Nonlinear Control Systems
5. - Stability in the Sense of Lyapunov

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Lyapunov Stability

- Consider a time-invariant system

$$\dot{x}(t) = f(x(t))$$

where \( f : D \rightarrow \mathbb{R}^n \) is locally Lipschitz in \( D \)

- We say \( x^* \in D \) is an equilibrium point if \( 0 = f(x^*) \).

- WLOG we can take \( x^* = 0 \) since we can introduce a change of variables \( y = x - x^* \) for which

$$\dot{y} = \dot{x} - \dot{x}^* = f(y + x^*) = g(y)$$

has an equilibrium at 0.

- We say \( x^* = 0 \) is stable if for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |x(0)| < \delta \) implies \( |x(t)| < \epsilon \) for all \( t \geq 0 \).

- We say the equilibrium is asymptotically stable if it is stable and if \( x(t) \to 0 \) as \( t \to \infty \). If it is not stable, then it is unstable.
Example

- pendulum

\[ m\ell \ddot{\theta} = -mg \sin \theta - k\ell \dot{\theta} \]

- Transform second order ODE into two first order ODEs

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{\ell} \sin x_1 - \frac{k}{m} x_2 \]

- The equilibria are solutions to \( x_2 = 0 \) and \( \sin x_1 = 0 \) so that \( x^* = (\pm n\pi, 0) \). Orbits may be seen as evolving on cylinder or plane.
Since pendulum is a mechanical system, we can examine potential/kinetic energy and see if total energy is decreasing over time.

\[
E(x) = \int_0^x \frac{g}{\ell} \sin y dy + \frac{1}{2}x_2^2 = \frac{g}{\ell} (1 - \cos x_1) + \frac{1}{2}x_2^2
\]

Take directional derivative of \( E \) assuming \( k = 0 \) (no damping)

\[
\dot{E} = \frac{\partial E}{\partial x_1} \dot{x}_1 + \frac{\partial E}{\partial x_2} \dot{x}_2
\]

\[
= \frac{f}{\ell} x_2 \sin x_1 + x_2 \left(-\frac{g}{\ell} \sin x_1\right) = 0
\]

With positive damping we can see that \( \dot{E} = -\frac{k}{m} x_2^2 \leq 0 \), so that energy is decreasing and must converge to a minimum level \( E^* \).
Lyapunov’s Direct Theorem

**Theorem 1**

*(Lyapunov Direct)* Let 0 be an equilibrium point for \( \dot{x} = f(x) \) where \( f : D \to \mathbb{R}^n \) is locally Lipschitz on domain \( D \subset \mathbb{R}^n \). Assume there exists a continuously differentiable function \( V : D \to \mathbb{R} \) such that

- \( V(0) = 0 \) and \( V(x) > 0 \) for all \( x \in D \) not equal to zero.
- \( \dot{V} = \frac{dV(x(t))}{dt} = \frac{\partial V}{\partial x} = [D_f V](x) \leq 0 \) for all \( x \in D \).

Then \( x = 0 \) is stable in the sense of Lyapunov.
Proof:

- For any $\epsilon > 0$ consider $B_r = \{ x : |x| < r \}$ where $r \in (0, \epsilon]$.
- Let $\alpha = \min_{|x|=r} V(x)$ and define set for $\beta \in (0, \alpha)$

$$\Omega_\beta = \{ x \in B_r : V(x) \leq \beta \}$$

so that $\Omega_\beta$ is contained in epsilon-neighborhood of 0 where $V$ is less than $\alpha$.

- For any solution of $\dot{x} = f(x)$ where $x(0) \in \Omega_\beta$ we can see that $V(x(t)) \leq V(x(0)) \leq \beta$ (because $\dot{V} \leq 0$), which means $\Omega_\beta$ is forward invariant.

- Since $V$ is continuous there exist $\delta > 0$ such that $|x| \leq \delta$ implies $V(x) < \beta$ and so there exists $B_\delta \subset \Omega_\beta \subset B_r$, which establishes Lyapunov stability. ♦
Theorem 2

\textbf{(Asymptotic Stability)} Under the hypotheses of theorem 1, if $\dot{V}(x) < 0$ for all $x \in D - \{0\}$, then the equilibrium is asymptotically stable.

\textbf{Proof}: Since $V(x(t))$ is monotone decreasing and bounded below there exists a real $c \geq 0$ such that $V(x(t)) \rightarrow c$. If $c$ is strictly greater than zero then $x$ can never enter $\{x : V(x) \leq c\}$. Let $-\gamma$ be maximum value rate of change outside of this set, then

$$V(x(t)) = V(x(0)) - \int_{0}^{t} \dot{V}(x(\tau))d\tau \leq V(x(0)) - \gamma t$$

which would imply that $V$ is eventually negative for large enough time - a contradiction so that $c$ must be zero. $\Diamond$
Lypuanov Stability Certificates

- The existence of a Lyapunov function certifies the stability of the equilibrium - so we call them stability certificates.
- Finding a certificate can be very difficult. We usually start from a candidate that is known to be a certificate for a related system, then see how we can modify it.
- Consider the system \( \dot{x} = -g(x) \) where \( g \) is locally Lipschitz on \((-a, a)\) with \( g(0) = 0 \). We assume that \( g \) is "odd". Clearly \( x = 0 \) is an equilibrium.
- Consider the function

\[
V(x) = \int_0^x g(y)\,dy
\]

(similar to a potential energy function). We can easily show that \( V \) is positive definite and its derivative is

\[
\dot{V} = \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} (-g) = -g^2(x) < 0
\]

So the origin is asymptotically stable.
We now return to the pendulum system

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{\ell} \sin x_1 \]

and note that it is similar to the scalar system in preceding slide. The difference is that we have two states not one.

So let us use the earlier certificate as a candidate this system by simply summing a certificate for each state

\[
V(x) = \int_0^{x_1} \frac{g}{\ell} \sin y \, dy + \int_0^{x_2} y \, dy
\]

\[
= \frac{g}{\ell} (1 - \cos(x_1)) + \frac{1}{2} x_2^2
\]

This is the same energy function we had before, and so \( \dot{V} = 0 \) which means we can only conclude the equilibrium is stable.
Lyapunov Stability Certificates

To ”asymptotically” stabilize this system we need to add damping. So let $k > 0$ and let us recompute $\dot{V}$ to get

$$\dot{V} = \frac{g}{\ell} x_2 \sin x_1 - \frac{g}{\ell} x_2 \sin x_1 - \frac{k}{m} x_2^2$$

$$= -\frac{k}{m} x_2^2$$

This is only negative semidefinite since $\dot{V} = 0$ for any state $x = (x_1, 0)^T$. So again we can only conclude Lyapunov stability.

