CHAPTER 4

Shock-Induced Regime Shifts

A shock-induced regime shift occurs when an impulsive disturbance causes the system state to jump into the region of attraction (ROA) of an alternative basic set (regime). An example of this was seen in Fig. 4 of chapter 2, where a lake was "shocked" into a eutrophic state by a nutrient pulse. These nutrient pulses may be modeled as a stochastic process that generates a sequence of pulses having random amplitudes and random arrival times. In the lake example, these pulses occur because intense rain events release bursts of storm runoff. Since the national climate assessment forecasts more intense and frequent spring storms in the midwestern U.S., there is interest in assessing the environmental impact this trend in storms will have on lake ecosystems.

The likelihood of a shock-induced regime shift may be characterized in terms of the system's *first passage time* (FPT); a random variable representing the first time when the system exits the nominal regime. In particular, let $\{x(t)\}$ denote a sample path of the process that takes values in an open bounded set $\mathcal{X}$. Let $\mathcal{X}_{\rm nom}$ denote the nominal regime, let $\mathcal{X}_{\rm roa}$ denote the ROA for the nominal regime, and let $\mathcal{X}_u = \mathcal{X} - \mathcal{X}_{\rm roa}$ denote a "forbidden" region outside of the ROA of the nominal regime. This chapter determines the probability distribution for the FPT generated by the system's first exit from $\mathcal{X}_{\rm roa}$. This exit probability is obtained by computing, for each stopping time τ, a constant $\gamma \in [0,1]$ such that

$$\mathbb{P}\left\{x(t) \in \mathcal{X}_u \text{ for some } 0 \leq t \leq \tau \,:\, x(0) \in \mathcal{X}_{\rm nom}\right\} \leq \gamma \tag{23}$$

The constant γ bounds the probability that the first passage time will be τ. By computing γ for a range of τ, we therefore obtain a function that is an upper bound on the FPT's probability distribution.

The algorithmic framework used to find the bound γ in equation (23) was established in [87]. That framework, however, computed the bound for regular diffusion processes that satisfy the stochastic differential equation $dx = f(x)dt + g(x)dw$ in which $\{w(t)\}$ is a Wiener process. This framework cannot be directly applied to the regime shift problem because systems driven by randomly arriving impulses are better modeled as *jump-diffusion processes* (JDP). This chapter, therefore, extends the framework in [87] to jump-diffusion processes. It then uses this extension to bound probability distributions for the FPT of shock-induced regimes shifts and demonstrates the method on an intra-guild predation system [29].

1. Jump Diffusion Processes

This section reviews results for jump-diffusion processes. Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete probability space and let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration over that space which is right continuous [89]. Consider an adapted jump-diffusion process (JDP), $\{x(t)\}_{t\geq 0}$, that satisfies the following stochastic differential equation

$$dx(t) = f(x(t))dt + g(x(t))dw(t) + dJ(t), \quad x(0) = x_0 \tag{24}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous functions, $\{w(t)\}_{t\geq 0}$ is a Wiener process, and $\{J(t)\}_{t\geq 0}$ is a shot noise process that takes values

$$J(t) = \sum_{\ell=1}^{N(t)} y_\ell e^{-\delta(t-\tau_\ell)}, \quad \ell \in \mathbb{Z}_{\geq 0} \tag{25}$$

in which $\{N(t)\}$ is a Poisson process with intensity ρ, $\{\tau_\ell\}$ are event times of the Poisson jumps, $\{y_\ell\}$ is an i.i.d. random process with distribution $F(y)$

describing the ℓ-th jump's size, and δ is a real positive constant representing the rate of exponential decay after a jump. The JDP in equation (24) is understood in Itô's sense and $\{w(t)\}$ is statistically independent of $\{J(t)\}$.

Consider the random process $\{Y(\tau_\ell, y_\ell)\}$ generated from $\{y_\ell\}$ and $\{\tau_\ell\}$ through the equation

$$Y(\tau_\ell, y_\ell) = y_\ell e^{\delta \tau_\ell}$$

then we may write the values taken by the shot noise process $\{J(t)\}$ as

$$J(t) = e^{-\delta t} \int_0^t \int_{\mathbb{R}^n} Y(\tau, y) N(d\tau, dy) \tag{26}$$

where $N(d\tau, dy)$ is a Poisson measure with $\mathbb{E}\{N(dt, dy)\} = \rho dt F(dy)$. We define the *increment* of $\{J(t)\}$ as

$$dJ(t) = J(t + dt) - J(t)$$

where dt is an infinitesimal time increment. We use equation (26) to expand out the increment, dJ, and retaining the first order terms in dt one rewrites the shot noise increment as

$$dJ(t) = -\delta J(t) dt + \int_{\mathbb{R}^n} y N(dt, dy). \tag{27}$$

The expression for the jump increment in equation (27) is inserted into the JDP of equation (24) and rewritten to obtain

$$\begin{aligned} dx(t) &= (f(x(t)) - \delta J(t))dt + g(x(t))dw(t) \\ &\quad + \int_{\mathbb{R}^n} y N(dt, dy), \quad x(0) = x_0 \end{aligned} \tag{28}$$

Since $\{J(t)\}$ and $\{w(t)\}$ in equation (28) are independent Markov processes and with the assumed conditions on the filtration, $\mathcal{F}_t$, one may conclude that the solution of the JDP in equation (28) is a Markov process with right continuous sample paths [89]. The JDP equation (28), therefore, generates a Markov process whose current state, $x(t)$, encapsulates all information needed to characterize its future sample paths.

