Stabilizing bit-rate of disturbed event triggered control systems ⋆

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Abstract: Event triggering is a sampling method where sampling occurs only if data 'novelty' exceeds a threshold. Prior work has demonstrated that event triggered systems have longer average sampling periods than periodic sampled systems with comparable system performance. Based on this fact, it is claimed that event triggered systems make more efficient use of communication resources than periodic sampled systems. If, however, we account for the number of bits in each sample and the maximum acceptable delay of this sample, it is possible that the bit-rates generated by event triggered systems are greater than that of periodic sampled systems. Our prior work in Li et al. [2012] has established, in noise-free cases, the condition under which the stabilizing bit-rates for quantized event-triggered systems converge asymptotically to a finite rate as the system approaches its equilibrium point. In some cases, this limiting bit-rate was shown to be 0. This paper extends that earlier work to quantized event-triggered systems with essentially bounded disturbances. Conditions on triggering event, quantization error and maximum delay are established to assure the input-to-state stability (ISS). The stabilizing bit-rate is, then, shown to be always bounded by a continuous, positive definite, increasing function with respect to the norm of the state. Since the system is ISS, the stabilizing bit-rate can be bounded from above by a function of time. This result provides a guide on how to assign communication resource to the control system. If we set external disturbance to be 0, the results in Li et al. [2012] are recovered.

1. INTRODUCTION

State-dependent event-triggered control systems are systems that transmit the system state over the feedback channel when the difference between the current state and last sampled-state exceeds a state-dependent threshold. These systems were originally viewed as embedded computational systems in Tabuada [2007]. In this case, one was interested in reducing how often the system state was sampled, as a means of reducing processor utilization. The concept of event-triggering can be easily extended to networked control systems and wireless sensor-actuator networks, in which case the sampled state is *transmitted* over a communication channel.

Early interest in event-triggered control was driven by experimental results suggesting that these systems could have longer inter-sampling intervals than comparably performing periodic sampled-data systems (see Sandee et al. [2007], Tabuada [2007], Wang and Lemmon [2009]). In extending this idea to networked control systems, one might suppose that event-triggering can also reduce the system's usage of the communication channel since it might reduce the frequency at which feedback states are transported across the channel. This extension, however, is complicated by the fact that the communication channel is discrete in nature. Sampled states must first be quantized into a finite number of bits before being transmitted across the channel. Moreover, the transmitted bits must be delivered with a delay that does not de-stabilize the system. So an accurate measure of channel usage is the bit rate as defined by the number of bits per sampled state divided by the acceptable delay in message delivery. It means that the system's *stabilizing bit rate* (i.e., the bit rate assuring closed-loop stability) rather than the inter-transmission interval (i.e. the time between consecutive transmissions of the sampled state) provides a more realistic measure of channel usage in event-triggered networked control systems.

Prior work in state-dependent event-triggered control has used two different techniques to bound the inter-transmission times and acceptable delays. The method used in Tabuada [2007] bounds the minimum inter-transmission as a function of the open-loop system's Lipschitz constant. This work goes on to show that system stability is preserved for sufficiently small delays. More accurate measures of inter-transmission intervals were obtained in Anta and Tabuada [2010] using scaling properties of homogeneous systems. Quantitative bounds on both the inter-transmission time and maximum acceptable delay were obtained for self-triggered $\mathcal{L}_2$ systems in Wang and Lemmon [2009]. The results in Wang and Lemmon [2009] are significant because they show how the delay and inter-transmission time scale as a function of the last sampled state. These scaling properties led to the characterization

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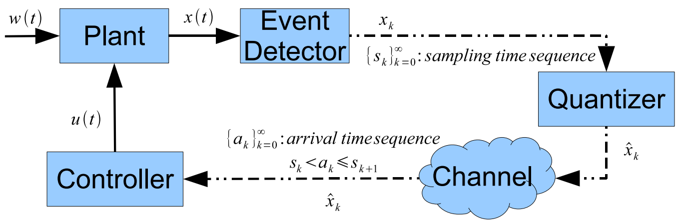

Fig. 1. Event-triggered control system with quantization

in Wang and Lemmon [2011] of event-triggered systems whose inter-transmission times exhibited *efficient attentiveness* (i.e. the inter-transmission intervals asymptotically approach infinity as the state approaches its equilibrium point). The approach used in this paper builds upon the techniques used in Wang and Lemmon [2011]. This new paper characterizes how stabilizing bit rates scale as the system state approaches the equilibrium point when there are disturbances.

This paper's bounds on stabilizing bit rates are reminiscent of earlier work on dynamic quantization. Prior work showed that static quantization maps required an infinite number of bits to achieve asymptotic stability (see Delchamps [1990]). With a finite number of bits, the best one can achieve is ultimate boundedness when using static maps (see Wong and Brockett [1999]). This led to the development of dynamic quantization maps in which the quantization map is dynamically varied to track state uncertainty (see Brockett and Liberzon [2000]). For linear systems, one was able to obtain bounds on the bit rate that were necessary and sufficient for stability, assuming a single sample delay (see Tatikonda and Mitter [2004]). In the case of nonlinear systems, lower bounds on the quantization rate were obtained (see Liberzon and Hespanha [2005]). The quantization maps developed in this paper are dynamic maps, similar to those used in Liberzon and Hespanha [2005]. The different thing is that our work is based on the event triggered sampling with state dependent delay while Liberzon and Hespanha [2005] considered periodic sampling with one period delay. This paper shows that the bit-rate sufficient to guarantee input-to-state stability for a nonlinear system with essentially bounded disturbance is always bounded from above by a continuous, positive definite, and increasing function with respect to the norm of state. It indicates that the farther the state is away from the origin, the higher the bit-rate may be used to stabilize the control system. If we set the disturbance to be 0, the results in Li et al. [2012] are recovered.

