Event-Triggered State Estimation in Vector Linear Processes

Lichun Li, Michael Lemmon and Xiaofeng Wang

***Abstract*— This paper considers a distributed estimation problem in which a sensor sporadically transmits information to a remote-observer. An event-triggered approach is used to trigger the transmission of information from the sensor to the remote-observer. The event-trigger is chosen to minimize the mean square estimation error at the remote-observer subject to a constraint on how frequently the information can be transmitted. This problem was studied by O.C. Imer et al. [1] and M. Rabi et al. [2] where the observed process was a scalar linear system over a finite time interval. This paper extends those earlier results by relaxing the prior assumption that the initial condition is zero-mean with no measurement noise. It extends those earlier results to vector linear systems through a computationally efficient way of computing sub-optimal event-triggering thresholds.**

I. INTRODUCTION

A major challenge faced by wireless sensor networks is that they have limited throughput capacity. Moreover, a wireless link's capacity may vary over time due to changes in the external environment. Time-varying link capacity may negatively impact overall system behavior. This is true in networked control systems, where the quality of the feedback data has a direct impact on the physical plant's stability and performance [3]. This is also true for embedded sensor networks where sensor measurements are transmitted over ad hoc wireless networks to a data fusion center [4].

Many networked control systems presume the periodic transmission of information. Periodic transmission, however, may consume more network bandwidth than necessary. Since the period is chosen prior to system deployment, it must be robust over all variations in network and system behavior and this *open-loop* approach to period selection can be overly conservative in its use of network bandwidth.

The recognition of the inherent conservatism in open-loop periodic transmission policies has led numerous researchers to move towards the *sporadic* transmission of information through *event-triggered* formalisms. Event-triggering has an agent transmit information to its neighbors when some measure of the novelty in that information exceeds a specified threshold. Early examples of event-triggering were used in relay control systems and recent work has looked at event-triggered PID controllers [5]. Early event-triggered controllers assumed event-triggers with constant triggering thresholds. It was recently shown that state-dependent event triggers could be used to enforce stability concepts such as input-to-state stability [6] or $\mathcal{L}_2$ stability [7]. Experimental evidence [8] suggests that event-triggering can reduce communication bandwidth while preserving overall system performance. Event-triggering therefore provides a useful approach for reducing an application's use of the communication network.

This paper considers a canonical problem that was recently studied in [1], [2]. This problem considers a discrete-time scalar linear process over a finite interval of time. The process is observed by a sensor that constructs local estimates of the process state and must decide when to transmit those local estimates to a remote-observer so that the mean square estimation error at the remote-observer is minimized. To keep the problem interesting, transmission decisions must satisfy a bandwidth constraint that limits the number of messages that the sensor can send to the remote-observer. This paper extends the earlier work [1], [2] by dropping the assumption of zero mean initial conditions with no measurement noise and developing an efficient way of computing event-triggers for vector systems.

II. PRIOR WORK

It has long been recognized that the sporadic flow of information can be incorporated into Kalman filters [9]. Rather than simply analyzing the impact that nondeterministic network artifacts have on estimator performance, one may *control* the way information is transmitted. In multi-sensor networks, for example, one may schedule sensor transmissions [10]. The potential benefits of controlling transmission time were experimentally documented in [11]. Formal analyses of this tradeoff were done in [12] for infinite horizon estimation problems. Finite horizon problems were treated in [1], [2]. This paper uses event-triggering to control transmission times across a single communication link.

Network nodes communicate with each other in the system architecture in [11]. Each node estimates the state of its neighboring nodes. When a node finds that the estimation error of its state is greater than a pre-specified threshold, the true local state is broadcast to its neighbors. It was shown through simulations that network bandwidth can be significantly reduced while the performance of the system is only slightly impacted. This paper therefore provides a good motivation for controlling transmission times.

Based on the same system architecture, an optimal event-trigger was derived in [12] that minimizes the sum of the mean square estimation error (MSEE) and communication cost over an infinite horizon. The optimal event-triggering

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threshold, however, was expensive to compute and computationally tractable approximations were proposed in [13].

A related problem was studied in [1], [2]. This work characterized event-triggers that minimized MSEE over a finite horizon subject to a constraint on the maximum number of transmissions. The work was confined to discrete-time scalar linear systems with zero initial condition and no measurement noise. The problem was solved in [1] using dynamic programming concepts. The problem was solved in [2] using optimal stopping concepts for the multiple transmission problem. While this work asserted that the extension to multiple transmissions and vector systems was relatively easy, those assertions were supported with only partial characterizations of the proofs and algorithms.