This system is actually asymptotically stable about the origin. To show this we consider a parameterized version of the original candidate Lyapunov certificate we used before. In particular we let

$$V(x) = \frac{g}{\ell} (1 - \cos(x_1)) + \frac{1}{2} x^2 P x$$

where $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ is a symmetric and positive definite matrix.
Lyapunov Stability Certificates

- **P** is positive definite if and only if
  \[ p_{11} > 0, \quad p_{22} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0 \]  
  (1)

- We recompute \( \dot{V} \) to get

  \[
  \dot{V} = \left[ p_{11}x_1 + p_{12}x_2 + \frac{g}{\ell} \sin x_1 \right] x_2 + (p_{12}x_1 + p_{22}x_2) \left[ -\frac{g}{\ell} \sin x_1 - \frac{k}{m} \right] \\
  = \frac{g}{\ell} (1 - p_{22}) x_2 \sin x_1 - \frac{g}{\ell} p_{12} x_1 \sin x_1 \\
  + \left[ p_{11} - p_{12} \frac{k}{m} \right] x_1 x_2 + \left[ p_{12} - p_{22} \frac{k}{m} \right] x_2^2
  \]

- We select the free terms \( p_{ij} \) to ensure \( \dot{V} \) is negative definite and \( V \) is positive definite. In particular, we select them to cancel out the indefinite terms in the \( \dot{V} \) expression.
Lyapunov Stability Certificates

- Selecting $p_{22} = 1$ cancels the term $x_2 \sin x_1$. Selecting $p_{11} = \frac{k}{m} p_{12}$ cancels the cross term $x_1 x_2$. So that

$$\dot{V} = -\frac{g}{\ell} p_{12} x_1 \sin x_1 + \left[ p_{12} - p_{22} \frac{k}{m} \right] x_2^2$$

- If we let $p_{12} > 0$ then the first term is negative. If we let $p_{22} = 1$ and $p_{12} < k/m$ then the second term is negative. Combining these considerations we select $P$ to be

$$0 < p_{12} < \frac{k}{m}, \quad p_{22} = 1, \quad p_{11} = \frac{k}{m} p_{12} < \frac{k^2}{m^2}$$

- and so this certifies that the origin is asymptotically stable. The example illustrates a significant limitation of Lyapunov’s method. It is only sufficient for stability and one must find the "right" certificate and that search might be difficult.
For example, a commonly used candidate certificate is $V(x) = x^T P x$ since this is the Lyapunov function for a "linear system".

But in our pendulum example we would et

$$
\dot{V} = -p_{22} \frac{g}{\ell} x_2 \sin x_1 - \frac{g}{\ell} p_{12} x_1 \sin x_1 \\
\quad + \left[ p_{11} - p_{12} \frac{k}{m} \right] x_1 x_2 + \left[ p_{12} - p_{22} \frac{k}{m} \right] x_2^2
$$

If we now tried to remove the cross term $x_2 \sin x_1$ by setting $p_{22} = 0$, then the resulting $P$ would not be positive definite.

Notice however that

$$x_2 \sin x_1 \approx x_2 (x_1 + o(|x_1|^2))$$
Lyapunov Stability Certificate

- If we insert this into our direction derivative we obtain

\[
\dot{V} = \left( p_{11} - p_{12} \frac{k}{m} - p_{22} \frac{g}{\ell} \right) x_1 x_2
\]

\[
- p_{22} \frac{g}{\ell} x_2 o(|x_1|^2) - \frac{g}{\ell} p_{12} x_1 \sin x_1 + \left[ p_{12} - p_{22} \frac{k}{m} \right] x_2^2
\]

- we then select \( P \) to kill the \( x_1 x_2 \) cross term, leave the \( o(|x_1|^2) \) term, and render the last two terms negative. This would give us

\[
\dot{V} = -W(x_1, x_2) + \text{error term}
\]

in which \( W \) is positive definite and \( O(|x|^2) \) (big O rather than little o).

- The error term is \( o(|x|^2) \) so it is dominated by \( W \) close to the origin and so we can establish local asymptotic stability, but for a very small neighborhood of the origin. The other choice would have established asymptotic stability globally.
Overview of Advanced Theorems and Methods

- Chetaev’s instability Theorem - a certificate for an unstable system
- Barbashin-Krasovskii - radially unbounded $V$ for global stability
- LaSalle’s Invariance Principle - Concluding asymptotic stability when $\dot{V} \leq 0$
- Necessary and Sufficient Conditions for LTI systems
- Lyapunov’s Indirect Method - inferring stability from the linearization
- Lyapunov Stability for Time-Varying Systems
- Converse Stability Theorems
- Computational methods for Finding Stability Certificates
(Chetaev Instability Theorem) Let \( x = 0 \) be an equilibrium for \( \dot{x} = f(x(t)) \) where \( f : D \to \mathbb{R}^n \) is locally Lipschitz on domain \( D \subset \mathbb{R}^n \). Let \( V : D \to \mathbb{R} \) by a \( C^1 \) function such that \( V(0) = 0 \) and \( V(x_0) > 0 \) for some \( x_0 \) with arbitrarily small \( |x_0| \). If there exists \( \delta > 0 \) such that \( \dot{V}(x) > 0 \) for all \( x \) in

\[
U = \{ x : |x| \leq \delta, \quad V(x) > 0 \}
\]

then \( x = 0 \) is unstable.
Proof of Chetaev’s Theorem

- Suppose there exists $\delta$ such that $\dot{V}(x) > 0$ for all $x \in U$. Let $x_0 \in \text{int}(U)$ and let $x(t; x_0)$ be the trajectory leaving $x_0$.
- Since $\dot{V} > 0$ and $U$ is compact there is a minimum rate of growth
  \[ \gamma = \min\{\dot{V}(x) : x \in U \text{ and } V(x) \geq V(x_0)\} \]
- So we can conclude
  \[ V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))ds \geq V(x_0) + \int_0^t \gamma ds = V(x_0) + \gamma t \]
  and there exists time $T > 0$ when $x(T)$ is on boundary of $U$. 
Proof of Chetaev’s Theorem

There are two parts to that boundary

\[ \partial_B U = \{ x : |x| = \delta, \quad V(x) < 0 \} \]
\[ \partial_A U = \{ x : |x| < \delta, \quad V(x) = 0 \} \]

- \( x(T) \) cannot be in \( \partial_A U \) since \( V(x) > 0 \) and \( \dot{V}(x) > 0 \) for all \( x \in U \). \( x(T) \) must therefore be in \( \partial_B U \)
- This means that if \( x_0 \) is arbitrarily close to 0, the trajectory leaves \( U \) and hence leaves a \( \delta \)-neighborhood of the origin - so equilibrium is unstable. ♦
Example 1 - Chetaev’s Theorem

- Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_1 + g_1(x) \\
\dot{x}_2 &= -x_2 + g_2(x)
\end{align*}
\]

where \( g_1 \) and \( g_2 \) satisfy

\[|g_1(x)| \leq k|x|^2, \quad |g_2(x)| \leq k|x|^2\]

- Consider a candidate Chetaev function

\[V(x) = \frac{1}{2}(x_1^2 - x_2^2)\]
Example 1 - Chetaev’s Theorem

- Select a point along $x_2 = 0$ since $V > 0$ for all such points and note that

$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

with the last term satisfying

$$|x_1 g_1(x) - x_2 g_2(x)| \leq |x_1||g_1(x)| + |x_2||g_2(x)| \leq 2k|x|^3$$

- This implies that

$$\dot{V} \geq |x|^2 - 2k|x|^3 = |x|^2(1 - 2k|x|)$$

- If we choose $\epsilon < \frac{1}{2k}$, then $\dot{V} > 0$ and so $V$ is a Chetaev function and the origin is unstable.
Example 2 - Chetaev Theorem

Consider the system

\[
\begin{align*}
\dot{x}_1 &= f_1(x) = -x_1^3 + x_2 \\
\dot{x}_2 &= f_2(x) = x_1^6 - x_2^3
\end{align*}
\]

This system has two equilibria at \((0, 0)\) and \((1, 1)\).