Now let $\{x(t)\}_{t\geq 0}$ be a Markov process with right continuous sample paths defined on a bounded open set $\mathcal{X} \subset \mathbb{R}^n$. Let $\tau < \infty$ be a stopping time for the process such that

$$\tau \leq \inf\{t : x(t) \notin \mathcal{X}\}$$

Let $V : \mathbb{R}^n \to \mathbb{R}$ be any function acting on x. The process $\{V(x(t))\}_{t\geq 0}$ is called a *supermartingale* with respect to process $\{x(t)\}$ if

(1) $V(x(t))$ is $\mathcal{F}_t$-measurable for all t
(2) $\mathbb{E}\{|V(x(t)|\} < \infty$
(3) and $\mathbb{E}\{V(x(t_2)) \,|\, V(x(t_1))\} \leq V(x(t_1))$ for all $0 \leq t_1 \leq t_2 \leq \tau$

The first two conditions are regularity conditions that can generally be satisfied by requiring $\mathcal{X}$ to be bounded and V to be sufficiently smooth. The third condition means that given $V(x(t_1))$, the average of future values of V will be smaller. In other words, the conditional expectation of $V(x(t))$ is a monotone decreasing function of time. If we think of V as the payoff from a gambling game, having $\{V(x(t))\}$ be a supermartingale means that our average winnings from the game are always decreasing.

Supermartingales are useful in extending the concept of asymptotic stability to Markov processes. In particular, when $\{V(x(t))\}$ is a supermartingale, then we can think of V as a "stochastic" Lyapunov function whose existence certifies that the equilibrium of the system is asymptotically stable with probability one [69]. The following theorem allows us to say a bit more about the probability of leaving the neighborhood of the equilibrium. This result will be used later in characterizing the probability of shock-induced regime shifts. The theorem is stated below and its proof can be found in [70, 69].

THEOREM 1. *Let $\{V(x(t))\}$ be a supermartingale with respect to the Markov process $\{x(t)\}$ where $x(t)$ takes values on a bounded open set $\mathcal{X}$ and τ is a stopping time for the process on $\mathcal{X}$. Let $V : \mathcal{X} \to \mathbb{R}$ be non-negative in $\mathcal{X}$, then for any constant $\theta > 0$ and any initial condition $x_0 \in \mathcal{X}$ we have*

$$\mathbb{P}\left\{ \sup_{0 \le t \le \tau} V(x(t)) \ge \theta \,\middle|\, x(0) = x_0 \right\} \le \frac{V(x_0)}{\theta} \tag{29}$$

The use of theorem 1, however, requires we find a way to certify that $\{V(x(t))\}$ is a supermartingale. This certification can be done using the JDP's *infinitesimal generator*. In particular, let $\{x(t)\}$ be a Markov process with right continuous sample paths and consider any function $V : \mathbb{R}^n \to \mathbb{R}$. The infinitesimal generator of $\{x(t)\}$ is an operator, $\mathcal{L}$, whose action on a function V takes values

$$\mathcal{L}[V](x(t)) = \lim_{h \downarrow 0^+} \frac{\mathbb{E}\left\{ V(x(t+h)) \,|\, V(x(t)) \right\} - V(x(t))}{h}$$

When the Markov process is generated by the JDP in equation (28) and V is a C^2 function with two continuous derivatives, then the generator can be written as [**81**]

$$\begin{aligned} \mathcal{L}[V](x(t)) \;\; = \;\; & \frac{\partial V(x(t))}{\partial x}(f(x(t)) - \delta J(t)) \\ & + \frac{1}{2} \mathrm{trace}\left(g^T(x(t)) \frac{\partial^2 V(x(t))}{\partial x^2} g(x(t)) \right) \\ & + \rho \int_0^\infty (V(x+y) - V(x)) dF(y) \end{aligned} \tag{30}$$

Note that the JDP's generator is similar to that of a regular diffusion, with the main difference being the integral term in the last line of equation (30).

We can now use the generator in equation (30) to obtain a sufficient condition certifying that $\{V(x(t))\}$ is a supermartingale. In the first place, we note that we require $\mathcal{X}$ to be bounded and we also restrict our attention to V that are C^2. These two restrictions ensure that a supermartingale's first two regularity conditions are satisfied. The third condition can be satisfied if we require

$$\mathcal{L}[V](x(t)) \leq 0 \tag{31}$$

for all x. To justify this assertion, we invoke Dynkin's formula for JDPs

THEOREM 2. (Dynkin's formula [81]) *Consider the JDP in equation (28) defined on an open bounded set $\mathcal{X} \in \mathbb{R}^n$ with smooth boundary $\partial\mathcal{X}$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function and let $\tau < \infty$ be a stopping time such that $\tau \leq \inf\{t : x(t) \notin \mathcal{X}\}$. Suppose*

$$\mathbb{E}\left\{|V(x(\tau))| + \int_0^\tau |\mathcal{L}[V](x(s))|ds\right\} < \infty$$

then

$$V(x(\tau)) = V(x(0)) + \mathbb{E}\left\{\int_0^\tau \mathcal{L}V(x(s))ds \,\middle|\, x(0)\right\} \tag{32}$$

This theorem asserts that $V(x(t))$ satisfies an integral equation where the generator $\mathcal{L}[V]$ lies inside the integral term. From this equation it is apparent that if condition 31 is satisfied then $V(x(\tau)) \leq V(x(0))$ for $\tau > 0$ which is sufficient to establish the third condition for a supermartingale. For our purposes, therefore, the hypotheses of theorem 1 are satisfied if we can find a non-negative V such that $\mathcal{L}[V](x) \leq 0$. Since we have an explicit expression for the JDP's generator, this condition can be checked for a given V thereby allowing us to bound the probability that $V(x(t))$ will be greater than θ. The next section uses this idea to bound the probability of a shock-induced regime shift occurring before the stopping time.

