Asymptotic Stability in Distributed Event-Triggered Networked Control Systems with Delays

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***Abstract*— This paper examines event-triggered broadcasting of state information in distributed networked control systems (NCS) with transmission delays. We provide positive bounds on transmission delays for asymptotic stability of NCS. Each agent can compute the bound on the delays in its state transmissions based on its local information. These results are important because they show the existence of strictly positive bounds on transmission delays in NCS, which guarantee asymptotic stability of the systems. Those bounds can be used to schedule data transmissions.**

I. Introduction

A distributed networked control system (NCS) consists of numerous loosely coupled systems, which are geographically distributed. In such a system, individual subsystems (also called *agents*) exchange information over a communication network. These networked systems are found throughout our national infrastructure with specific examples being the electrical power grid and transportation networks. Not only does networking refers to the communication infrastructure supporting feedback control, it also refers to the fact that individual subsystems may be interconnected in a way that can be modeled as a network. The networking of control effort can be advantageous in terms of lower system costs due to streamlined installation and maintenance costs.

The introduction of communication network infrastructure, however, raises important issues regarding the impact that such communication has on the control system's performance. In practice, communication networks, especially wireless communication networks, can only broadcast data in discrete packets and packet loss often happens. Moreover, the communication media is a resource that is usually accessed in a mutually exclusive manner by neighborhood agents. This means that the throughput capacity of such networks is limited that can cause delays in message delivery [1]. Such delay can have a major impact on overall system stability. So one important issue in the implementation of such systems is to identify methods for more effectively using the limited network bandwidth available for transmitting state information.

For this reason, some researchers began investigating the timing issue in networked control. In other words, how frequently should subsystems communicate such that the NCS still maintains a desired level of performance? Early work analyzing scheduling of real-time network traffic was presented in [2] and [3]. However, the impact of communication constraints on system performance was not addressed in this work. [4], [5], and [6] noticed the harmful effect of communication delay on system stability and considered the one packet transmission problem, in which all of the system outputs were packaged into a single packet. Therefore, agents in the network do not have to compete for channel access. One packet transmission strategies, however, require a supervisor to gather all subsystem data into this huge packet. As a result, such schemes may be impractical for large-scale systems with limited network bandwidth.

Asynchronous transmissions were considered in [7]. In this work, several sensors and actuators request access to the channel at the same time, but only one of them can get through, depending on the network protocols. The basic idea is that one first designs the controllers under the assumption of perfect communication and then determines the *maximum allowable transfer interval* (MATI) between two subsequent message transmissions under the network protocol so that closed loop system stability can still be maintained. A bound on the MATI was derived in [7] so that system stability can be guaranteed. It led to scheduling methods [8] that were able to assure the MATI was not violated. Further work was done in [9], [10] to tighten bounds on the MATI. All this work confined its attention to control area network (CAN) buses where centralized computers are used to coordinate the information transmission with protocols. The length of the MATI heavily relies on the choice of network protocols.

In all of the aforementioned work, however, the computation of the MATI and the execution of the corresponding protocols must be done in a highly centralized manner, which is impractical in large-scale systems due to its poor scalability. Moreover, because the MATI is computed before the system is deployed, it must ensure adequate behavior over a wide range of possible input disturbances. As a result, the MATI may be conservative. Consequently, the bandwidth of the network may have to be higher than necessary to ensure the MATI is not violated. These limitations suggest a great need of distributed approaches to address this timing issue in way that enables the networked control system to use network bandwidth in an extremely frugal manner [11].

Recent work in [12], [13] considering event-triggered feedback sampled-data systems shows that to maintain system stability, the sampling rates under event-triggering are well below those in periodic task models because the system can adaptively adjust the rates in a manner that is sensitive

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The authors gratefully acknowledge the partial financial support of the National Science Foundation (NSF CNS-0720457)

to what is currently happening within the system. By event-triggering, an agent broadcasts its state information to its neighbors only when some measure of the subsystem's local state error exceeds a specified threshold. Following this idea, distributed event-triggered feedback schemes were proposed in [14] and [15] for linear and nonlinear systems, respectively. An implementation of event-triggering in sensor-network was introduced in [16] that is done in a centralized manner. Most recently, transmission delay in distributed event-triggered NCS was considered in [17]. This work shows that if the transmission delay is nonzero, then the resulting NCS is globally uniformly ultimately bounded, provided that the system dynamic is bounded. These results, however, are conservative.

In this paper, we extend the work in [17]. Strictly positive delay is allowed and the assumption of bounded dynamic is relaxed. We provide state-based bounds on transmission delays (also known as *maximal allowable delay* or *deadline*) in distributed event-triggered NCS that are always greater than a positive constant. Each agent can predict the bound on the delays in its state transmissions based on its local information. As long as the delays are less than these bounds, asymptotic stability can be guaranteed. These results are important because they show that, in asynchronous transmission frameworks, strictly positive bounds on transmission delays exist, with which asymptotic stability of NCS can be guaranteed.

The paper is organized as follows: section II formulates the problem; a distributed scheme is introduced in section III to predict the deadlines for transmission delay; simulation results are presented in section IV; in section V, the conclusions are drawn.

II. Problem Formulation

Consider a distributed NCS containing N subsystems (also called "agents"). These N agents are coupled together and each agent can receive information from some of other agents. Let $\mathcal{N} = \{1, 2, \cdots, N\}$ and

- $Z_i \subseteq \mathcal{N}$ denotes the set of agents that agent i can get information from;
- $D_i \subseteq \mathcal{N}$ denotes the set of agents that directly drive agent i's dynamics;
- $S_i \subseteq \mathcal{N}$ denotes the set of agents who are directly driven by agent i;
- $U_i \subseteq \mathcal{N}$ denotes the set of agents that can receive agent i's information;
- for any set $\Sigma_i \subseteq \mathcal{N}$, $|\Sigma_i|$ denotes the number of the elements in Σ_i and $\bar{\Sigma}_i = \Sigma_i \cup \{i\}$;
- $\mathbb{R}^+$ denotes all the positive real numbers and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$;
- $\|\cdot\|_2$ denotes 2-norm of a vector, $\|\cdot\|$ denotes the matrix norm, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimal and maximal eigenvalues of a symmetric matrix A, respectively.

