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# Weakly Coupled Transmissions in Event Triggered Output Feedback Systems

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**Abstract** This paper examines event triggered output feedback control where there are separate links between the sensor-to-controller and controller-to-actuator. The proposed triggering events only rely on local information so that the transmissions from the sensor and controller subsystems are not necessarily synchronized. This represents an advance over recent work in event-triggered output feedback control where transmission from the controller subsystem was tightly coupled to the receipt of event-triggered sensor data. The paper presents an upper bound on the optimal cost attained by the closed-loop system. Simulation results demonstrate that transmissions between sensors and controller subsystems are not tightly synchronized. These results are also consistent with derived upper bounds on overall system cost.

**Keywords** Weakly coupled transmissions · Event triggering · Output feedback control

## 1 Introduction

Event triggering can be seen as a communication protocol where information is transmitted only if some event occurs. In particular, information is transmitted when a measure of data 'novelty' exceeds a specified threshold. In contrast to more commonly used periodic transmission schemes, event-triggering tends to generate traffic patterns that are *sporadic* in nature. Prior experimental results have demonstrated that event-triggering can use fewer communication resources than periodic transmission schemes with comparable performance levels [10, 3, 9, 2, 7]. The reason for this more efficient use of communication

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resources is that event-triggering makes use of on-line information in making transmission decisions. This method, therefore, can adapt its usage of the communication channel to the importance of the data it must transmit.

Most prior work in the event triggering literature discusses state feedback control and state estimation. This work has traditionally assumed a single feedback link in the system. It has only been very recently that researchers have turned to study event-triggered output feedback control where there are separate communication channels from sensor-to-controller and controller-to-actuator. If we design triggering events for both communication channels, an interesting question to ask is how these two triggering events are coupled with each other.

Some of the work in event triggered output feedback systems hid this question by assuming that only part of the control loop was closed over communication channel, i.e. either sensor-to-controller link or controller-to-actuator link is connected directly [4,11,8]. Another work [1] assumed very strong coupling between the triggering rules of sensor-to-controller and controller-to-actuator links. They required that the transmission in one link triggered the transmission in the other link, so the transmissions of the two communication channels are synchronized.

This synchronization is not necessary. This paper proposes weakly coupled event-triggers. We attempt to find the optimal event-triggers which minimize the mean square cost of the system state discounted by the communication cost in both links. The optimal event-trigger in the sensor-to-controller link which minimizes the mean square state estimation error discounted by the communication cost in sensor-to-controller link is given first. With this optimal event trigger in the sensor-to-controller link, we then derive the optimal event-trigger in the controller-to-actuator link which minimizes the mean square state estimate discounted by the communication cost in the controller-to-actuator link. It turns out that the optimal cost of the output feedback control system is bounded from above by the sum of the optimal costs of the state estimation problem and the state feedback problem. Because the optimal event-triggers are very difficult to calculate, quadratic event triggers and upper bounds on the costs are derived.

This paper is an extended version of our paper [5,6], and is organized as the following. Section 2 and 3 introduce mathematical preliminaries and problem setup, respectively. The main results are in section 4, 5 and 6. Section 7 gives the simulation results to demonstrate our main theorems in section 4, 5 and 6. Conclusion can be found in section 8.

## 2 Preliminaries

Let $\mathbb{R}^n$ be the $n$ dimensional real space, and $\mathbb{Z}^+$ indicate the set including all non-negative integers.

A sequence $\{\tau^l\}_{l=0}^{\infty}$ is *forward progressing* if for any $k \geq 0$, there always exists an $l \in \mathbb{Z}^+$ such that $\tau^l \geq k$.

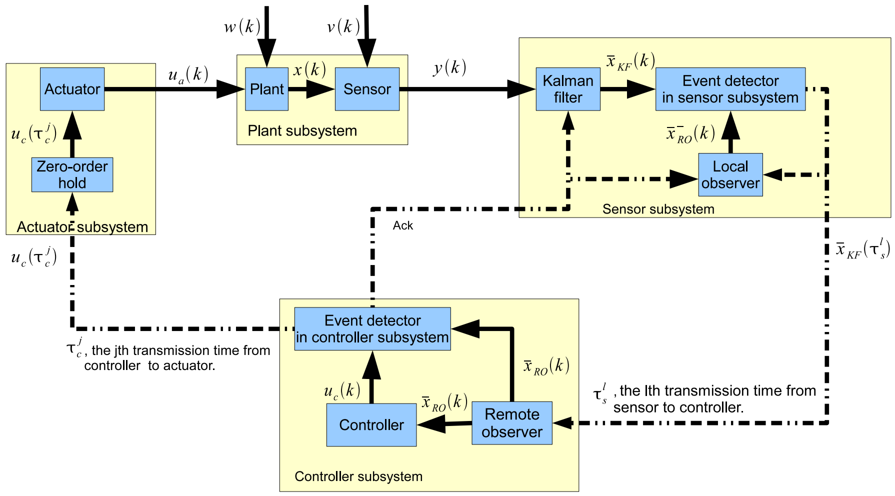


**Fig. 1** Structure of the event triggered output feedback control systems

## 3 Problem Statement

A block diagram of the closed loop system is shown in Figure 1. This closed loop system consists of four components: a *plant subsystem*, a *sensor subsystem*, a *controller subsystem* and an *actuator subsystem*.

