Efficiently Attentive Quantized Event-Triggered Systems

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Sampled Data Control System

- Encoder - quantizes and samples system state
- Channel - Lossy with finite bit-rate
- Decoder - zero-order hold
- Plant/Controller - stable under perfect state feedback

Minimum Information for Stabilization?

\[ \{ s_k \}_{k=0}^{\infty} = \text{sampling instants} \]

- Plant/Controller state, \( x(t) \) satisfies
  \[ \dot{x}(t) = f(x(t), u(t), w(t)); \quad x(0) = x_0 \]

- Sampled and Quantized state, \( \hat{x}_k = Q(x(s_k)) \)

- Reconstructed state \( \hat{x}(t) = \hat{x}_k \) for \( t \in [s_k, s_{k+1}) \)
Dynamically Quantized Feedback

- Discrete-time Linear System with one sample delay
  - state is periodically sampled with quantization\textsuperscript{[1,2]}

\[
R \geq R = \sum_{i=1}^{n} \max(0, \log_2 |\lambda_i|)
\]

Uncertainty sets
at time instant $k$

Uncertainty sets
at time instant $k+1$

- Necessary and sufficient bit rate for asymptotic stability
Emulation Method

- “Emulation Method” for Sampled System Design
  - design controller, $K$, that leaves “continuously” sampled system input-to-state stable (ISS)\(^3\).

There exist functions $\beta \in KL$ and $\gamma \in K$ such that

$$|\dot{z}(t)| < \max\{\beta(|z_0|, t), \gamma(|e|_{\mathcal{L}_\infty})\}$$

$e \in \mathcal{L}_\infty$ and $z(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ satisfies

$$\dot{z}(t) = f(z(t), K(z(t) + e(t)))$$

- Select Sampling Instants to guarantee that the sporadically sampled system is also ISS.
Event-Triggered Feedback\cite{4}

- State is sporadically sampled with no quantization

- Since the ”continuously” sampled system is ISS, there exists a positive definite function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}^+$ and functions $\alpha, \gamma \in \mathcal{K}$ such that

$$\frac{\partial V}{\partial x} f(x, K(x + e)) \leq -\alpha(|x|) + \gamma(|e|)$$

- Select sampling instants, $\{s_k\}_{k=0}^\infty$ so that for all time, $t$,

$$|e(t)| \leq \gamma^{-1} ((1 - \sigma)\alpha(|x(t)|)) \equiv \xi(|x(t)|)$$

where $0 < \sigma < 1$
Resource Utilization in Event-Triggered Systems

Prior work has suggested that event-triggered systems have lower resource utilization than comparable periodically sampled systems\cite{5,6,7,8}.

But this may not always be the case.
Consider the following system
\[ \dot{x}(t) = f(x(t)) + u(t) \text{ with } u(t) = -2f(\hat{x}_k) \]
where sublinear \( f(x) = \text{sgn}(x) \sqrt{|x|} \) and superlinear \( f(x) = x^3 \).

Given sampling instant \( s_k \), the next sampling instant \( s_{k+1} \) is the first time when the state leaves
\[ \Omega_k = \{ x \in \mathbb{R}^n : (\hat{x}_k - x)^2 \leq |x|^2 \equiv \theta(|x|) \} \]

- **Zeno-sampling**: infinite samples over a finite interval
- **Efficiently Attentive**: Intersampling Intervals get larger as system state approach origin.
Dynamic Quantization versus Event-Triggering

● Differences
  - Single sample delay (Q) versus small delay (ET)
  - Utilization measured by bit rate versus inter-sampling time
  - Dynamic quantizers achieve minimum stabilizing bit rate, but lower resource utilization not guaranteed for event-triggered systems

● Similarities
  - both attempt to reduce “information” over channel.
  - quantization in “time” versus “space.
  - both discretize a continuous-time control system

● Objectives:
  - Unification quantization and event-triggering
  - Design Event-triggers to control resource utilization by ensuring efficient attentiveness
Quantized Event-Triggered Networked System\cite{11,12}

- **Sampling Instants**, \(\{s_k\}_{k=0}^{\infty}\), and **Arrival Instants** \(\{a_k\}_{k=0}^{\infty}\).