Consider the set

\[
\Omega = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\} \quad (2)
\]
Example 2 - Chetaev’s Theorem

- On the "A" boundary, we know \( x_2 = x_1^2 \), \( f_2(x) = 0 \), and \( f_1(x) > 0 \) and the vector field points into \( \Omega \).

- On the "B" boundary, \( x_2 = x_1^3 \), \( f_2(x) > 0 \) and \( f_1(x) = 0 \), and the vector field again points into \( \Omega \).

Select a candidate Chetaev function

\[
V(x) = (x_1^2 - x_2)(x_2 - x_1^3)
\]

So \( V > 0 \) and \( \dot{V} > 0 \) on \( \Omega \), then \( V \) is a Chetaev function and the origin is unstable.
Global Stability

- Consider the dynamical system

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) = -\frac{6x_1}{(1 + x_1^2)^2} + 2x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) = -2\frac{x_1 + x_2}{(1 + x_1^2)^2}
\end{align*}
\]

- The following

\[
V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2
\]

is a Lyapunov function since

\[
\dot{V}(x) = \frac{2x_1(1 + x_1^2) - 2x_1^3}{(1 + x_1^2)^2} \dot{x}_1 + 2x_2 \dot{x}_2
\]

\[
= -\frac{12x_1^2}{(1 + x_1^2)^4} - \frac{4x_2^2}{(1 + x_1^2)^2} < 0
\]

- So the equilibrium is asymptotically stable
Global Stability

Consider the hyperbola \( x_2 = \frac{2}{x_1 - \sqrt{2}} \) and note that

\[
\frac{f_2}{f_1} \Bigg|_{\text{hyperbola}} > \text{slope of hyperbola's tangents}
\]

So there is a region of \( \mathbb{R}^2 \) that can never enter the attracting region about the origin.
We say $V : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded if $V(x) \to \infty$ as $|x| \to \infty$.

**Theorem 4**

*(Barbashin-Krasovskii Theorem)* Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ where $f$ is locally Lipschitz. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a radially unbounded $C^1$ function such that $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$ and $\dot{V}(x) < 0$ for all $x \neq 0$. Then the equilibrium point $x = 0$ is globally asymptotically stable.
Consider a dynamical system $\phi$ and recall that an orbit of $\phi$ with respect to $x$ is

$$\gamma(x) = \{y \in \mathbb{R}^n : y = \phi(t, x), \quad t \in \mathbb{R}\}$$

A point $p$ is a $\omega$ or **positive limit point** of $x$ if there exists an increasing sequence of times $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$ and

$$\lim_{t_n \to \infty} \phi(t_n, x) = p.$$  

The point is an $\alpha$ or **negative limit point** of $x$ if $\lim_{t_n \to \infty} \phi(-t_n, x) = p$.

The set of all positive limit points of $x$ is denoted as $\omega(x)$ and union of all such limit points is the system’s **positive limit set**

$$\Omega = \bigcup_{x \in D} \omega(x).$$
Positive Limit Set Theorem

- Consider a dynamical system $\phi$ then a set $M \subset \mathbb{R}^n$ is (positively) invariant if there exists $x \in M$ implies $\phi(t, x) \in M$ for all $t \geq 0$.
- $M$ is attracting if for all $\epsilon > 0$ and any $x \in D$ there exists $T > 0$ such that $\inf_{y \in M} |\phi(t, x) - y| < \epsilon$ for all $t > T$.

**Theorem 5**

*(Positive Limit Set Theorem)* Consider the dynamical system $\dot{x}(t) = f(x(t))$ where $f : D \to \mathbb{R}^n$ is locally Lipschitz on a compact set $D \subset \mathbb{R}^n$. Assume that all $x : \mathbb{R} \to \mathbb{R}^n$ satisfying $\dot{x} = f(x)$ belong to $D$ for all $t \geq 0$. Then the system’s $\omega$-limit set, $\Omega$, is non-empty, compact, invariant and attracting.
(Invariance Principle) Consider the system $\dot{x} = f(x)$ where $f : D \rightarrow \mathbb{R}^n$ is locally Lipschitz on $D \subset \mathbb{R}^n$. Let $K \subset D$ be a compact invariant set with respect to $f$. Let $V : D \rightarrow \mathbb{R}$ be a $C^1$ function such that $\dot{V}(x) \leq 0$ on $K$. Let $M$ be the largest invariant set in $E = \{x \in K : \dot{V}(x) = 0\}$. Then $M$ is attracting for all trajectories starting in $K$. 
Proof of Invariance Principle

- Since $\dot{V} \leq 0$ and $V(x(t))$ is bounded, we know that $V(x(t)) \to a \geq 0$.
- For any $p$ in the system’s positive limit set $\Omega$, there exists $\{t_n\}$ such that $x(t_n) \to p$ as $t_n \to \infty$. This implies that

$$a = \lim_{n \to \infty} V(x(t_n)) = V(\lim_{n \to \infty} x(t_n))$$

We can therefore conclude that $V(x) = a$ for all $x \in \Omega$.
- From the earlier limit-set theorem, we know that $\Omega$ is attracting and invariant. Since $V(x) = a$ on $\Omega$ this implies that $\dot{V}(x) = 0$ on $\Omega$ so we can conclude that $\Omega \subset M$ (since it is invariant) and $\Omega \subset E$ (since $\dot{V} = 0$).
- So we can conclude that

$$\Omega \subset M \subset E \subset K$$

and since $\Omega$ is attracting the set $M$ must also be attracting. ♦
Asymptotic Stability using Invariance Principle

This is a direct consequence of the invariance principle in theorem 6. This theorem’s hypothesis states that the origin is the largest invariant set in the set where $\dot{V}(x) = 0$ and so the origin is attracting all trajectories in $D$.

**Theorem 7**

*(Asymptotic Stability - invariance theorem)* Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ where $f : D \to \mathbb{R}^n$ is locally Lipschitz on $D \subset \mathbb{R}^n$. Let $V : D \to \mathbb{R}$ be a $C^1$ positive definite function on $D$ containing $x = 0$ such that $\dot{V}(x) \leq 0$ on $D$. If the origin, $\{0\}$, is the largest invariant set in the set $\{x \in D : \dot{V}(x) = 0\}$, then the origin is asymptotically stable.
Consider the system

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = -g(x_1) - h(x_2)
\]

where \( g \) and \( h \) are locally Lipschitz such that \( g(0) = 0, h(0) = 0, \ yg(y) > 0 \) and \( yh(y) > 0 \).

The equilibria satisfy

\[
0 = x_2, \quad 0 = -g(x_1) - h(x_2)
\]

which implies the origin is the equilibrium point.

Consider the candidate Lyapunov function

\[
V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2
\]
Application of Invariance Principle

- We know that $V$ is positive definite and its directional derivative is

$$
\dot{V}(x) = g(x_1)x_2 + x_2(-g(x_1) - h(x_2))
$$

$$
= -x_2h(x_2) \leq 0
$$

So $\dot{V} \leq 0$ and is only negative semi-definite.

- The set $E$ is

$$
E = \{x ; \dot{V}(x) = 0\} = \{x : x_2 = 0\}
$$

- Let $x(t; x_0)$ start in $E$, this implies $\dot{x}_1(0) = x_2(0) = 0$ which means $x_1(t)$ is constant.

- But if $x_1(t) \neq 0$, then

$$
\dot{x}_2(t) = -g(x_1(t)) - h(x_2(t)) \neq 0
$$

which would force $x(t)$ to leave set $E$, So the origin is the largest invariant set in $E$ and so must be asymptotically stable.
Another Application of Invariance Principle

- consider the Lyapunov equation

\[ A^T P + PA = -C^T C \]

associated with the linear dynamical system

\[
\dot{x} = Ax(t) \\
y = Cx(t)
\]

- Let \( V(x) = x^T P x \) where \( P \) is positive definite and symmetric.