The plant subsystem consists of two parts: a plant and a sensor, satisfying the following difference equation

$$
\begin{aligned}
x(k) &= Ax(k-1) + Bu_a(k-1) + w(k-1),\\
y(k) &= Cx(k) + v(k),
\end{aligned}
$$

for $k = 1, 2, \cdots$, where $x : \mathbb{Z}^+ \to \mathbb{R}^n$ is the system state, the initial state $x(0)$ is a Gaussian random variable with mean $\mu_0$ and variance $\Pi_0$, $y : \mathbb{Z}^+ \to \mathbb{R}^p$ is a corrupted output with additive white Gaussian noise. Matrix $A$, $B$ and $C$ are in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{p\times n}$, respectively. Besides, $(A, B, C)$ is controllable and observable. $u_a(k-1) \in \mathbb{R}^m$ is the actual control input applied to the plant which will be further explained when the actuator subsystem is introduced. $w$ and $v$ are independent zero mean white Gaussian noise processes with variance $W$ and $V$, taking values in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. The corrupted measurement $y$, then, is fed into the sensor subsystem which decides when to transmit information to the controller subsystem.

The sensor subsystem consists of a *Kalman filter*, a *local observer* and an *event detector in the sensor subsystem*. The *Kalman filter* generates a filtered state $\overline{x}_{KF} : \mathbb{Z}^+ \to \mathbb{R}^n$ that minimizes the weighted mean square estimation error (MSEE), i.e.

$$
\overline{x}_{KF}(k) = \arg\min_{\overline{x}_{KF}(k)} \mathbb{E}\left[\|x(k) - \overline{x}_{KF}(k)\|_Z^2 \mid \{y(0), y(1), \cdots, y(k)\}\right]
$$

where $Z \geq 0$ is a symmetric positive semi-definite weighting matrix, and $\|\theta\|_Z^2 = \theta^T Z \theta$. For the process under study the filter equation is

$$\overline{x}_{KF}(k) = A\overline{x}_{KF}(k-1) + Bu_a(k-1) + L\left[y(k) - C(A\overline{x}_{KF}(k-1) + Bu_a(k-1))\right],$$

where $L = AXC^T(CXC^T+V)^{-1}$, and $X$ satisfies the discrete linear Riccati equation

$$AXA^T - X - AXC^T(CXC^T+V)^{-1}CXA^T + W = 0.$$

Let $Z = P_z^T P_z$. The steady state estimation error $\overline{e}_{KF}(k) = x(k) - \overline{x}_{KF}(k)$ is a Gaussian random variable with zero mean and variance

$$\mathbb{E}(\overline{e}_{KF}\overline{e}_{KF}^T) = Q = (I - LC)X.$$

Let $\{\tau_s^l\}_{l=0}^{\infty}$ denote a sequence of increasing and forward progressing times when information is transmitted from the sensor to the controller subsystem. Let $\overline{\mathcal{X}}(k) = \left\{\overline{x}_{KF}(\tau_s^1), \overline{x}_{KF}(\tau_s^2), \cdots, \overline{x}_{KF}(\tau_s^{l(k)})\right\}$ denote the filtered state estimates that are transmitted to the controller subsystem by step $k$ where $l(k) = \max\left\{l : \tau_s^l \leq k\right\}$. The *local observer* first generates an *a priori* estimate of state $\overline{x}_{RO}^- : \mathbb{Z}^+ \to \mathbb{R}^n$ that minimizes the weighted MSEE based on the information received up to step $k-1$ (i.e. $\mathbb{E}\left[\|x(k) - \overline{x}_{RO}^-(k)\|_Z^2 \,|\, \overline{\mathcal{X}}(k-1)\right]$), and then generates an *a posteriori* estimate $\overline{x}_{RO} : \mathbb{Z}^+ \to \mathbb{R}^n$ that minimizes the weighted MSEE based on the information received up to step $k$ (i.e. $\mathbb{E}\left[\|x(k) - \overline{x}_{RO}(k)\|_Z^2 \| \overline{\mathcal{X}}(k)\right]$). These estimates take the following form

$$\overline{x}_{RO}^-(k) = A\overline{x}_{RO}(k-1) + Bu_a(k) \tag{1}$$

$$\overline{x}_{RO}(k) = \begin{cases} \overline{x}_{RO}^-(k), \text{ if } e_{KF,RO}^-(k) \in S_s; \\ \overline{x}_{KF}(k), \text{ otherwise }, \end{cases} \tag{2}$$

where $e_{KF,RO}^-(k) = \overline{x}_{KF}(k) - \overline{x}_{RO}^-(k)$ is the gap between the filtered state estimate and the *a priori* remote state estimate, $S_s \subseteq \mathbb{R}^n$, the triggering set in the sensor subsystem, is a compact set including the origin. $\overline{x}_{RO}^-(0) = \mu_0$.

The *event detector in the sensor subsystem* monitors the *a priori* gap $e_{KF,RO}^-(k)$ and compares the gap with the triggering set $S_s$. If the gap is inside the triggering set $S_s$, then no data is transmitted. Otherwise, the filtered state $\overline{x}_{KF}(k)$ is sent to the controller subsystem.