Notice that $i \notin Z_i \cup D_i \cup S_i \cup U_i$.

The state equation of agent i is

$$\begin{aligned} \dot{x}_i(t) &= f_i(x_{\bar{D}_i}, u_i) \\ u_i(t) &= g_i(x_{\bar{Z}_i}) \\ x_i(t_0) &= x_{i0} \end{aligned}$$

where $x_i : \mathbb{R}_0^+ \to \mathbb{R}^n$ is the state trajectory of agent i, $u_i : \mathbb{R}_0^+ \to \mathbb{R}^m$ is a control input, $g_i : \mathbb{R}^{n|\bar{Z}_i|} \to \mathbb{R}^m$ is the feedback strategy of agent i satisfying $g_i(0) = 0$, $f_i : \mathbb{R}^{n|\bar{D}_i|} \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^n$ is continuous and locally Lipschitz satisfying $f_i(0, 0) = 0$, and $x_{\bar{D}_i} = \{x_j\}_{j \in \bar{D}_i}$, $x_{\bar{Z}_i} = \{x_j\}_{j \in \bar{Z}_i}$. For notational convenience, we assume that the states/inputs/disturbances of agents have the same dimension. The results in this paper can be easily extended to the case where the dimensions of agents' states/inputs/disturbances are different from each other.

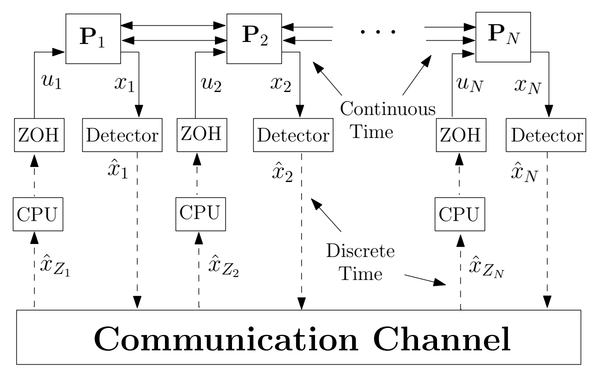

Fig. 1. The infrastructure of the real-time NCS

This paper considers a real-time implementation of this distributed NCS. The infrastructure of such an implementation is plotted in Figure 1. In such a system, agent i can only detect its own state, x_i. If the local "error" signal exceeds some given threshold, which can be detected by hardware detectors, agent i will sample and broadcast its state information to those agents in U_i through a real-time network. Meanwhile, agent i's control, u_i, at time t is computed based on the states broadcasted by agents in $\bar{Z}_i$. These broadcasted states are denoted as $\hat{x}_{\bar{Z}_i}(t)$. The control signal used by agent i is held constant by a zero-order hold (ZOH) unless one of the agents in $\bar{Z}_i$ makes a successful broadcast. This means that the state equation of agent i can be written as

$$\begin{aligned} \dot{x}_i &= f_i(x_{\bar{D}_i}, u_i) \\ u_i &= g_i(\hat{x}_{\bar{Z}_i}). \end{aligned} \tag{1}$$

Agent i's broadcast is characterized by two monotone increasing sequences of time instants: the broadcast release time, $\{b_k^i\}_{k=1}^{\infty}$, and the broadcast finishing time $\{f_k^i\}_{k=1}^{\infty}$. The time b_k^i denotes the time instant when the kth broadcast of agent i is released. At this time, we assume there is no delay between sampling and broadcast release. The time f_k^i denotes the time instant when the kth broadcasted data of agent i is received by its neighbors. Notice that $\hat{x}_i(t) =$

$x_i(b_k^i)$ for all $t \in [f_k^i, f_{k+1}^i)$. For notational convenience, we define $e_i : \mathbb{R}_0^+ \to \mathbb{R}^n$ by $e_i(t) \triangleq x_i(t) - \hat{x}_i(t)$ for $\forall t \geq 0$.

The objective of this paper is to develop distributed event-triggering schemes to identify $\{b_k^i\}_{k=1}^{\infty}$, and $\{f_k^i\}_{k=1}^{\infty}$ such that the NCS defined in equation (1) is asymptotically stable.

III. Event-Triggered NCS with Delays

In this section, we study the NCS with transmission delays. We provide state-based deadlines on these delays. The NCS is asymptotically stable as long as the delay in each transmission is less than the give bound. We also show that these bounds are always greater than a positive constant.

Before we present the main result, we would like to introduce three lemmas. The first lemma (Lemma 3.1) is the result in [17], which provides sufficient condition for asymptotic stability in event-triggered NCS. The second lemma (Lemma 3.4) shows that if there is a bound on the delay, then the resulting NCS is at least globally uniformly ultimately bounded. The third lemma (Lemma 3.6) shows that, if the bound on the delay is appropriately chosen, then the state trajectory will fall into some known compact set. The proofs of lemmas can be found in the appendix. For notational convenience, we let $L_{f_i}V = \frac{\partial V}{\partial x_i} f_i(x_{\bar{D}_i}, u_i)$ with some function $V : \mathbb{R}^{nN} \to \mathbb{R}_0^+$.