- Note that \( \dot{V}(x) = -x^T C^T C x \leq 0 \) This means that \( \dot{V} \) is only negative semi-definite unless \( C^T C \) has full rank.
The set where $\dot{V} = 0$ is the set where $Cx = 0$. So our set $E$ is

$$E = \{x : Cx = 0\}$$

If the null space is trivial, then clearly $E$ is just the origin, which would mean that the origin is asymptotically stable.

In most cases, however, we would not expect $\ker(C)$ to be trivial. But let us assume the pair $(A, C)$ is observable and $x_0 = 0$. So the only trajectory that can remain in $E$ for all time is the one that starts at the origin.

We can therefore conclude under the observability condition that the largest invariant set in $E$ and hence the origin is asymptotically stable.
Lyapunov Stability for LTI systems

Consider a candidate Lyapunov function of the form, \( V(x) = x^T P x \)

The directional derivative of \( V \) is

\[
\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x
\]

If \( Q = Q^T > 0 \) and \( P = P^T > 0 \) satisfy the Lyapunov equation

\[
A^T P + PA + Q = 0
\]

then the origin of the LTI system is asymptotically stable.

But this is only a sufficient condition.
Eigenvalue Test for Asymptotic Stability

Since the system is LTI, we know that

\[ x(t) = \sum_{i=1}^{\sigma} \sum_{k=1}^{n_i} A_{ik} t^k e^{\lambda_i t} x_0 \]

Which implies the origin is stable if and only if the eigenvalues of \( A \) all have negative real parts.

**Theorem 8**

Given matrix \( A \in \mathbb{R}^{n \times n} \) the following statements are equivalent

1. All eigenvalues of \( A \) have negative real parts.
2. There exists a positive definite matrix \( Q \) such that the Lyapunov equation has a unique solution.
3. For every positive definite matrix \( Q \), the Lyapunov equation has a unique solution.
Proof

- (3) ⇒ (2) is obvious and (2) ⇒ (1) follows from Lyapunov’s direct theorem.

- The interesting part of the proof is that (1) implies (3) which is established by considering the following solution to the Lyapunov equation

\[
P = \int_0^\infty e^{A^T t} Q e^{A t} dt
\]

and computing the Lyapunov equation

\[
A^T P + PA = \int_0^\infty \frac{d}{dt} \left( e^{A^T t} Q e^{A t} \right) = e^{A^T t} Q e^{A t} \bigg|_{t=0}^\infty
\]

Because A is Hurwitz this goes to Q as \( t \to \infty \).

- One can easily show that P is positive definite and symmetric.
Proof

- To establish uniqueness of solution, assume it is not unique and there exists $\mathbf{P}$ such that $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = 0$.
- Take the difference and to get

$$\mathbf{A}^T(\mathbf{P} - \mathbf{P}) + (\mathbf{P} - \mathbf{P})\mathbf{A} = 0$$

- pre-multiply by $e^{\mathbf{A}^T \mathbf{t}}$ and post multiply by $e^{\mathbf{A} \mathbf{t}}$ to get

$$0 = e^{\mathbf{A}^T \mathbf{t}} \mathbf{A}^T(\mathbf{P} - \mathbf{P})e^{\mathbf{A} \mathbf{t}} + e^{\mathbf{A}^T \mathbf{t}}(\mathbf{P} - \mathbf{P})\mathbf{A}e^{\mathbf{A} \mathbf{t}}$$

$$= \frac{d}{dt} \left( e^{\mathbf{A}^T \mathbf{t}}(\mathbf{P} - \mathbf{P})e^{\mathbf{A} \mathbf{t}} \right)$$

- So $e^{\mathbf{A}^T \mathbf{t}}(\mathbf{P} - \mathbf{P})e^{\mathbf{A} \mathbf{t}}$ is constant for all $\mathbf{t}$. Since $e^{\mathbf{A} \mathbf{t}} \to 0$ as $\mathbf{t} \to \infty$, this means that constant is zero, so $\mathbf{P} = \mathbf{P}$. 
Exponential Stability for LTI System

- The existence of a Lyapunov function is also necessary and sufficient for LTI asymptotic stability.
- In fact LTI systems are *exponentially stable*. From the Lyapunov equation we know

\[
\dot{V} = x^T (A^T P + PA)x \leq -x^T Q x
\]

- We also know that

\[
\lambda(P)|x|^2 \leq x^T P x \leq \bar{\lambda}(P)|x|^2
\]

- which means that

\[
\dot{V} = -x^T Q x \leq -\lambda(Q)|x|^2 \leq -\frac{\lambda(Q)}{\lambda(P)} V(x(t))
\]

- and so by the comparison principle

\[
V(t) \leq e^{-\frac{\lambda(Q)}{\lambda(P)} t} V(0) \quad \Rightarrow \quad |x(t)| \leq \frac{\lambda(P)}{\lambda(P)} |x(0)|^2 e^{-\frac{\lambda(Q)}{\lambda(P)} t}
\]
Value of Lyapunov Condition?

- Checking eigenvalues is easier than solving Lyapunov equation, so what is value in the Lyapunov condition for LTI systems?
- Consider a perturbed LTI system \( \dot{x} = (A + \Delta)x \) where origin of unperturbed system is asymptotically stable.
- \( \dot{V} \) for perturbed system is
  \[
  \dot{V} = x^T(A^T P + PA + \Delta^T P + P\Delta)x \leq x^T(-Q + \Delta^T P + P\Delta)x
  \]
- This will be negative definite provided
  \[
  x^T(\Delta^T P + P\Delta)x < x^TQx
  \]
- Using earlier eigenvalue bounds for matrices we see that this means
  \[
  2\lambda(P)\lambda(\Delta) < \lambda(Q), \quad \Rightarrow \quad \lambda(\Delta) \leq \frac{\lambda(Q)}{2\lambda(P)}
  \]
  which bounds the largest perturbation before the system becomes unstable (robust stability).
Consider a controlled system

\[ \dot{x} = Ax + Bu \]

where \( A \) is not necessarily Hurwitz.

Consider a control \( u = Kx \), if there exists \( K \) that stabilizes the origin then by the "converse" nature of Lyapunov function, there must exist a Lyapunov function \( V(x) = x^T P x \) such that

\[ (A + BK)^T P + P(A + BK) + Q \]

for any \( Q > 0 \).

So we use above equation to solve for the controller \( K \).
This result shows that about a hyperbolic equilibrium the asymptotic stability of the origin is determined by its linearization.

**Theorem 9**

*(Lyapunov’s Indirect Method)* Let \( \dot{x} = Ax \) be the linearization of nonlinear system \( \dot{x} = f(x) \) about the nonlinear system’s equilibrium point. Let \( \{\lambda_i\}_{i=1}^n \) denote the eigenvalues of matrix \( A \). If \( \text{Re}(\lambda_i) < 0 \) for all \( i \) then the nonlinear system’s equilibrium is asymptotically stable. If there exists \( i \) such that \( \text{Re}(\lambda_i) > 0 \), then the origin is unstable.

So one extremely useful approach to evaluate local asymptotic stability is to see whether or not the linearization is asymptotically stable.
Proof - stability part

- So assuming $A$ is Hurwitz, there exists positive definite $P$ that satisfies $A^T P + PA + Q = 0$ for any $Q = Q^T > 0$.