2. PROBLEM STATEMENT

The system under study is a networked event-triggered control system with quantization. Figure 1 is a block diagram showing the components of this system.

The *plant*'s state trajectory $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ is an absolutely continuous function satisfying the initial value problem,

$$\dot{x}(t) = f(x(t), u(t), w(t)), \quad x(0) = x_0 \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$ is locally Lipschitz in x, u and w with $f(0,0,0) = 0$.The control signal $u(\cdot) : \mathbb{R}^+ \to \mathbb{R}^m$ is generated by the *controller* in figure 1. The disturbance $w(\cdot) : \mathbb{R}^+ \to \mathbb{R}^q$ is an $\mathcal{L}_\infty$ disturbance with $\|w\|_{L_\infty} = \bar{w}$. The vector x_0 is the plant's initial condition.

The system state, $x(t)$, at time t is measured by the *event detector*. The event detector decides when to hand over the system state to the *quantizer*, and the *quantizer* converts this real vector into a finite bit representation. This quantized state is denoted as $\hat{x}_k \in \mathbb{R}^n$. Let $e_k(t) = x(t) - \hat{x}_k$ be the *gap* between the current state and quantized state, and $\|\cdot\|$ indicate the infinity norm of a vector. The sampling times $\{s_k\}$ are generated by the event trigger so that the gap is always less than a state-dependent *threshold function*

$$\|e_k(t)\| < \theta(\|\hat{x}_k\|, \bar{w}) \tag{2}$$

for all $t \in (s_k, s_{k+1}]$ where $k = 0, 1, \ldots, \infty$. The function $\theta(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing with respect to both arguments and satisfying $\theta(0,0) = 0$. For notational convenience, the kth consecutively sampled state $x(s_k)$ will be denoted as x_k. The kth *inter-sampling* interval is defined as $T_k = s_{k+1} - s_k$.

Upon the violation of equation (2), the *quantizer* converts the system state x_k into the quantized state $\hat{x}_k$ with the quantization error $x_k - \hat{x}_k$ satisfies

$$\|x_k - \hat{x}_k\| \leq \bar{e}_q(\|e_{k-1}(s_k)\|, \bar{w}), \tag{3}$$

where the *quantization error function* $\bar{e}_q(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing with respect to both arguments and satisfying $\bar{e}_q(0,0) = 0$. Notice that

$$\bar{e}_q(\|e_{k-1}(s_k)\|, \bar{w}) = \bar{e}_q(\theta(\|\hat{x}_{k-1}\|, \bar{w}), \bar{w}). \tag{4}$$

This paper uses the following *quantization map* when the event detector decides to transmit data. Since at sampling time s_k, both sensor and controller understand that $\|e_{k-1}(s_k)\| = \theta(\|\hat{x}_{k-1}\|, \bar{w})$, we only need to quantize the surface of the box $\{e_{k-1} : \|e_{k-1}(s_k)\| \leq \theta(\|\hat{x}_{k-1}\|, \bar{w})\}$. First, we use $\lceil \log_2 2n \rceil$ bits to identify which side e_{k-1} lies on, and then we cut this side uniformly into $\left\lceil \frac{\theta(\|\hat{x}_{k-1}\|, \bar{w})}{\bar{e}_q(\|e_{k-1}(s_k)\|, \bar{w})} \right\rceil^{n-1}$ parts. If $e_{k-1}(s_k)$ lies on one of the small parts, then $e_{k-1}(s_k)$ will be quantized as the center of this part, and $\hat{x}_k$ can be calculated as the sum of $\hat{x}_{k-1}$ and the quantized $e_{k-1}(s_k)$. In all, the number of bits used at the kth sampling is

$$N_k = \lceil \log_2 2n \rceil + \left\lceil \log_2 \left\lceil \frac{\theta(\|\hat{x}_{k-1}\|, \bar{w})}{\bar{e}_q(\|e_{k-1}(s_k)\|, \bar{w})} \right\rceil^{n-1} \right\rceil \tag{5}$$

We should notice that the number of bits transmitted at each time can be different, since we fix the quantization error function instead of the number of bits.

We assume that the quantized state, $\hat{x}_k$, is always successfully delivered to the controller. The channel, however, is assumed to introduce a finite delay into the delivery time. In particular, the arrival time of the kth sampled state $\hat{x}_k$ at the controller is denoted as $a_k \in \mathbb{R}^+$. This time is strictly greater than s_k. The delay of the kth message is $D_k = a_k - s_k$. We need to assume some orderliness to the transmission and delivery of such messages. In particular, we require that the transmission times, s_k, and arrival times, a_k, satisfy the following order $s_k < a_k \leq s_{k+1}$ for $k = 0, 1, \ldots, \infty$. Such a sequence of transmissions and arrivals will be said to be *admissible*.