Lemma 3.1 ([17]): Consider the N-agent NCS in equation (1). Assume that there exist a smooth, positive-definite function $V : \mathbb{R}^{nN} \to \mathbb{R}_0^+$ and positive constants $p \geq 1$, α_i, $\beta_i \in \mathbb{R}^+$ for all $i \in \mathcal{N}$ such that

$$\sum_{i\in\mathcal{N}} L_{f_i}V \leq \sum_{i\in\mathcal{N}} -\alpha_i\|x_i\|_2^p + \beta_i\|e_i\|_2^p. \tag{2}$$

If for any $i \in \mathcal{N}$,

$$c_i\|e_i(t)\|_2 \leq \rho_i\|\hat{x}_i(t)\|_2 \tag{3}$$

holds for all $t \geq 0$, where $\rho_i \in (0, 1)$ and

$$c_i = 1 + \left(\frac{\beta_i}{\alpha_i}\right)^{\frac{1}{p}}, \tag{4}$$

then the NCS is asymptotically stable.

Remark 3.2: The event for agent i is only associated with e_i and x_i. Agent i just needs to use the violation of equation (12) to trigger the broadcast such that the inequality in equation (12) holds. If there are no transmission delay and data dropouts, the system stability of NCS can be guaranteed.

Remark 3.3: If for given $\zeta_i, \delta_i \in \mathbb{R}^+$, agent i can find a continuous, positive-definite function $V_i : \mathbb{R}^n \to \mathbb{R}_0^+$, a positive constant $\eta_i \in \mathbb{R}^+$, and control law $g_i : \mathbb{R}^{n|\bar{Z}_i|} \to \mathbb{R}^m$ satisfying

$$L_{f_i}V_i \leq -\eta_i\|x_i\|_2^p + \sum_{j\in D_i\cup Z_i} \zeta_j\|x_j\|_2^p + \sum_{j\in\bar{Z}_i} \delta_j\|e_j\|_2^p \tag{5}$$

$$\eta_i - |S_i \cup U_i|\zeta_i > 0, \tag{6}$$

then

$$V(x) = \sum_{i\in\mathcal{N}} V_i(x_i) \tag{7}$$

$$\alpha_i = \eta_i - |S_i \cup U_i|\zeta_i \tag{8}$$

$$\beta_i = \delta|\bar{U}_i| \tag{9}$$

satisfy equation (2). Equation (5) suggests agent i is $\mathcal{L}_p$ stable from $\{x_j\}_{j\in D_i\cup Z_i}$, $\{e_j\}_{j\in\bar{Z}_i}$ to x_i. The satisfaction of equation (6) requires β_i to be small. This implies weak coupling among agents. By solving these two equations, agent i can locally find the pair (α_i, β_i). In general, however, it is not easy to solve them.

Lemma 3.4: Consider the N-agent NCS in equation (1). Suppose that (2) holds. Also assume that

$$\underline{L}\|x\|_2^q \leq V(x) \leq \bar{L}\|x\|_2^q \quad \text{and} \tag{10}$$

$$\|f_i\left(x_{\bar{D}_i}(t), g_i\left(\hat{x}_{\bar{Z}_i}(t)\right)\right)\|_2 \leq \theta_i, \tag{11}$$

hold for all $t \geq t_0$ with some $\theta_i \in \mathbb{R}^+$. If for any $i \in \mathcal{N}$,

$$c_i\|x_i(t) - x_i(b_k^i)\|_2 \leq \rho_i\|x_i(b_k^i)\|_2 \tag{12}$$

holds for all $t \in [b_k^i, b_{k+1}^i)$ for some $\rho_i \in (0, 1)$ and the delay in the kth transmission satisfies

$$f_k^i - b_k^i \leq \frac{1-\rho_i}{c_i\theta_i}\max\left\{\frac{\|x_i(b_{k-1}^i)\|_2}{2}, \Delta\right\} \tag{13}$$

with a given $\Delta \in \mathbb{R}^+$, then for any $\bar{\phi} \in \mathbb{R}^+$ satisfying $\bar{\phi} > \phi$, there exists $T \geq t_0$ such that

$$\|x(t)\|_2 \leq \left(\frac{\bar{L}}{\underline{L}}\right)^{\frac{1}{q}} N^{\frac{1}{2}}\bar{\phi}\Delta$$

holds for all $t \geq T$, where

$$\phi = \left(\frac{\sum_{i\in\mathcal{N}}\alpha_i(1-\rho_i)}{\min_{i\in\mathcal{N}}\alpha_i(1-\varsigma_i)}\right)^{\frac{1}{p}} \tag{14}$$

$$\varsigma_i = \max\left\{\left(\frac{1+\rho_i}{2}\right)^p,\ \rho_i\right\}. \tag{15}$$

Remark 3.5: Lemma 3.4 suggests that, with a given Δ, the overall system is globally uniformly ultimately bounded.

Suppose equation (2) holds and f_i, g_i are locally Lipschitz for all $i \in \mathcal{N}$. Then we can define a compact set, $\Lambda \subset \mathbb{R}^{nN}$, as

$$\Lambda \triangleq \left\{x \in \mathbb{R}^{nN} \mid V(x) \leq V(x_0)\right\} \tag{16}$$

and then find positive constants, $\theta_i, L_i \in \mathbb{R}^+$ for $i = 1, 2, \cdots, N$ such that

$$\|f_i(x_{\bar{D}_i}, g_i(\hat{x}_{\bar{Z}_i}))\|_2 \leq L_i(\|x\|_2 + \|\hat{x}\|_2), \quad \forall x, \hat{x} \in \Lambda \tag{17}$$

$$\underline{L}\|x\|_2^q \leq V(x) \leq \bar{L}\|x\|_2^q, \quad \forall x \in \Lambda \tag{18}$$

$$\theta_i = 2L_i\left(\frac{V(t_0)}{\underline{L}}\right)^{\frac{1}{q}} \tag{19}$$

with some $q \geq 1$. For the notational convenience, we use $V(t)$ to denote $V(x(t))$ for all $t \geq t_0$.