This paper uses dynamic programming to solve the finite-horizon multi-sample problem treated in [2]. We recover the original results in [2] that determine an optimal time-varying event-triggering threshold. We also generalize the results in [2] to cases where the initial state is non-zero mean and the sensor data is corrupted by measurement noise. This paper's results apply to the scalar systems treated in [2] as well as more general vector linear systems. Solving for the event-triggers in vector systems, however, has a computational complexity that is exponential in the state space dimension. This paper, therefore, introduces a computationally tractable method for determining event-triggers using families of quadratic forms to bound the problem's value function.

III. Problem Statement

Consider a sensor that is observing a linear discrete-time process over a finite horizon of length $M+1$. The process state $x : [0, 1, \ldots, M] \to \mathbb{R}^n$ satisfies the difference equation

$$x_{k+1} = Ax_k + w_k$$

for $k \in [0, 1, \ldots, M]$ where A is an $n \times n$ real matrix , $w : [0, 1, \ldots, M] \to \mathbb{R}^n$ is a zero mean white noise process with covariance matrix Q. The initial state, x_0, is assumed to be a Gaussian random variable with mean μ_0 and variance Π_0. The sensor generates a measurement $y : [0, 1, \ldots, M] \to \mathbb{R}^m$ that is a corrupted version of the process state. The sensor measurement at time k is

$$y_k = Cx_k + v_k$$

for $k \in [0, 1, \ldots, M]$ and where $v : [0, 1, \ldots, M] \to \mathbb{R}^m$ is another zero mean white noise process with variance R that is uncorrelated with the process noise w. We assume that (A, C) is observable. The process and sensor blocks are shown in figure 1. In this figure, the output of the sensor feeds into a transmission subsystem that decides when to transmit information to a remote-observer.

The transmission subsystem consists of three components; an *event-detector*, a *filter*, and a *local-observer*. The event-detector decides when to transmit information at $B \in [0, 1, \ldots, M+1]$ time instants to the remote-observer. So B represents the total number of transmissions that the sensor is allowed to make to the remote-observer. We let $\{\tau^l\}_{l=1}^B$ denote a sequence of increasing times ($\tau^l \in [0, 1, \ldots, M]$) when information is transmitted from the sensor to the remote-observer. The decision to transmit is based on estimates that are generated by the filter and local-observer.

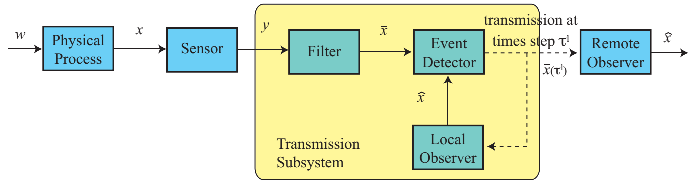

Fig. 1. Structure of event triggered networked state estimator

Let $\mathcal{Y}_k = \{y_0, y_1, \cdots, y_k\}$ denote the measurement information available at time k. The *filter* generates a state estimate $\overline{x} : [0, 1, \ldots, M] \to \mathbb{R}^n$ that minimizes the mean square estimation error $E\left[(x_k - \overline{x}_k)^2 \,|\, \mathcal{Y}_k\right]$ at each time step conditioned on all of the sensor information received up to and including time k. These estimates are computed using a Kalman filter. The filter equations for the system are,

$$\begin{aligned} \overline{x}_k &= E\left[x_k \,|\, \mathcal{Y}_k\right] = A\overline{x}_{k-1} + L_k(y_k - CA\overline{x}_{k-1}) \\ \overline{P}_k &= E\left[(x_k - \overline{x}_k)^2 \,|\, \mathcal{Y}_k\right] && (1) \\ &= A\overline{P}_{k-1}A^T + Q - L_kC(A\overline{P}_{k-1}A^T + Q) && (2) \end{aligned}$$

where L_k is the Kalman filter gain and $k = 1, 2, \ldots, M$. The initial condition $\overline{x}_0$ is the first a posteriori update based on y_0 and $\overline{P}_0$ is the covariance of this initial estimate.

The event-detector uses the filter's state estimate, $\overline{x}$, and another estimate generated by a *local-observer* to decide when to transmit the filtered state $\overline{x}$ to the *remote-observer*. Given a set of transmission times $\{\tau^l\}_{l=1}^B$, let $\overline{\mathcal{X}}_k = \{\overline{x}_{\tau^1}, \overline{x}_{\tau^2}, \ldots, \overline{x}_{\tau^{\ell(k)}}\}$ denote the filter estimates that were transmitted to the remote-observer by time k where $\ell(k) = \max\left\{\ell : \tau^\ell \le k\right\}$. This is the *information set* available to the remote-observer at time k. The remote-observer generates an a posteriori estimate $\hat{x} : [0, 1, \ldots, M] \to \mathbb{R}^n$ of the process state that minimizes the MSEE, $E\left[(x_k - \hat{x}_k)^2 \,|\, \overline{\mathcal{X}}_k\right]$, at time k conditioned on the information received up to and including time k. The a priori estimate of the remote-observer, $\hat{x}^- : [0, 1, \ldots, M] \to \mathbb{R}^n$, minimizes $E\left[(x_k - \hat{x}_k)^2 \,|\, \overline{\mathcal{X}}_{k-1}\right]$, the MSEE at time k conditioned on the information received up to and including time $k-1$. These estimates take the form