Upon the arrival of the kth quantized state, $\hat{x}_k$, at the controller, a control input is computed and then held until the next quantized state is received. In other words, the control signal takes the form

$$u(t) = u_k = K(\hat{x}_k) \tag{6}$$

for $t \in [a_k, a_{k+1})$. The function $K(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz, and satisfies $K(0) = 0$. As has been done in Tabuada [2007], this paper assumes that K is chosen so the system

$$\dot{x}(t) = f(x(t), K(x(t) + e(t)), w(t)), \tag{7}$$

is input-to-state stable with respect to the signals $e, w \in \mathcal{L}_\infty$. This means, of course, that there exists a function $V(\cdot) : \Upsilon \to \mathbb{R}^+$ satisfying

$$\underline{\alpha}(\|x\|) \le V \le \overline{\alpha}(\|x\|) \tag{8}$$

$$\frac{\partial V}{\partial x} f(x, w) \le -\alpha(\|x\|) + \gamma_1(\|e\|) + \gamma_2(\|w\|), \tag{9}$$

for all $x \in \Upsilon$ where $\Upsilon \subseteq \mathbb{R}^n$ is a domain containing the origin, α, γ_1 and γ_2 are class $\mathcal{K}$ functions. Note that this can be a very restrictive assumption since such K may not always exist (see Angeli et al. [2000]).

3. INPUT-TO-STATE STABILITY

This section characterizes a threshold function, a quantization error function and a maximum delay such that the event-triggered system described in section 2 is ISS.

With equation 9, we first give a sufficient condition of ISS.

Lemma 1. If for all $t \in [a_k, a_{k+1})$ and all $k = 0, 1, \cdots, \infty$,

$$\|e_k(t)\| \le \xi(\|x(t)\|, \bar{w}) = \gamma_1^{-1}(\varsigma\alpha(\|x(t)\|) + \gamma_3(\bar{w})), \tag{10}$$

where $\varsigma \in (0, 1)$ and γ_3 is a $\mathcal{K}$ function, then the system is ISS, i.e. there exist some class $\mathcal{KL}$ function β and class $\mathcal{K}$ function γ such that

$$\|x(t)\| \le \max\{\beta(\|x_0\|, t), \gamma(\bar{w})\}, \forall t \ge 0.$$

Proof. Apply equation (10) into equation (9), we have

$$\frac{\partial V}{\partial x} f(x, w) \le -(1 - \varsigma)\alpha(\|x\|) + \gamma_2(\bar{w}) + \gamma_3(\bar{w}), \forall t \ge 0.$$

Since $\varsigma \in (0, 1)$ and $\gamma_2(\bar{w}) + \gamma_3(\bar{w})$ is a class $\mathcal{K}$ function of $\bar{w}$, the system is ISS. □

With Lemma 1, it's natural to consider a threshold function $\theta(\|\hat{x}_k\|, \bar{w})$ which is always bounded from above by $\xi(\|x(t)\|, \bar{w})$. Noticing that θ and ξ depends on different variables, we have the following lemma.

Lemma 2. Assume that $\xi(s, \bar{w})$ is locally Lipschitz with respect to s over the compact set $[0, \eta]$ where $\eta = \max\{\beta(\|x_0\|, 0), \gamma(\bar{w})\}$. For any $\xi(\|x(t)\|, \bar{w})$, there must exist a positive definite function $\underline{\xi}(s, \bar{w})$ such that if

$$\|e_k(t)\| \le \underline{\xi}(\|\hat{x}_k\|, \bar{w}), \forall t \in [a_k, a_{k+1}), \tag{11}$$

for all $k = 0, 1, \cdots, \infty$, then equation (10) holds, and hence the system is ISS. Moreover, $\underline{\xi}(s, \bar{w})$ is locally Lipschitz with respect to s over the compact set $[0, \eta]$ and monotonically increasing with respect to two arguments.

Proof. Let $\underline{\xi}(s, \bar{w}) = \rho\xi(s, \bar{w})$ for some $\rho \in (0, 1)$. Equation (11), then, takes the form of

$$\|e_k(t)\| \le \rho\xi(\|\hat{x}_k\|, \bar{w}) \le \rho\xi(\|x(t)\| + \|e_k(t)\|, \bar{w}). \tag{12}$$

Since $\xi(s, \bar{w})$ is locally Lipschitz with respect to s over $[0, \eta]$, there exists $L \ge 0$ such that for all $\delta > 0$

$$|\xi(s + \delta, \bar{w}) - \xi(s, \bar{w})| \le L\delta.$$

Apply the above equation into 12, we have

$$\|e_k(t)\| \le \rho\xi(\|x(t)\|, \bar{w}) + \rho L\|e_k(t)\|.$$

If we choose ρ such that $\rho \le \frac{1}{1+L}$, then

$$\|e_k(t)\| \le \frac{\rho}{1 - \rho L}\xi(\|x(t)\|, \bar{w}) \le \xi(\|x(t)\|, \bar{w}).$$

According to Lemma 1, the above equation implies the system is ISS. It is easy to see that $\underline{\xi}(s, \bar{w})$ is locally Lipschitz with respect to s over $[0, \eta]$ and monotonically increasing with respect to two arguments. □

Lemma 2 indicates that to guarantee the system is ISS, we should choose our threshold function θ satisfying

$$\theta(\|\hat{x}_k\|, \bar{w}) < \underline{\xi}(\|\hat{x}_k\|, \bar{w}). \tag{13}$$

To make sure that the inter-sampling interval T_k is always positive, we choose $\bar{e}_q$ satisfying

$$\bar{e}_q(\|e_{k-1}(s_k)\|, \bar{w}) < \theta(\|\hat{x}_k\|, \bar{w}). \tag{14}$$

With the threshold function θ and quantization error function $\bar{e}_q$, we first analyze the inter-sampling interval T_k, which will be used to characterize the maximum delay.