- The derivative of $V$ w.r.t. nonlinear system is

$$
\dot{V} = x^T P f(x) + f^T(x) P x
$$

$$
= x^T P (Ax + g(x)) + \left[ x^T A^T + g^T(x) \right] P x
$$

$$
= x^T (PA + A^T P)x + 2x^T P g(x)
$$

$$
= -x^T Q x + 2x^T P g(x)
$$

where $g(x) = o(|x|)$.

- So for all $\gamma > 0$ there exists $r$ such that $|g(x)| < \gamma |x|$ when $|x| < r$. This means $\dot{V}$ satisfies

$$
\dot{V} < -x^T Q x + 2\gamma \|P\| |x|^2
$$

$$
< - (\lambda(Q) - 2\gamma \|P\|) |x|^2
$$

So choose $\gamma < \frac{\lambda(Q)}{2\|P\|}$ to ensure $\dot{V} < 0$ when $|x| < r$. 

Now $A$ has at least one eigenvalue with a positive real part. Split $A$ into its stable and unstable parts

$$
T^{-1} = \begin{bmatrix} -A_1 & 0 \\ 0 & A_2 \end{bmatrix}
$$

where $A_1$ and $A_2$ are both Hurwitz. The system equation are now

$$
\dot{z}_1 = -A_1 z_1 + g_1(z), \quad \dot{z}_2 = A_2 z_2 + g_2(z)
$$

where $z_2 \to 0$.

Let $Q_1$ and $Q_2$ be two PD matrices and consider $P_1$ and $P_2$ that solve the associated Lyapunov equations and show that

$$V(z) = z_1^T P_1 z_1 - z_2^T P_2 z_2$$

is a Chetaev function.
What if Linearization is not Hyperbolic?

- Let us assume $\dot{x} = f(x)$ can be written as

$$
\begin{align*}
\dot{z} &= Az + f(z, y) \\
\dot{y} &= By + g(z, y)
\end{align*}
$$

where $A$ has eigenvalues with zero real parts and $B$ is Hurwitz.

- We know that the dynamics of the system restricted to the center manifold

$$
\dot{u} = Au + f(u, h(u))
$$

where $h : \mathbb{R}^c \to \mathbb{R}^s$ is graph of center manifold is sufficient to determine stability of the equilibrium.

- So we can *reduce* the dimensionality of the stability problem by always just focusing on the dynamics of the center manifold.
The Lyapunov stability concept must apply uniformly for time-varying systems

\[ \dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \]

In particular, we say the origin is **uniformly stable** if for all \( \epsilon > 0 \) there exists \( \delta > 0 \) that is independent of \( t_0 \) such that \( |x(t_0)| \leq \delta \) implies \( |x(t)| < \epsilon \) for all \( t \geq t_0 \).

The equilibrium is **uniformly asymptotically stable** (UAS) if it is uniformly stable (US) and there exists \( \delta \) independent of \( t_0 \) such that for all \( \epsilon > 0 \) there exists \( T > 0 \) also independent of \( t_0 \) where \( |x(t)| \leq \epsilon \) for all \( t \geq t_0 + T(\epsilon) \) and for all \( |x(t_0)| < \delta \).

Global notions of US and UAS require \( \delta(\epsilon) \to \infty \) as \( \epsilon \to \infty \).
Why we need a uniform stability concept?

- Consider the system \( \dot{x} = (6t \sin t - 2t)x \) with solution
  
  \[ x(t) = x(t_0) \exp \left( 6 \sin t - 6t \cos t - t^2 - 6 \sin t_0 + 6t_0 \cos t_0 + t_0^1 \right) \]

- For fixed \( t_0 \), the quadratic term \(-t^2\) will eventually dominate and there is a constant \( c(t_0) \) where \(|x(t)| < c(t_0)|x(t_0)|\) for all \( t \geq t_0 \). But constant \( c(t_0) \) is a function of the initial time.

- Consider sequence of initial times \( \{2n\pi\}_{n=0}^{\infty} \) and note that
  
  \[ x(t_0n + \pi) = x(t_0) \exp((4n + 2)(6 - \pi)\pi) \]

- This means
  
  \[ \frac{x(t_0n + \pi)}{x(t_0n)} \to \infty \text{ as } n \to \infty \]

In other words the systems becomes “less” stable for later initial times.
Why we need a uniform stability concept?

- Consider the system \( \dot{x} = -\frac{x}{1+t} \) whose solution is

\[
x(t) = x(t_0) \frac{1 + t_0}{1 + t}
\]

- The origin is asymptotically stable, so for any \( \epsilon > 0 \), we can show that \( |x(t)| < \epsilon \) for all \( t \geq t_0 + T \) where

\[
T > t_0 \left( \frac{|x(t_0)|}{\epsilon} - 1 \right) - 1
\]

- So \( T \) is a function of \( t_0 \) and as \( t_0 \to \infty \) then \( T \to \infty \).

- In other words the system’s rate of approach to the origin can be made arbitrarily slow by taking the initial time \( t_0 \) large enough.
Comparison Functions

- A continuous function $\alpha : [0, a) \to [0, \infty)$ belongs to $\mathcal{K}$ if it is strictly increasing and if $\alpha(0) = 0$.
- A function $\alpha : [0, a) \to [0, \infty)$ belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and it is radially unbounded (i.e. $\alpha(r) \to \infty$ as $r \to \infty$).
- A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is class $\mathcal{KL}$ if
  - for all fixed $s$, then $\beta(r, s)$ is class $\mathcal{K}$ with respect to $r$.
  - for all fixed $r$, $\beta(r, s)$ is decreasing with respect to $s$ such that $\beta(r, s) \to 0$ as $s \to \infty$.

**Theorem 10**

(*Comparison Function Properties*)

1. If $\alpha \in \mathcal{K}$ over $[0, a)$ then $\alpha^{-1} \in \mathcal{K}$ over $[0, \alpha(a))$.
2. If $\alpha \in \mathcal{K}_\infty$, then $\alpha^{-1} \in \mathcal{K}_\infty$.
3. If $\alpha_1, \alpha_2 \in \mathcal{K}$, then $\alpha_1 \circ \alpha_2 \in \mathcal{K}$.
4. If $\alpha_1, \alpha_2 \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ then $\alpha_1(\beta(\alpha_2(r), s)) \in \mathcal{KL}$.
Theorem 11

(Class $\mathcal{K}$ version of Comparison Principle) Let $\dot{y} = -\alpha(y)$ where $t \in \mathbb{R}$ and $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz class $\mathcal{K}$ function on $[0, a)$. For all $0 \leq y_0 < a$, this equation has a unique solution satisfying the initial value problem $\dot{y} = -\alpha(y)$ with $y(0) = y_0$ that we denote as $y(t) = \phi(y, t_0)$ where $\phi$ is class $\mathcal{KL}$. 
Proof

- By integration we have \(- \int_{y_0}^y \frac{dx}{\alpha(x)} = \int_t^{t_0} ds\) and for notational convenience we let \(\eta(y) = - \int_{y_0}^y \frac{dx}{\alpha(x)}\) for any \(b < a\).