$$\begin{aligned} \hat{x}_k^- &= E\left[x_k \,|\, \overline{\mathcal{X}}_{k-1}\right] = A\hat{x}_{k-1} \\ \hat{x}_k &= E\left[x_k \,|\, \overline{\mathcal{X}}_k\right] = \begin{cases} \hat{x}_k^- & \text{don't transmit at step } k \\ \overline{x}_k & \text{transmit at step } k \end{cases} \end{aligned}$$

where $\hat{x}_0^- = \mu_0$.

The event-triggering strategy that is used to select the transmission times τ^l is based on observing the *gap*, $e_k^- = \overline{x}_k - \hat{x}_k^-$ between the filter's estimate $\overline{x}$ and the remote-observer's a priori estimate $\hat{x}^-$. Note that even though the gap is a function of the remote-observer's estimate, this signal will be available to the sensor. This is because the sensor has access to all of the information, $\overline{\mathcal{X}}_k$, that it sent to the remote-observer. As a result, the sensor can use a local-observer to construct a copy of $\hat{x}$ that can be locally accessed by the event-detector to compute the gap. This local-observer is shown as part of the transmission subsystem in figure 1. The event-detector's transmission is triggered when the estimate's gap e_k^- leaves the time-varying trigger set $S_k^{p_k}$ where $k \in [0, 1, \ldots, M]$ and p_k is the number of transmissions that are remaining at step k. As noted in [2], this type of decision logic treats the transmission time as a random variable that forms a stopping time of the stochastic process being monitored. These sets can be computed using optimal stopping theory. The result, however, is a backward recursion that bears great similarity to dynamic programming recursions. So this paper uses stochastic dynamic programming to obtain these trigger sets.

For later convenience, the following notational conventions are used throughout this paper, The estimation error at step k is $\hat{e}_k = x_k - \hat{x}_k$ and the filter's error at step k is $\overline{e}_k = x_k - \overline{x}_k$. We let $e_k^- = \overline{x}_k - \hat{x}_k^-$ and $e_k = \overline{x}_k - \hat{x}_k$ denote the a priori and a posteriori gap, respectively. The a priori information available to the event-detector at time k is denoted as $I_k^- = (e_k^-, p_k)$, an ordered pair consisting of the a priori gap and the number of transmissions remaining to be made. The a posteriori information available to the event-detector is $I_k = (e_k, p_{k+1})$. We let $\mathcal{S}_r^b(k) = \left\{ S_k^{\max\{0, b-k+r\}}, \ldots, S_k^{\min\{b, M+1-k\}} \right\}$ denote the triggering sets to be used at step k when there are b transmissions remaining at step $r \le k$. We let $\mathcal{S}_r^b = \left\{ \mathcal{S}_r^b(r), \ldots, \mathcal{S}_r^b(M) \right\}$.

We are now in a position to formally state the problem being addressed in this paper. Consider a cost function of the form

$$J_M(B; \mathcal{S}_0^B) = E\left[\sum_{k=0}^{M} \hat{e}_k^2 \,|\, p_0 = B \right] \qquad (3)$$

where the expectation is taken over $\hat{e}_0, \ldots, \hat{e}_M$ and $\tau^1, \cdots, \tau^B$. The objective is to find the optimal trigger sets minimizing the cost function:

$$J_M(B^*) = \min_{\mathcal{S}_0^B} J_M(B; \mathcal{S}_0^B). \qquad (4)$$

IV. Characterization of Value Function

The problem in equation (4) may be treated as the optimal control of a stochastic process. The control variable is the trigger set S_r^b. We use a stochastic version of Bellman's principle of optimality to obtain a backward recursion that generates the value function for our problem. The value function characterizes the cost (as measured by the MSEE at the remote-observer) from any initial system state.

The problem's value function is defined as

$$v(\zeta, b; r) = \min_{\mathcal{S}_r^b} E\left(\sum_{k=r}^{M} \hat{e}_k^2 \quad | \quad I_r^- = (\zeta, b) \right), \qquad (5)$$

which is the minimal expected cost conditioned on the information $I_r^- = (e_r^-, p_r)$ at time r. The optimal value satisfies $J_M(B^*) = E\left(v(\zeta, B; 0)\right)$.

It can be shown that the information sequence $\{I_0^-, I_0, ..., I_M^-, I_M\}$ is Markov, so the value function in equation (5) is only conditioned on the current information, rather than all past information. This section's main result is a theorem characterizing the backward recursion used to calculate the value function. The theorem's proof is given in the appendix.