Lemma 3.6: Consider the N-agent NCS in equation (1). Suppose that equation (2), (17), and (18) hold. Given $\rho_i \in (0, 1)$ for all $i \in \mathcal{N}$ and $\bar{\phi} > \phi$, where ϕ is defined by equation (14), if for any $i \in \mathcal{N}$, equation (12) holds for all $t \in [b_k^i, b_{k+1}^i)$ with some $\rho_i \in (0, 1)$ and the delay in the kth transmission satisfies

$$f_k^i - b_k^i \leq \max\left\{\frac{(1-\rho_i)}{2c_i\theta_i}\|x_i(b_{k-1}^i)\|_2, \frac{(1-\rho_i)\underline{L}^{\frac{1}{q}}}{2c_iL_iN^{\frac{1}{2}}\bar{\phi}\bar{L}^{\frac{1}{q}}}\right\}, \tag{20}$$

then the state $x(t)$ is always in the set Λ, defined in equation (16), for all $t \geq t_0$.

Remark 3.7: Lemma 3.6 shows that if the bound on the delay is small enough, the system dynamic will be in a compact set Λ. This lemma helps us to relax the assumption in (11).

With these lemmas, we now can present the main theorem.

Theorem 3.8: Consider the N-agent NCS in equation (1). Suppose that equation (2), (17), and (18) hold. Given $T \in \mathbb{R}^+$, $\rho_i \in (0,1)$ for all $i \in \mathcal{N}$ and $\bar{\phi} > \phi$, where ϕ is defined by equation (14), if for any $i \in \mathcal{N}$, the $k+1$st broadcast is released by the violation of

$$E_1 \wedge E_2 \tag{21}$$

for some $\rho_i \in (0,1)$, where E_1 is the inequality in equation (12) and

$$E_2 : t \leq b_k^i + T, \tag{22}$$

and the delay in the kth transmission satisfies equation (20), then the NCS is asymptotically stable.

Proof: By Lemma 3.6, we know the state trajectory $x(t) \in \Lambda$ for all $t \geq t_0$. Therefore, by equation (18),

$$\underline{L}\|x(t)\|_2^q \leq V(t) \leq V(t_0), \ \ \forall t \geq t_0 \tag{23}$$

holds.

According to equation (17), we have

$$f_i\left(x_{\bar{D}_i}(t), g_i(\hat{x}_{\bar{Z}_i}(t))\right) \leq 2L_i \left(\frac{V(t_0)}{\underline{L}}\right)^{\frac{1}{q}} = \theta_i \tag{24}$$

for all $t \geq t_0$.

Let $\hat{\phi} = \frac{\phi + \bar{\phi}}{2}$. Since the hypotheses of Lemma 3.4 are satisfied with $\Delta := \Delta_1 = \frac{\theta_i \underline{L}^{\frac{1}{q}}}{2L_i N^{\frac{1}{2}} \bar{\phi} \bar{L}^{\frac{1}{q}}}$, we know that there exists a positive number $t_1 > t_0$, such that

$$\|x(t)\|_2 \leq \left(\frac{\bar{L}}{\underline{L}}\right)^{\frac{1}{q}} N^{\frac{1}{2}} \hat{\phi} \Delta_1 = \frac{\hat{\phi}}{\bar{\phi}} \left(\frac{V(t_0)}{\underline{L}}\right)^{\frac{1}{q}}, \ \ \forall t \geq t_1 \tag{25}$$

By E_2, we know each agent will broadcast after t_1. Let s_1 be the time when each agent in $\mathcal{N}$ broadcasts at least once after t_1. Then

$$\|\hat{x}(t)\|_2 \leq \frac{\hat{\phi}}{\bar{\phi}} \left(\frac{V(t_0)}{\underline{L}}\right)^{\frac{1}{q}}, \ \ \forall t \geq s_1 > t_1 \tag{26}$$

holds. Applying the preceding two equations into equation (17) yields

$$f_i\left(x_{\bar{D}_i}(t), g_i(\hat{x}_{\bar{Z}_i}(t))\right) \leq \frac{\hat{\phi}}{\bar{\phi}} \theta_i, \ \ \forall t \geq s_1. \tag{27}$$

We now set $\Delta := \Delta_2 = \frac{\hat{\phi}}{\bar{\phi}} \Delta_1$ and use the preceding equation to bound the behavior of f_i over $[s_1, \infty)$. Then Lemma 3.4 suggests that there exists $t_2 \geq s_1$ such that

$$\|x(t)\|_2 \leq \left(\frac{\bar{L}}{\underline{L}}\right)^{\frac{1}{q}} N^{\frac{1}{2}} \hat{\phi} \Delta_2 = \left(\frac{\hat{\phi}}{\bar{\phi}}\right)^2 \left(\frac{V(t_0)}{\underline{L}}\right)^{\frac{1}{q}}, \ \ \forall t \geq t_2.$$

Let s_2 be the time when each agent in $\mathcal{N}$ broadcasts at least once after t_2. Then

$$\|\hat{x}(t)\|_2 \leq= \left(\frac{\hat{\phi}}{\bar{\phi}}\right)^2 \left(\frac{V(t_0)}{\underline{L}}\right)^{\frac{1}{q}}, \ \ \forall t \geq s_2.$$

holds. With the preceding equation, we can re-compute the bound for f_i over $[s_2, \infty)$ and re-apply Lemma 3.4 to get new bounds on $\|x(t)\|_2$ and $\|\hat{x}(t)\|_2$, so on and so forth. Then there exists $s_k > t_0$ such that

$$\|x(t)\|_2 \leq \left(\frac{\hat{\phi}}{\bar{\phi}}\right)^k \left(\frac{V(t_0)}{\underline{L}}\right)^{\frac{1}{q}} \tag{28}$$

$$f_i\left(x_{\bar{D}_i}(t), g_i(\hat{x}_{\bar{Z}_i}(t))\right) \leq \left(\frac{\hat{\phi}}{\bar{\phi}}\right)^k \theta_i \tag{29}$$

hold for all $t \geq s_k$.