2. Shock-Induced Regime Shift Certificates

A function whose existence is sufficient for a dynamical system to have a specified property is called a *certificate*. Certificates have been developed for a number of system properties such as Lyapunov stability, input-to-state stability, and passivity. The certificate used in [**87**] is sometimes called a *barrier certificate*. But the existence of this function is sufficient to ensure that the forbidden set $\mathcal{X}_u$ can be reached in finite time from the initial set $\mathcal{X}_{\rm nom}$. So in this regard, perhaps they are more accurately described as reachability certificates.

This section is concerned with certificates for shock-induced regime shifts, or what we will refer to as a *regime shift certificate* conditioned on a given probability γ and stopping time τ. The existence of these functions means that a shock-induced regime shift will occur with probability less than γ within the stopping time τ. For V to be a regime shift certificate, it must satisfy certain inequality constraints on V and the generator, $\mathcal{L}[V]$. The formal statement of these conditions simply restates the main theorem of [**87**] in terms of the nominal and alternative regimes (basic set).

THEOREM 3. *Let $\{x(t)\}$ be a Markov process generated by the JDP in equation (28) where $x(t)$ takes values on an open bounded set $\mathcal{X}$ and $\tau < \infty$ is a stopping time with respect to $\mathcal{X}$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function such that $\int_0^\tau \phi(t)dt < \infty$ and let $\mathcal{X}_{\rm nom}$ be the basic set associated with the deterministic system's ($\dot{x} = f(x)$) nominal regime. Let $\mathcal{X}_{\rm roa}$ be the region of attraction for $\mathcal{X}_{\rm nom}$ whose closure is contained in the interior of $\mathcal{X}$. Define the forbidden set $\mathcal{X}_u = \mathcal{X} - \overline{\mathcal{X}}_{\rm roa}$. If there exists a C^2 function $V : \mathbb{R}^n \to \mathbb{R}$ and a constant $\gamma \in [0,1]$ such that*

$$
\begin{array}{rcll}
V(x) & \leq & \gamma, & \forall x \in \mathcal{X}_{\rm nom} \\
V(x) & \geq & 1, & \forall x \in \mathcal{X}_u \\
V(x) & \geq & 0, & \forall x \in \mathcal{X} \\
\mathcal{L}[V](x(t)) & \leq & \phi(\tau), & \forall x(t) \in \mathcal{X}
\end{array} \tag{33}
$$

then

$$
\mathbb{P}\left\{ x(t) \in \mathcal{X}_u \text{ for some } 0 \leq t \leq \tau \,\middle|\, x(0) \in \mathcal{X}_{\rm nom} \right\} \leq \gamma
$$

The proof of this theorem is based on the probability bound given in theorem 1 of the preceding section. It is instructive to go through it here to see how the regime shift concepts are carried over into the proof. The generator in equation (33) is the JDP generator in equation (30). Since we assumed the domain $\mathcal{X}$ is bounded, $V \in C^2$, and $\mathcal{L}[V](x) \leq 0$, we know that $\{V(x(t)\}$ is a supermartingale. From Kushner's theorem (1) we know

$$\mathbb{P}\left\{ \sup_{0 \leq t < \tau} V(x(t)) \geq \theta \,\middle|\, x(0) = x_0 \right\} \leq \frac{V(x_0)}{\theta} \tag{34}$$

The initial states are taken in the nominal basic set $\mathcal{X}_{\text{nom}}$ where we know $V(x) \leq \gamma$. On the forbidden set $\mathcal{X}_u$, we know that $V(x) \geq 1$. So we take θ to be one in equation (34) and get

$$\begin{aligned} \gamma &\geq V(x(0)) \\ &\geq \mathbb{P}\left\{ \sup_{0 \leq t \leq \tau} V(x(t) \geq 1 \,\middle|\, x(0) \in \mathcal{X}_{\text{nom}} \right\} \\ &\geq \mathbb{P}\left\{ x(t) \in \mathcal{X}_u \,\middle|\, x(0) \in \mathcal{X}_{\text{nom}} \right\} \end{aligned}$$

thereby completing the proof.

A function V that satisfies the conditions in theorem 3 will be called a *regime-shift certificate* conditioned on probability γ and stopping time τ. In particular, the existence of the function certifies that within time τ, a *shock-induced* regime shift will occur with probability less than γ. Fig. 15(a) will be used to help justify this description of V. This figure shows the phase space for a deterministic system whose orbits are confined to a compact set $\overline{\mathcal{X}}$ whose boundary, $\partial\mathcal{X}$, is a negative limit set of the system. We already know that the positive limit set for this system can be partitioned into a mutually disjoint collection of *basic sets*. For the system in Fig. 15 these basic sets are $\{\partial\mathcal{X}, [R]_1, [R]_2\}$. We take the nominal regime as the basic set $[R]_1 := \mathcal{X}_{\text{nom}}$. The "forbidden" set, $\mathcal{X}_u$, is shown in red in Fig. 15(a).