Theorem 4.1: The value function (5) satisfies the backward recursive equation:

$$v(\zeta, b; r) = \min\left\{ v_{nt}(\zeta, b, r), v_t(\zeta, b, r) \right\}, \qquad (6)$$

where

$$\begin{aligned} v_{nt}(\cdot) &= \operatorname{tr}\left(\overline{P}_r\right) + \|\zeta\|_2^2 + E\left(v(e_{r+1}^-, b; r+1) | I_r = (\zeta, b)\right) \\ v_t(\cdot) &= \operatorname{tr}\left(\overline{P}_r\right) + E\left(v(e_{r+1}^-, b-1; r+1) | I_r = (0, b-1)\right) \end{aligned}$$

with initial conditions

$$v(\zeta, 0; r) = \zeta^T \Lambda_{r,1}^0 \zeta + c_{r,1}^0 \qquad (7)$$

$$v(\zeta, b; M+1-b) = \rho_{M+1-b}^b, \qquad (8)$$

in which

$$\Lambda_{r,1}^0 = \sum_{k=r}^{M} (A^T)^{k-r} A^{k-r},$$

$$c_{r,1}^0 = \sum_{k=r}^{M} \left(\sum_{j=r+1}^{k} \operatorname{tr}\left(\Sigma_{j-1} L_j^T (A^T)^{k-j} A^{k-j} L_j\right) + \operatorname{tr}(\overline{P}_k) \right);$$

$$\rho_{M+1-b}^b = \operatorname{tr}\left(\sum_{k=M+1-b}^{M} \overline{P}_k \right),$$

and $\Sigma_j = CA\overline{P}_j A^T C^T + CQC^T + R$. The optimal triggering set is

$$S_r^{b*} = \{\zeta : v_{nt}(\zeta, b, r) \le v_t(\zeta, b, r)\}, \qquad (9)$$

with $S_r^{0*} = \mathbb{R}^n$ for all $r = B, B+1, \ldots, M$ and $S_{M+1-b}^{b*} = \emptyset$ for all $b = 1, 2, \ldots, B$.

What should be apparent in examining equation (6) is that the optimal cost at time step r is based on the choice between the costs of transmitting (i.e., $v_t(\cdot)$) or not transmitting (i.e. $v_{nt}(\cdot)$) at step r. The actual values that those two costs take is conditioned on the value, ζ, that the a priori gap, e_r^-, takes at time step r. This means we can use the choice in equation (6) to identify two mutually disjoint sets; the trigger set S_r^{b*} and its complement. If e_r^- is not in the set S_r^{b*}, then we trigger a transmission otherwise the sensor decides not to transmit its information.

Equation (6) recurses over two set of indices; the time steps, r, and the remaining transmissions b. The value function, $v(\zeta, b; r)$ is computed from the value functions, $v(\zeta, b; r+1)$ and $v(\zeta, b-1; r+1)$. The initial conditions for this recursion are given in equations (7) and (8). Equation (7) specifies the value function when at time step $r \in [B, B+1, \ldots, M]$ there are no transmissions remaining ($b = 0$). These initial conditions are computed as the total MSEE assuming no further measurement updates. Equation (8) specifies the value function when there are $b \in [1, 2, \ldots, B]$ transmissions remaining between time step $M+1-b$ and M. This initial condition equals the MSEE assuming an update at each remaining time step. We may picture the recursion as shown in figure 2. This picture plots the indices (b, r) and

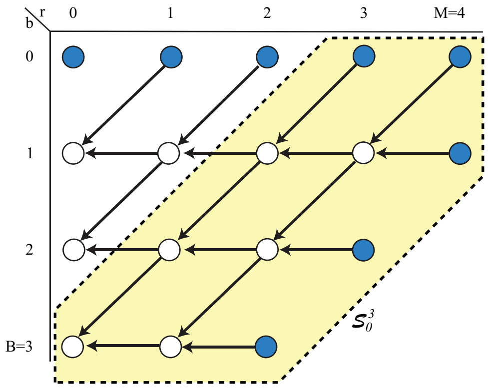

Fig. 2. Order of calculating value function with M=4, B=3

identifies the initial conditions and the order of computation. The filled-in circles are the indices for value functions in equations (7) and (8). The arrows show the computational dependencies in the recursion.

Some properties of the value function and optimal triggering sets are stated in the following corollaries. The proofs for these corollaries are omitted due to space limitations.

