The Optimal Exploitation of Renewable Resource Stocks: Problems of Irreversible Investment
Author(s): Colin W. Clark, Frank H. Clarke and Gordon R. Munro
Source: *Econometrica*, Vol. 47, No. 1 (Jan., 1979), pp. 25-47
Published by: Econometric Society
Stable URL: http://www.jstor.org/stable/1912344
Accessed: 07-10-2015 21:19 UTC

Econometrica, Vol. 47, No. 1 (January, 1979)

THE OPTIMAL EXPLOITATION OF RENEWABLE RESOURCE STOCKS: PROBLEMS OF IRREVERSIBLE INVESTMENT[1]

BY COLIN W. CLARK, FRANK H. CLARKE, and GORDON R. MUNRO

This paper studies the effects of irreversibility of capital investment upon optimal exploitation policies for renewable resource stocks. It is demonstrated that although the long-term optimal sustained yield is not affected by the assumption of irreversibility (except in extreme cases), the short-term dynamic behavior of an optimal policy may depend significantly upon the assumption. It is suggested that the results may have profound implications for problems of rehabilitation of overexploited fisheries and other renewable resource stocks.

1. INTRODUCTION

THIS PAPER IS CONCERNED with the problem of "non-malleability" of capital and the implications thereof for the optimal exploitation of renewable resource stocks over time. We use the term "non-malleability" to refer to the existence of constraints upon the disinvestment of capital assets utilized in exploiting the resource stock (cf. Arrow [**1**], Arrow and Kurz [**2**]).

Previous studies of renewable resource economics have assumed, either explicitly [**3**] or implicitly [**8**], that capital stocks were perfectly malleable. It is easy to see that this implies that the variable representing the capital stock can be eliminated from the analysis. This is no longer possible when capital is assumed to be non-malleable, however, with the result that the corresponding optimization problem becomes considerably more complex, since it necessarily involves a minimum of two state variables. It will be shown that in fact the non-malleability assumption has a significant influence on the form of optimal exploitation policies. This has long been suspected on intuitive grounds; for example, numerous discussions of the practical problems of fishery management created by the non-malleability of capital (both physical and human) can be found in the literature (e.g. [**5**, p. 222; **19**]).

In Section 2 we describe the model to be used as the basis for our investigations. This model, associated with the names of Gordon [**11**] and Schaefer [**17**], has often been used in the study of commercial fisheries [**8**, **10**]. In spite of its somewhat specialized nature, we are confident that the results will remain qualitatively valid for a wide choice of alternative models of renewable resource exploitation.

In Section 3 we review briefly the case in which capital is perfectly malleable. It will be verified that in this case the capital stock variable can indeed be eliminated from the analysis and that the capital input can be treated as a flow. The model thus reduces to the single-state-variable model studied earlier [**8**, **10**].

[1] This research has been supported in part by the National Research Council of Canada (Grant numbers A-3990 and A-9082), and also by the Donner Canadian Foundation as part of a project on "Canada and the International Management of the Oceans" sponsored by the Institute of International Relations, University of British Columbia, Canada, along with the University of B.C. Programme in Natural Resource Economics, financed by the Canada Council. The suggestions of the referees are gratefully acknowledged.

In Sections 4–6 we introduce three alternative assumptions of non-malleability of capital. In Section 4 we deal with the case of perfectly non-malleable capital in which the depreciation rate is equal to zero and in which the capital has a negligible scrap value. In Section 5 we continue to assume that the scrap value is zero, but allow for a positive depreciation rate. In this case we say that capital is "quasi-malleable." In Section 6 we allow for a positive unit scrap value, which is significantly below the replacement price.

The material presented in Sections 4–6 is of a descriptive nature. A rigorous analytic proof of all our results appears in Section 7. Our proof is based upon a method due to Carathéodory [**4**], which does not seem to have been employed previously in economic analysis.

Inasmuch as our model involves two stock variables (resource stock, capital stock) and two control variables (harvest rate, investment rate), it is structurally similar to a number of two-sector models of economic growth that have appeared in the literature [**12**, **14**]. Such models are known to give rise to relatively complex optimal trajectories, frequently involving multiple switches of the control variables. Our model is no exception to this rule. On the other hand, our model is much less abstract than these growth models. It is an eight-parameter model, each parameter representing a measurable biological or economic variable. Because of the concrete nature of the model, the bio-economic reasons underlying the form of optimal exploitation policies become quite transparent, particularly since we are able to express the solution explicitly in synthesized (feedback) form: see Figures 1–3.

2. THE GENERAL MODEL

For the sake of explicitness we shall henceforth restrict our attention to a bio-economic model of the commercial fishery under sole ownership. The biological basis of the model is the general production model developed by Schaefer [**17**] and Pella and Tomlinson [**15**]; the economic basis stems from the work of Gordon [**11**], Crutchfield and Zellner [**10**], and others (see Clark and Munro [**8**]).

The population dynamics of the fishery resource is modeled by the equation

$$\frac{dx}{dt} = F(x) - qEx, \quad x(0) = x^0, \tag{2.1}$$

where $x(t)$ is population biomass at time t, $F(x)$ is the natural growth function, q is the catchability coefficient (constant), and $E(t)$ is fishing effort at time t. Regarding the natural, or biological, growth function $F(x)$, we shall assume that

$$F(0) = F(\bar{x}) = 0, \quad F(x) > 0, \quad F''(x) < 0 \quad \text{for} \quad 0 < x < \bar{x}. \tag{2.2}$$

In equation (2.1) the rate of harvest $h(t)$ is of the form

$$h(t) = qE(t)x(t); \tag{2.3}$$

this particular form of the Cobb-Douglas production function is the traditional harvest production function employed in fishery models.

The variables $x(t)$ and $E(t)$ of equation (2.1) are subject to the constraints

$$x(t) \geqslant 0 \tag{2.4}$$

and

$$0 \leqslant E(t) \leqslant E_{\max} = K(t) \tag{2.5}$$

where $E_{\max}$ is maximum effort capacity and $K(t)$ is the amount of capital invested in the fishery at time t. We shall think of $K(t)$ as representing the number of "standardized" fishing vessels available to the fishery. Equation (2.5) then asserts that the maximum effort capacity equals the number of vessels available, and that the actual level of effort employed at any time cannot exceed $E_{\max}$.

Possible adjustments to the level of capital are modeled by the equation

$$\frac{dK}{dt} = I - \gamma K, \quad K(0) = K^0, \tag{2.6}$$

where $I(t)$ is gross investment rate at time t (expressed in physical terms) and γ is the rate of depreciation (constant). The fish biomass $x(t)$ and capital stock $K(t)$ are subject to the constraints

$$x(t) \geqslant 0; \quad K(t) \geqslant 0. \tag{2.7}$$

The assumption of non-malleability is embodied in the following constraint:

$$0 \leqslant I(t) \leqslant +\infty. \tag{2.8}$$

The case $I(t) = +\infty$ allows for instantaneous jump increases in the level of capital. Admitting this possibility simplifies the analysis and lets us concentrate on the phenomenon of non-malleability of capital.

As our objective function we employ the discounted net cash flow:

$$J = \int_0^\infty e^{-\delta t}\{ph(t) - cE(t) - \pi I(t)\}\, dt \tag{2.9}$$

where δ is the instantaneous rate of discount (constant), p is price of landed fish (constant), c is operating cost per unit effort (constant), and π is price (purchase or replacement) of capital (constant). All the parameters of our model, viz q, γ, δ, p, c, and π, are taken as given constants, and all are assumed to be positive, except for the depreciation rate γ, which we merely assume to be $\geqslant 0$. We shall refer to $\gamma = 0$ as the case of perfect non-malleability, and to $\gamma > 0$ as the case of quasi-malleability of capital.

The problem we face is that of determining the optimal effort and investment policies $E(t)$, $I(t)$, leading to the maximization of the objective (2.9). In the next three sections we describe the solution to this problem, and discuss a variety of policy implications of the model. The rigorous justification of this solution is provided in Section 7.