The controller subsystem has three components: a *remote observer*, a *controller* and an *event detector in the controller subsystem*. The remote observer has the same behavior as the local observer. The *a posteriori* state estimate $\overline{x}_{RO}(k)$ is fed into the controller. The controller generates a control input

$$u_c(k) = K\overline{x}_{RO}(k),$$

where $K$ is the controller gain.

Let's define an increasing and forward progressing time sequence $\{\tau_c^j\}_{j=1}^{\infty}$, where $\tau_c^j$ is the $j$th time when the control input is sent to the actuator subsystem from the controller subsystem. The *event detector in the controller* *subsystem* transmits the current control input $u_c(k)$ to the actuator subsystem when $[\overline{x}_{RO}(k)\ u_a(k)]^T$ lies outside of a compact set $S_c$ which includes the origin. Therefore, $S_c$ is called the triggering set in the controller subsystem. Once the current control input is sent to the actuator, an acknowledgement is transmitted to the sensor subsystem to let it know that the control input has been updated. When the sensor subsystem receives the acknowledgement, it uses $\overline{x}_{RO}(k)$ generated by the local observer to obtain the new control input to the actuator subsystem.

The actuator subsystem has two parts: a *zero order hold* and an *actuator*. Let $u_a(k)$ denote the actual control input applied to the plant. When $u_c(\tau_c^j)$ is transmitted, the actuator subsystem updates $u_a(k)$ to be $u_c(\tau_c^j)$, and holds this value until the next transmission occurs. $u_a(k)$, therefore, takes the form of

$$u_a(k) = u_c(\tau_c^j), \forall k \in [\tau_c^j, \tau_c^{j+1}).$$

The average cost is defined as

$$J(S_s, S_c) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}\left(c(x(k), S_s, S_c)\right),$$

where the cost function

$$c(x(k), S_s, S_c) = \|x(k)\|_Z^2 + \lambda_s 1(e^-_{KF,RO}(k) \notin S_s) + \lambda_c 1\left(\begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \notin S_c\right),$$

$\lambda_s$ and $\lambda_c$ are the communication prices for transmissions over the sensor-to-controller link and controller-to-actuator link, respectively. Let $\omega$ be a statement, and $\Omega$ be a set of statements. $1(\cdot) : \Omega \to \{0, 1\}$ is the characteristic function defined as

$$1(\omega) = \begin{cases} 1, \text{if statement } \omega \text{ is true;} \\ 0, \text{otherwise.} \end{cases}$$

Our objective is to design the triggering sets $S_s$ and $S_c$ which minimize the average cost $J(S_s, S_c)$, i.e.

$$J^* = \min_{S_s, S_c} J(S_s, S_c).$$

## 4 State estimation cost and control cost

With the problem setup given in the last section, this section shows that the average cost $J$ can be expressed as the sum of the state estimation cost and the control cost.

Let $\overline{e}_{RO}(k) = x(k) - \overline{x}_{RO}(k)$ be the remote state estimation error. The key point is that the remote state estimation error $\overline{e}_{RO}(k)$ is uncorrelated from the remote state estimate $\overline{x}_{RO}(k)$. This property allows us to express the average cost as the sum of the state estimation cost and the control cost.

**Lemma 1** *$\overline{x}_{RO}(k)$ and $\overline{e}_{RO}(k)$ are uncorrelated with each other.*

*Proof* From the dynamics of the closed system, we can derive that

$$\overline{e}_{RO}^{-}(k) = A\overline{e}_{RO} + w(k-1)$$
$$\overline{e}_{RO}(k) = \begin{cases} \overline{e}_{RO}^{-}(k), \ e_{KF,RO}^{-} \in S_s; \\ \overline{e}_{KF}(k), \text{otherwise.} \end{cases}$$

Let $\tau_s^{l(k)}$ be the last sampling time from sensor-to-controller no later than step $k$. From the equations above, we can see that $\overline{e}_{RO}(k)$ is a linear combination of $\overline{e}_{KF}(\tau_s^{l(k)})$, $w(\tau_s^{l(k)})$, $w(\tau_s^{l(k)}+1)$, $\cdots$, $w(k)$, i.e.

$$\overline{e}_{RO}(k) = \alpha_0 \overline{e}_{KF}(\tau_s^{l(k)}) + \sum_{j=\tau_s^{l(k)}+1}^{k} \alpha_j w(j),$$

where $\alpha_j$ are some matrices with proper dimensions.

From equation (1) and (2), we can see that $\overline{x}_{RO}(k)$ is a linear combination of $\overline{x}_{KF}(\tau_s^{\ell(k)})$, $\overline{x}_{KF}(\tau_s^{\ell(k)-1})$, $\cdots$, $\overline{x}_{KF}(\tau_s^{1})$, i.e.

$$\overline{x}_{RO}(k) = \sum_{j=1}^{l(k)} \beta_j \overline{x}_{KF}(\tau_s^{j}), \tag{3}$$

where $\beta_j$ are some matrices with proper dimensions.

Since $\overline{e}_{KF}(\tau_s^{l(k)})$, $w(\tau_s^{l(k)}+1)$, $\cdots$, $w(k)$ is uncorrelated with $\overline{x}_{KF}(\tau_s^{l'})$ for any $l' \leq l(k)$, we can conclude that $\overline{x}_{RO}(k)$ and $\overline{e}_{RO}(k)$ are uncorrelated with each other. □

Note that if the system is nonlinear or the noise processes are not Gaussian, then the remote state estimation error and the estimate may be correlated.