Corollary 4.2: With b and r fixed, the value function $v(\zeta, b; r)$ is symmetric about the origin and nondecreasing with respect to $\|\zeta\|_2$ in the same direction, i.e.

$$v(\zeta, b; r) = v(-\zeta, b; r);$$
$$v(\alpha_1 d, b; r) \geq v(\alpha_2 d, b; r), \forall \alpha_1 \geq \alpha_2 \geq 0, d \in \mathbb{R}^n.$$

Corollary 4.3: Given any direction $d \in \mathbb{R}^n$, the optimal triggering set S_r^{b*} lying in this direction is in the form of $[-\theta_r^b(d), \theta_r^b(d)]$.

With corollary 4.3, the triggering event becomes $|e_r^-| > \theta_r^b$. For the scalar case one may search for the optimal threshold θ_r^b, instead of finding the optimal set S_r^{b*}. A similar strategy can be used in the vector case, where we search for the threshold along some ray extending away from the origin.

V. Computation of Event-Triggers

This section discusses the complexity of computing the value function and event-triggers. Direct computation of the value function scales in an exponential manner with state-dimension. This fact has made it difficult to extend earlier results in [1], [2] beyond scalar systems. This section introduces a computationally tractable method that bounds the value function with a family of quadratic approximations. This allows us to determine event-triggers for vector linear systems. This result is demonstrated through a simulation.

Theorem 4.1 computes the value function $v(\zeta, b; r)$ as the minimum of two functions $v_t(\zeta, b; r)$ and $v_{nt}(\zeta, b; r)$. The event-triggering threshold, θ_r^b, occurs at those points where $v_t(\zeta, b; r) = v_{nt}(\zeta, b; r)$. Moreover, corollaries 4.2 and 4.3 imply that we can search for the threshold θ_r^b along rays extending out from the origin. The number of rays, however, that would need to be considered is an exponential function of the process' state space dimension. As a result, it has proven impractical to compute these optimal thresholds for state dimensions any larger than $n = 2$.

As suggested in [13], this problem may be circumvented by using quadratic functions to approximate the value function. The following theorem bounds the value function $v(\zeta, b; r)$ from above with a family of quadratic forms, $\{\zeta^T \Lambda_{r,j}^b \zeta + c_{r,j}^b\}_{j=1}^{M+1-b-r}$ where $\Lambda_{r,j}^b$ is a symmetric positive definite matrix and $c_{r,j}^b$ is a constant that are computed recursively over the indices r and b. The proof for this theorem is in the appendix.

Theorem 5.1: The value function (5) is bounded above by

$$\bar{v}(\zeta, b; r) = \min\{\bar{v}_{nt}(\zeta, b, r), \bar{v}_t(\zeta, b, r)\},$$

where

$$\bar{v}_{nt}(\zeta, b, r) = \min_{j=1,\cdots,\ell_r^b} \{\zeta^T \Lambda_{r,j}^b \zeta + c_{r,j}^b\}, \text{if } b \neq 0 \quad (10)$$
$$\bar{v}_t(\zeta, b, r) = \rho_r^b. \quad (11)$$

$\Lambda_{r,j}^b$,$c_{r,j}^b$ and ρ_r^b are computed recursively as

$$\Lambda_{r,j}^b = \begin{cases} A^T \Lambda_{r+1,j}^b A + I, & j < M+1-b-r; \\ I, & j = M+1-b-r, \end{cases}$$
$$c_{r,j}^b = \begin{cases} \sigma_{r+1,j}^b + \mathrm{tr}(\overline{P}_r), & j < M+1-b-r; \\ \rho_{r+1}^b + \mathrm{tr}(\overline{P}_r), & j = M+1-b-r, \end{cases}$$
$$\rho_r^b = \begin{cases} \mathrm{tr}(\overline{P}_r) + \sigma_{r+1,1}^{b-1}, \text{if } b = 1; \\ \mathrm{tr}(\overline{P}_r) + \min\{\sigma_{r+1,1}^{b-1}, \cdots, \sigma_{r+1,\ell_r^b}^{b-1}, \rho_{r+1}^{b-1}\}, \text{else.} \end{cases}$$

where $\sigma_{r+1,j}^b = \mathrm{tr}(\Sigma_r L_{r+1}^T \Lambda_{r+1,j}^b L_{r+1}) + c_{r+1,j}^b$ and $\ell_r^b = M+1-b-r$. The initial conditions for $\bar{v}_{nt}$ and $\bar{v}_t$ are described by equations (7) and (8) respectively. The sub-optimal triggering sets are

$$S_r^{b+} = \{\zeta : \bar{v}_{nt}(\zeta, b, r) \leq \bar{v}_t(\zeta, b, r)\}$$

with $S_r^{0+} = \mathbb{R}^n$ for all $r = B, \cdots, M$ and $S_{M+1-b}^{b+} = \emptyset$ for all $b = 1, \cdots, B$.