In Section 6 we shall discuss an alternative model, in which disinvestment is unconstrained, but in which unwanted capital can be sold only as scrap. Let

$\pi_s =$ unit scrap value of capital (constant);

we shall assume that

$$(2.10)\qquad 0<\pi_s<\pi.$$

For this model we suppose that gross investment is unrestricted:

$$(2.11)\qquad -\infty\leq I(t)\leq +\infty.$$

However, the objective functional is now replaced by

$$(2.12)\qquad J=\int_0^\infty e^{-\delta t}\{ph(t)-cE(t)-\phi(I(t))\}\,dt$$

where

$$(2.13)\qquad \phi(I)=\begin{cases}\pi I & \text{if}\quad I>0,\\ \pi_s I & \text{if}\quad I<0.\end{cases}$$

The corresponding control problem is as before.

3. PERFECTLY MALLEABLE CAPITAL

To set the stage for the more difficult problems generated by non-malleability of capital we review briefly the case in which capital is perfectly malleable, in the sense that

$$(3.1)\qquad -\infty\leq I(t)\leq +\infty$$

and that $\pi=\pi_s$. The relevant objective functional is (2.9).

Under this assumption it is clear that for an optimal policy there will never be excess harvesting capacity, i.e., we will always have

$$(3.2)\qquad K(t)=E(t)$$

since any unused capacity can immediately be disposed of at the purchase price π. Consequently $I(t)=\dot{K}+\gamma K=\dot{E}+\gamma E$ and (2.9) becomes, after integration by parts,

$$(3.3)\qquad \begin{aligned} J&=\int_0^\infty e^{-\delta t}\{ph-cE-\pi(\delta+\gamma)K\}\,dt+\pi K^0\\ &=\int_0^\infty e^{-\delta t}\{pqx-c_{\text{total}}\}E\,dt+\text{constant}\end{aligned}$$

where

$$(3.4)\qquad c_{\text{total}}=c+(\delta+\gamma)\pi$$

represents the total cost per unit of fishing effort. In this expression the term c

denotes unit operating cost, while $(\delta+\gamma)\pi$ can be viewed as the unit "rental" cost of capital.

Thus when capital is assumed to be perfectly malleable, the stock variable K can be eliminated entirely from the model (as pointed out earlier by Beddington, Watts, and Wright [**3**]; see also Hadley and Kemp [**12**, Chapter 6]). The problem reduces to the maximization of expression (3.3) subject to the biomass equation (2.1) and to constraints (2.4) and (2.5) on $x(t)$ and $E(t)$. But in fact this is precisely the problem of the received dynamic model of the commercial fishery [**8**, **10**]. We see therefore that the received theory assumes, whether explicitly or implicitly, that capital is perfectly malleable. The consequences of relaxing this assumption will become clear in the sequel.

The maximization problem for equation (3.3) subject to (2.1) has a particularly simple solution [**6**, **8**]. Namely, there exists an optimal biomass level $x=x^*$, determined by the "modified golden rule":

$$(3.5) \qquad F'(x^*)-\frac{c'_{\text{total}}(x^*)F(x^*)}{p-c_{\text{total}}(x^*)}=\delta$$

where $c_{\text{total}}(x)$ denotes unit harvesting costs:

$$(3.6) \qquad c_{\text{total}}(x)=\frac{c_{\text{total}}}{qx}.$$

The left-hand side of equation (3.5) is simply the own rate of interest of the resource biomass x^*; the second term is referred to as the marginal stock effect [**8**]. As an alternative, equation (3.5) can be written in the form

$$(3.7) \qquad \frac{1}{\delta}\left[\frac{d}{dx^*}\{(p-c_{\text{total}}(x^*))F(x^*)\}\right]=p-c_{\text{total}}(x^*)$$

where the left-hand side can be interpreted as the shadow price or imputed demand price of the biomass and the right-hand side as the supply price of the biomass (see [**8**, pp. 95–96]).

The optimal approach to x^* from a non-optimal initial biomass level $x^0\neq x^*$ is the "most rapid" [**18**] or "bang-bang" [**8**] approach:

$$(3.8) \qquad E(t)=\begin{cases}E_{\max} & \text{whenever} \quad x(t)>x^*,\\ E_{\min} & \text{whenever} \quad x(t)<x^*,\end{cases}$$

where in our present model $E_{\min}=0$ and $E_{\max}$ is an ad hoc upper bound.

It is interesting to note in passing the rather extreme policy implications of (3.8). Suppose, for example, that the fishery being subject to an optimal management policy has hitherto been an open-access fishery with the consequence that $x(0)\ll x^*$. According to (3.8) the appropriate policy would be the drastic one of shutting the fishery down entirely until the biomass has recovered to the level x^*. (If for political or other reasons not considered here, it is practically impossible to shut the fishery down entirely, then condition (3.8) prescribes the reduction of E to $E_{\min}>0$ until the biomass grows to x^*.)

Finally we consider for the sake of future reference the (limiting) case in which capital is "free" in the sense that $\pi = \pi_s = 0$. In this case the optimal biomass level becomes $x = \tilde{x}$, where $\tilde{x}$ is determined from the equation

$$F'(\tilde{x}) - \frac{c'(\tilde{x})F(\tilde{x})}{p - c(\tilde{x})} = \delta \tag{3.9}$$

or

$$\frac{1}{\delta} \cdot \frac{d}{d\tilde{x}}\{(p - c(\tilde{x}))F(\tilde{x})\} = p - c(\tilde{x}) \tag{3.10}$$

where

$$c(x) = \frac{c}{qx}. \tag{3.11}$$

Thus operating costs alone are relevant to the determination of $\tilde{x}$. Assuming that x^* and $\tilde{x}$ are uniquely determined by these equations we have

$$\tilde{x} < x^*. \tag{3.12}$$

The role of the two biomass levels $\tilde{x}$ and x^* will become clear in the following discussions.

4. NON-MALLEABLE CAPITAL

We turn now to the main problem, that of delineating the optimal harvest and investment policies under the assumption that capital is non-malleable; see equation (2.8). As will become apparent, the form of the solution is complicated significantly by this assumption. In this and the next two sections we present a description of the optimal solution and a discussion of some of its implications, without attempting to give a proof of optimality. The proof is discussed in detail, however, in Section 7.

We commence with the easiest of the non-malleable capital cases, that in which capital is perfectly non-malleable. In this section, therefore, we shall assume that

$$\gamma = 0. \tag{4.1}$$

A. *The Unexploited Fishery*

First we consider the case of an initially unexploited fishery resource:

$$x^0 = \bar{x} \quad \text{and} \quad K^0 = 0. \tag{4.2}$$

We also assume that $\bar{x} > x^*$, where x^* is defined by equation (3.5), for otherwise no development of the fishery proves to be worthwhile.

Let us refer to Figure 1, which constitutes a feedback control diagram (in the $x - K$ state space) for the optimal harvest and investment policy, in the case of perfectly non-malleable capital ($\gamma = 0$; $I \geqslant 0$). The optimal values of E and I are given in this figure as functions of the current state variables $x = x(t)$ and

$K=K(t)$. The arrows in Figure 1 show the time motions (trajectories) of an optimally controlled biomass-capital system.

Notice that there are three classes of optimal policy, which are utilized, respectively, in the three subregions R_1, R_2 (shaded region), R_3, viz.

$$\begin{aligned} &\text{in } R_1: && E=I=0, \\ (4.3) \qquad &\text{in } R_2: && E=E_{\max}=K, \quad I=0, \\ &\text{in } R_3: && I=+\infty. \end{aligned}$$

In addition there are certain equilibrium positions, indicated by the heavy curve BCD in the figure. The biomass levels $\tilde{x}$ and x^* (and the corresponding levels of capital $\tilde{K}=F(\tilde{x})/q\tilde{x}$ and $K^*=F(x^*)/qx^*$) play an important role in the solution.