Let's define a compact set Ω_k as

$$\Omega_k = \{x \in \mathbb{R}^n : \|x\| \le \|\hat{x}_k\| + \underline{\xi}(\|\hat{x}_k\|, \bar{w})\},$$

and a positive definite function $\bar{f} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\bar{f}(x, u, \bar{w}) = \|f(x, u, 0)\| + L^w_{\Omega_k}\bar{w}, \tag{15}$$

where $L^w_{\Omega_k}$ is the Lipschitz constant of f with respect to w over compact set Ω_k. With $\bar{f}$, it's easy to show that

$$\|f(x, u_{k-1}, w)\| \le \bar{f}(\hat{x}_k, u_{k-1}, \bar{w}) + L^x_{\Omega_k}\|e_k\|, \tag{16}$$

where $L^x_{\Omega_k}$ is the Lipschitz constant with respect to x over compact set Ω_k. With these preliminaries defined, we have the following lemma providing a lower bound on the inter-sampling interval T_k.

Lemma 3. Assume that the sequence of transmissions and arrivals are admissible, i.e. $s_k < a_k \le s_{k+1}$, and equation (13) and (14) hold. With the triggering event defined as in (2) and the maximum quantization error function defined as in (3), the inter-sampling interval T_k satisfies

$$T_k \ge \underline{T}_k = \frac{1}{L^x_{\Omega_k}}\left(\ln\left(1 + \frac{L^x_{\Omega_k}\theta(\|\hat{x}_k\|, \bar{w})}{\hat{f}(\hat{x}_k, u_k, u_{k-1}, \bar{w})}\right) - \ln\left(1 + \frac{L^x_{\Omega_k}\bar{e}_q(\|e_{k-1}(s_k)\|, \bar{w})}{\hat{f}(\hat{x}_k, u_k, u_{k-1}, \bar{w})}\right)\right), \tag{17}$$

where

$$\hat{f}(\hat{x}_k, u_k, u_{k-1}, \bar{w}) = \bar{f}(\hat{x}_k, u_{k-1}, \bar{w}) + |\bar{f}(\hat{x}_k, u_k, \bar{w}) - \bar{f}(\hat{x}_k, u_{k-1}, \bar{w})|.$$

$\underline{T}_k$ is called the *minimum inter-sampling interval.*

Proof. Since $s_k < a_k \le s_{k+1}$, there are two different dynamic behaviors during interval $[s_k, s_{k+1})$. The first one is during interval $[s_k, a_k)$ when u_{k-1} is the control input, and the second one is during interval $[a_k, s_{k+1})$ when u_k is the control input. We will analyze the dynamic behavior of the system during these two intervals, respectively.

From equation (13), it's easy to show that $x(t)$ lies in Ω_k for all $t \in [s_k, s_{k+1})$. Therefore, during $[s_k, a_k)$, the derivative of $\|e_k(t)\|$ satisfies

$$\frac{d\|e_k(t)\|}{dt} \leq \|\dot{e}_k(t)\| \leq \bar{f}(\hat{x}_k, u_{k-1}, \bar{w}) + L^x_{\Omega_k}\|e_k(t)\|.$$

According to the comparison principle, we have

$$\begin{aligned}\|e_k(a_k)\| \leq & \frac{\bar{f}(\hat{x}_k, u_{k-1}, \bar{w})}{L^x_{\Omega_k}}(e^{L^x_{\Omega_k} D_k} - 1) \\ & + \bar{e}_q(\|e_{k-1}(s_k)\|, \bar{w})e^{L^x_{\Omega_k} D_k}.\end{aligned}$$

For interval $[a_k, s_{k+1})$, the derivative of $\|e_k(t)\|$ satisfies

$$\frac{d\|e_k(t)\|}{dt} \leq \|\dot{e}_k(t)\| \leq \bar{f}(\hat{x}_k, u_k, \bar{w}) + L^x_{\Omega_k}\|e_k(t)\|. \quad (18)$$

With $\|e_k(a_k)\|$ as the initial condition, we have

$$\begin{aligned}\|e_k(s_{k+1})\| \leq & \frac{\hat{f}(\hat{x}_k, u_k, u_{k-1}, \bar{w})}{L^x_{\Omega_k}}(e^{L^x_{\Omega_k} T_k} - 1) \\ & + \bar{e}_q(\|e_{k-1}(s_k)\|, \bar{w})e^{L^x_{\Omega_k} T_k}. \quad (19)\end{aligned}$$

From $\|e_k(s_{k+1})\| = \theta(\|\hat{x}_k\|, \bar{w})$, it can be derived that $T_k \geq \underline{T}_k$. □

Remark 4. Equation (14) guarantees that the minimum inter-sampling interval $\underline{T}_k$ is always positive, and hence avoid the zeno behavior.

The maximum delay which guarantees the input-to-state stability is given in the next theorem.