The sub-optimal triggering set is the union of the ellipsoidal sets $\{\zeta \in \mathbb{R}^n : \zeta^T \Lambda_{r,j}^b \zeta + c_{r,j}^b \leq \rho_r^b\}$ for $j = 1, 2, \ldots, M+1-b-r$. Given r and b, this set may be computed using the $M+1-b-r$ quadratic forms in $\overline{v}_{nt}(\zeta, b; r)$. Computing the value function only requires the evaluation of a quadratic form on the order of n^3 multiplies. The complexity, therefore associated with evaluating the bounds $\overline{v}(\zeta, b; r)$ is on the order of $(M+1-B)(M-B)n^3$, which is cubic in the state space dimension.

We now consider a comparsion between the thresholds, θ_b^r, computed using the value function $v(\zeta, b; r)$ and the bound $\overline{v}(\zeta, b; r)$. For $n = 2$, it is possible to compute $v(\zeta, b; r)$ and its associated thresholds. let $A = \begin{bmatrix} 0 & -1 \\ 1 & \sqrt{2} \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$. The mean and variance of initial condition are $[1, 0]^T$ and I (identity matrix), respectively. The variance of w and v are $[1\ 2; 2\ 5]$ and 1, respectively. The terminal step is chosen to be $M = 4$ with only one allowed transmission, $B = 1$. The value functions and their bounds are computed using theorems 4.1 and 5.1. The results from these computations are shown in figure 3.

The left side of this figure shows cross-sectional plots of the value functions and their upper bounds. The red and

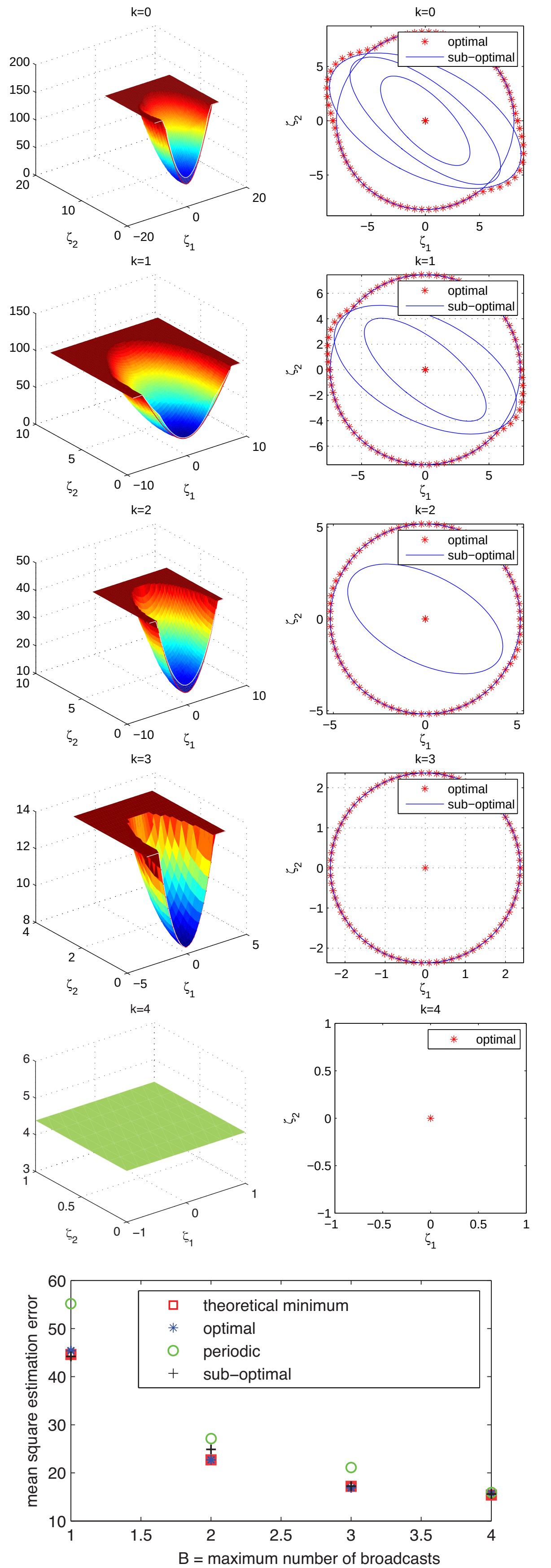

Fig. 3. Top plots show value functions, upper bounds, optimal, and sub-optimal triggering sets. Bottom plots shows experimental results

white lines in the cross-section represent the value function and its upper bound, respectively. One can see that the differences between these two lines are small, especially at the points where v_{nt} and v_t are equal. These points are important, because they form the edge of the optimal triggering set. The right side shows the optimal and sub-optimal triggering sets. We can see that the union of the ellipses, which is the sub-optimal triggering set, over-approximates the optimal triggering set very closely.