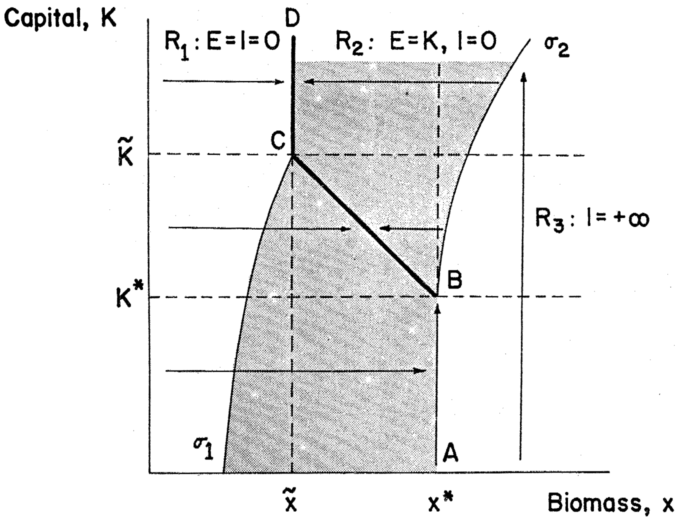

FIGURE 1

The initial point (4.2) lies in region R_3. Consequently the optimal policy requires a once-and-for-all jump increase in the level of capital, up to the level K specified by the "switching curve" σ_2. By assumption, this investment is completed instantaneously at time $t=0$. Note that $K>K^*$, so that the optimal level of capital exceeds the long-run optimum level that would prevail under conditions of perfect malleability (but not necessarily the short-run optimum).

Once the investment occurs at $t=0$, capital costs cease to be relevant. The optimal biomass level thus becomes the "free capital" level, $\tilde{x}$. If the optimal capital stock is $K>\tilde{K}$, we set $E(t)=K$, reducing the biomass stock, $x(t)$, as rapidly as possible, until reaching the optimal stock level $\tilde{x}$. Upon reaching $\tilde{x}$, we then set $E=\tilde{K}$ and henceforth harvest $h=F(\tilde{x})$ on a sustained-yield basis. In a sense the fishery is then "overcapitalized," in that the excess capital $K-\tilde{K}$ is redundant.

The overcapitalization, however, is only *ex post* and not *ex ante*, since the initial level of capital K is assumed to have been optimally determined.

Whether in fact $K > \tilde{K}$, given that $x^0 = \bar{x}$, depends on the parameters of the model. If $K < \tilde{K}$, then the optimal biomass level $\tilde{x}$ is not attainable at the effort rate $E = K$. The optimal policy calls for setting $E = K$ for all $t > 0$, and an equilibrium level lying between $\tilde{x}$ and x^* (see Figure 1) will thus be approached. In a sense we are "trapped" in that it neither pays to expand the capital stock sufficiently to make $\tilde{x}$ a feasible equilibrium biomass level, nor does it pay to build up the biomass level to x^*.

How is the switching curve σ_2, which determines the optimal level of capitalization (given that $x^0 > x^*$), determined? Consider a point (x, K) in the state space, with $K > K^*$ and such that (x, K) lies to the right of the line BCD. Let $S(x, K)$ denote the "return function," starting at time $t = 0$ at the point (x, K), and using the policy just described, i.e., $I = 0$ for all $t \geqslant 0$, and $E = K$ as long as $x > \tilde{x}$, otherwise $E = \tilde{K}$. Thus $S(x, K)$ is simply the present-value integral

$$S(x, K) = \int_0^\infty e^{-\delta t}\{pqx - c\}E(t)\,dt,$$

where $E(t)$ is as above, and $x(t)$ is the corresponding biomass, given by equation (2.1). The switching curve σ_2 is then determined by the equation

$$\frac{\partial S(x, K)}{\partial K} = \pi. \tag{4.4}$$

Equation (4.4) is simply the usual Keynesian investment rule.

B. *The Overexploited Fishery*

Consider now the case of an initially overexploited fishery:

$$x^0 < \tilde{x}. \tag{4.5}$$

Clearly a policy of recovery, or rehabilitation, of the fish stock is indicated in these circumstances. The only questions to be answered are the extent of the rehabilitation and the desired speed of recovery. Not surprisingly in light of the previous discussion the answers will depend upon the initial level of capital, K^0.

Let it be supposed first that $K^0 > \tilde{K}$. The cost of initial investment is a bygone and is thus irrelevant to future decisions. Moreover, the capital stock K^0 is abundant in the sense that there is more than sufficient capital to permit us to harvest $F(\tilde{x})$ on a sustained basis. Hence the optimal harvest policy is to set $E = 0$ until the biomass has grown to $\tilde{x}$ and then to set $E = \tilde{K}$, i.e., to harvest $F(\tilde{x})$ on a sustained basis. From our discussion in Section 3 we recognize that at $x = \tilde{x}$ the demand price of the biomass is equal to its supply price, given that the only relevant effort costs are operating costs.

Next let it be supposed that $K^* < K^0 < \tilde{K}$. (To consider a practical example, the fishery may hitherto have been subject to uncontrolled international exploitation, but is now encompassed by a coastal state's exclusive economic zone. Internal political pressures compel the coastal state to exclude all foreign vessels. The remaining domestic fleet constitutes K^0.) The initial stock of capital is no longer abundant, because it is insufficient to harvest $F(\tilde{x})$ on a sustained basis. The capital stock can be built up to $\tilde{K}$, but only at a cost of π per unit. Yet, as will be proven in Section 7, at every point to the left of x^* the demand price of capital, which we have characterized as $\partial S(x, K)/\partial K$, will be less than the supply price. Hence investment is non-optimal, so that we will be prevented from harvesting on a sustained basis at $\tilde{x}$. We will thus be confronted with an enforced "conservationist" policy in which the biomass will necessarily rise above $\tilde{x}$ to an equilibrium level lying in the "trap" between $\tilde{x}$ and x^*. Moreover, as a consequence of the coming enforced conservation policy, it will no longer be optimal to refrain from harvesting until the biomass has grown to the level $\tilde{x}$. Rather, the optimal policy calls for switching from zero harvesting to maximum harvesting as $x(t)$ crosses the switching curve σ_1 (see Figure 1). This premature switching phenomenon is a common occurrence in linear optimal control problems.

Finally in the case that $K^0 < K^*$ we see that the biomass level eventually recovers to the level $x = x^*$. At this moment it becomes optimal to (suddenly) increase capital to the level K^*, thus establishing a long-run equilibrium at the point (x^*, K^*).

The switching curve σ_1 can be determined by the same method used above for σ_2. Specifically, for an initial position (x, K) to the left of line ABC, and below $K = \tilde{K}$, let $S(x, K)$ denote the return function corresponding to the policy $I = 0$, $E = K$ as long as x remains below x^*, together with an impulse jump in K to K^* if and when $x(t) = x^*$. The switching curve σ_1 is then given by the solution of the equation

$$\frac{\partial S(x, K)}{\partial x} = p - \frac{c}{qx}, \tag{4.6}$$

where the left-hand side and right-hand side represent the demand price (or shadow price) and the supply price of the biomass, respectively. The reason for premature switching thus becomes transparent. If we commence at a point to the left of σ_1 with $K < \tilde{K}$, it will pay to increase the biomass, but not to the same extent that it would if capital were abundant.

It is instructive to compare the above rehabilitation policy with the corresponding policy in the case of perfect malleability of capital, in which it is optimal to shut the fishery down entirely until the biomass has grown to x^*. If harvesting capital is perfectly non-malleable, the optimal policy is radically altered. It will then never pay to refrain from harvesting once the biomass reaches the lower level $\tilde{x}$. If $K < \tilde{K}$, it will be optimal to commence harvesting at the maximum rate even before $x(t)$ reaches $\tilde{x}$. The ultimate optimal equilibrium biomass level depends upon the initial stock of capital K^0.

If capital depreciates at a positive rate, these conclusions must be altered. Let us now turn to this more interesting case.