Based on the uncorrelation between the remote state estimation error and the remote state estimate, it is easy to show that the average cost is the sum of state estimation cost and the control cost.

**Theorem 1** *The average cost*

$$J(S_s, S_c) = J_s(S_s, \{\overline{e}_{RO}(k)\}_{k=0}^{\infty}) + J_c(S_c, \{\overline{x}_{RO}(k)\}_{k=0}^{\infty}),$$

*where*

$$J_s(S_s, \{\overline{e}_{RO}(k)\}_{k=0}^{\infty}) = \lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{E}\left[ \|\overline{e}_{RO}(k)\|_Z^2 + \lambda_s 1(e_{KF,RO}^{-}(k) \notin S_s) \right],$$
$$J_c(S_c, \{\overline{x}_{RO}(k)\}_{k=0}^{\infty}) = \lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{E}\left[ \|\overline{x}_{RO}(k)\|_Z^2 + \lambda_c 1\left( \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k) \end{bmatrix} \notin S_c \right) \right].$$

*Proof* According to Lemma 1, the average cost $J(S_s, S_c)$ is rewritten as

$$\begin{aligned}J(S_s, S_c) = &\lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{E}\left( \|\overline{e}_{RO}(k)\|_Z^2 + \lambda_s 1(e^-_{KF,RO}(k)) \right. \\ &\left. + \|\overline{x}_{RO}(k)\|_Z^2 + \lambda_c 1\left( \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \notin S_c \right) \right) \\ =&J_s(S_s, \{\overline{e}_{RO}(k)\}_{k=0}^{\infty}) + J_c(S_c, \{\overline{x}_{RO}(k)\}_{k=0}^{\infty}).\end{aligned}$$

□

$J_s(S_s, \{\overline{e}_{RO}(k)\}_{k=0}^{\infty})$ relies on the remote state estimation error and the communication price between sensor and controller, and hence is called the *state estimation cost*. $J_c(S_c, \{\overline{x}_{RO}(k)\}_{k=0}^{\infty})$ relies on the remote state estimate and the communication price between controller and actuator, and hence is called the *control cost*.

*Remark 1* Both the state estimation cost and the control cost depend on the triggering set $S_s$ in the sensor subsystem. It is easy to see that the state estimation cost $J_s$ relies on $S_s$. The control cost $J_c$ also relies on $S_s$, because $J_c$ must be computed with respect to the probability distribution of the remote state estimate, $\overline{x}_{RO}(k)$. From equation (2), we can see that the distribution of $\overline{x}_{RO}(k)$ is a function of $S_s$, the triggering set in the sensor subsystem. Therefore, the control cost $J_c(S_c, \{\overline{x}_{RO}(k)\}_{k=0}^{\infty})$ relies on $S_s$, and hence is coupled with $J_s(S_s, \{\overline{e}_{RO}(k)\}_{k=0}^{\infty})$. To emphasize the dependence of the state estimation cost and control cost on the triggering set $S_s$ in the sensor subsystem , we rewrite the state estimation cost and the control cost as

$$\begin{aligned}J_s(S_s) &= J_s(S_s, \{\overline{e}_{RO}(k)\}_{k=0}^{\infty}), \\ J_c(S_c, S_s) &= J_c(S_c, \{\overline{x}_{RO}(k)\}_{k=0}^{\infty}),\end{aligned}$$

respectively.

Let $S_s^\dagger$ be the optimal sensor triggering set that minimizes the state estimation cost $J_s$, and the corresponding optimal state estimation cost is $J_s^\dagger$. Let $S_c^\dagger$ be the controller's event-triggering strategy that minimizes the controller cost $J_c$ assuming the sensor uses the event-trigger $S_s^\dagger$, and the corresponding controller's cost becomes $J_c^\dagger(S_s^\dagger)$. Since $J_s$ and $J_c$ are coupled, we can see that the minimum cost $J^*$ is bounded above by

$$J^* \leq J(S_s^\dagger, S_c^\dagger) = J_s^\dagger + J_c^\dagger(S_s^\dagger). \tag{4}$$

Note that $S_s^\dagger$ and $S_c^\dagger$ may not be the optimal triggering sets that minimize the overall cost $J$. To emphasize this fact, we call $S_s^\dagger$ and $S_c^\dagger$ the *suboptimal* triggering set in the sensor subsystem and the controller subsystem, respectively. The next two sections present methods to derive the suboptimal and quadratic triggering sets in the sensor subsystem and the controller subsystem, respectively.

## 5 The suboptimal and quadratic triggering sets in the sensor subsystem

This section first provides the suboptimal triggering set $S_s^\dagger$. Determining $S_s^\dagger$ has high complexity both in terms of computation and space (memory). We therefore present a Quadratic triggering set, $S_s$, that is an approximation of the sub-optimal trigger, $S_s^\dagger$. The cost obtained by this quadratic trigger is an upper bound on the optimal state estimation cost, $J_s^\dagger$. This quadratic triggering set is more tractable in computation and space.