- One can show \(\lim_{y \to 0} \eta(y) = \infty\) and that its inverse exists. So for any \(y_0 > 0\)

  \[
  \eta(y(t)) - \eta(y_0) = t - t_0, \quad \Rightarrow y(t) = \eta^{-1}(\eta(y_0) + t - t_0)
  \]

- Define the function

  \[
  \sigma(r, s) = \begin{cases} 
  \eta^{-1}(\eta(r) + s) & r > 0 \\
  0 & r = 0 
  \end{cases}
  \]

  and note \(y(t) = \sigma(t, t_0)\). We can also show that \(\sigma\) is class \(\mathcal{KL}\). 

Lyapunov Stability for Time-Varying Systems

Theorem 12

(Lyapunov Stability for Time-Varying System) Let $x = 0$ be an equilibrium point for $\dot{x} = f(t, x)$ and let $V: [0, \infty \times D \rightarrow \mathbb{R}$ be a $C^1$ function over $D \subset \mathbb{R}^n$ such that

$$\alpha(|x|) \leq V(t, x) \leq \bar{\alpha}(|x|)$$

If

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha(x)$$

for all $t \geq 0$ for all $x \in D$ where $\alpha$, $\bar{\alpha}$, and $\alpha$ are class $\mathcal{K}$ functions on $D$, then $x = 0$ is uniformly asymptotically stable.
Proof: Consider any $\epsilon > 0$ and define $\delta = \overline{\alpha}^{-1}(\alpha(\epsilon))$

From knowledge of $V$ and $\dot{V}$ we can show that if $|x(t_0)| < \delta$ then

$$\alpha(|x(t)|) \leq V(x(t)) \leq V(x(t_0))$$

$$\leq \overline{\alpha}(|x(t_0)|) \leq \alpha(|x(t_0)|) \leq \overline{\alpha}(\delta) = \alpha(\epsilon)$$

So we can conclude that $|x(t)| \leq \epsilon$ for all $t \geq t_0$ which implies the Lyapunov stability of the equilibrium point.

Now let $V(t) = V(x(t))$ where $x$ is a trajectory satisfying the system's differential equation. Let $\theta(r) = \alpha(\overline{\alpha}^{-1}(r))$ and observe that $\dot{V} \leq -\alpha(|x|)$ implies that

$$\dot{V} \leq -\alpha(\overline{\alpha}(V(t))) = -\theta(V(t))$$
Proof - part 2

- $\theta$ is class $\mathcal{K}$ and locally Lipschitz so there exists a unique solution $y(t)$ to the differential equation $\dot{y} = -\theta(y)$ such that $y(t_0) = V(t_0)$ and $y(t) = \phi(V(t_0), t - t_0)$ for some $\mathcal{KL}$ function $\phi$.

- By the comparison lemma we therefore know that $V(t) \leq \phi(V(t_0), t - t_0)$ which implies that

$$|x(t)| \leq \alpha^{-1}(\phi(\alpha(|x(t_0)|), t - t_0))$$

- Since the right hand side is a class $\mathcal{KL}$ function, we know $|x(t)| \to 0$ as $t \to \infty$, which is sufficient to assure that the equilibrium is UAS. 

$\diamondsuit$
Any positive definite function can be bounded above and below by class $\mathcal{K}$ functions. This fact allows us to establish a version of Lyapunov’s direct method for time-varying systems.

**Theorem 13**

(Lyapunov Direct Method for Time-Varying System) Let $x = 0$ be an equilibrium for $\dot{x}(t) = f(t, x)$, let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and let $\underline{W} : \mathbb{R}^n \to \mathbb{R}$, $\overline{W} : \mathbb{R}^n \to \mathbb{R}$, and $W : \mathbb{R}^n \to \mathbb{R}$ be continuous positive definite functions such that

$$\underline{W}(x) \leq V(t, x) \leq \overline{W}(x)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W(x)$$

for all $t \geq 0$ and $x \in D$, then the origin is uniformly asymptotically stable.
Theorem 14

(Converse Theorem - exponential stability:) Let $x = 0$ be an equilibrium point for $\dot{x}(t) = f(t, x)$ where $f$ is a $C^1$ function whose Jacobian, $\left[ \frac{\partial f}{\partial x} \right]$ is bounded on $D$. Let $k$ and $\gamma$ be positive constants such that

$$\|x(t)\| \leq k \|x(t_0)\| e^{-\gamma(t-t_0)}$$

for all $t \geq t_0$. Then there is a function $V : [0, \infty) \times D \to \mathbb{R}$ such that

$$c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -c_3 \|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\|$$

for some positive constants $c_1, c_2, c_3, \text{ and } c_4$. 
Theorem 15

(UAS - Converse Theorem) Let $x(t) = 0$ be the equilibrium point of $\dot{x} = f(t, x)$ where $f$ is continuously differentiable on $D = \{x \in \mathbb{R}^n : \|x\| < r\}$ and the Jacobian matrix is bounded uniformly in $t$. Let $\beta$ be a class $\mathcal{KL}$ function and let $r_0$ be a constant such that $\beta(r_0, 0) < r$. Assume that the system trajectory satisfies, $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$ for all $x(t_0) \in D$ and all $t \geq t_0 > 0$. Then there is a continuously differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha$, $\bar{\alpha}$, $\alpha$, and $\omega$ such that

$$\alpha(\|x\|) \leq V(t, x) \leq \bar{\alpha}(\|x\|)$$

$$\dot{V} \leq -\alpha(\|x\|)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq \omega(\|x\|)$$
Lyapunov’s direct method certifies stability by checking if $V > 0$ and $\dot{V} \leq 0$. It provides, however, no guidance, on how to find such a function $V$.

One way of finding a Lyapunov function is to take the Lyapunov function of a closely related system, parameterize this function and then search for parameters for which $V > 0$ and $\dot{V} \leq 0$.

This can be a daunting task if the search is done through analysis.

In recent years, however, convex optimization methods have advanced to the point where it is now possible to have the computer search for the Lyapunov function.

One of the main stumbling blocks in the computational approach is that deciding whether a multi-variation function is positive semidefinite is undecidable.

We get around this problem by relaxing our Lyapunov conditions so that we require $V$ and $-\dot{V}$ to be sum-of-squares or SOS polynomials.
SOS polynomials

- If a polynomial $V \in \mathbb{R}[x]$ has even degree, then it is positive semidefinite.
- So an easy condition to test for is whether we can write $V$ as a sum of squares

$$V(x) = \sum_i v_i^2(x)$$

where $v_i \in \mathbb{R}[x]$.
- How conservative is this necessary condition? It is exact for polynomials of one variable, quadratic/quartic polynomials in two variables. Other than that this is Hilbert’s 17th problem (still open).
SOS polynomials - how to find?

- Assume $V \in \mathbb{R}[x]$ is degree $2d$ and we can write it as a quadratic form in all monomials

$$V(x) = v^T Q v, \quad v^T = [1, x_1, x_2, \ldots, x_n, x_1 x_2, \ldots, x_n^d]$$

- If $Q$ positive semidefinite then $V$ has an SOS decomposition and so is non-negative.

- The matrix $Q$ is not unique and forms an affine variety of a linear subspace is the space of symmetric matrices.

- If the intersection of this affine subspace with the positive semidefinite matrix cone then $V$ is guaranteed to be SOS.
SOS polynomials - how to find?