- **State Equation** is 
  \[ \dot{x}(t) = f(x(t), k(\hat{x}_k)) \]
  where \(\hat{x}_k\) is the quantized state and \(t \in [a_k, a_{k+1})\).

- **Quantized state**, \(\hat{x}_k\), satisfies 
  \[ |x(s_k) - \hat{x}_k| < E(|x(s_k)|) \]
  where \(E \in \mathcal{K}\) is the **Quantization Error**.

- **Controller**, \(K\), ensures "continuously" sampled system is ISS with respect to the **gap** \(e_k(t) = x(t) - \hat{x}_k\).
Modeling of Channel Delay and Quantization\[^{[13,14,15]}\]

- Quantization Error, $E(|x_k|)$
  
  Delay = $D_k = a_k - s_k$

  Inter-sampling Interval = $T_k = s_{k+1} - s_k$

  Admissible Sampling: $s_k < a_k < s_{k+1}$

- What are dynamics of gap function, $e_k(t) = x(t) - \tilde{x}_k$?

\[ E(|x_k|) = \text{quantization error} \]
Dynamics of the Gap Function\textsuperscript{[11,12]}

Lipschitz on Compacts: \( L_k > 0 \) such that

\[
|f(x(t), K(\hat{x}_k))| \leq |f(\hat{x}_k, K(\hat{x}_k))| + L_k |e_k(t)|
\]

for all \( x(t) = \hat{x}_k + e_k(t) \in \Omega_k \) where

\[
\Omega_k = \{ x \in \mathbb{R}^n : |x| \leq |\hat{x}_k| + \xi(|\hat{x}_k|) \}
\]

and \( \xi(s) = \sup \{ r : \xi(s - r), s < r \} \)

- Since \( x(t) \in \Omega_k \), the gap, \( e_k(t) \), satisfies the following differential inequality for \( t \in [s_k, a_{k+1}) \)

\[
\frac{d|e_k|}{dt} \leq |\dot{e}_k(t)| \leq |f(\hat{x}_k, K(\hat{x}_k))| + L_k |e_k(t)|, \quad e_k(s_k) < E(|x_k|)
\]

- This is a linear differential inequality and we can use the comparison principle to bound the gap \( |e_k(t)| \)
Lower Bound on Inter-sampling Interval$^{[10,11,13]}$

- $k$th Gap at $k$th transmission time is $|e_k(s_k)| < E(|x_k|)$
- $k$th Gap at $k + 1$st transmission time is

$$|e_k(s_{k+1})| \leq E(|x_k|)e^{L_k T_k} + \frac{\Psi(\hat{x}_k, \hat{x}_{k-1})}{L_k} (e^{L_k T_k} - 1)$$

where $\Psi(\hat{x}, \hat{x}_{k-1}) = |f(\hat{x}_k, K(\hat{x}_k))| + 2|f(\hat{x}_k, K(\hat{x}_{k-1}))|$
Lower Bound on Inter-sampling Interval

\[
\theta(|x_k|) \\
\text{event-trigger} \\
e_k(s_{k+1}) \\
e_k(s_k) \\
D_k \\
T_k \\
\]

- \( s_{k+1} \), is the first time when \(|e_k(t)| = \theta(|\hat{x}_k|)\). We refer to \( \theta(\cdot) : \mathbb{R} \to \mathbb{R} \) as the \textit{event-triggering function}.