Before talking about the suboptimal triggering set in sensor subsystem, let us first analyze the remote state estimation error $\overline{e}_{RO}$. Let $e_{KF,RO}(k) = \overline{x}_{KF}(k) - \overline{x}_{RO}$ be the *a posteriori* gap between the filtered state estimate and the remote state estimate. We notice that

$$\overline{e}_{RO}(k) = \overline{e}_{KF}(k) + e_{KF,RO}(k), \tag{5}$$

and the filtered state error $\overline{e}_{KF}(k)$ is uncorrelated with the gap between the filtered state estimate and the remote state estimate $e_{KF,RO}(k)$. This property is stated and proved in the following lemma.

**Lemma 2** *The filtered state error, $\overline{e}_{KF}(k)$, and the gap between filtered state and the remote state estimate, $e_{KF,RO}(k)$, are uncorrelated.*

*Proof* Let $\tau_s^{l(k)}$ be the last sampling time from sensor-to-controller no later than step $k$. From equation (3), we have

$$e_{KF,RO}(k) = \overline{x}_{KF}(k) - \sum_{j=1}^{l(k)} \beta_j \overline{x}_{KF}(\tau_s^j), \tag{6}$$

where $\beta_j$ are some matrices with proper dimensions.

Since $\overline{e}_{KF}(k)$ is uncorrelated with $\overline{x}_{KF}(k)$ and $\overline{x}_{KF}(\tau_s^{l'})$ for any $l' \leq l(k)$, from equation (6), we conclude that $\overline{e}_{KF}(k)$ is uncorrelated with $e_{KF,RO}(k)$. □

From equation (5) and Lemma 2, the state estimation cost $J_s(S_s)$ takes another form which is stated in the next lemma.

**Lemma 3**

$$J_s(S_s) = tr(QZ) + \hat{J}_s(S_s), \tag{7}$$

*where*

$$\hat{J}_s(S_s) = \lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{E}\left[c_s(e^-_{KF,RO}(k), S_s)\right],$$

*and*

$$\begin{aligned} c_s(e^-_{KF,RO}(k), S_s) &= \|e_{KF,RO}(k)\|_Z^2 + \lambda_s 1(e^-_{KF,RO}(k) \notin S_s) \\ &= \|e^-_{KF,RO}(k)\|_Z^2 1(e^-_{KF,RO}(k) \in S_s) + \lambda_s 1(e^-_{KF,RO}(k) \notin S_s). \end{aligned}$$

It is easy to see that the suboptimal triggering set $S_s^\dagger$ that minimizes $\hat{J}_s(S_s)$ also minimizes the state estimation cost $J_s(S_s)$.

Now, we are ready to analyze the suboptimal triggering set in sensor subsystem $S_s^\dagger$.

**Theorem 2** *If there exists a piece-wise continuous and bounded function $h_s : \mathbb{R}^n \to \mathbb{R}$ and a finite number $J_s'$ such that*

$$J_s' + h_s(e^-_{KF,RO}(k)) = G_s\left(e^-_{KF,RO}(k)\right) \tag{8}$$

*where*

$$G_s(\theta) = \min_{S_s}\left\{E(h_s(e^-_{KF,RO}(k+1))|e^-_{KF,RO}(k)=\theta) + c_s(\theta, S_s)\right\},$$

*then the optimal average cost of remote state estimation is*

$$J_s^\dagger = J_s' + tr(QZ), \tag{9}$$

*and the suboptimal triggering set in sensor subsystem*

$$\begin{aligned} S_s^\dagger = \Big\{\theta : &\mathbb{E}(h_s(e^-_{KF,RO}(k+1))|e_{KF,RO}(k)=\theta) + \|\theta\|_Z^2 \\ &\leq \lambda_s + \mathbb{E}(h_s(e^-_{KF,RO}(k+1))|e_{KF,RO}(k)=0)\Big\}. \end{aligned} \tag{10}$$

*Proof* From equation (8), we have

$$J_s' + h_s(e^-_{KF,RO}(k)) \leq \mathbb{E}(h_s(e^-_{KF,RO}(k+1))|e^-_{KF,RO}(k)) + c_s(e^-_{KF,RO}(k), S_s).$$

Taking expectation of both sides, the inequality above becomes

$$J_s' + \mathbb{E}(h_s(e^-_{KF,RO}(k))) \leq \mathbb{E}(h_s(e^-_{KF,RO}(k+1))) + \mathbb{E}(c_s(e^-_{KF,RO}(k), S_s)).$$

Adding the inequalities from step 0 to $N-1$ and taking the limit of N as it goes to infinity, we have

$$J_s' \leq \hat{J}_s(S_s).$$

From Lemma 3, we have equation (9).

The equality holds when $S_s = S_s^\dagger$. □

It is very difficult to find a constant $J_s'$ and a function $h_s$ satisfying equation (8), and hence the suboptimal triggering set described in equation (10) is hard to find. Because of the computation complexity associated with computing the suboptimal triggering set $S_s^\dagger$, we derive a quadratic sensor triggering set that is an approximation of the sub-optimal triggering set. Moreover, an upper bound on the cost achieved by this quadratic sensor triggering set is derived.

To find the quadratic triggering set in the sensor subsystem, we first give the following lemma which provides a way to search for the quadratic triggering set in the sensor subsystem and bounding its cost.