Theorem 5. Assume equation (13) and (14) hold. If the delay of the $k+1$st transmission D_{k+1} satisfies

$$D_{k+1} \leq \Delta_{k+1} = \min\{\underline{T}_{k+1}, \hat{D}_{k+1}\},$$

where $\underline{T}_k$ is defined in (17) and

$$\begin{aligned}\hat{D}_k = & \frac{1}{L^x_{\Omega_k}}\left(\ln\left(1 + L^x_{\Omega_k}\frac{\underline{\xi}(\|\hat{x}_k\|, \bar{w})}{\bar{f}(\hat{x}_k, u_k, \bar{w})}\right)\right. \\ & \left. - \ln\left(1 + L^x_{\Omega_k}\frac{\theta(\|\hat{x}_k\|, \bar{w})}{\bar{f}(\hat{x}_k, u_k, \bar{w})}\right)\right), \quad (20)\end{aligned}$$

then the system is ISS.

Proof. During interval $[a_k, s_{k+1})$, we know that equation (2) holds. From (13), it's easy to show that

$$\|e_k(t)\| \leq \underline{\xi}(\|\hat{x}_k\|, \bar{w}), \forall t \in [a_k, s_{k+1}). \quad (21)$$

For interval $[s_{k+1}, a_{k+1})$, we first notice that $s_{k+1} < a_{k+1} \leq s_{k+2}$. This is derived from $D_{k+1} \leq \underline{T}_{k+1}$ using proof by contradiction. So, during interval $[s_{k+1}, a_{k+1})$, the dynamic behavior of $\|e_k(t)\|$ still satisfies (18). With the initial condition $\|e_k(s_{k+1})\| = \theta(\|\hat{x}_k\|, \bar{w})$, we have

$$\begin{aligned}\|e_k(a_{k+1})\| \leq & \frac{\bar{f}(\hat{x}_k, u_k, \bar{w})}{L^x_{\Omega_k}}(e^{L^x_{\Omega_k} D_{k+1}} - 1) \\ & + \theta(\|\hat{x}_k\|, \bar{w})e^{L^x_{\Omega_k} D_{k+1}}.\end{aligned}$$

To guarantee that

$$\|e_k(a_{k+1})\| \leq \underline{\xi}(\|\hat{x}_k\|, \bar{w}), \forall t \in [s_{k+1}, a_{k+1}), \quad (22)$$

$D_{k+1} \leq \hat{D}_{k+1}$ where $\hat{D}_{k+1}$ is defined in (20).

According to lemma 2, with equation (21) and (22), the system is ISS. □

Remark 6. Equation (13) guarantees that $\hat{D}_k$ is positive. Since $\underline{T}_k$ is also positive, the maximum delay Δ_k is always positive.

4. STABILIZING BIT-RATES

A stabilizing bit-rate is the bit-rate which is sufficient to guarantee the ISS stability of the system. This section shows that the stabilizing bit-rates are always bounded from above by a continuous increasing function with respect to the norm of the state.

From theorem 5, we know that as long as the network delay D_k is no greater than the maximum delay Δ_k, the closed loop system is ISS. If we define $\underline{r}_k$ as

$$\underline{r}_k = \frac{N_k}{\Delta_k}, \quad (23)$$

$\underline{r}_k$ is sufficient to stabilize the system, and we call it the stabilizing bit-rate of the kth transmission.

For convenience of the rest of this paper, we give some preliminaries first. Let's define $f_c(\|\hat{x}_k\|)$ as a class $\mathcal{K}$ function satisfying

$$\|f(\hat{x}_k, K(\hat{x}_k), 0)\| \leq \bar{f}_c(\|\hat{x}_k\|), \quad (24)$$

and $\bar{K}(\|\hat{x}_k\|)$ as a class $\mathcal{K}$ function satisfying

$$u_k = \|K(\hat{x}_k)\| \leq \bar{K}(\|\hat{x}_k\|). \quad (25)$$

We say a function g has *non-negative* order, if $\lim_{s\to 0} g(s) < \infty$. With this definition, we have the following lemma.

Lemma 7. Let $g : [0, \sigma] \to \mathbb{R}^+$ be a continuous, positive definite function with non-negative order for some $\sigma \geq 0$. There must exist continuous, positive definite, increasing functions $\underline{h}$ and $\overline{h}$ defined on $[0, \sigma]$ such that

$$\underline{h}(s) \leq g(s) \leq \overline{h}(s), \forall s \in [0, \sigma],$$
$$\lim_{s\to 0} g(s) = \lim_{s\to 0} \underline{h}(s) = \lim_{s\to 0} \overline{h}(s).$$

Proof. See Lemma 4.3 in Khalil and Grizzle [1992]. □

With these preliminaries, we show that the stabilizing bit-rate is bounded from above by an increasing function with respect to (w.r.t.) the norm of the state. This is done by showing that N_k is bounded from above by an increasing function, and that Δ_k is bounded from below by a decreasing function.