Since $\frac{\hat{\phi}}{\bar{\phi}} \in (0,1)$, as $k \to \infty$, the preceding equation implies $x(t) \to 0$, which means the NCS is asymptotically stable. ■

Remark 3.9: The introduction of T is the safety requirement of systems. It requires each agent broadcast at least every T unit-time so that some accidents can be detected. T is arbitrarily chosen.

Remark 3.10: It is easy to see that the state-based bounds on the delays defined in equation (20) are always greater than a positive constant,

$$\frac{(1-\rho_i)\underline{L}^{\frac{1}{q}}}{2c_i L_i N^{\frac{1}{2}} \bar{\phi} \bar{L}^{\frac{1}{q}}}. \tag{30}$$

which is known as the *worst-case execution time* (WCET) of agent i.

Remark 3.11: From equation (20), we can see that the farther the state is away from the equilibrium, the larger the bound on the delay can be. The WCET dominates the deadline only when the state is very close to the equilibrium.

Let us now re-visit the distributed scheme described by (5) and (6). In this distributed approach, agent i may require

$$\underline{L}_i \|x_i\|_2^q \leq V_i(x_i) \leq \bar{L}_i \|x_i\|_2^q, \ \ \forall x \in \Lambda. \tag{31}$$

Since the inequality

$$\left(\sum_i \|x_i\|_2^b\right)^{\frac{1}{b}} \leq \left(\sum_i \|x_i\|_2^a\right)^{\frac{1}{a}} \leq N^{\frac{1}{a}-\frac{1}{b}} \left(\sum_i \|x_i\|_2^b\right)^{\frac{1}{b}}$$

holds for any $1 \leq a \leq b < \infty$, we have

$$\begin{aligned} \underline{L}\|x\|_2^q &\leq \min_{i\in\mathcal{N}} \underline{L}_i \sum_{i\in\mathcal{N}} \|x_i\|_2^q \leq V(x) \\ &\leq \max_{i\in\mathcal{N}} \bar{L}_i \sum_{i\in\mathcal{N}} \|x_i\|_2^q \leq \bar{L}\|x\|_2^q \end{aligned}$$

where

$$\underline{L} = \begin{cases} \frac{\min_{i\in\mathcal{N}} \underline{L}_i}{N^{\frac{q}{2}-1}} & q \geq 2 \\ \min_{i\in\mathcal{N}} \underline{L}_i & 1 \leq q \leq 2 \end{cases} \tag{32}$$

$$\bar{L} = \begin{cases} \max_{i\in\mathcal{N}} \bar{L}_i & q \geq 2 \\ N^{1-\frac{q}{2}} \max_{i\in\mathcal{N}} \bar{L}_i & 1 \leq q \leq 2. \end{cases} \tag{33}$$

Therefore, for a distributed consideration, agent i is required to find $\underline{L}_i$ and $\bar{L}_i$ satisfying equation (31) and $\bar{L}$, $\underline{L}$ are defined in equation (33) and (32), respectively.

IV. Simulations

This section presents simulation results demonstrating the distributed event-triggering scheme. The system under study is a collection of carts coupled by softening springs [18] (Figure 2). The ith subsystem state is the vector $x_i = \begin{bmatrix} y_i & \dot{y}_i \end{bmatrix}^T$ where y_i is the ith cart's position. We assume that at the equilibrium of the system, all springs are unstretched.

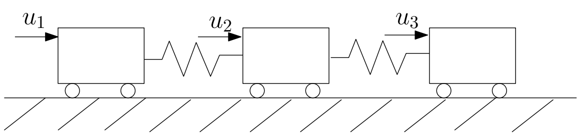

Fig. 2. Three carts coupled by springs

The state equation for the ith cart is equation (1), where

$$\dot{x}_i = \begin{bmatrix} \dot{y}_i \\ u_i + \kappa_i^1 \tanh(y_{i+1} - y_i) + \kappa_i^2 \tanh(y_{i-1} - y_i) \end{bmatrix}.$$

Here $\kappa_1^2 = \kappa_N^1 = 0$. Otherwise, $\kappa_i^1 = \kappa_i^2 = 1$. The control input of agent i is

$$u_i = K_i \hat{x}_i - \kappa_i^1 \tanh(\hat{y}_{i+1} - \hat{y}_i) - \kappa_i^2 \tanh(\hat{y}_{i-1} - \hat{y}_i),$$

where $K_i = \begin{bmatrix} -2 & -2 \end{bmatrix}$ for $i = 1, \cdots, N$.