5. QUASI-MALLEABLE CAPITAL

In this section we consider the case

$$(5.1) \qquad \gamma > 0;$$

we continue to impose the constraint of non-negative gross investment, $I \geq 0$. Figure 2 is the feedback optimal control diagram for this case. The similarity with Figure 1 is apparent, but there are also significant differences.

As before, the state plane is divided into three control regions R_1–R_3, and the control law (4.3) again applies. The biomass level $\tilde{x}$ is the same as before, but x^* (which is determined by the total cost $c_{\text{total}} = c + \pi(\delta + \gamma)$) has moved to the right. The most important new feature of Figure 2, however, concerns the dynamic behavior of the system. Because of the depreciation of capital, all trajectories

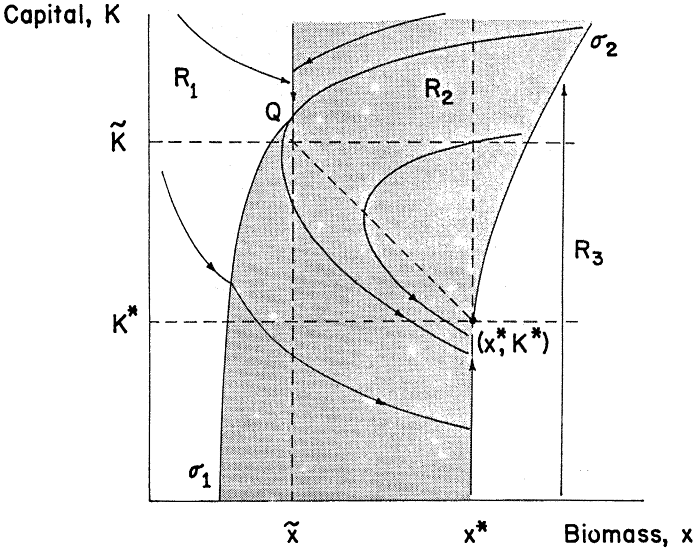

FIGURE 2

(outside of R_3) now move downwards with the passage of time. It follows that, in contrast to the perfectly non-malleable case, the system now tends to a uniquely determined long-run equilibrium, at (x^*, K^*). For this reason, it is appropriate to refer to x^* as the *long-run optimum* biomass level, and to $\tilde{x}$ as a (possible) *short-run optimum* biomass level.

The description and interpretation of the optimal harvest and investment policies from any initial position is now straightforward. Two aspects of the optimal policy bear emphasizing, however. First note that the switching curve σ_1 now meets the line $x = \tilde{x}$ at the point Q above $K = \tilde{K}$. The implication of this is as follows. Consider the system in short-run equilibrium with $x = \tilde{x}$ and (x, K) lying above Q. Although the existing stock of capital is abundant, the abundance is strictly temporary, because of depreciation. Anticipation of the coming "shortage" of capital—in the sense that the singular solution $x(t)=\tilde{x}$ will eventually become non-feasible—leads to a switch in the harvesting policy at the point Q. The switch is to a policy of harvesting at maximum rate ($E=K$), *even though this switch will cause* $x(t)$ *to fall temporarily below the short-run optimum* $\tilde{x}$.

The second and more important aspect concerns the general problem of restoration of an overexploited fishery. Although the optimal restoration policy ultimately leads to a long-run equilibrium at $x=x^*$ and $K=K^*$, and although this equilibrium is the same as for the case of perfectly malleable capital, the restoration policies in the two cases are notably different. In the malleable case, the optimal restoration policy requires a complete moratorium. In the case that capital is quasi-malleable, however, the optimal approach to x^* is far more gradual. Indeed, after a certain point it will be optimal to harvest at the maximum rate with the existing stock of capital. In other words, the disruptive consequences of a fishing moratorium can only be considered optimal in the case that fishing vessels have viable alternative uses, except that a brief moratorium may be optimal if the fish stock is very severely depleted (i.e., to the left of σ_1).

6. A MARKET FOR SCRAP

We turn next to the alternative model specified by equations (2.10)–(2.13). Whereas the original model was linear in both control variables E, I, this alternative model displays a minor but significant nonlinearity, inasmuch as the function $\phi(I)$ consists of two linear segments of unequal slope π_s and π, respectively. As shown elsewhere [7] in a simpler setting, this nonlinearity gives rise to a new type of optimal control ("corner control"), which persists for time intervals during which $I(t)\equiv 0$.

Let x_s denote the solution to the equation

$$(6.1)\qquad F'(x)-\frac{c_s'(x)F(x)}{p-c_s(x)}=\delta$$

where

$$(6.2)\qquad c_s(x)=\frac{c+(\delta+\gamma)\pi_s}{x}.$$

Thus x_s represents the optimal biomass level for a model of perfectly malleable capital with price (purchase price *and* unit scrap value) equal to π_s. Since $0<\pi_s<\pi$ we have

$$(6.3)\qquad \tilde{x}<x_s<x^*.$$

The feedback control diagram for this model is given in Figure 3. This diagram is almost identical to that of Figure 2 except for the presence of an additional "overlay" disinvestment region R_4, bounded by the line $x=x_s$ and a new switching curve σ_3. (Depending on parameter values, σ_3 may cross σ_1 as shown, or the two curves may fail to intersect.) The optimal harvest and investment policies are also identical with those of the previous model, except for (a) initial positions (x, K) lying within R_4 and (b) initial positions that lead to trajectories that penetrate R_4, i.e., that hit the line $x = x_s$ above the point Q_s.

In case (a) the optimal policy requires immediate disinvestment, down to the level given by σ_3. Following this initial disinvestment, the policy given in Section 5 is optimal. In case (b), an impulse disinvestment must be undertaken, to the level Q_s, at the instant when $x(t)$ reaches x_s.

The optimal disinvestment policy is thus essentially symmetric to the optimal investment policy of region R_3. This of course reflects the close symmetry of the present model with respect to investment and disinvestment. (Region R_2, by the way, is an instance of "corner" control mentioned above.)

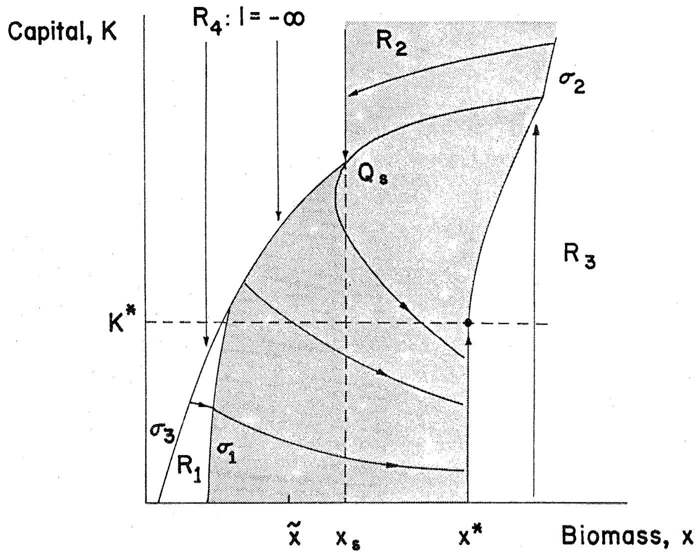

FIGURE 3

The switching curve σ_3 is determined from the equation $\partial S/\partial K = \pi_s$.

The rationale behind this solution is quite straightforward. At any given time, the decision must be taken whether to disinvest, to invest, or to do neither. Just as investment is indicated when there are "too few" vessels and "too many" fish, so is disinvestment indicated only when these conditions are reversed. Like investment, disinvestment turns out to be a once-and-for-all decision (there is an exception regarding investment, when $x(t)$ returns to x^*). If the decision is taken not to disinvest, then the optimal policy goes ahead *exactly as if the disinvestment opportunity did not exist.* Disinvestment may occur later, namely if the trajectory later hits $x = x_s$ above Q_s.