Theorem 8. If all the conditions in theorem 5 hold, and

$$\lim_{s\to 0} \frac{s}{\bar{e}_q(s, 0)} < \infty, \quad (26)$$

then there exists an increasing function $\overline{N}_k(\|\hat{x}_{k-1}\|)$ such that

$$N_k \leq \overline{N}_k(\|\hat{x}_{k-1}\|). \quad (27)$$

Proof. If $\bar{w} > 0$, it's easy to show that

$$\lim_{x\to 0} \frac{\theta(\|\hat{x}_{k-1}\|, \bar{w})}{\bar{e}_q(\|e_{k-1}(s_k)\|, \bar{w})} < \infty. \quad (28)$$

If $\bar{w} = 0$, with equation (4) and (26), we have

$$\begin{aligned}\lim_{x\to 0} \frac{\theta(\|\hat{x}_{k-1}\|, 0)}{\bar{e}_q(\|e_{k-1}(s_k)\|, 0)} &= \lim_{x\to 0} \frac{\theta(\|\hat{x}_{k-1}\|, 0)}{\bar{e}_q(\theta(\|\hat{x}_{k-1}\|, 0), 0)} \\ &= \lim_{s\to 0} \frac{s}{\bar{e}_q(s, 0)} < \infty.\end{aligned}$$

Since in both cases equation (28) holds, according to Lemma (7), it's easy to show that there exists an increasing function $\overline{N}_k(\|\hat{x}_{k-1}\|)$ such that $N_k \leq \overline{N}_k(\|\hat{x}_{k-1}\|)$. □

Remark 9. From equation (5) and (4), it's easy to see that equation (26) guarantees that N_k is finite as x goes to 0.

We show that there exists a decreasing lower bound w.r.t. $\|\hat{x}_{k-1}\|$ on Δ_k by showing that both $\hat{D}_k$ and $\underline{T}_k$ have decreasing lower bounds w.r.t. $\|\hat{x}_{k-1}\|$.

Lemma 10. If all the conditions in theorem 5 hold and

$$\lim_{s\to 0}\frac{\bar{f}_c(s)}{\theta(s,0)}<\infty, \tag{29}$$

then there exists a positive definite, decreasing function $\underline{\hat{D}}_k(\|\hat{x}_{k-1}\|)$ such that

$$\hat{D}_k \geq \underline{\hat{D}}_k(\|\hat{x}_{k-1}\|). \tag{30}$$

Proof. From equation (20), we have

$$\hat{D}_k \geq \frac{\ln\left(1+L^x_{\Omega_{k-1}}\frac{1-\frac{\theta(\|\hat{x}_{k-1}\|,\bar{w})}{\underline{\xi}(\|\hat{x}_{k-1}\|,\bar{w})}}{\frac{\bar{f}_c(\|\hat{x}_{k-1}\|)+L^x_{\Omega_{k-1}}\bar{w}}{\underline{\xi}(\|\hat{x}_{k-1}\|,\bar{w})}+L^x_{\Omega_{k-1}}\frac{\theta(\|\hat{x}_{k-1}\|,\bar{w})}{\underline{\xi}(\|\hat{x}_{k-1}\|,\bar{w})}}\right)}{L^x_{\Omega_{k-1}}}.$$

Equation (13) indicates that there exists a constant $c_1\in(0,1)$ such that $\frac{\theta(s,\bar{w})}{\underline{\xi}(s,\bar{w})}\leq c_1$. So, we have

$$\hat{D}_k \geq \frac{1}{L^x_{\Omega_{k-1}}}\ln\left(1+L^x_{\Omega_{k-1}}\frac{1-c_1}{\frac{\bar{f}_c(\|\hat{x}_{k-1}\|)+L^x_{\Omega_{k-1}}\bar{w}}{\underline{\xi}(\|\hat{x}_{k-1}\|,\bar{w})}+L^x_{\Omega_{k-1}}c_1}\right).$$

According to Lemma 7, from equation (29) and (13), no matter whether $\bar{w}$ equals to 0 or not, there always exists a positive definite increasing function $h_1(s,\bar{w})$ such that

$$\frac{\bar{f}_c(s)+L^w_{\Omega_{k-1}}\bar{w}}{\underline{\xi}(s,\bar{w})}\leq h_1(s,\bar{w}).$$

Therefore, we have

$$\hat{D}_k \geq \frac{1}{L^x_{\Omega_{k-1}}}\ln\left(1+L^x_{\Omega_{k-1}}\frac{1-c_1}{h_1(\|\hat{x}_{k-1}\|,\bar{w})+L^x_{\Omega_{k-1}}c_1}\right).$$

It's easy to show that the left hand side is a positive definite decreasing function with respect to $\|\hat{x}_{k-1}\|$. ☐

Lemma 11. Assume all the conditions in theorem 5 and equation (29) are satisfied. If

$$\lim_{s\to 0}\frac{\bar{K}(s)}{\theta(s,0)}<\infty, \tag{31}$$

then there exists a positive definite decreasing function $\underline{\hat{T}}_k(\|\hat{x}_{k-1}\|)$ such that

$$\underline{T}_k \geq \underline{\hat{T}}_k(\|\hat{x}_{k-1}\|). \tag{32}$$

Proof. From equation (17), we have

$$\underline{T}_k = \frac{1}{L^x_{\Omega_k}}\ln\left(1+\frac{L^x_{\Omega_k}\left(1-\frac{\bar{e}_q(\|e_{k-1}(s_k)\|,\bar{w})}{\theta(\|\hat{x}_k\|,\bar{w})}\right)}{\frac{\hat{f}(\hat{x}_k,u_k,u_{k-1},\bar{w})}{\theta(\|\hat{x}_k\|,\bar{w})}+L^x_{\Omega_k}\frac{\bar{e}_q(\|e_{k-1}(s_k)\|,\bar{w})}{\theta(\|\hat{x}_k\|,\bar{w})}}\right).$$