According to the distributed design scheme in [17], we have

$$c_i = 10.2852, \quad L_i = 8.3630, \quad \underline{L}_i = 2.2214, \quad \bar{L}_i = 10.7523$$

for $i = 2, \cdots, N-1$ and

$$c_i = 5.9638, \quad L_i = 6.6310, \quad \underline{L}_i = 5.2826, \quad \bar{L}_i = 18.4751$$

for $i = \{1, N\}$. With $\rho_i = 0.5$, the triggering events are

$$-0.5\|\hat{x}_i(t)\|_2 + 5.9638\|e_i(t)\|_2 = 0, \text{ for } i = 1, N$$
$$-0.5\|\hat{x}_i(t)\|_2 + 10.2852\|e_i(t)\|_2 = 0, \text{ otherwise.}$$

We set $N = 3$ and ran the event-triggered NCS for 6 seconds. We assume that transmission delay is equal to the predicted deadline (the bound on the delay in (20)). The initial state x_0 was randomly generated satisfying $\|x_0\|_\infty \leq 1$. From the top plot of Figure 3, we can see that the system is asymptotically stable. The successful broadcast periods of agent 1 (cross), agent 2 (diamond), and agent 3 (dot) are shown in the middle plot of Figure 3 that vary in a wide range. It demonstrates the ability of event-triggering in adjusting broadcast periods in response to variations in the system's states. The bottom plot in Figure 3 shows the history of agents' predicted deadlines, which were reduced to fixed constants as the state is very close to the equilibrium . This is because as the states get small, the WCET dominates the deadlines as shown in equation (20). The detailed data of this simulation is listed in table I.

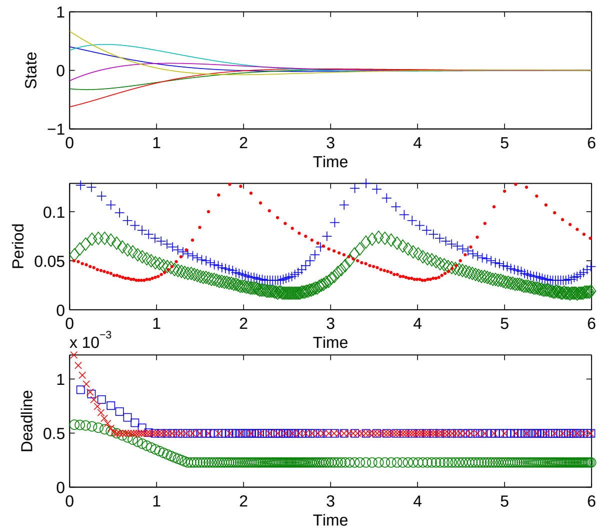

Fig. 3. State trajectory, broadcast periods, and predicted deadlines in an event-triggered NCS

TABLE I
Results on Running a Distributed Event-Triggered NCS

	Agent 1	Agent 2	Agent 3
Number of Broadcasts	111	189	107
Average Broadcast Period	0.0541	0.0317	0.0561
Maximal Predicted Deadline	0.0009	0.0006	0.0012
WCET	0.0005	0.0002	0.0005

We then considered the relation between the WCET and the average broadcast period as the parameter ρ_i changes. In this simulation, we assume that the delays in agents are equal to their WCETs. ρ_1 varied from 0.001 to 0.99. $\rho_2 = \rho_3$ are set to be 0.5. For each ρ_1, the system ran for 6 seconds. Figure 4 shows the simulation results. As ρ_1 increases, the average period increases and the WCET decreases. This simulation, therefore, suggests a tradeoff between broadcast periods and transmission deadlines.

V. Conclusions

This paper examines event-triggered broadcasting of state information in distributed networked control systems with transmission delays. State-based bounds on transmission delays for asymptotic stability of the overall system are provided. These bounds are always greater than a positive constant and can be derived in a distributed manner.

Appendix

Proof: [Proof of Lemma 3.4]

Consider the derivative of $\left\|x_i(t) - x_i(b_k^i)\right\|_2$ over the time interval $[b_k^i, f_k^i)$.

$$\begin{aligned} \frac{d}{dt}\left\|x_i(t) - x_i(b_k^i)\right\|_2 &\leq \|\dot{x}_i(t)\|_2 \\ &= \|f_i\left(x_{\bar{D}_i}, g_i\left(\hat{x}_{\bar{Z}_i}\right)\right)\|_2 \leq \theta_i \end{aligned}$$

holds for all $t \in [b_k^i, f_k^i)$.

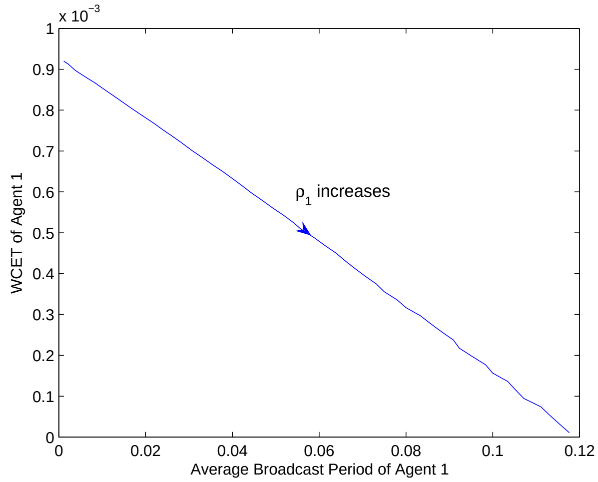

Fig. 4. Agent 1's average broadcast period versus the WCET when ρ_1 varies from 0.001 to 0.99

Solving the preceding inequality with the initial condition $\left\|x_i(t) - x_i(b_k^i)\right\|_2 |_{t=b_k^i} = 0$ implies

$$\begin{aligned} & \|x_i(t) - x_i(b_k^i)\|_2 \leq \theta_i(t - b_k^i) \\ \leq \quad & \frac{1-\rho_i}{c_i} \max\left\{ \frac{\|x_i(b_{k-1}^i)\|_2}{2}, \Delta \right\} \end{aligned} \quad (34)$$

holds for all $t \in [b_k^i, f_k^i)$, where the second inequality is obtained by applying equation (13).

We know

$$\|x_i(t) - x_i(b_{k-1}^i)\|_2 \leq \frac{\rho_i}{c_i} \|x_i(b_{k-1}^i)\|_2. \quad (35)$$

for all $t \in [b_{k-1}^i, b_k^i)$.