Let's now vary the number of allowed transmissions, B, between 1 to 4 and calculate the optimal and sub-optimal triggering sets. These sets were used in a simulation of the system whose results are shown at the bottom of figure 3. This figure plots the MSEE as a function of B where event-triggering was done using the optimal or sub-optimal thresholds. The figure shows that the suboptimal event-triggers perform are only slightly worse than the optimal event-triggering thresholds. Simulations were also run for periodic transmssions with comparable periods. These results show that the sub-optimal and optimal event-triggers have smaller MSEE than comparable periodic transmission schemes. Finally, we determine the actual MSEE that should have been achieved and this value matches what was achieved using the optimal event-triggers.

VI. Summary

This paper discussed the design of optimal event-triggers for distributed multi-dimensional state estimation problems with finite terminal time and a fixed number of transmissions. This paper extends the results in [2] to vector linear systems with nonzero mean initial conditions and measurement noise. The paper provides a computationally tractable approach for determining the event-triggering thresholds, thereby suggesting that these event-triggering approach can be used on multi-dimensional linear systems, not just the scalar systems that have been usually studied in the past.

VII. Appendix

Proof of Theorem 4.1: The value function may be written as

$$\begin{aligned} v(\zeta,b;r) &= \min_{\mathcal{S}_r^b} E\left(\sum_{k=r}^{M} \|\hat{e}_k\|_2^2 \,|\, I_r^- = (\zeta,b)\right) \\ &= \min_{S_r^b} g(\zeta,b,S_r^b) \end{aligned}$$

where $g(\zeta,b,S_r^b) = \min_{\mathcal{S}_r^b - S_r^b} E\left(\sum_{k=r}^{M} \|\hat{e}_k\|_2^2 | I_r^- = (\zeta,b)\right)$.

We calculate $g(\zeta,b,S_r^b)$ for the two cases: $\zeta \in S_r^b$ and $\zeta \notin S_r^b$. Here, the first case is explained explicitly. Because the second case can be derived similarly, we only give the final result.

If $\zeta \in S_r^b$,

$$\begin{aligned} & g(\zeta,b,S_r^b) \\ = & \min_{S_r^b(r+1),\cdots,S_r^b(M)} E\left(\sum_{k=r}^{M} \|\hat{e}_k\|_2^2 | e_r^- = \zeta \in S_r^b, p_r = b\right). \end{aligned}$$

Because the condition $e_r^- = \zeta \in S_r^b, p_r = b \Leftrightarrow e_r = \zeta, p_{r+1} = b$ and $e_r^- = \zeta, p_r = b$,

$$\begin{aligned} g(\zeta, b, S_r^b) &= \min_{\mathcal{S}_r^b - S_r^b} E\left(\sum_{k=r}^{M} \|\hat{e}_k\|_2^2 | I_r = I_r^- = (\zeta, b)\right) \\ &= \min_{\mathcal{S}_r^b - S_r^b} E\left(\sum_{k=r}^{M} \|\hat{e}_k\|_2^2 | I_r = (\zeta, b)\right). \end{aligned}$$

Since $p_{r+1} = b$ which means b transmissions remaining at step $r+1$, only $\mathcal{S}_{r+1}^b$ can influence the value of the expectation.

$$\begin{aligned} & g(\zeta, b, S_r^b) \\ = \;& \min_{\mathcal{S}_{r+1}^b} E\left(\sum_{k=r}^{M} \|\hat{e}_k\|_2^2 | I_r = (\zeta, b)\right) \\ = \;& tr(\overline{P}_r) + \|\zeta\|_2^2 + \min_{\mathcal{S}_{r+1}^b} E\left(\sum_{k=r+1}^{M} \|\hat{e}_k\|_2^2 | I_r = (\zeta, b)\right) \\ = \;& tr(\overline{P}_r) + \|\zeta\|_2^2 + \min_{\mathcal{S}_{r+1}^b} E\left(E\left(\sum_{k=r+1}^{M} \|\hat{e}_k\|_2^2 | I_{r+1}^- = \right.\right. \\ & \left.\left. (e_{r+1}^-, b), I_r = (\zeta, b)\right) | I_r = (\zeta, b)\right) \\ = \;& tr(\overline{P}_r) + \|\zeta\|_2^2 + E\left(\min_{\mathcal{S}_{r+1}^b} E\left(\sum_{k=r+1}^{M} \|\hat{e}_k\|_2^2 | I_{r+1}^- = \right.\right. \\ & \left.\left. (e_{r+1}^-, b)\right) | I_r = (\zeta, b)\right) \\ = \;& tr(\overline{P}_r) + \|\zeta\|_2^2 + E\left(v(e_{r+1}^-, b; r+1) | I_r = (\zeta, b)\right) \\ = \;& v_{nt}(\zeta, b, r). \end{aligned}$$

The fourth equality holds because the information set sequence $\{I_k^-, I_k\}_{k=0}^M$ is Markov and e_{r+1}^- is independent with $\mathcal{S}_{r+1}^b$.