Equation (14) indicates that there exists a constant $c_2\in(0,1)$ such that $\frac{\bar{e}_q(\|e_{k-1}(s_k)\|,\bar{w}))}{\theta(\|\hat{x}_k\|,\bar{w})}\leq c_2$. So,

$$\underline{T}_k \geq \frac{1}{L^x_{\Omega_k}}\ln\left(1+L^x_{\Omega_k}\frac{1-c_2}{\frac{\hat{f}(\hat{x}_k,u_k,u_{k-1},\bar{w})}{\theta(\|\hat{x}_k\|,\bar{w})}+L^x_{\Omega_k}c_2}\right).$$

Let $L^u_{\Omega_k}$ be the Lipschitz constant of f with respect to u over the compact set Ω_k. From equation (24) and (25), we have

$$\frac{\hat{f}(\hat{x}_k,u_k,u_{k-1},\bar{w})}{\theta(\|\hat{x}_k\|,\bar{w})}\leq\frac{3\bar{f}_c(\|\hat{x}_k\|)}{\theta(\|\hat{x}_k\|,\bar{w})}+\frac{2L^u_{\Omega_k}\bar{K}(\|\hat{x}_k\|)}{\theta(\|\hat{x}_k\|,\bar{w})}+\frac{L^w_{\Omega_k}\bar{w}}{\theta(\|\hat{x}_k\|,\bar{w})}+\frac{2L^u_{\Omega_k}\bar{K}(\|\hat{x}_{k-1}\|)}{\theta(\|\hat{x}_k\|,\bar{w})}$$

It's easy to show that the first three terms have non-negative order. According to Lemma 7, there exists a positive definite increasing function $h_2(\|\hat{x}_k\|,\bar{w})$ such that

$$\frac{\hat{f}(\hat{x}_k,u_k,u_{k-1},\bar{w})}{\theta(\|\hat{x}_k\|,\bar{w})}\leq h_2(\|\hat{x}_k\|,\bar{w})+\frac{2L^u_{\Omega_k}\bar{K}(\|\hat{x}_{k-1}\|)}{\theta(\|\hat{x}_k\|,\bar{w})}.$$

From equation (14), we have

$$\frac{\bar{K}(\|\hat{x}_{k-1}\|)}{\theta(\|\hat{x}_k\|,\bar{w})}\leq\frac{\bar{K}(\|\hat{x}_{k-1}\|)}{\bar{e}_q(\theta(\|\hat{x}_{k-1}\|,\bar{w}),\bar{w})}=\frac{\bar{K}(\|\hat{x}_{k-1}\|)}{\theta(\|\hat{x}_{k-1}\|,\bar{w})}\frac{\theta(\|\hat{x}_{k-1}\|,\bar{w})}{\bar{e}_q(\theta(\|\hat{x}_{k-1}\|,\bar{w}),\bar{w})}$$

From equation (31) and (26), it's easy to see that $\frac{\bar{K}(\|\hat{x}_{k-1}\|)}{\theta(\|\hat{x}_k\|,\bar{w})}$ also has non-negative order. So, there must exist a positive definite increasing function $h_3(\|\hat{x}_{k-1}\|,\bar{w})$ such that $\frac{\bar{K}(\|\hat{x}_{k-1}\|)}{\theta(\|\hat{x}_k\|,\bar{w})}\leq h_3(\|\hat{x}_{k-1}\|,\bar{w})$. Therefore,

$$\frac{\hat{f}(\hat{x}_k,u_k,u_{k-1},\bar{w})}{\theta(\|\hat{x}_k\|,\bar{w})}\leq h_2(\|\hat{x}_k\|,\bar{w})+2L^u_{\Omega_k}h_3(\|\hat{x}_{k-1}\|,\bar{w})$$

So,

$$\underline{T}_k \geq \frac{\ln\left(1+\frac{L^x_{\Omega_k}(1-c_2)}{h_2(\|\hat{x}_k\|,\bar{w})+2L^u_{\Omega_k}h_3(\|\hat{x}_{k-1}\|,\bar{w})+L^x_{\Omega_k}c_2}\right)}{L^x_{\Omega_k}}. \tag{33}$$

Equation (2) and (3) imply that

$$\begin{aligned}\|\hat{x}_k\| &\leq\|x_k\|+\bar{e}_q(\theta(\|\hat{x}_{k-1}\|,\bar{w}),\bar{w})\\ &\leq\|\hat{x}_{k-1}\|+\theta(\|\hat{x}_{k-1}\|,\bar{w})+\bar{e}_q(\theta(\|\hat{x}_{k-1}\|,\bar{w}),\bar{w}).\end{aligned}$$

Apply the above equation into (33), it's easy to see that there must exist a positive definite decreasing function $\underline{\hat{T}}_k(\|\hat{x}_{k-1}\|)$ such that equation (32) holds. ☐

With lemma 10 and 11, we have the following theorems.