Combining equation (34) and (35) implies

$$\begin{aligned} & \|e_i(t)\|_2 = \|x_i(t) - x_i(b_{k-1}^i)\|_2 \\ \leq \quad & \|x_i(t) - x_i(b_k^i)\|_2 + \|x_i(b_k^i) - x_i(b_{k-1}^i)\|_2 \\ \leq \quad & \frac{1-\rho_i}{c_i} \max\left\{ \frac{\|x_i(b_{k-1}^i)\|_2}{2}, \Delta \right\} + \frac{\rho_i}{c_i} \|x_i(b_{k-1}^i)\|_2 \\ \leq \quad & \max\left\{ \tfrac{(1+\rho_i)\|x_i(b_{k-1}^i)\|_2}{2c_i}, \tfrac{(1-\rho_i)\Delta + \rho_i \|x_i(b_{k-1}^i)\|_2}{c_i} \right\} \end{aligned}$$

holds for $t \in [f_{k-1}^i, f_k^i)$.

The preceding equation then suggests that

$$\begin{aligned} & \left(\frac{\beta_i}{\alpha_i}\right)^{\frac{1}{p}} \|e_i(t)\|_2 = (c_i - 1)\|x_i(t) - x_i(b_{k-1}^i)\|_2 \\ \leq \quad & \max\left\{ \tfrac{(1+\rho_i)\|x_i(b_{k-1}^i)\|_2}{2}, (1-\rho_i)\Delta + \rho_i \|x_i(b_{k-1}^i)\|_2 \right\} \\ & - \|x_i(t) - x_i(b_{k-1}^i)\|_2 \\ \leq \quad & \max\left\{ \frac{1+\rho_i}{2} \|x_i(t)\|_2, (1-\rho_i)\Delta + \rho_i \|x_i(t)\|_2 \right\} \end{aligned}$$

holds for $t \in [f_{k-1}^i, f_k^i)$. Therefore,

$$\begin{aligned} & \frac{\beta_i}{\alpha_i} \|e_i(t)\|_2^p \\ \leq \quad & \max\left\{ \left(\tfrac{1+\rho_i}{2}\right)^p \|x_i(t)\|_2^p, ((1-\rho_i)\Delta + \rho_i \|x_i(t)\|_2)^p \right\} \\ \leq \quad & \max\left\{ \left(\tfrac{1+\rho_i}{2}\right)^p \|x_i(t)\|_2^p, (1-\rho_i)\Delta^p + \rho_i \|x_i(t)\|_2^p \right\} \end{aligned} \quad (36)$$

We now consider $\dot{V}$ for any $t \geq 0$. Equation (2) implies that

$$\dot{V} \leq \sum_{i \in \mathcal{N}} -\alpha_i \|x_i(t)\|_2^p + \beta_i \|e_i(t)\|_2^p$$

Applying equation (36) into the preceding equation yields

$$\begin{aligned} & \dot{V} \leq \sum_{i \in \mathcal{N}} \alpha_i (-\|x_i(t)\|_2^p + \\ & \max\left\{ \left(\tfrac{1+\rho_i}{2}\right)^p \|x_i(t)\|_2^p,\ (1-\rho_i)\Delta^p + \rho_i \|x_i(t)\|_2^p \right\}). \end{aligned}$$

Let $\Omega_t = \{i \in \mathcal{N} \mid \left(\frac{1+\rho_i}{2}\right)^p \|x_i(t)\|_2^p > (1-\rho_i)\Delta^p + \rho_i \|x_i(t)\|_2^p\}$. Therefore, the preceding equation is equivalent to

$$\begin{aligned} \dot{V} \quad \leq \quad & \sum_{i \in \Omega_t} \alpha_i \left(\left(\frac{1+\rho_i}{2} \right)^p - 1 \right) \|x_i(t)\|_2^p \\ & + \sum_{i \in \mathcal{N} \setminus \Omega_t} (1-\rho_i)\alpha_i \Delta^p + \sum_{i \in \mathcal{N} \setminus \Omega_t} \alpha_i (\rho_i - 1) \|x_i(t)\|_2^p \end{aligned}$$

According to equation (15), the preceding equation implies

$$\begin{aligned} & \dot{V} \leq \sum_{i \in \mathcal{N} \setminus \Omega_t} \alpha_i (1-\rho_i)\Delta^p + \sum_{i \in \mathcal{N}} \alpha_i (\varsigma_i - 1) \|x_i(t)\|_2^p \\ \leq \quad & \sum_{i \in \mathcal{N}} \alpha_i (1-\rho_i)\Delta^p - \sum_{i \in \mathcal{N}} \alpha_i (1-\varsigma_i) \|x_i(t)\|_2^p \quad (37) \\ \leq \quad & \sum_{i \in \mathcal{N}} \alpha_i (1-\rho_i)\Delta^p - \min_{i \in \mathcal{N}} \alpha_i (1-\varsigma_i) \sum_{i \in \mathcal{N}} \|x_i(t)\|_2^p \\ = \quad & \min_{i \in \mathcal{N}} \alpha_i (1-\varsigma_i) \left(\phi^p \Delta^p - \sum_{i \in \mathcal{N}} \|x_i(t)\|_2^p \right) \end{aligned}$$

where ϕ is defined in equation (14). This inequality means that if there exists $i \in \mathcal{N}$ such that $\|x_i(t)\|_2 \geq \Delta\bar{\phi}$ for any $\bar{\phi} > \phi$, then

$$\dot{V} \leq \min_{i \in \mathcal{N}} \alpha_i (1-\varsigma_i)(\phi^p - \bar{\phi}^p)\Delta^p < 0$$

holds. This implies that the preceding inequality holds when

$$\|x(t)\|_2^q = \left(\sum_{i \in \mathcal{N}} \|x_i(t)\|_2^2 \right)^{\frac{q}{2}} \geq N^{\frac{q}{2}} \Delta^q \bar{\phi}^q$$

since the inequality above implies $\|x_i(t)\|_2 \geq \Delta\bar{\phi}$.