- As example consider

\[ V(x, y) = 2x^4 + 2x^3y - x^2y^2 + 5y^4 \]

and let us find its SOS decomposition

- We let \( v^T = [x^2, y^2, xy] \) and rewrite \( V \) as

\[
V(x, y) = \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix} = q_{11}x^4 + q_{22}y^4 + (q_{33} + 2q_{12})x^2y^2 + 2q_{13}x^3y + 2q_{23}xy^3
\]
If we then equate coefficients, we obtain the following system of linear equations

\[
\begin{bmatrix}
2 \\
5 \\
-1 \\
2 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
q_{11} \\
q_{22} \\
q_{33} \\
q_{12} \\
q_{13} \\
q_{23}
\end{bmatrix}
\]

The set of all solutions to this system of linear inequalities can be readily shown to be

\[
\begin{bmatrix}
q_{11} & q_{22} & q_{33} & q_{12} & q_{13} & q_{23}
\end{bmatrix}
= 
\begin{bmatrix}
2 & 5 & -1 - 2\lambda & \lambda & 1 & 0
\end{bmatrix}
\]

where \( \lambda \in \mathbb{R} \) is any real value.
SOS polynomials - how to find?

- So $V$ can be written as

$$V(x, y) = v^T \begin{bmatrix} 2 & \lambda & 1 \\ \lambda & 5 & 0 \\ 1 & 0 & -1 - 2\lambda \end{bmatrix} v$$ (3)

$$= v^T Q(\lambda) v = v^T (Q_0 + \lambda Q_1) v$$ (4)

where $Q_0 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ and $Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

- To see if $V$ has an SOS decomposition, we need to find $\lambda$ such that $Q(\lambda) = Q_0 + \lambda Q_1$ is a positive semidefinite matrix.

- This has the form of a nonstrict linear matrix inequality or LMI
The “standard form” for a “strict” linear matrix inequality (LMI) is an affine matrix-valued function of the form,

\[ Q(\lambda) = Q_0 + \sum_{i=1}^{m} \lambda_i Q_i > 0 \]

where \( \lambda \in \mathbb{R}^m \) are decision variables and \( Q_i = Q_i^T \in \mathbb{R}^{n \times n} \) are symmetric matrices for \( i = 1, 2, \ldots, m \).

The LMI feasibility problem is given symmetric matrices, \( \{Q_i\}_{i=1}^{m} \), determine where there exists a vector \( \lambda \in \mathbb{R}^m \) such that the LMI \( Q(\lambda) > 0 \).

The LMI feasibility problem is one of those matrix problems which are computationally tractable.
Solving LMI feasibility problems

- LMI feasibility is efficiently solved (polynomial time) using “interior-point” techniques that revolutionized the solution of linear programs back in the mid 1980’s.
- IP solvers for strict LMIs appeared in early 1990’s with polynomial time complexity. In many cases the number of recursions is relatively constant with respect to the number of problem decision variables.
- Algorithms solving nonstrict LMIs (also called semidefinite programs) appeared in 1996 (Gahinet) with freely available SDP solvers (SDPT3) appearing around 2000.
One of the main issues in using such SDP solvers is that their user interfaces are not in a form that is easy to use directly.

This has led to the development of a number of toolkits that essentially translate LMI expressions that are in the form of matrix inequalities, into the standard form that the solvers then work with.

One of the first widely used toolkits that was developed specifically for SOS programming was SOSTOOLS.

The interface for SOSTOOLS can be somewhat clumsy to work with and so a more recent interface toolkit known as YALMIP (Yet Another LMI Program) has been gaining widespread acceptance across the community.
We will now use YALMIP to see if the polynomial in our example has an SOS decomposition.

Recall that this involves finding a real $\lambda$ such that $(Q_0 + \lambda Q_1)$ is positive semidefinite.

We first declare the state variables and form the polynomial

\[
\begin{align*}
x & = \text{sdpvar}(1,1); \\
y & = \text{sdpvar}(1,1); \\
V & = (2x^4)+(2x^3y)-(x^2y^2)+(5y^4);
\end{align*}
\]

We then form the vector and monomials and construct the quadratic form

\[
\begin{align*}
v & = \text{monolist}([x \ y], \text{degree}(V)/2); \\
Q & = \text{sdpvar}(\text{length}(v)); \\
V_{\text{sos}} & = v' \cdot Q \cdot v;
\end{align*}
\]
We then form the set of SOS constraints that are to be based to the solver.

These constraints require $Q$ to be PSD and the coefficients of the SOS polynomial to match the coefficients of the specified $V$.

We then call the SOS solver

```matlab
F = [coefficients(V-V_sos,[x y])==0, Q>=0];
sol=optimize(F);
if sol.problem==0
    value(Q);
end
```

If `sol.problem` is zero, then the SDP solve found the solution, which in this case is

$$Q(\lambda) = \begin{bmatrix} 2 & -1.4476 & 1 \\ -1.4476 & 5 & 0 \\ 1 & 0 & 1.8952 \end{bmatrix} = \begin{bmatrix} 2 & \lambda & 0 \\ \lambda & 5 & 0 \\ 1 & 0 & -1 - 2\lambda \end{bmatrix}$$

for $v^T = [x^2, y^2, xy]$ with a value of $\lambda = 1.4476$ for which $Q \succeq 0$. 
YALMIP and our problem

- We can verify this result by computing eigenvalues of $Q(-1.4476)$.
- To find the SOS decomposition, we take the square root $L^T L = Q$
  (cholesky) to get $L = \begin{bmatrix} 1.2927 & -0.4202 & 0.3903 \\ -0.4202 & 2.1957 & 0.0467 \\ 0.3903 & 0.0467 & 1.3193 \end{bmatrix}$.
- The SOS decomposition of $V$ is then

  $V(x) = \sum_{i=1}^{3} v_i(x)$

  $= (1.2927x^2 - 0.4202y^2 - 0.3903xy)^2$
  $+(-0.4202x^2 + 2.1957y^2 + 0.0467xy)^2$
  $+(0.3903x^2 + 0.0467y^2 + 1.3193xy)^2$

  $= 2x^4 + 2x^3y - x^2y^2 + 5y^4$
YALMIP also provides a more direct way of doing this through the command `sos` that streamlines the task of forming an SOS constraint and then using the command `solvesos` to compute the decomposition and actually find the $Q$ matrices. Alternatively, one could use the command `sosd` to just return the SOS decomposition.

```matlab
x = sdpvar(1,1); y = sdpvar(1,1);
V = (2*x^4)+(2*x^3*y)-(x^2*y^2)+(5*y^4);
F = sos(V);
[sol,u,Q,res] = solvesos(F);
if sol.problem==0
    sdisplay(u{1})
    value(Q{1})
v = sosd(F);
sdisplay(v)
end;
```

This returns a slightly different decomposition than we obtained doing the long way, but it still forms an SOS decomposition for $V$, merely emphasizing the fact that these decompositions are not unique.
Consider system

\[
\begin{align*}
\dot{x}_1 &= -ax_1 + x_2 \\
\dot{x}_2 &= x_1 - x_2
\end{align*}
\]

which is unstable for \( a < 1 \).