It is the complement of $[R]_1$'s region of attraction. Given a family of impulse disturbances, theorem 3 asserts that if there exists a V satisfying the conditions in equation (33) then there is a sample path that will reach the boundary of $\mathcal{X}_{\rm nom}$'s region of attraction with probability γ before the time τ. Once $x(t)$ reaches its boundary it is, of course, in the ROA of an alternative basic set (regime) and so unless there are further disturbances the system state remains in the ROA of the alternative regime. Because that regime must be attracting, the orbit will asymptotically approach the alternative regime's basic set and a shock-induced regime shift will have occurred. The existence of the regime shift certificate, V, is actually guaranteed by Conley's decomposition theorem 5 which is formally stated in the next chapter. The only challenge that remains is finding the regime shift certificate, V, so we can quantitatively assess the likelihood of exiting the nominal regime.

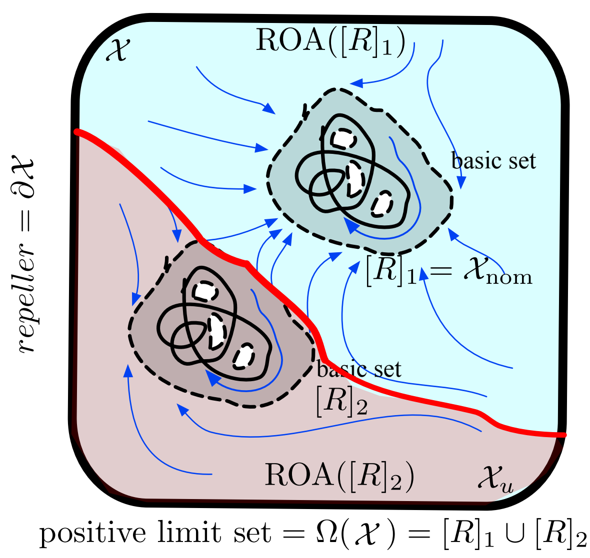

FIGURE 15. Morse Decomposition of Positive Limit Sets and Shock-induced Regime Shifts

3. Sum-of-Squares Regime Shift Certificates

Theorem 3 provides sufficient conditions for function, V, whose existence "certifies" that a regime shift will occur with a probability less than γ within the stopping time τ. We can use this certificate to construct an upper bound on the probability distribution for the system's first passage time out of the nominal regime; i.e. a probability distribution on the time a shock-induced regime shift occurs. This is done by embedding the conditions in theorem 3 in an optimization problem that seeks for a specified τ, the minimum γ for which the theorem's conditions are satisfied. The optimization problem we want to solve, therefore, has the following form,

$$
\begin{array}{rll}
\text{minimize:} & \gamma & \\
\text{with respect to:} & V(x) & \\
\text{subject to:} & 0 \leq \gamma - V(x), & \forall x \in \mathcal{X}_{\text{nom}} \\
& 0 \leq V(x) - 1, & \forall x \in \mathcal{X}_{\text{u}} \\
& 0 \leq V(x), & \forall x \in \mathcal{X} \\
& 0 \leq -\mathcal{L}[V](x) + \phi(\tau), & \forall x \in \mathcal{X}
\end{array} \tag{35}
$$

where τ is a fixed stopping time, ϕ is any function such that $\int_0^\tau \phi(t)dt < \infty$, the regions $\mathcal{X}$, $\mathcal{X}_u$, and $\mathcal{X}_{\text{nom}}$ are known, and $\mathcal{L}[V]$ is the JDP generator in equation (30). Minimizing γ is done through a bisection search in which theorem 3 acts as an algorithmic oracle. The outcome of this search, $\gamma^*(\tau)$, is an upper bound on the probability that the first passage time out of the nominal regime is τ. So by solving the optimization problem for a range of stopping times, τ, we obtain an upper bound on the FPT's probability distribution, thereby providing a useful characterization of the shock-induced regime shift.

Solving the optimization problem in equation (35) can be very challenging. In the first place, all of the constraints require that some function of x be positive semidefinite. In general the problem of deciding whether a multi-variate function is positive semidefinite is undecidable. Secondly, the constraints in equation (35) are restricted to a subset of the state space. We need a way of translating these constrained inequality condition into an unconstrained condition if we are to solve the problem. The following subsections (3.1 and 3.2) show that sum-of-squares (SOS) relaxations can be used to address both challenges when the system and the constraint sets are defined using polynomials in $\mathbb{R}[x]$. The use of SOS relaxations raises an additional issue because of the integral term in the JDP generator (30). The final subsection (3.3) shows how to handle this issue.

3.1 Certifying Unconstrained Polynomials are Positive Semidefinite: The problem of verifying the positivity of a multi-variate function is undecidable. If we restrict our attention to functions that are polynomials in $\mathbb{R}[x]$, however, we find this is still NP-hard. So at first glance this suggests that theorem 3 is of little practical value even when the system is polynomial. One may circumvent this complexity issue by relaxing the requirement that the function be positive semidefinite to simply requiring that it satisfy a sufficient condition for positivity that is relatively easy to check. We refer to this as a *relaxation* of the original certification problem. In recent years one particular relaxation has emerged that is both sufficient for positivity and easy to compute. This relaxed problem requires the certificates to be sum-of-squares or SOS polynomials, rather than positive semidefinite polynomials. Satisfying the SOS condition clearly ensures the function is positive. Checking if a function is SOS is relatively easy because it takes the form of a convex optimization problem that can be efficiently solved using recent numerical advances in interior point optimization.