**Lemma 4** *Given the triggering set $S_s$, if there exists a piece-wise continuous and bounded function $f_s : \mathbb{R}^n \to \mathbb{R}$ and a finite constant $\overline{J}_s$ such that for any $k \in \mathbb{Z}^+$,*

$$\mathbb{E}\left( f_s(e^-_{KF,RO}(k+1)) | e^-_{KF,RO}(k) = \theta, S_s \right) + c_s\left(\theta, S_s\right) \le \overline{J}_s + f(\theta) \tag{11}$$

*then*

$$\hat{J}_s(S_s) \le \overline{J}_s \tag{12}$$

*Proof* Using the same technique as shown in the proof of Lemma 2, we have Lemma 4. □

Based on Lemma 4, we derive a triggering set which is in quadratic form. Moreover, the upper bound on this quadratic triggering set is also given.

**Theorem 3** *Given a quadratic triggering set*

$$S_s = \{e^-_{KF,RO}(k) : \|e^-_{KF,RO}(k)\|^2_{H_s} \le \lambda_s - \zeta_s\}, \tag{13}$$

*where the $n \times n$ matrix $H_s \ge 0$ satisfies the Lyapunov inequality*

$$\frac{A^T H_s A}{1+\delta_s^2} - H_s + \frac{Z}{1+\delta_s^2} \le 0, \tag{14}$$

*for some $\delta_s^2 \ge 0$, and $\zeta_s = \frac{\delta_s^2 \lambda_s + tr(H_s R)}{1+\delta_s^2}$, where*

$$R = L(CAQA^T C^T + CWC^T + V)L^T,$$

*then $J_s(S_s)$ is bounded from above by*

$$J_s(S_s) \le \min\{tr(H_s R) + \zeta_s, \lambda_s\} + tr(QZ) \tag{15}$$

*Proof* To find an upper bound on the cost of triggering set defined in equation (13), we need to find a bounded function $h_s$ and a finite constant $\overline{J}_s$ such that equation (11) is satisfied. With Lemma 4 and 3, we can derive that $J_s(S_s) \le \overline{J}_s + tr(QZ)$.

Now, let's define $f_s$ as

$$f_s(e^-_{KF,RO}(k)) = \min\{\|e^-_{KF,RO}(k)\|^2_{H_s} + \zeta_s, \lambda_s\},$$

and $\overline{J}_s$ as

$$\overline{J}_s = \mathbb{E}(f_k(e^-_{KF,RO}(k+1)) | e_{KF,RO}(k) = 0). \tag{16}$$

In the case of $\|e^-_{KF,RO}(k)\|^2_{H_s} \leq \lambda_s - \zeta_s$, no transmission occurs at step $k$, so the left hand side of equation (11) satisfies the following equations.

$$\begin{aligned}
&\mathbb{E}\left(f_s(e^-_{KF,RO}(k+1))|e^-_{KF,RO}(k), S_s\right) + c_s\left(e^-_{KF,RO}, S_s\right)\\
&= \mathbb{E}\left(f_s(e^-_{KF,RO}(k+1))|e_{KF,RO}(k) = e^-_{KF,RO}(k)\right) + \|e^-_{KF,RO}\|^2_Z\\
&\leq \|e^-_{KF,RO}\|^2_{A^T H_s A} + tr(H_s R) + \zeta_s + \|e^-_{KF,RO}\|^2_Z\\
&\leq \|e^-_{KF,RO}(k)\|^2_{H_s} + \|e^-_{KF,RO}(k)\|^2_{A^T H_s A - H_s + Z} + \zeta_s + tr(H_s R)\\
&\leq \|e^-_{KF,RO}(k)\|^2_{H_s} + \zeta_s + \delta_s^2(\lambda_s - \zeta_s) + tr(H_s R)\\
&= f_s(e^-_{KF,RO}(k)) + \zeta_s\\
&\leq f_k(e^-_{KF,RO}(k)) + \overline{J}_s.
\end{aligned}$$

The second step above makes use of the fact that $E(\min(f,g)) \leq \min(E(f), E(g))$, the fourth step is derived from equation (14) and the fact that $\|e^-_{KF,RO}(k)\|^2_{H_s} \leq \lambda_s - \zeta_s$, and the fifth step is derived from how we define the $\zeta_s$.

In the case of $\|e^-_{KF,RO}(k)\|^2_{H_s} > \lambda_s - \zeta_s$, a transmission occurs, and the left hand side of inequality (11) satisfies

$$\begin{aligned}
&\mathbb{E}\left(f_s(e^-_{KF,RO}(k+1))|e^-_{KF,RO}(k), S_s\right) + c_s\left(e^-_{KF,RO}, S_s\right)\\
&= \mathbb{E}\left(f_s(e^-_{KF,RO}(k+1))|e_{KF,RO}(k) = 0\right) + \lambda_s\\
&= \overline{J}_s + f_s(e^-_{KF,RO}(k))
\end{aligned}$$

Since inequality (11) holds in any condition, from Lemma 4, we know that $\hat{J}_s(S_s)$ is bounded from above by $\overline{J}_s$ defined in (16), i.e.

$$\hat{J}_s(S_s) \leq \overline{J}_s.$$

From the fact that $E(\min(f,g)) \leq \min(E(f), E(g))$, it's easy to show that

$$\overline{J}_s \leq \min\{tr(H_s R) + \zeta_s, \lambda_s\}.$$

From Lemma 3, we have

$$J_s(S_s) = \hat{J}_s(S_s) + tr(QZ) \leq \min\{tr(H_s R) + \zeta_s, \lambda_s\} + tr(QZ).$$

□

## 6 The suboptimal and quadratic triggering sets in the controller subsystem

This section first studies the suboptimal triggering set for the controller subsystem. As was found in the preceding subsection, direct computation of the suboptimal triggering set is complex. We therefore introduce a quadratic event-trigger and bound the performance obtained using this trigger. The theorem below provides the suboptimal triggering set $S_c^\dagger$ and the optimal cost $J_c^\dagger$.