- With \(|e_k(s_{k+1})| = \theta(|\hat{x}_k|)\), our earlier constraint becomes

\[
T_k > \frac{1}{L_k} \left( \ln \left( 1 + \frac{L_k \theta(|\hat{x}_k|)}{\Psi(\hat{x}_k, \hat{x}_{k-1})} \right) - \ln \left( 1 + \frac{L_k E(|x_k|)}{\Psi(\hat{x}_k, \hat{x}_{k-1})} \right) \right)
\]
Using similar techniques, we find

\[ |e_k(a_{k+1})| \leq \theta(|\hat{x}_k|)e^{L_k D_{k+1}} + \frac{|f(\hat{x}_k, K(\hat{x}_k))|}{L_k} (e^{L_k D_{k+1}} - 1) \]

Asymptotic stability requires \( |e_k(a_{k+1})| \leq \xi(|\hat{x}_k|) \), so the stabilizing delay satisfies

\[ D_{k+1} \leq \frac{1}{L_k} \left( \ln \left( 1 + L_k \frac{\xi(|\hat{x}_k|)}{|f(\hat{x}_k, K(\hat{x}_k))|} \right) - \ln \left( 1 + L_k \frac{\theta(|\hat{x}_k|)}{|f(\hat{x}_k, K(\hat{x}_k))|} \right) \right) \]
Asymptotic Stability\textsuperscript{[11,12]}

Assume that the event-triggering threshold $\theta$ satisfies

\[ E(s) \leq \theta(s) \leq \xi(s) \]

for any $s \in \mathbb{R}^+$ and assume the delay $D_k < \min\{T_k, \overline{D}_k\}$ where

\[ T_k = \frac{1}{L_k} \left( \ln \left( 1 + \frac{L_k \theta(|\hat{x}_k|)}{\Psi(\hat{x}_k, \hat{x}_{k-1})} \right) - \ln \left( 1 + \frac{L_k E(|x_k|)}{\Psi(\hat{x}_k, \hat{x}_{k-1})} \right) \right) \]

\[ \overline{D}_k = \frac{1}{L_{k-1}} \left( \ln \left( 1 + L_{k-1} \frac{\xi(|\hat{x}_{k-1}|)}{|f(\hat{x}_{k-1}, K(\hat{x}_{k-1}))|} \right) - \ln \left( 1 + L_{k-1} \frac{\theta(|\hat{x}_{k-1}|)}{|f(\hat{x}_{k-1}, K(\hat{x}_{k-1}))|} \right) \right) \]

Then the closed-loop quantized event-triggered system is asymptotically stable with an inter-sampling interval $T_k$ which is always bounded below by $T_k > 0$. 
Stabilizing Bit Rate[11,12]

- \( N_k \) = Number of Bits used to represent sampled state \( \hat{X}_k \)
- Assuming \(|x|\) is the sup-norm, then
  \[
  N_k = \lceil \log_2 2n \rceil + \left( n - 1 \right) \left\lfloor \log_2 \left( \frac{\theta(\hat{x}_{k-1})}{E(|x_k|)} \right) \right\rfloor \text{ bits}
  \]
- To guarantee the asymptotic stability of the closed-loop system, these bits must be delivered within the delay
  \[
  D_k \leq \overline{D}_k \equiv \frac{1}{L_{k-1}} \left( \ln \left( 1 + L_{k-1} \frac{\xi(\hat{x}_{k-1})}{|f(\hat{x}_{k-1}, K(\hat{x}_{k-1}))|} \right) - \ln \left( 1 + L_{k-1} \frac{\theta(\hat{x}_{k-1})}{|f(\hat{x}_{k-1}, K(\hat{x}_{k-1}))|} \right) \right)
  \]
- A bit rate stabilizing this system must therefore be
  \[
  R_k = \frac{N_k}{D_k} > R_k \equiv \frac{L_k}{\ln 2} \left( A(|\hat{x}_k|)(n - 1) + B(|\hat{x}_k|) \right)
  \]
Conditions for Zero Bit Rate\cite{11}

- In some cases, we can show that the stabilizing bit rate goes to zero as the system approaches its equilibrium point.