Let $\mathbb{R}[x]$ denote the set of all polynomials in the indeterminate variables $x = \{x_1, \ldots, x_n\}$ with real coefficients. If a polynomial $V \in \mathbb{R}[x]$ is positive semidefinite (PSD) then an obvious necessary condition for positivity is that its degree is even. A simple sufficient condition for the positivity of

V, therefore, is the existence of a sum-of-squares (SOS) decomposition of the function,

$$V(x) = \sum_{i=1}^{m} v_i^2(x) \tag{36}$$

where $v_i \in \mathbb{R}[x]$ for $i = 1, 2, \ldots, m$. If one can find V that satisfies equation (36), then we can conclude V is positive semidefinite.

It is relatively easy to computationally search for such SOS decompositions of V. Let us consider $V \in \mathbb{R}[x]$ of degree 2d and let us assume it can be written as a quadratic form in all monomials of degree less than or equal to d

$$V(x) \quad = \quad v^T \mathbf{Q} v \tag{37}$$

where

$$v^T(x) = \begin{bmatrix} 1 & x_1 & x_2 & \ldots & x_n & x_1 x_2 & \ldots & x_n^d \end{bmatrix}$$

and $\mathbf{Q}$ is a constant real-valued matrix. The length of the monomial vector, $v(x)$, is $\binom{n+d}{d}$. If the matrix $\mathbf{Q}$ is positive semidefinite, then $V(x)$ has an SOS decomposition and so is non-negative. Note that the matrix $\mathbf{Q}$ is not unique and so $\mathbf{Q}$ may be PSD for some representations and not for others. By expanding the right hand side of equation (37) and matching coefficients of x, one can show that the set of matrices satisfying this equation form an affine variety of a subspace in the linear space of symmetric matrices. If the intersection of this affine variety with the cone of positive semidefinite matrices is nonempty, then the function V is SOS.

Since the set of matrices satisfying equation (37) form an affine variety, one may write any matrix in that variety as a linear matrix inequality (LMI) of the form

$$\mathbf{Q}(\lambda) = \mathbf{Q}_0 + \sum_{i=1}^{m} \lambda_i \mathbf{Q}_i \geq 0$$

where $\lambda \in \mathbb{R}^m$ is a vector parameterizing the matrices in that variety and $\mathbf{Q}_i = \mathbf{Q}_i^T \in \mathbb{R}^{n\times n}$ are symmetric matrices for $i = 1, 2, \ldots, m$. When stated in this way, the problem of certifying if V is SOS devolves to finding a parameter vector λ such that $\mathbf{Q}(\lambda) \geq 0$. When formulated in this way, it should be apparent that the problem of certifying whether a polynomial V is SOS is equivalent to solving an LMI feasibility problem.

The LMI feasibility problem is one of those matrix problems that is computationally tractable. This problem is efficiently solved using interior point technique that revolutionized the solution of linear programs back in the mid 1980's [**1**]. The development of interior point solvers for strict LMI problems appeared in the early 1990's [**40**]. These solvers are recursive codes with polynomial time complexity. Algorithms that solve the nonstrict LMI problems are sometimes called *semidefinite programs* (SDP)[**118**]. Freely available SDP solvers began to appear around 2000 [**116, 107**] and now represent an essential tool for control systems engineering.

Remark: It can be cumbersome to directly use SDP solvers such as SDPT3 or SEDUMI. This has led to the development of a number of toolkits that translate LMI expressions into the standard form that these solvers work with. One of the first widely used tools kits for SOS certificates was SOSTOOLS [**88**]. Another well known tool kit that many use today is YALMIP [**73**].

3.2 Certifying Constrained Polynomials are Positive Semidefiite: The preceding subsection showed how LMI's could be used to certify whether a function V is SOS (PSD) for all $x \in \mathbb{R}^n$. Note, however, that the conditions in the optimization problem (35) only require positive definiteness over a subset of $\mathbb{R}^n$. This means that to solve that optimization problem, we need to find a way to certify that the *constrained* polynomial function is SOS.

So our basic problem is to certify

$$V(x) \in \mathbb{R}[x] \text{ is SOS for all } x \in \mathcal{X} \subset \mathbb{R}^n$$

We will require the constraint set, $\mathcal{X}$ to be semi-algebraic, meaning that it can be represented using equality and inequality constraints of the form,

$$\mathcal{X} = \{x \in \mathbb{R}^n \,:\, p_i(x) \geq 0 \text{ for } i = 1, \dots, k \text{ and } q_j(x) = 0 \text{ for } j = 1, 2, \dots, m\}$$

with p_i and q_j being polynomials in x. With this charcterization one can use the following theorem from algebraic geometry to formulate our constrained SOS problem. This theorem is called the *positivstellensatz* theorem [**105**] and may be seen as a generalization of the S-produre that is used for constrained LMI's [**9**]. There are several versions of this theorem, the one given below is due to Putinar [**90**].

THEOREM 4. **(Putinar's Positivstellensatz)** *Suppose*

$$\mathcal{X} = \{x \in \mathbb{R}^n \,:\, p_i(x) \geq 0 \textit{ for } i = 1, \dots, k \textit{ and } q_j(x) = 0 \textit{ for } j = 1, 2, \dots, m\}$$

has compact level sets. If there exist SOS polynomials $\sigma_i(x)$ ($i = 0, 1, 2, \dots, k$) and polynomials $\beta_j(x)$ ($j = 1, 2, \dots, m$) such that

$$V(x) = \sigma_0(x) + \sum_{i=1}^{k} \sigma_i(x) p_i(x) + \sum_{j=1}^{m} \beta_j(x) q_j(x)$$

then the polynomial $V(x)$ is positive semidefinite (SOS) for all $x \in \mathcal{X}$.