**Theorem 4** *Define* $C_c$ *as*

$$C_c\left(\overline{x}_{RO}(k), u_a(k-1), S_c\right) = \|\overline{x}_{RO}(k)\|_Z^2 + \lambda_c 1\left(\begin{bmatrix}\overline{x}_{RO}(k)\\ u_a(k)\end{bmatrix} \notin S_c\right).$$

*Given* $S_s^\dagger$*, if there exists a piece-wise continuous and bounded function* $h_c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$*, and a bounded function* $J_c' : \mathbb{S}^n \to \mathbb{R}$ *(*$\mathbb{S}^n$ *indicates the collection of all subsets of* $\mathbb{R}^n$*) such that*

$$\begin{aligned} J_c'(S_s^\dagger) + h_c(\overline{x}_{RO}(k), u_a(k-1)) = \min_{S_c} \{ & C_c(\overline{x}_{RO}(k), u_a(k-1), S_c) \\ & + E\left(h_c(\overline{x}_{RO}(k+1), u_a(k)) | \overline{x}_{RO}(k), u_a(k-1), S_c\right)\}, \end{aligned} \tag{17}$$

*then*

$$J_c^\dagger(S_s^\dagger) = J_c'(S_s^\dagger), \tag{18}$$

*and the suboptimal triggering set in the controller subsystem is*

$$\begin{aligned} S_c^\dagger = \Bigg\{ \begin{bmatrix}\theta\\ \eta\end{bmatrix} : E\left[h_c\left(\begin{bmatrix}\overline{x}_{RO}(k+1)\\ u_a(k)\end{bmatrix}\right)\middle| \begin{bmatrix}\overline{x}_{RO}(k)\\ u_a(k-1)\end{bmatrix} = \begin{bmatrix}\theta\\ \eta\end{bmatrix}\right] \le \\ E\left[h_c\left(\begin{bmatrix}\overline{x}_{RO}(k+1)\\ u_a(k)\end{bmatrix}\right)\middle| \begin{bmatrix}\overline{x}_{RO}(k)\\ u_a(k-1)\end{bmatrix} = \begin{bmatrix}\theta\\ K\theta\end{bmatrix}\right] + \lambda_c \Bigg\} \end{aligned}$$

*Proof* With the same technique as shown in the proof of Theorem 2, Theorem 4 is proven. □

$S_c^\dagger$ can only be computed numerically to obtain a concrete representation for the controller's event-trigger, $S_c^\dagger$. Due to its concrete representation, the event-trigger would require a great deal of memory to store. We therefore introduce a quadratic triggering set for the controller subsystem which is an approximation of the suboptimal trigging set..

Before introducing the quadratic triggering set for the controller subsystem and its upper bound, a lemma providing the basis for finding an upper bound on the quadratic triggering set is given. This lemma is similar to Lemma 4, so the detailed proof is not given here.

**Lemma 5** *Given any* $S_c$*, if there exists a piece-wise continuous function* $f_c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *bounded from below and a finite constant* $\overline{J}_c'$ *such that*

$$\begin{aligned} & C_c\left(\overline{x}_{RO}(k), u_a(k-1), S_c\right) + E\left[f_c\left(\overline{x}_{RO}(k+1), u_a(k)\right) | \overline{x}_{RO}(k), u_a(k-1)\ , S_c\right] \\ & \le \overline{J}_c' + f_c\left(\overline{x}_{RO}(k), u_a(k-1)\right) \end{aligned} \tag{19}$$

*then* $J_c(S_c) \le \overline{J}_c'$.

The quadratic triggering set in the controller subsystem and its upper bound are given in the following theorem.

**Theorem 5** *Let $S_s$ in equation (13) be the triggering set in sensor subsystem.* $A_u = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}$, $A_c = \begin{bmatrix} A+BK & 0 \\ K & 0 \end{bmatrix}$, $Z_a = \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}$, *and* $H_s = P_{Hs}^T P_{Hs}$. *Given a quadratic triggering set of controller subsystem*

$$S_c = \left\{ \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} : \left\| \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \right\|_{H_c}^2 + \zeta_c \le \|\overline{x}_{RO}(k)\|_Z^2 + \lambda_c \right\}, \qquad (20)$$