Theorem 12. Assume all the condition in theorem 5 hold. If equation (29) and (31) are satisfied, then there exists a positive definite function $\underline{\Delta}_k(s,\bar{w})$ decreasing with respect to s such that

$$\Delta_k \geq \underline{\Delta}_k(\|\hat{x}_{k-1}\|,\bar{w}).$$

With theorem 8 and 12, it is easy to find an upper bound on $\underline{r}_k$ which is increasing w.r.t. $\|\hat{x}_{k-1}\|$.

Theorem 13. Assume all the condition in theorem 5 hold. If equation (26), (29) and (31) are satisfied, then there exists a positive definite function $\overline{r}_k(s,\bar{w})$ increasing with respect to s such that

$$\underline{r}_k \leq \overline{r}_k(\|\hat{x}_{k-1}\|).$$

Remark 14. Theorem 13 indicates that the further the state is away from the origin, the higher the stabilizing bit-rate will be. Moreover, since there exist some class $\mathcal{KL}$ function $\hat{\beta}$ and class $\mathcal{K}$ function $\hat{\alpha}$ such that $\|\hat{x}_{k-1}\|\leq\max\{\hat{\beta}(\|x_0\|,t),\hat{\alpha}(\bar{w})\}$, we are also able to find how the upper bound on the stabilizing bit-rate, which measures the maximum of the minimum communication resource sufficient to stabilize the system, varies with respect to time. This piece of information gives us a guide on how to

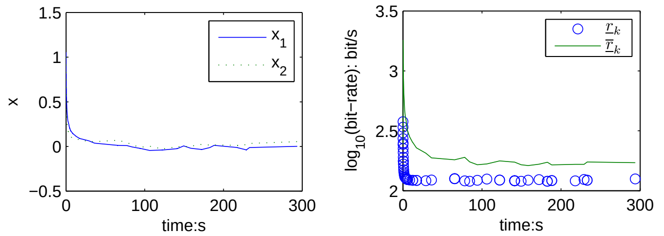

Fig. 2. State trajectory and stabilizing bit-rate

assign communication resource to the control system. We should notice that this assignment can be time varying.

Remark 15. If there is no disturbance, the results in Li et al. [2012] are recovered.

5. SIMULATION RESULTS

This section uses a nonlinear system to demonstrate theorem 5 and 13.

Now, let's consider a nonlinear system

$$\dot{x}_1 = x_1^3 + x_2^3 + u_1 + w_1$$
$$\dot{x}_2 = -x_1^3 + x_2^3 + u_2 + w_2$$

with $u_1 = -3\hat{x}_1^3$, $u_2 = -3\hat{x}_2^3$, $\bar{w} = 0.5$ and $x_0 = [1; 1]$. It's easy to see that $\bar{f}_c(s) = 3s^3$, $\bar{K}(s) = 3s^3$ and $L = 14\|x\|$. We give the ISS-Lyapunov function as $V = x_1^4 + x_2^4$. and it can be derived that $\xi(s, \bar{w}) = 0.13s^1.5 + 0.12\bar{w}$. From Lemma 2, we choose $\underline{\xi}(s, \bar{w}) = 0.115s^{1.5} + 0.1\bar{w}$. From equation (13) and (14), the threshold function is chosen to be $\theta(s, \bar{w}) = 0.058s^1.5 + 0.05\bar{w}$, and the quantization error function $\bar{e}_q(s, \bar{w}) = 0.25s$. So, according to equation (5), the number of bits is always 6. The upper bound on the stabilizing bit-rate is given below

$$\overline{r}_k(s) = \frac{6}{\min\left\{\hat{\underline{T}}_k(s), \hat{\underline{D}}_k(s)\right\}} \tag{34}$$

where

$$\hat{\underline{T}}_k(s) = \frac{1}{L^x_{\Omega_{k-1}}} \ln\left(1 + \frac{0.75L^x_{\Omega_{k-1}}}{190s^{1.5} + 20 + 0.25L^x_{\Omega_{k-1}}}\right),$$
$$\hat{\underline{D}}_k(s) = \frac{1}{L^x_{\Omega_{k-1}}} \ln\left(1 + \frac{0.5L^x_{\Omega_{k-1}}}{27s^{1.5} + 10 + 0.5L^x_{\Omega_{k-1}}}\right).$$

We ran the system with disturbance for 300 seconds, and always used Δ_k as the delay in the communication network. The state trajectories and stabilizing bit-rates are shown in figure 2. The top plot shows the state trajectory. The x-axis indicates time, and the y-axis indicates x. We can see that the state gradually stay in a neighborhood of the origin, and hence the system is ISS. The bottom plot shows the stabilizing bit-rates with x-axis indicating time, and y-axis indicating $\log_{10} \underline{r}_k$. The stabilizing bit-rate and its upper bound are expressed by circles and solid line, respectively. We can see that the stabilizing bit-rate is always bounded from above by the upper bound. Moreover, when the system stays in a neighborhood of the origin, the upper bound is only about 2 times of the stabilizing bit-rate. Therefore, we can say that this is equation (34) gives a tight upper bound on the stabilizing bit-rate.

6. FUTURE WORK

The results in this paper can be used as a foundation to study the scheduling problem in networked control systems. This paper provides the maximum delay Δ_k and the stabilizing bit-rate $\underline{r}_k$. With this information, communication channel can assign the communication resource to different control systems. Interesting topics includes the necessary and/or sufficient bandwidth to stabilize all control systems in the network, and the scheduling policy to achieve the necessary and/or sufficient bandwidth.

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