Combining this with equation (10) is sufficient to show that there exists $T \geq t_0$ such that

$$\|x(t)\|_2^q \leq \frac{\bar{L}}{\underline{L}} N^{\frac{q}{2}} \bar{\phi}^q \Delta^q$$

holds for all $t \geq T$, as shown in [19]. ■

Proof: [Proof of Lemma 3.6]

Consider the set

$$\Gamma = \left\{x \in \Lambda \mid \bar{L}\|x\|_2^q \leq V(x_0)\right\}. \qquad (38)$$

It is easy to see that equation (18) implies $\Gamma \subseteq \Lambda$ and $\frac{\bar{L}}{\underline{L}} \geq 1$.

We now show that $V(t) \leq V(t_0)$ holds for all $t > t_0$. We prove it by contradiction. Suppose that there is time instant $\hat{t} > t_0$ such that $V(\hat{t}) > V(t_0)$.

Notice that before the first time the inequality in equation (12) is violated, the inequality

$$\dot{V} \leq \sum_{i\in\mathcal{N}} -(1-\rho_i^p)\alpha_i\|x_i\|_2^p$$

holds. Therefore, there must exist time instant $\bar{t} > t_0$ such that $V(t) < V(t_0)$ for all $t \in (t_0, \bar{t}]$. Since $V(t)$ is continuous and $V(\hat{t}) > V(t_0)$, we know there must exist at least one time interval $(s-\epsilon_1, s+\epsilon_1) \subset (\bar{t}, \hat{t})$ such that

$$V(s) = V(t_0) \qquad (39)$$

$$\dot{V}(t) \geq 0, \quad \forall t \in (s-\epsilon, s). \qquad (40)$$

Assume that s is the first time in $(t_0, \hat{t})$ satisfying equation (39), (40) with a parameter ϵ. Then we have

$$t_0 < \bar{t} < s < \hat{t} \qquad (41)$$

$$V(t) \leq V(t_0), \quad \forall t \in [t_0, s). \qquad (42)$$

Equation (42) implies

$$x(t) \in \Lambda \quad \text{and} \quad \|x(t)\|_2^q \leq \frac{V(t_0)}{\underline{L}} \qquad (43)$$

for all $t \in [t_0, s)$ according to equation (17). Combining this inequality with equation (17), we have

$$f_i\left(x_{\bar{D}_i}(t), g_i(x_{\bar{Z}_i}(t))\right) \leq 2L_i\left(\frac{V(t_0)}{\underline{L}}\right)^{\frac{1}{q}} = \theta_i$$

for all $t \in [t_0, s)$. Also equation (20) means

$$f_k^i - b_k^i \leq \max\left\{\frac{(1-\rho_i)}{2c_i\theta_i}\|x_i(b_{k-1}^i)\|_2, \frac{(1-\rho_i)}{c_i\theta_i}\Delta\right\}. \qquad (44)$$

with

$$\Delta = \frac{\theta_i \underline{L}^{\frac{1}{q}}}{2L_i N^{\frac{1}{2}}\bar{\phi}\bar{L}^{\frac{1}{q}}}. \qquad (45)$$

Therefore, following the same reasoning in Lemma 3.4, we have

$$\dot{V} \leq \min_i \alpha_i(1-\varsigma_i)\}\left[\Delta^p\phi^p - \sum_{i\in\mathcal{N}}\|x_i(t)\|_2^p\right] \qquad (46)$$

for all $t \in [t_0, s)$, where ς_i is defined in equation (15).

Since $\dot{V}(t) \geq 0$ for all $t \in (s-\epsilon, s)$ according to equation (40), we know by equation (46) that

$$\Delta^p\phi^p \geq \sum_{i\in\mathcal{N}}\|x_i(t)\|_2^p \geq \|x_i(t)\|_2^p, \quad \forall t \in (s-\epsilon, s), \qquad (47)$$

holds, which implies

$$\begin{aligned} N^{\frac{q}{2}}\Delta^q\phi^q &= \left(N\Delta^2\phi^2\right)^{\frac{q}{2}} \geq \left(\sum_{i\in\mathcal{N}}\|x_i(t)\|_2^2\right)^{\frac{q}{2}} \\ &= \|x(t)\|_2^q, \quad \forall t \in (s-\epsilon, s), \end{aligned} \qquad (48)$$

and therefore, implementing equation (45) into the preceding equation implies

$$N^{\frac{q}{2}}\Delta^q\phi^q = \frac{V(t_0)\phi^q}{\bar{\phi}^q\bar{L}} \geq \|x(t)\|_2^q, \quad \forall t \in (s-\epsilon, s). \qquad (49)$$

Since $x(t)$ is continuous, equation (49) implies

$$\frac{V(t_0)\phi^q}{\bar{\phi}^q\bar{L}} \geq \lim_{t\to s}\|x(t)\|_2^q = \|x(s)\|_2^q.$$

Because $\bar{\phi} > \phi$, we have

$$\frac{V(t_0)}{\bar{L}} > \frac{V(t_0)\phi^q}{\bar{\phi}^q\bar{L}_i} \geq \|x(s)\|_2^q,$$

which implies that

$$V(t_0) > \bar{L}\|x(s)\|_2^q \geq V(s), \qquad (50)$$

which contradicts equation (39). Therefore, we conclude $V(t) \leq V(t_0)$ holds for all $t \geq t_0$. ■

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