \frac{2}{3}(x + 1)^{3/2} \) is the only extremal for this problem.

Problem 3.3: Consider a simple planar pendulum as shown in example 2.5 of the textbook. Use the necessary conditions for constrained optimality to derive the equations of motion

\[ \ddot{\theta} = -\frac{g}{\ell} \sin \theta \tag{2} \]

Problem 3.4: Consider the functional

\[ J(y) = \int_{0}^{b} [y'(x)]^2 \, dx \]

where \( y(0) = 0 \) and \( y(b) = K \). Use Legendre’s second order necessary condition to verify that the extremal satisfying the problem’s Euler-Lagrange equation is indeed a local minimum.

Problem 3.5: Use Taylor’s theorem with remainder to show that the \( o(\alpha^2) \) term in equation (2.57) of the text can be written in the form,

\[ o(\alpha^2) = \int_{a}^{b} \left( \overline{P}(x, \eta(x), \eta'(x), \alpha) (\eta'(x))^2 + \overline{Q}(x, \eta(x), \eta'(x), \alpha) (\eta(x))^2 \right) \, dx \cdot \alpha^2 \]

where \( \overline{P}, \overline{Q} \to 0 \) as \( \alpha \to 0 \). Moreover, show that this convergence is uniform over \( \eta \) with respect to the 1-norm. What is the relevance of this problem to Legendre’s second order condition?

Problem 3.6: Show that externals for the functional

\[ J(y) = \int_{a}^{b} [y'(x)]^2 + yy' + y^2 \, dx \]

with fixed endpoints \( y(a) = y_a \) and \( y(b) = y_b \) can have no corners.

Problem 3.7: Determine the externals for the functional

\[ J(y) = \int_{0}^{4} [y'(x) - 1]^2 [y'(x) + 1]^2 \, dx \]

which has only one corner. The boundary conditions are \( y(0) = 0 \), \( y(4) = 2 \).

Problem 3.8: Consider the problem of minimizing

\[ J(x) = \int_{0}^{1.5} \sqrt{1 + (y')^2} \, dx \]
Subject to the constraints that $y(0) = 1.5$ and $y(1.5) = 0$ and such that there exists only one point $c$ such that $y(c) = -x + 2$. Figure 1 shows the geometry of this problem. Determine a solution to this problem.

**Figure 1. Problem 3.8**

**Problem 3.9:** Consider the problem to minimize the functional,

$$J(y) = \int_a^b L(x, y(x), y'(x))dx$$

on $D = \{ y \in C^1[a, b] : y(a) = y_0, y(b) = y_1 \}$. Suppose that the Lagrangian $L(x, y, z)$ is $C^1$ and convex in $(y, z)$. Prove that if $y^*$ satisfies the Euler-Lagrange equation, then $y^*$ is a global minimizer for $J$ on $D$.

**Problem 3.10:** Use the Weierstrass necessary condition to prove that a piecewise $C^1$ strong minimum must satisfy the Weierstrass-Erdmann corner conditions.