The positivstellenstaz theorem provides a way to certify the constraints in theorem 3. In particular, let us assume there are polynomial functions p, p_{nom}, and p_u mapping $\mathbb{R}^n$ onto $\mathbb{R}$ such that the constraint sets in theorem 3 are semi-algebraic

$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^n \,:\, p(x) \geq 0\} \\ \mathcal{X}_{\text{nom}} &= \{x \in \mathbb{R}^n \,:\, p_{\text{nom}}(x) \geq 0\} \\ \mathcal{X}_u &= \{x \in \mathbb{R}^n \,:\, p_u(x) \geq 0\} \end{aligned}$$

Theorem 4 allows us to recast the regime-shift certificate problem (35) as

$$
\begin{aligned}
&\text{minimize} && \gamma \\
&\text{with respect to:} && \text{SOS functions } V, \sigma, \sigma_{\text{nom}}, \sigma_u \\
&\text{subject to:} && -V(x) + \gamma - \sigma_0(x) p_0(x) \text{ is SOS} \\
& && V(x) - \sigma_u(x) p_u(x) - 1 \text{ is SOS} \\
& && V(x) - \sigma(x) p(x) - \epsilon \text{ is SOS} \\
& && -\mathcal{L}[V](x) - \sigma(x) p(x) + \phi(\tau) \text{ is SOS}
\end{aligned}
\tag{38}
$$

where τ is a selected stopping time, $\int_0^\tau \phi(t)dt < \infty$, and $\epsilon > 0$ is chosen small to simply ensure V is sufficiently positive definite (not just positive semidefinite). The SOS program in equation (38) can be efficiently solved using the SDP solvers discussed in the preceding subsection.

3.3 Polynomial Representations for JDP Generator: There is still one final issue with the SOS program in equation (38) that needs to be dealt with before demonstrating its use in the next section. The last constraint certifies that $\mathcal{L}[V](x) \leq 0$ on $\mathcal{X}$ and is needed to verify that $\{V(x(t))\}$ is a supermartingale. The problem here is that our expression for the generator in equation (30) has an integral term in it that does not fit easily handled by SOS solvers. This section addresses that issue by showing how to reduce that integral term into a polynomial.

Multi-indices will be of use in reducing the integral term in equation (30). Given a multi-index, $\alpha = (\alpha_1, \ldots, \alpha_n)$, we let $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$. The sum/difference of two multi-indices is the component-wise sum/difference and $\alpha \geq \beta$ is also defined in a component-wise manner. The binomial coefficient of two multi-indices is $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$ The multi-index binomial theorem says

$$
(x+y)^{[\alpha]} = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} x^{[\alpha-\beta]} y^{[\beta]}
$$

If $V : \mathbb{R}^n \to \mathbb{R}$ is a real valued function and given an n-dimensional multi-index, α, then the αth order partial derivative of V is defined as

$$\partial^{[\alpha]} V = \frac{\partial^{\alpha_1} V}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2} V}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n} V}{\partial x_1^{\alpha_n}}$$

and finally it can be shown that for $V(x) = x^{[\beta]}$ that its derivative is

$$\partial^{[\alpha]} x^{[\beta]} = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} x^{[\beta-\alpha]} & \text{if } \alpha \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

We use the preceding notational conventions to reduce the integral term in the JDP generator. Consider a polynomial $V \in \mathbb{R}[x]$ and write out $V(x+y)$ as

$$\begin{aligned} V(x+y) &= \sum_{|\alpha| \leq p} k_\alpha (x+y)^{[\alpha]} \\ &= \sum_{|\alpha| \leq p} k_\alpha \sum_{0 \leq |\beta|, \beta \leq \alpha} \binom{\alpha}{\beta} x^{[\alpha-\beta]} y^{[\beta]} \end{aligned}$$

where k_α is a set of real valued coefficients. The above expression can be used to write out the difference $V(x+y) - V(x)$ that appears in the JDP generator's integral

$$\begin{aligned} V(x,y) - V(x) &= \sum_{|\alpha| \leq p} k_\alpha \sum_{1 \leq |\beta|, \beta \leq \alpha} \frac{1}{\beta!} \left(\partial^{[\beta]} x^{[\alpha]} \right) y^{[\beta]} \\ &= \sum_{1 \leq |\beta| \leq p} \frac{1}{\beta!} \left(\partial^{[\beta]} V(x) \right) y^{[\beta]} \end{aligned}$$

Integrating both sides with respect to $F(y)$ gives

$$\int (V(x,y) - V(x)) dF(y) = \sum_{1 \leq |\beta| \leq p} \frac{1}{\beta!} \left(\partial^{[\beta]} V(x) \right) \mathbb{M}^{|\beta|} \tag{39}$$

where $\mathbb{M}^{|\beta|} = \int y^{|\beta|} dF(y)$ is the $|\beta|$th order moment of y. Equation (39) is the integral term in the JDP generator equation (30). Substituting this into that equation allows us to explicitly write out the generator in a form that can be used in the SOS solvers.

4. First Passage Times for Intra-Guild Predation System

This section demonstrates the computation of the FPT probability distribution for an intraguild predation system originally studied in [29] for freshwater lakes that have both bass and crayfish. Bass-crayfish interactions form an intraguild predation system in which both species compete for the same resource while also predating on each other. The system has two stable equilibria (regimes); one in which the bass dominate the ecosystem and the other in which the crayfish dominate the ecosystem. An outbreak of crayfish is undesirable as it can suppress the bass population. If such an outbreak does occur, the environmental resource manager can adopt management policies that trigger a regime shift from the crayfish-dominated regime to the bass-dominated regime.