Let us also note that, because of the disinvestment opportunity, there is now no short-run optimal equilibrium biomass level. (In control-theoretic language, x_s is not a singular solution for this model, although $\tilde{x}$ is such for the previous model.) From an arbitrary initial position the system ultimately converges to the same long-run equilibrium solution (x^*, K^*) as in the case of quasi-malleable capital.

The proof of optimality for this model is outlined in the next section.

7. PROOF OF OPTIMALITY

In this section we shall prove that the policy described above is indeed optimal for the problem at hand. The method we shall employ is one of verification rather than deduction. The solution was arrived at by the study of necessary conditions, a knowledge of the solution when K is constant, and a certain amount of trial and error. However, we shall not give a detailed account of this procedure; instead we simply prove optimality. A word about the necessary conditions: it is not possible to apply the Pontryagin maximum principle to the problem, since this principle is not valid when the permissible values of the controls (E, I) depend on the current value of the state variables (x, K), as they do here $(E \leqslant K)$. We used instead the more general formulation of "differential inclusions" and some recent results (see **[9]** for a complete discussion) to first observe the existence of the two distinguished ("singular") values $\tilde{x}$ and x^* (see Section 3 above).

We remark in passing that the methods developed here should prove useful in other control-theoretic problems for which the maximum principle may prove inappropriate. For example, an exhaustible resource optimization model similar to our model has been discussed by Puu **[16]**; our method can be used to complete and correct the proposed solution for this model. A recent model of capital accumulation and durable goods production (Kamien and Schwartz **[13]**) also has a similar structure—and similar properties of optimal policies—to our model; it seems likely that our approach could be applied to models of this kind.

The verification technique we shall employ would be recognized by the expert as an adaptation of a classical approach in the calculus of variations sometimes labeled "the royal road of Carathéodory" (see, for example, **[4]**). It has the advantage of being simple-minded and elementary. Of course, it depends upon knowing the answer in the first place! The basic idea is the following: let a harvesting-investment policy be specified from any initial value (x, K), and let the resulting net discounted return (see (2.9)) be denoted $S(x, K)$. Suppose that for all values of (x, K), for all E in $[0, K]$ and $I \geqslant 0$, the following inequality holds:

$$\begin{aligned}(7.1)\qquad \delta S(x, K)+E[qxS_x(x, K)-pqx+c]-F(x)S_x(x, K)+\gamma KS_K(x, K)\\ +I(\pi-S_K(x, K))\geqslant 0\end{aligned}$$

where S_x and S_K denote partial derivatives. Under these conditions, we claim that the given policy is optimal. To see this, let any other control policy $E(t)$, $I(t)$ be given, with I finite. Then, if (x_0, K_0) is the starting point,

$$\begin{aligned}
&\int_0^\infty e^{-\delta t}\{(pqx-c)E-\pi I\}\,dt - S(x_0, K_0)\\
&\quad=\int_0^\infty e^{-\delta t}\{(pqx-c)E-\pi I\}\,dt+\int_0^\infty \frac{d}{dt}\{e^{-\delta t}S(x(t), K(t))\}\,dt\\
&\quad=\int_0^\infty e^{-\delta t}\{(pqx-c)E-\pi I-\delta S(x, K)+S_x\dot{x}+S_K\dot{K}\}\,dt\\
&\quad=-\int_0^\infty e^{-\delta t}\{\delta S+E[qxS_x-pqx+c]-F(x)S_x+\gamma KS_K+I(\pi-S_K)\}\,dt\\
&\quad\leqslant 0 \quad \text{(since the integrand is always nonnegative).}
\end{aligned}$$

But this says that the return from any other policy does not exceed the return from our given policy (it suffices to know this for policies with finite investment rates to conclude optimality).

In summary, one can verify the optimality of a policy by producing a function S with the properties mentioned above. We shall now do just this for the policy described in the preceding sections. There remains also the task of precisely defining the switching curves, and there will be an added complication due to the fact that S will sometimes fail to be differentiable along these curves, but the underlying idea remains unchanged.

We assume henceforth that δ, $\gamma>0$ (the case $\gamma=0$ is simpler and can be treated with minor modifications). The constraint $I\geqslant 0$ is assumed. We also set $q=\pi=1$, which merely amounts to a scaling of the variables E and K, but simplifies the notation. We assume that F is twice continuously differentiable and satisfies $F''<0$ in the interval $(0, \bar{x})$, in which F is positive. This says that F is strictly concave, and assures among other things that $K=F(x)/x$ defines K as a strictly decreasing function of x.

Now let (see (3.5), (3.9))

$$\tilde{\phi}(x)=\delta-F'(x)+c'(x)F(x)/[p-c(x)],$$

$$\phi^*(x)=\delta-F'(x)+c'_{\text{total}}(x)F(x)/[p-c_{\text{total}}(x)].$$

We assume the following: the equation $\tilde{\phi}(x)=0$ has a unique solution $\tilde{x}$ in the interval $(x_\infty, \bar{x})$; for $x>\tilde{x}$ we have $\tilde{\phi}(x)>0$, and for $x<\tilde{x}$, $\tilde{\phi}(x)<0$; $p\tilde{x}-c>0$. These conditions are easily seen to hold if the marginal stock effect (last term on the right-hand side) is a decreasing function of x. We make similar assumptions regarding ϕ^*, denoting the solution to $\phi^*(x)=0$ by x^*, which is necessarily greater than $\tilde{x}$. We let $\tilde{K}=F(\tilde{x})/\tilde{x}$, $K^*=F(x^*)/x^*$.

Let us remark before proceeding to the systematic construction of the optimal policy and corresponding return that all our hypotheses are satisfied (for a suitable range of parameters) if F is the familiar logistic growth function, i.e., if

$$F(x)=rx(\bar{x}-x).$$

Finally we note the interpretation of (2.9) when $I=\infty$ is allowed: set $t_0=0$, and suppose jumps in the value of K occur at times $t_0, t_1, t_2, \ldots$. If we denote the values immediately before and after these jumps by $K(t_i-)$ and $K(t_i+)$, respectively, and if a finite investment rate $I(t)$ is employed between the jumps, then (2.9) is given by

$$\int_0^\infty e^{-\delta t}\{(px-c)E-I\}\,dt-\sum_{i=0}^{\infty} e^{-\delta t_i}[K(t_1+)-K(t_i-)]. \tag{7.2}$$

We let C_1 be the locus of points (x_0, K_0) in the part of the (x, K)-plane $x \geqslant x^*$, $K \geqslant K^*$ such that the trajectory $x(t)$, $K(t)$ originating from (x_0, K_0) with $E(t)=K(t)$, $I(t)=0$ passes through (x^*, K^*) (see Figure 4). It follows that for any (x, K) such that $0<x<x^*$, $K>0$, or such that $x \geq x^*$ and (x, K) lies above C_1, the policy $E=K$, $I=0$ will result after finite time in arriving at $x(t)=x^*$, $0<K(t)\leqslant K^*$ (a sample trajectory is indicated in Figure 4). Let this arrival time be denoted $\tau(x_0, K_0)$, and let K be increased to K^* at time τ; for $t \geqslant \tau$ we remain at (x^*, K^*) by setting $E=E^*$, $I=\gamma K^*$. We denote the net discounted return from this policy $S(x, K)$.