- System Equations
  \[
  \dot{x}_1 = x_1^3 + 2x_2^3 + u \\
  \dot{x}_2 = -x_1^3 - x_2^3
  \]

- Feedback Control
  \[
  u = -2\hat{x}_1 - \hat{x}_2^3
  \]

- Switching Condition: \[|e_k(t)| = \theta(|\hat{x}_k|) = 0.015|\hat{x}_k|\]
Efficiently Attentive Stabilizing Bit Rates\textsuperscript{[10,12]}

- It can be very difficult to determine the minimal stabilizing bit rate. In such cases, a reasonable option is to require the bit rate to be efficiently attentive.

- A bit rate is efficiently attentive if it is an increasing function of state.

Assume that the delay $D_k < \bar{D}_k$, and the event-triggering satisfies

$$E(s) \leq \theta(s) \leq \xi(s)$$

Let $\phi_c, \phi_u \in \mathcal{K}$ such that $|f(x, K(x))| \leq \phi_c(|x|)$ and $|K(x)| \leq \phi_u(|x|)$.

If we also know that

$$\lim_{s \to 0} \frac{\theta(s)}{E(s)} < \infty, \quad \lim_{s \to 0} \frac{\phi_c(s)}{\theta(s)} < \infty, \quad \lim_{s \to 0} \frac{\phi_u(s)}{\theta(s)} < \infty$$

then there exists a continuous, positive definite, increasing function $\underline{R}(|\hat{x}_k|)$ such that if the actual bit rate is greater than $\underline{R}$, then the system is asymptotically stable.
Simulation Example \cite{12,18}

- This example extends prior work to essentially bounded disturbances

**System Equations**

\[
\begin{align*}
\dot{x}_1 &= x_1^3 + 2x_2^3 + u_1 + w_1 \\
\dot{x}_2 &= -x_1^3 - x_2^3 + u_2 + w_2
\end{align*}
\]

**Controller**

\[
\begin{align*}
u_1 &= -3\dot{x}_1 \\
u_2 &= -3\dot{x}_2
\end{align*}
\]

- Event Trigger and Quantization Map

\[
\begin{align*}
\theta_1(s, w) &= 0.075s^{1.5} + 0.05w \\
E_1(s, w) &= 0.025s^{1.5} + 0.017w
\end{align*}
\]

- Predicted bit-rates are conservative bound on actual bit-rates

![Graph showing predicted and actual bit-rates](image.png)
Event-triggering gives rise to sporadic message streams in wireless networked control systems.

After an impulsive disturbance is applied to middle cart, one can bound the future inter-sampling times and bit-rate requirements of all controllers.

Can we use this information to reschedule controller transmission to maintain both overall physical system performance while staying within communication network’s capacity limits?
Safety-Critical Systems

- One concern with event-triggered systems is that they are ill suited for safety-critical systems.
- In the presence of disturbances, however, event-triggered solutions must be implemented with a minimum sampling frequency whose size is determined by the “disturbance”.
- With efficiently attentive systems, event frequency increases in the presence of impulsive disturbances.

http://www.nd.edu/~lemmon/projects/NSF-05-1518/heli-movie/
Future Directions

- Relax **conservativeness** of bounds\textsuperscript{[19]}.
- Event-trigger design is based on existence of ISS controller (input disturbances). This may not always be possible. One solution may be to extend framework to **iISS controllers**\textsuperscript{[16]}.
- These results provide some guidance on the selection of controllers, quantizers, and event-triggers. We still need to formalize this guidance into a **design procedure**.
- Similarities between dynamic quantization and the quantized event-triggers. When do we achieve the known **necessary and sufficient stabilizing bit rates for linear systems**?
- Efficiently attentive systems provide a basis for co-design of communication/controller in a deterministic setting. Is it possible to extend these ideas to a stochastic setting? One possible approach would involve the use of stochastic ISS concepts\textsuperscript{[17]}. 

References