One commonly used management policy is to permit the harvesting of crayfish by anglers. In general, this harvesting process can be modeled as a jump process in which the size and timing of the harvesting events are parameters that the environmental resource manager needs to set. The management policy therefore seeks to trigger a shock-induced regime shift from the lake's current (nominal) crayfish dominated condition to the more desirable (alternative) bass dominated condition. The manager can determine how best to set the intensity and frequency of crayfish harvests by using regime-shift certificates to estimate the FPT probability distribution for a selected harvesting strategy. This section demonstrates how that FPT probability distribution is computed and compares the result against a Monte Carlo simulation for the system.

A model for crayfish (x) and bass (y) interaction under harvesting satisfies the following consumer resource equations

$$
\begin{aligned}
dx(t) &= x(1-x-0.7y) - \frac{0.08yx^2}{0.01+x^2} + \sigma dw_1(t) - \sum_{i=1}^{N(t)} z_i \delta(t-\tau_i), \\
dy(t) &= 1.5y(1-y-0.9x) + \frac{0.01yx^2}{0.01+x^2} + \sigma dw_2(t).
\end{aligned}
\tag{40}
$$

Equation (40) is a consumer-resource system which is driven by Wiener processes $\{w_i(t)\}$, $(i = 1, 2)$ driving the system. The crayfish equation, x, has an additional term that models the harvesting of the crayfish, x, as a compound Poisson process in which the harvest size $\{z_i\}_{i=1}^{N_t}$ and the harvest times $\{\tau_i\}_{i=1}^{N_t}$ are i.i.d. with exponential distribution of intensity μ and λ, respectively, and $N(t)$ is the number of harvest events in the interval $[0, t]$. In the absence of the stochastic disturbances $w(t)$ and $N(t)$, this model has three equilibria (two stable and one unstable) in $\mathbb{R}^2_+$.

Figure 16 plots identifies the two stable equilibria with E_{bass} denoting the bass-dominated equilibrium and E_{cray} denoting the crayfish-dominated equilibrium. The regions of attraction (ROA) for both equilibria are separated by the separatrix also marked in Fig. 16. We let $\mathcal{X}_{\text{nom}}$ denote the initial regime the system starts in be a neighborhood of the crayfish dominated equilibrium. The forbidden set, $\mathcal{X}_u$, is the ROA for the bass-dominated equilibrium which is shaded in green. Note that the state space in Fig. 16 has been translated so the crayfish dominated equilibrium, E_{cray}, is at the origin.

To setup the SOS optimization problem used in bounding the FPT's probability distribution, we first set the harvesting policy's parameters, μ and λ, and the value of the stopping time we're testing for. We then need to obtain semi-algebraic characterizations of the sets, $\mathcal{X}$, $\mathcal{X}_{\text{nom}}$, $\mathcal{X}_u$. We take for $\mathcal{X}$ a unit square region in $\mathbb{R}^2$. $\mathcal{X}_{\text{nom}}$ can be expressed in terms of a quadratic

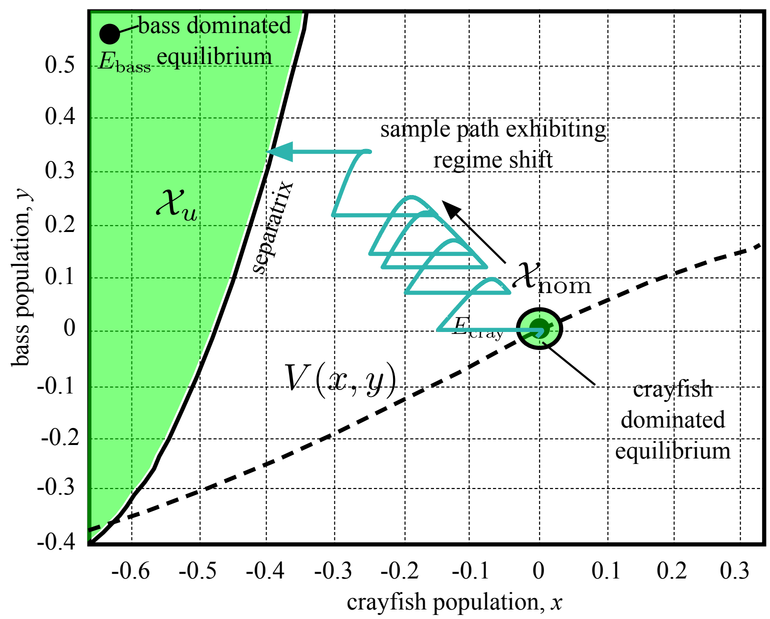

FIGURE 16. State Space of Bass-Crayfish System with the sets $\mathcal{X}_{\text{nom}}$, $\mathcal{X}_u$ marked along with the level curve of a certificate, $V(x,y)$, that triggers a regime shift with probability one when the jump parameters are $\mu = 0.075$ and $\lambda = 0.2$.