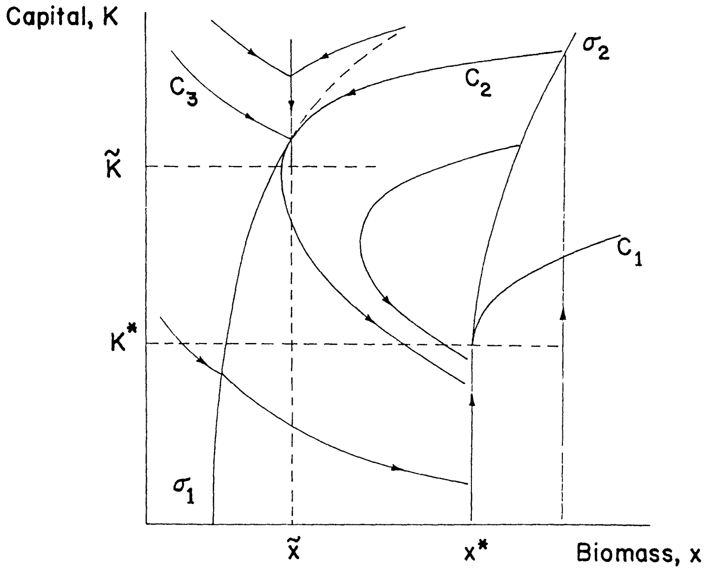

FIGURE 4

LEMMA 1: *In the interior of its domain of definition, S is given by*

$$S(x, K)=K+\int_0^\infty e^{-\delta t}\{px(t)-c-\delta-\gamma\}K(t)\,dt;$$

$S(x, K)$ *is twice continuously differentiable and satisfies*

$$\delta S+(Kx-F)S_x+\gamma K S_K-K(px-c)=0.$$

PROOF: Using the notation established for (7.2), we have by integration by parts:

$$\int_0^\infty e^{-\delta t} I\,dt = \int_0^\infty e^{-\delta t}(\dot{K} + \gamma K)\,dt$$
$$= \int_0^\infty e^{-\delta t}(\delta + \gamma)K(t)\,dt + \sum_{i \geqslant 1} e^{-\delta t_i} K(t_i -)$$
$$- \sum_{i \geqslant 1} e^{-\delta t_{i-1}} K(t_{i-1}+).$$

When this is substituted into (7.2), along with $E(t) = K(t)$, we obtain the stated expression (all the discrete summation terms cancel except $e^{-\delta t_0}K(t_0 -) = K$).

It is a consequence of the implicit function theorem that τ is a twice continuously differentiable function of (x, K), and it is also known that $x(t)$ and $K(t)$ are (for each $t \leqslant \tau$) twice continuously differentiable functions of their initial value (x, K). When we make use of the preceding formula to write

$$S(x, K) = K + \int_0^\tau e^{-\delta t}\{px - c - \delta - \gamma\}K\,dt + e^{-\delta \tau}\{px^* - c - \delta - \gamma\}K^*/\delta,$$

the twice continuous differentiability of S becomes apparent.

If we proceed to observe the identity

$$e^{-\delta t}S(x(t), K(t)) = \int_0^t e^{-\delta s}\{px - c\}K(s)\,ds + S(x, K)$$

(which holds for $t < \tau$), differentiate with respect to t, and then set $t = 0$, we obtain the required partial differential equation. *Q.E.D.*

We shall show presently that the equations

$$S_x = p - c/x,$$
$$S_K = 1,$$

define two curves σ_1 and σ_2, respectively, located essentially as indicated in Figure 4.

LEMMA 2: $S_K < 1$ *for* $x < x^*$, *and for points* (x^*, K) *with* $K > K^*$.

PROOF: If in the formula for S derived in the preceding lemma we substitute $K(t) = F(x(t)) - \dot{x}(t)$, we see that

$$S(x, K) = K + \int e^{-\delta t}\{p - (c + \delta + \gamma)/x\}F(x)\,dt$$
$$- e^{-\delta t}\{p - (c + \delta + \gamma)/x\}\,dx,$$

where the line integral is taken in the (t, x)-plane along the curve $x = x(t)$, $t \geqslant 0$. If we now increase K to $K + \Delta$, the graph of the resulting $x(t)$ will lie below that of the original $x(t)$ for all t, and using Green's theorem we obtain

$$S(x, K+\Delta)-S(x, K)=\Delta+\iint e^{-\delta t}\{p-(c+\delta+\gamma)/x\}\phi^*(x)\,dt\,dx,$$

where the double integral is taken over the (compact) region in the (t, x)-plane between the graphs. Since $\phi^*(x)<0$ in that region, we obtain immediately $S_K(x, K)\leqslant 1$. An elementary estimate shows that the double integral is bounded above by a term $-\varepsilon\Delta$ ($\varepsilon>0$), so that we have in fact $S_K<1$. *Q.E.D.*

LEMMA 3: *$S_x<p-c/x$ for $x\geqslant\tilde{x}$, $K\leqslant\tilde{K}$.*

PROOF: Let the trajectory originating from (x, K) be denoted as usual $x(t)$, $K(t)$, and let the trajectory originating from $(x+\Delta, K)$ ($\Delta>0$) be denoted $x_1(t)$, $K_1(t)$. If we proceed as in the previous lemma, keeping account of the line segment from x to $x+\Delta$ in the (t, x)-plane which together with the graphs of $x(t)$ and $x_1(t)$ comprise our closed curve, we obtain

$$\begin{aligned} S(x, K)-S(x+\Delta, K) &= \oint e^{-\delta t}\{p-c/x\}F(x)\,dt-e^{-\delta t}\{p-c/x\}\,dx \\ &\quad+\int_0^\infty e^{-\delta t}(\delta+\gamma)(K_1(t)-K(t))\,dt \\ &\quad+\int_x^{x+\Delta}-(p-c/x)\,dx \\ &\geqslant\iint e^{-\delta t}\{p-c/x\}\tilde{\phi}(x)\,dt\,dx-\int_x^{x+\Delta}(p-c/x)\,dx, \end{aligned}$$

since $K_1(t)\geqslant K(t)$.

This implies

$$-S_x\geqslant-(p-c/x),$$

since $\tilde{\phi}>0$ in the region in question. Strict inequality follows as in Lemma 2. *Q.E.D.*

From Lemma 3 and the facts that S_x is bounded (as is easily shown) and $p-c/x$ tends to $-\infty$ as x decreases to 0, we deduce that the locus of points satisfying $S_x=p-c/x$ defines a curve σ_1 which lies between the lines $x=0$ and $x=\tilde{x}$ for $K\leqslant\tilde{K}$. We shall prove later (see Lemma 4, Corollary) that σ_1 crosses the line $x=\tilde{x}$ at one point $(\tilde{x}, \bar{K})$ with $\bar{K}>\tilde{K}$.

LEMMA 4: *Along σ_1, we have*

$$(F(x)-Kx)(S_{xx}-c/x^2)-\gamma KS_{xK}=(p-c/x)\tilde{\phi}(x),$$

and any trajectory employing $E=K$, $I=0$ meets σ_1 at most once for $x<\tilde{x}$.

PROOF: The above equation follows immediately upon differentiating the equation of Lemma 1 with respect to x and using the equality $S_x=p-c/x$. Since

$\tilde{\phi}(x)<0$ when $x<\tilde{x}$, the above equality implies that whenever any trajectory as stated meets σ_1, we have at that point

$$\frac{d}{dt}[S_x(x(t), K(t))-p+c/x(t)]<0.$$

Thus the quantity in brackets can only vanish once along the trajectory. *Q.E.D.*

COROLLARY: *σ_1 crosses the line $x=\tilde{x}$ at a unique point $(\tilde{x}, \bar{K})$ with $\bar{K}>\tilde{K}$; when $Kx>F(x)$ the graph of σ_1 is that of a function $K=g(x)$.*

PROOF: Let σ_1 cross the graph of $K=F(x)/x$ at $x=x_1$. It follows from Lemma 4 that S_{xK} is positive at that point. If one recalls that the normal vector to σ_1 (in the direction of increasing S_x) is $[S_{xx}-c/x^2, S_{xK}]$, we see from Lemma 4 that for $x>x_1$, σ_1 defines a function $K=g(x)$, since its tangent vector can never be vertical.

It follows that g cannot grow to ∞, but must have a finite value $g(\tilde{x})=\bar{K}>\tilde{K}$ at $\tilde{x}$ (otherwise some trajectory $x(t)$, $K(t)$ would meet σ_1 twice).