If $\zeta \notin S_r^b$, we can show that $g(\zeta, b, S_r^b) = tr(\overline{P}_r) + E\left(v(e_{r+1}^-, b-1; r+1) | I_r = (0, b-1)\right) = v_t(b, r)$

With the value function in both cases, we conclude that $g(\zeta, b, S_r^b) = v_{nt} \mathbf{1}_{\zeta \in S_r^b} + v_t(b, r) \mathbf{1}_{\zeta \notin S_r^b}$, and the value function $v(\zeta, b; r) = \min\{v_{nt}(\zeta, b, r), v_t(b, r)\}$ with $S_r^{b*} = \{\zeta : v_{nt}(\zeta, b, r) \le v_t(b, r)\}$.

There are two initial conditions for the recursive equation. One is the case when there is no remaining transmissions, $v(\zeta, 0; r)$. The other is the case when the number of remaining transmissions is the same as the remaining steps, $v(\zeta, b; r)$ for $b \in [1, B]$ and $r = M+1-b$. Both of them can be calculated directly. ■

Proof of Theorem 5.1: The initial conditions for $\bar{v}_t = v_t$ and $\overline{v}_{nt} = v_{nt}$ satisfy equations (11) and (10), respectively, where $\Lambda_{r,j}^0 = \mathbf{0}$, $c_{r,j}^0 = \infty$ and $\rho_r^0 = \infty$ for $j = 2, \cdots, \ell_r^0$. Now assume that

$$\begin{aligned} \bar{v}_{nt}(\zeta, k, r+1) &= \min_{j=1,\cdots,M-k-r} (\zeta^T \Lambda_{r+1,j}^k \zeta + c_{r+1,j}^k) \\ \bar{v}_t(\zeta, k, r+1) &= \rho_{r+1}^k \end{aligned}$$

are upper bounds for $v_{nt}(\zeta, k, r+1)$ and $v_t(\zeta, k, r+1)$ when $k = b$ and $b-1$. Let $\Omega_0 = \{1, 2, \ldots, M-b-r\}$ and let $\Omega_1 = \{1, 2, \ldots, M+1-b-r\}$. The cost of not transmitting, $v_{nt}(\cdot)$, can be bounded as

$$\begin{aligned} & v_{nt}(\zeta, b, r) \\ \le \;& \mathrm{tr}(\overline{P}_r) + \zeta^T \zeta + E(\bar{v}(\zeta, b; r+1) | I_r = (\zeta, b)) \\ \le \;& \mathrm{tr}(\overline{P}_r) + \zeta^T \zeta + \min\{ \\ & \min_{j \in \Omega_0} E(e_{r+1}^{-T} \Lambda_{r+1,j}^b e_{r+1}^- + c_{r+1,j}^b | e_r = \zeta), \rho_{r+1}^b\} \\ = \;& \min\{\min_{j \in \Omega_0} (\zeta^T (A^T \Lambda_{r+1,j}^b A + I) \zeta + \sigma_{r+1}^b + \mathrm{tr}(\overline{P}_r)) \\ & , \zeta^T \zeta + \rho_{r+1}^b + \mathrm{tr}(\overline{P}_r)\} \\ = \;& \bar{v}_{nt}(\zeta, b, r). \end{aligned}$$

In a similar way we can show that $v_t(\cdot)$ is bounded as

$$\begin{aligned} & v_t(\zeta, b, r) \\ \le \;& \mathrm{tr}(\overline{P}_r) + E(\bar{v}(\zeta, b-1; r+1) | I_r = (0, b-1)) \\ \le \;& \mathrm{tr}(\overline{P}_r) + \min\{ \\ & \min_{j \in \Omega_1} E(e_{r+1}^{-T} \Lambda_{r+1,j}^{b-1} e_{r+1}^- + c_{r+1,j}^{b-1} | e_r = 0), \rho_{r+1}^{b-1}\} \\ = \;& \mathrm{tr}(\overline{P}_r) + \min\{\sigma_{r+1,1}^{b-1}, \cdots, \sigma_{r+1,M+1-b-r}^{b-1}, \rho_{r+1}^{b-1}\} \\ = \;& \begin{cases} \mathrm{tr}(\overline{P}_r) + \sigma_{r+1,1}^{b-1}, \text{if } b = 1; \\ \mathrm{tr}(\overline{P}_r) + \min\{\sigma_{r+1,1}^{b-1}, \cdots, \sigma_{r+1,\ell_r^b}^{b-1}, \rho_{r+1}^{b-1}\}, \text{else.} \end{cases} \\ = \;& \bar{v}(\zeta, b, r) \end{aligned}$$

The fourth equality holds because $c_{r,j}^0 = \rho_r^0 = \infty$. ■

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