form. For this example, we can explicitly compute a polynomial expression for the separatrix and use that to describe X_{u} as a semi-algebraic set. In general, however, one would usually use SOS methods to find a semi-algebraic set characterizing the alternative regime's region of attraction as has been done in [**113**]. The particular semi-algebraic representations for this problem's constraint sets are

$$
\begin{aligned}
\mathcal{X} &= \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} (x+0.672)(0.328-x) \geq 0, \\ (y+0.4)(0.6-y) \geq 0 \end{array} \right\} \\
\mathcal{X}_{\text{nom}} &= \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, (0.075)^2 - x^2 - y^2 \geq 0 \right\} \\
\mathcal{X}_u &= \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} -(x+0.672)(0.39+x) \geq 0 \\ (y+0.4)((0.5-y) \geq 0 \\ y + 0.85x^3 - 5.6x^2 - 9.45x - 3.16 \geq 0 \end{array} \right\}
\end{aligned}
$$

The SOSTOOLS were used to minimize the FPT probability for a stopping time $\tau = 10^4$ (time steps) assuming the Brownian motion intensity was $\sigma = 0.05$, and the jump parameters $\mu = 0.075$ and $\lambda = 0.2$. The dashed line in Fig. 16 plots a level 0 curve (i.e. $V(x,y) = 0$) in $\mathbb{R}^2$ for the fourth order certificate function $V(x,y)$ obtained using the SOSTOOLS. One may observe that this regime shift certificate intersects $\mathcal{X}_u$ which verifies that under these parameters one can trigger a shock-induced regime shift to the alternative bass-dominated equilibrium with a probability one provided one waits long enough.

The analysis shown in Fig. 16 was done for a single stopping time $\tau = 10^4$ using the function $\phi(t) = t$. If we repeat this analysis for a range of stopping times we can obtain a more complete picture of the probability distribution for the FPT obtained under this given harvesting strategy (i.e. $\mu = 0.075$ and $\lambda = 0.2$). The results of this more complete analysis are shown in Fig. 17 which plots the computed probability γ against the stopping time τ. The solid bullets show values obtained using the SOSTOOLS. The green squares show values obtained from Monte Carlo simulations of the system with 500 samples. The Monte Carlo generated results may be seen as more accurate estimates of the FPT probabilities. These results indeed show that the SOSTOOLS provide an upper bound on the FPT probabilities. For stopping times $\tau \leq 10^2$, the difference between the SOSTOOL's bound and the MC result is about 0.3. This difference decreases significantly at longer stopping times, $\tau > 10^3$. The quality of the bounds obtained using the supermartingale technique will depend on the chosen parameterization of the function $V(x, y)$. The main advantage of this approach, however, is that it can prove the reachability of the process without relying on excessive simulations for computing the sample paths explicitly.

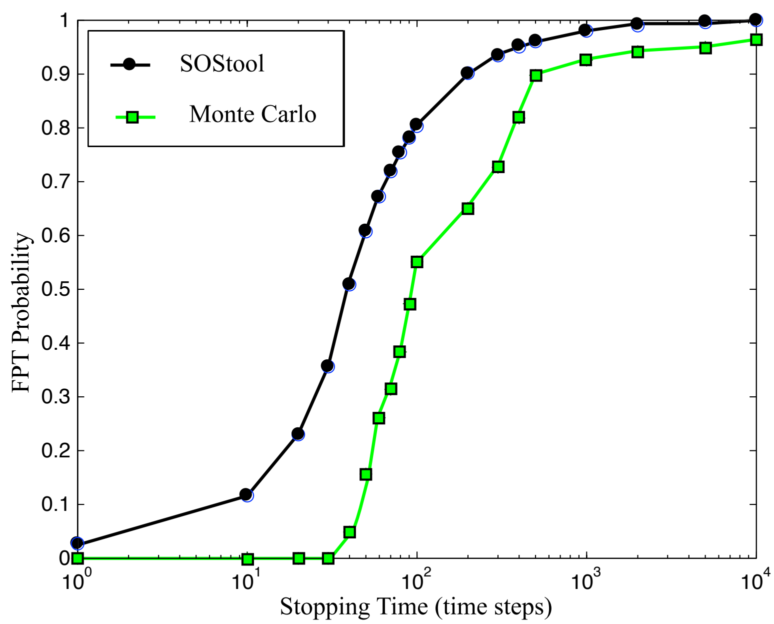

FIGURE 17. Estimates of FPT probability distribution computed using SOSTOOLS and Monte Carlo simulation with the jump parameters $\mu = 0.075$ and $\lambda = 0.2$

5. Summary and Further Reading

Shock-induced regime shifts occur when impulsive disturbances cause the system state to jump out of the nominal regime and into the ROA of an alternative regime. This chapter demonstrated how one could extend the framework for stochastic safety in [87] to systems that can be modeled as jump-diffusion processes, thereby allowing one to quantitatively assess the likelihood of a shock-induced regime shift occurring for a given strength and frequency of random impulses.

The FPT problem for the intra-guild predation system in the last section was originally studied in [29]. The methods used in this chapter were reported in [111] and provide a more precise quantitative characterization of the FPT's probability distribution than was possible in [29].

SOS programming for both the constrained and unconstrained case is now widely used to computationally search for certificates verifying Lyapunov stability [83], input-to-state stability [59], passivity [74], safety [123], region-of-attraction analysis [113], and so on. The development of these methods represents a remarkable synthesis of algebraic geometry and semidefinite programming [84].

A fundamental problem with the use of these tools is that the size of the SOS problem grows exponentially in the dimensionality of the state space. This effectively limits these methods to systems with state dimensions no greater than 4 or 5. Note that prior work [110] has also used SOS method to solve the D2B problem of the preceding section, but again these methods are limited to problems with only 4-5 parameters. This was why the last chapter used affine parameter dependent Lyapunov methods to address the D2B problem.