We let C_2 be the locus of points $x(t)$, $K(t)$ (for $x(t)\geqslant\tilde{x}$) lying on the trajectory which has $E=K$, $I=0$ and which passes through $(\tilde{x}, \bar{K})$. The next result follows from Lemma 4 as did the above Corollary.

LEMMA 5: *C_2 lies below σ_1 (for $x\geqslant\tilde{x}$).*

Let C_3 (see Figure 4) be the locus of points $x(t)$, $K(t)$ (for $x(t)\leqslant\tilde{x}$) lying on the trajectory which has $E=0$, $I=0$ and which passes through $(\tilde{x}, \bar{K})$. We define the policy above C_3 as follows: employ $E=0=I$ until $x(t)=\tilde{x}$, then use $E=F(\tilde{x})/\tilde{x}$ and $I=0$ until $K=\bar{K}$, then proceed according to the earlier policy. The resulting return is $S(x, K)$ on the region in question.

LEMMA 6: *Above C_3, S is twice continuously differentiable, $S_K\leqslant 1$, $S_x\geqslant p-c/x$, and S satisfies*

$$\delta S-F(x)S_x+\gamma KS_K=0.$$

PROOF: The smoothness of S and the equation that it satisfies follow as in Lemma 1. The inequality involving S_x may be proven by the method of Lemma 3, leaving only the verification that $S_K\leqslant 1$.

For each (x, K) in the region in question, let $T(x, K)$ denote the time t at which the trajectory $x(t)$, $K(t)$ beginning at x, K and using $E=0$, $I=0$ arrives at $x(t)=\tilde{x}$. Then we have

$$S(x, K)=e^{-\delta T}S(\tilde{x}, K e^{-\gamma T}).$$

Consequently we find (note $\partial T/\partial K=0$)

$$S_K(x, K)=e^{-\delta T}S_K(\tilde{x}, K e^{-\gamma T})\, e^{-\gamma T}.$$

Since it is easy to see that $S_K \leq 1$ when $x = \tilde{x}$, we conclude that $S_K(x, K) \leq 1$ for all x, K. *Q.E.D.*

Our next redefinition occurs in the region bounded by $x = 0$, σ_1 and C_3. There we employ $E = 0$, $I = 0$ until we reach σ_1, and after that we switch to $E = K$, $I = 0$ and proceed as before. Note that Lemma 4 assures that this policy is well defined. The proof of the following (which parallels that of Lemma 6) is omitted.

LEMMA 7: *In the above region, S is twice continuously differentiable, $S_K \leq 1$, $S_x \geq p - c/x$, and*

$$\delta S - F(x)S_x + \gamma K S_K = 0.$$

We next study the situation in the region lying to the right of $x = x^*$.

LEMMA 8: *Along C_1 we have*

$$S(x, K) - [K - F(x)/x] > (px - c - \gamma)(F(x)/x)/\delta.$$

PROOF: We first observe that the left-hand side is the return obtained from starting at $(x, F(x)/x)$ if we immediately increase the number of boats to K and use our stated policy from then on, while the right-hand side is the return obtained from using $E = F(x)/x$, $I = \gamma K$ and thus staying at (x, K). Proceeding as in Lemma 2, we may express the difference between these two returns as

$$\iint e^{-\delta t}\{p - (c + \delta + \gamma)/x\}\phi^*(x)\, dt\, dx,$$

which is positive in the region in question since $x > x^*$. *Q.E.D.*

LEMMA 9: *Along C_1 ($x > x^*$) we have $S_K > 1$.*

PROOF: It is possible to use classic theorems in differential equations to calculate $\partial x(t)/\partial K_0$, where $x(t)$ is the value at time t of the solution of our differential equation (with $E = K$, $I = 0$) and K_0 is the initial value of K. We remark only that at $t = 0$, this partial derivative is known to be zero. If we differentiate (with respect to K_0) the expression for S given in Lemma 1, it then follows that we obtain an expression which along C_1 is of the form

$$1 + \int_0^\tau f(t)\, dt,$$

where the integrand is recognizably positive for sufficiently small t. It follows that $S_K > 1$ when (x_0, K_0) lies on C_1, $x > x^*$, and τ is sufficiently small (i.e., when we are close to (x^*, K^*)).

Now let σ_2 be the curve $S_K = 1$. From Lemma 2 and the above, it follows that σ_2 lies strictly between C_1 and the line $x = x^*$, at least in a neighborhood of (x^*, K^*).

If we differentiate the equation of Lemma 1 with respect to K, set $S_K = 1$ and use the inequality of Lemma 8 (which also holds along σ_2 because $S_K \geqslant 1$ between σ_2 and C_1) we obtain

$$S_{Kx}(F(x)-Kx)-S_{KK}\gamma K<0$$

along σ_2. This shows that no trajectory can intersect σ_2 twice; in particular we conclude that σ_2 is strictly above C_1 for $x>x^*$. *Q.E.D.*

As noted above, we have shown:

COROLLARY: *Any trajectory using $E=K$, $I=0$ meets σ_2 at most once.*

Let us observe in passing that Lemma 9 shows that an optimal policy always results in temporary "overcapacity" under the circumstances that $x_0>x^*$.

There are now two cases that present themselves, depending on whether σ_2 intersects C_2 at a point P having $x<\bar{x}$, or whether σ_2 lies below C_2 in the region $x^*<x<\bar{x}$. We shall discuss the latter case, where it suffices to discuss the definition of S below σ_2 and above C_2. (In the former case the redefinition of S above C_2 and the left of P necessitates a redefinition of the switching curve σ_2 above C_2.)

Above C_2, we redefine our policy, and hence S, as follows: employ $E=K$, $I=0$ until $x=\tilde{x}$, then proceed from that point as previously defined (preceding Lemma 6). The resulting net discounted return is $S(x, K)$ for that region. Below σ_2, we immediately increase the number of boats to the value placing us on σ_2, and then we proceed as per our previously adopted policy. Note that by the preceding Corollary, we do not encounter σ_2 again.

LEMMA 10: *Above C_2, we have $S_x \leqslant p-c/x$, $S_K \leqslant 1$, and $\delta S+(Kx-F)S_x+\gamma KS_K-K(px-c)=0$. Below σ_2, we have $\delta S+(Kx-F)S_x+\gamma KS_K-K(px-c)\geqslant 0$, $S_x<p-c/x$ and $S_K=1$.*

PROOF: The first two inequalities may be proven by much the same arguments as in Lemmas 2 and 3, and the equation also follows as before. To prove the latter set of inequalities, we use the fact that σ_2 defines a function $K=h(x)$ for $x>x^*$. (This follows from the inequality $S_{KK}<0$, which says that the marginal value of boats decreases as the number of boats increases.) Thus below σ_2, S is given by

$$S(x, K)=S(x, h(x))-(h(x)-K),$$

and we see immediately $S_x<p-c/x$, $S_K=1$. The remaining inequality is then seen to be equivalent to

$$\delta(S(x, h(x))-h(x)+K)+(Kx-F)S_x(x, h(x))+\gamma K-K(px-c)\geqslant 0,$$

which we now establish. The derivative of the left-hand side with respect to K is $\delta+\gamma+c-px+xS_x(x, h(x))$. The inequality of Lemma 8 (which holds also along σ_2 as noted previously) and the equation of Lemma 1 imply that this last term is nonpositive. Thus it suffices to prove the required inequality when $K = h(x)$. But in that case it is already known (in fact, equality holds by Lemma 1). *Q.E.D.*

We have now defined a function S everywhere in the (x, K)-plane, and a corresponding policy of harvest and investment for which S is the return. We have seen that S is a smooth function except possibly along a finite number of curves (where its method of definition changes), and S is everywhere continuous. If we consider the optimality argument following (7.1), we see that it is unimpaired as long as the trajectories $x(t)$, $K(t)$, being otherwise arbitrary, are such that $S(x(t), K(t))$ is differentiable with respect to t except for a finite number of points t. But the trajectories $x(t)$, $K(t)$ which only cross the curves where S is non-differentiable a finite number of times (and never travel along these curves) are dense in the space of all admissible trajectories (i.e., any trajectory can be approximated to any degree of closeness by one of this kind).