Set up the workspace with \( a = 2 \) (should be stable)

```matlab
clear all;
yalmip('clear');
sdpvar x1 x2;
x = [x1; x2];
a = 2;
f = [-a*x1+x2; x1-x2];
```

We then for the constraint requiring \( V \geq 0 \)

```matlab
P = sdpvar(length(x));
V = x'*P*x;
F = [P>=0]+[sos(V)];
```
Form the second SOS constraint for $-\dot{V} > 0$. Note that the actual constraint we are checking to be SOS is $-\dot{V} - \epsilon(x_1^2 + x_2^2) \geq 0$. The second part of this inequality forces $-\dot{V}$ to be strictly positive definite since the SDP solver only works with nonstrict inequality constraints.

```
negVdot = -jacobian(V,x)*f;
eps = 0.1;
F = F + [sos(negVdot-eps*(x'*eye(2,2)*x))];
```

Solve the SOS decomposition

```
[sol,u,Q] = solvesos(F);
if sol.problem == 0
disp('Constraints are SOS');
sdisplay(u{1}'*Q{1}*u{1})
else
disp('Constraints FAILED');
end
```

which return solution status (sol), a vector of monomials (u) and the symmetric matrix, Q.
The Lyapunov function returned from this has the form

\[ V(x_1, x_2) = 5.2779x_1^2 + 6.7222x_2^2 + 2.8886x_1x_2 \]

\[ = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 5.2779 & 1.443 \\ 1.443 & 6.722 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ = x^T P x \]

We can readily check to see that \( P \) is indeed positive definite and symmetric with real eigenvalues 4.3852 and 7.6148.

We can also verify that it satisfies the Lyapunov equation

\[ A^TP + PA = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5.2779 & 1.443 \\ 1.443 & 6.722 \end{bmatrix} + \begin{bmatrix} 5.2779 & 1.443 \\ 1.443 & 6.722 \end{bmatrix} \]

\[ = \begin{bmatrix} -24 & 7.6671 \\ 7.6671 & -7.6671 \end{bmatrix} \]

which has eigenvalues \(-27.0352\) and \(-4.6320\) and so is negative definite as expected.
consider the system

\[
\begin{align*}
\dot{x}_1 &= -x_1^3 - x_1 x_3^2 \\
\dot{x}_2 &= -x_2 - x_1^2 x_2 \\
\dot{x}_3 &= -x_3 - \frac{3x_3}{x_3^2 + 1} + 3x_1^2 x_3
\end{align*}
\]

Right hand side is rational, not polynomial, which cannot be directly solved using SOS decompositions.

Introduce a time scale change \( dt = d\tau(x_3^2 + 1) \), then the ODEs become polynomial

\[
\begin{align*}
\dot{x}_1 &= (-x_1^3 - x_1 x_3^2)(x_3^2 + 1) \\
\dot{x}_2 &= (-x_2 - x_1^2)(x_3^2 + 1) \\
\dot{x}_3 &= (-x_3 + 3x_1^2 x_3)(x_3^2 + 1) - 3x_3
\end{align*}
\]

So this is what we write our YALMIP script for.
This example requires $V > 0$ and $-\dot{V} \leq 0$ and returns the following Lyapunov function

$$V(x) = 16.1618x_1^2 + 10.2512x_2^2 + 3.8279x_3^2$$
Alternate representation for SOS polynomials does not presume a quadratic form and keeps all monomials generated by monolist.

\[ V(x) = (38x_1^2 + 16x_2^2 + 7.58x_3^2) \times 10^{-4} \]

Basically the same as the other one. Use this approach in HW6
Global versus Local Stability

- Note that finding an SOS function in the way we did above establishes \textit{global} stability of the equilibrium.

- Of course, if the equilibrium is only locally asymptotically stable, then the SOS search will \textbf{fail}, which tells us nothing.

- So is it possible to search for a local Lyapunov function by constraining the set over which we require $\dot{V}$ to be negative definite.

- This is a \textit{constrained SOS} search. It can be used for establishing local asymptotic stability, estimating regions of attractions, and finding "safety" certificates for nonlinear systems.
Local Positivity

- Given a polynomial $h \in \mathbb{R}[x]$, is $h(x) \geq 0$ for $x \in K$ where $K \subset \mathbb{R}^n$?
- This is the problem in integer programming
  \[
  \text{minimize } \gamma \\
  \gamma \geq f_i(y) \\
  \forall y \in \{-1, 1\}^n
  \]
- This occurs in Local Lyapunov Stability Analysis
  \[
  V(x) \geq |x|^2 \quad \forall |x| \leq 1 \\
  \frac{\partial V}{\partial x} f(x) \leq 0 \\
  \forall |x| \leq 1
  \]
- This occurs in safety problems - make sure all state trajectories starting in $\Omega_0$ never reach the forbidden set $\Omega_u$
  \[
  V(x) \leq 0 \quad \forall x \in \Omega_0 \\
  V(x) > 0 \quad \forall x \in \Omega_u \\
  \frac{\partial V}{\partial x} f(x) \leq 0 \quad \forall x \in D
  \]
Semi-algebraic Sets and SOS Positivity

- How are these "sets" represented?

- A set \( D \subset \mathbb{R}^n \) is semi-algebraic if it can be represented using equality and inequality constraints

\[
D := \{ x : p_i(x) \geq 0 \text{ for } i = 1, \ldots, k \text{ and } q_j(x) = 0 \text{ or } j = 1, \ldots, m \}
\]

- S-procedure:

\[ z^T Q z \geq 0 \text{ for all } z \in S := \{ x \in \mathbb{R}^n : x^T P x \geq 0 \} \text{ if there exists a scalar } \tau \geq 0 \text{ such that } Q - \tau P \succeq 0, \text{ an LMI} \]

- If we confine our attention to SOS polynomials this becomes:

\[ \text{Theorem 16} \]

Suppose \( \tau(x) \) is SOS, if \( f(x) - \tau(x) g(x) \) is SOS then \( f(x) \geq 0 \) for all \( x \in S := \{ x : g(x) \geq 0 \} \)
The main theorem we use to formulate our local Lyapunov search is based on a generalization of the S-procedure known as the Postivestellensatz theorem.

Below we give one version that applies when the semi-algebraic set has compact level set.

**Theorem 18**

(*Putinar’s Postivestellensatz*) Suppose \( S = \{ x : g_i(x) \geq 0, h_j(x) = 0 \} \) has compact level sets. If \( f(x) > 0 \) for all \( x \in S \), then there exists SOS polynomials \( s_i(x) \) and polynomials \( t_j(x) \) such that

\[
f = s_0 + \sum_i s_i(x)g_i(x) + \sum_j t_j(x)h_j(x)
\]
Consider the "test set"

\[ X := \{ x : p_i(x) \geq 0 \text{ for } i = 1, \ldots, k \} \]

Suppose there exists a polynomial \( V(x) \), a constant \( \epsilon > 0 \) and SOS polynomials, \( s_0(x) \), \( s_i(x) \), \( t_0(x) \), and \( t_i(x) \) such that

\[
V(x) - \sum_i s_i(x)p_i(x) - s_0(x) - \epsilon x^T x = 0
\]

\[- \frac{\partial V}{\partial x} f(x) - \sum_i t_i(x)p_i(x) - t_0(x) - \epsilon x^T x = 0\]

Then the system is exponentially stable for any

\[ \Omega_\gamma := \{ x : V(x) \leq \gamma \} \text{ where } \Omega_\gamma \subset X. \]

We use a bisection search to find the large \( \Omega_\gamma \). This provides an estimate on the region of attraction for the system.
Local Stability Analysis

Step 1: Use bisection to find the largest ball on which you can find a Lyapunov function.

Step 2: Use bisection to find largest level set of that Lyapunov function on which there is a Lyapunov function.

Homework 6 problem 3 asks you to write a YALMIP script solving this local stability analysis for a Van der pol oscillator. Note if you already know $V$ is SOS over all of space, you don’t need additional constraint on $V$. So for a single scalar constraint $\Omega = \{x : g(x) \geq 0\}$ you need to check that $\dot{V} - s \* g - \epsilon |x|^2$ is SOS.