Consequently it suffices to know that our stated policy yields a return as good as any of these special trajectories; that is, it suffices to know that S satisfies (7.1) wherever it is differentiable. Thus the following concludes the proof:

LEMMA 11: *S satisfies* (7.1) *at all points not on the curves σ_1, σ_2, C_1, C_2, C_3 or the lines $x = \tilde{x}$, $x = x^*$.*

PROOF: In the region $x > x^*$ below σ_2, this is a consequence of Lemma 10. A perusal of all other cases will show that S always satisfies the equation

$$\delta S + K \min\{0, xS_x - px + c\} - FS_x + \gamma K S_K = 0$$

and that S_K is always less than or equal to 1. A moment's thought suffices to see that this implies (7.1). *Q.E.D.*

Finally we shall consider the alternative model discussed in Section 6. For this case, equation (7.1) must be modified by replacing the term $I(\pi - S_K)$ by

$$\psi(I) - IS_K = \begin{cases} I(\pi - S_K) & \text{for} \quad I > 0, \\ I(\pi_s - S_K) & \text{for} \quad I < 0. \end{cases}$$

To establish this modified inequality, we first use the equation

$$\frac{\partial S}{\partial K} = \pi_s$$

to define a new switching curve σ_3, the geometrical properties of which are established as in Lemma 4 above. The return function $S(x, K)$ is then redefined in region R_4 and also at all points (x, K) from which trajectories ultimately penetrate R_4.

We must then re-establish all of the inequalities proved above, and also verify that for all (x, K)

$$\pi_s \leqslant \frac{\partial S}{\partial K} \leqslant \pi.$$

However, the present function $S(x, K)$ is identical to the previous function $S(x, K)$ except for R_4 and points influenced by R_4. Hence it is only in these regions that further verifications are required. These additional verifications are sufficiently similar to those already discussed that we can safely leave them to the reader.

8. SUMMARY AND CONCLUSIONS

This paper has investigated the implications of restricted malleability of capital for the optimal exploitation of renewable resource stocks. While the study has been carried out on the basis of a specific model of the commercial fishing industry, we believe that the qualitative nature of our results will prove to be robust.

Under the non-malleability assumption the dynamics of the optimally controlled fishery can be described in terms of short-run versus long-run behavior. Over the long run (unless capital is perfectly non-malleable) the fishery reaches an equilibrium state corresponding to "optimum sustained yield," for which the relevant cost function incorporates the full cost of fishing, i.e., operating plus capital costs. Following the initial development of the fishery, however, there is a short-run phase during which capital is excessive (from the long-run viewpoint), and only operating costs are relevant to the management decision. The development of the fishery thus follows a complex pattern of expansion, "overcapacity," and gradual contraction via depreciation, leading ultimately to the OSY equilibrium. We emphasize again that this pattern is an optimal one under the assumptions of our model.

In deriving these results we have been forced to adopt several simplifying assumptions. Perhaps the most serious of these lies in the autonomous nature of our model. Practically speaking, variability of economic parameters over time is more the rule than the exception in renewable resource industries. We make no attempt to analyze the effects of such variations here (the malleable case has been discussed in [**8**]). Some information can be gleaned from a comparative dynamics approach, i.e., by studying the sensitivity of the solution to the parameters of the model. For example, it is easy to verify that the purchase price of capital π has no effect on the short-run equilibrium $\tilde{x}$, but affects x^* positively and also affects the switching curve σ_2 in a negative sense. Thus higher capital costs have no effect on "bygones," but decrease the optimal (ex ante) level of capitalization, and result in lower levels of exploitation over the long run. The effects of varying other parameters are also easily worked out.

Finally, the policy implications of our study are sufficiently clear from a qualitative viewpoint. On the one hand, the analysis supports the accepted belief that excessive capitalization is likely to occur during the initial development of a common-property resource, although a certain degree of overcapitalization is now shown to be generally acceptable. On the other hand, the analysis shows that extreme policies of stock rehabilitation (e.g., fishing moratoria), may be unwarranted unless the stock has become very severely depleted. The less transferable are capital assets, the more important this latter consideration becomes. (Along these lines, it is clear that non-transferability of labor would have similar implications.) The application of these findings to explicit resource-management problems will require additional research.

University of British Columbia

Manuscript received April, 1977; revision received November, 1977

REFERENCES

[1] ARROW, K. J.: "Optimal Capital Policy with Irreversible Investment," in *Value, Capital, and Growth: Papers in Honour of Sir John Hicks*, ed. by J. N. Wolfe. Edinburgh: Edinburgh University Press, 1968, pp. 1–20.

[2] ARROW, K. J., AND M. KURZ: *Public Investment, the Rate of Return, and Optimal Fiscal Policy*. Baltimore: Johns Hopkins Press, 1970.

[3] BEDDINGTON, J. R., C. M. K. WATTS, AND W. D. C. WRIGHT: "Optimal Cropping of Self-Reproducible Natural Resources," *Econometrica*, 43 (1975), 789–802.

[4] CARATHÉODORY, C.: *Calculus of Variations and Partial Differential Equations of the First Order* (English translation). San Francisco: Holden-Day, 1966.

[5] CHRISTY, F. T., JR., AND A. D. SCOTT: *The Common Wealth in Ocean Fisheries*. Baltimore: Johns Hopkins University Press, 1965.

[6] CLARK, C. W.: *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*. New York: Wiley-Interscience, 1976.

[7] ———: "Optimal Capital Policy and Scrap Value," to appear.

[8] CLARK, C. W., AND G. R. MUNRO: "The Economics of Fishing and Modern Capital Theory: A Simplified Approach," *Journal of Environmental Economics and Management*, 2 (1975), 92–106.

[9] CLARKE, F. H.: "Necessary Conditions for a General Control Problem," in *Calculus of Variations and Control Theory*, ed. by D. L. Russell. New York: Academic Press, 1976.

[10] CRUTCHFIELD, J. A., AND A. ZELLNER: "Economic Aspects of the Pacific Halibut Fishery," Fishery Industrial Research, Vol. 1, No. 1. Washington: U.S. Department of the Interior, 1962.

[11] GORDON, H. S.: "The Economic Theory of a Common-Property Resource: the Fishery," *Journal of Political Economy*, 62 (1954), 124–142.

[12] HADLEY, G., AND M. C. KEMP: *Variational Methods in Economics*. Amsterdam: North-Holland, 1971.

[13] KAMIEN, M. I., AND N. L. SCHWARTZ: "Optimal Capital Accumulation and Durable Goods Production," *Zeitschrift für Nationalökonomie*, 37 (1977), 25–43.

[14] NAGATANI, K., AND P. A. NEHER: "On Adjustment Dynamics—An Exercise in Traverse," in *Equilibrium and Disequilibrium in Economic Theory*, ed. by G. Schwödiauer. Dordrecht, Holland: D. Reidel, 1977, pp. 369–396.

[15] PELLA, J. J., AND P. K. TOMLINSON: "A Generalized Stock Production Model," *Bulletin of the Inter-American Tropical Tuna Commission*, 13 (1969), 421–496.

[16] PUU, T.: "On the Profitability of Exhausting Natural Resources," *Journal of Environmental Economics and Management*, 4 (1977), 185–199.

[17] SCHAEFER, M. B.: "Some Aspects of the Dynamics of Populations Important to the Management of the Commercial Marine Fisheries," *Bulletin of the Inter-American Tropical Tuna Commission*, 1 (1954), 25–56.

[18] SPENCE, M., AND D. STARRETT: "Most Rapid Approach Paths in Accumulation Problems," *International Economic Review*, 16 (1975), 388–403.

[19] UNITED NATIONS, FOOD AND AGRICULTURAL ORGANIZATION: *Economic Aspects of Fisheries Regulations*. Rome: F.A.O. Fisheries Report, No. 5, 1962.