*where $H_c \ge Z_a$ and controller gain $K$ satisfy*

$$A_u^T H_c A_u + (1+\delta_c^2)(Z_a - H_c) \le 0, \qquad (21)$$

$$A_c^T H_c A_c + (1-\rho_c^2)(Z_a - H_c) \le 0, \qquad (22)$$

*for some constant $\delta_c^2 \ge 0$ and $0 \le \rho_c^2 \le 1$, and*

$$\zeta_c = \frac{\delta_c^2 + \rho_c^2 - 1}{\delta_c^2 + \rho_c^2} \lambda_c, \qquad (23)$$

*the optimal controller cost is bounded from above by*

$$J_c(S_s, S_c) \le \overline{J}_c(S_c, S_s) = \frac{\delta_c^2}{\delta_c^2 + \rho_c^2} \lambda_c + \overline{\sigma}((P_{Hs}^T)^{-1} H_{c,lu} P_{Hs}^{-1})(\lambda_s - \zeta_s), \qquad (24)$$

*where $\overline{\sigma}(\cdot)$ indicates the greatest singular value, and $H_{c,lu}$ is the left upper $n \times n$ sub-matrix of $H_c$.*

*Proof* According to Lemma 5, as long as we can find a function $f_c$ bounded from below such that the inequality (19) is satisfied with $\overline{J}_c' = \overline{J}_c$, Theorem 5 is true.

Let's define $f_c$ as

$$f_c(\overline{x}_{RO}(k), u_a(k-1)) = \left\| \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \right\|_{H_c}^2 + \zeta_c.$$

First, we consider the case when $\left\| \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \right\|_{H_c}^2 + \zeta_c \le \|\overline{x}_{RO}(k)\|_Z^2 + \lambda_c$. In this case, the controller subsystem doesn't transmit. The left hand side of equation (19) satisfies

$$\begin{aligned}
&C_c(\overline{x}_{RO}(k), u_a(k-1), S_c) + E\left[ f_c(\overline{x}_{RO}(k+1), u_a(k)) \,|\, \overline{x}_{RO}(k), u_a(k-1)\,, S_c \right] \\
&\le \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}^T A_u^T H_c A_u \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} + \overline{\sigma}((P_{Hs}^T)^{-1} H_{c,lu} P_{Hs}^{-1})(\lambda_s - \zeta_s) \\
&\quad + \zeta_c + \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}^T Z_a \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \\
&\le \overline{J}_c + f\left( \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \right)
\end{aligned}$$

The first inequality is from equation (13), and the second inequality is from equation (21) and (23).

The second case is when $\left\|\begin{bmatrix}\overline{x}_{RO}(k)\\ u_a(k-1)\end{bmatrix}\right\|^2_{H_c} + \zeta_c > \|\overline{x}_{RO}(k)\|^2_Z + \lambda_c$. In this case, the controller subsystem transmits information. So the left hand side of equation (19) satisfies

$$\begin{aligned}
&C_c\left(\overline{x}_{RO}(k), u_a(k-1), S_c\right) + E\left[f_c\left(\overline{x}_{RO}(k+1), u_a(k)\right)|\overline{x}_{RO}(k), u_a(k-1)\ , S_c\right] \\
&\quad\le \begin{bmatrix}\overline{x}_{RO}(k)\\ u_a(k-1)\end{bmatrix}^T A_c^T H_c A_c \begin{bmatrix}\overline{x}_{RO}(k)\\ u_a(k-1)\end{bmatrix} + \overline{\sigma}((P_{Hs}^T)^{-1} H_{c,lu} P_{Hs}^{-1})(\lambda_s - \zeta_s) \\
&\qquad + \zeta_c + \begin{bmatrix}\overline{x}_{RO}(k)\\ u_a(k-1)\end{bmatrix}^T Z_a \begin{bmatrix}\overline{x}_{RO}(k)\\ u_a(k-1)\end{bmatrix} + \lambda_c \\
&\quad\le \overline{J}_c + f\left(\begin{bmatrix}\overline{x}_{RO}(k)\\ u_a(k-1)\end{bmatrix}\right).
\end{aligned}$$

The first inequality is from equation (13), and the second inequality is from equation (22) and (23).

Since in both cases, equation (19) holds, we conclude that the control cost of the quadratic triggering set $J_c(S_c, S_s)$ is bounded from above by $\overline{J}_c(S_c, S_s)$. □

From the results in equation (4), Theorem 2 and 4, we can give the suboptimal weakly coupled triggering sets in sensor and controller subsystems, and an upper bound on the optimal cost.

**Theorem 6** *The suboptimal triggering set in sensor subsystem $S_s^\dagger$ defined in Theorem 2 minimizes $J_s(S_s)$, and the suboptimal triggering set in controller subsystem $S_c^\dagger$ defined in Theorem 4 minimizes $J_c(S_c, S_s^\dagger)$. The optimal cost of the closed loop system $J^*$ is bounded from above by $J^\dagger = J_s^\dagger + J_c^\dagger(S_s^\dagger)$, where $J_s^\dagger$ and $J_c^\dagger(S_s^\dagger)$ are described in equation (9) and (18), respectively.*

From the analysis following Theorem 2 and 4, we know that the suboptimal triggering set $S_s^\dagger$ and $S_c^\dagger$ are hard to compute and store. So quadratic triggering sets and an upper bound on the cost of closed loop system triggered by these triggering sets are derived, which are computationally effective and easy to store. From the results in Theorem 1, 3 and 5, we have the following theorem.

**Theorem 7** *Given the triggering set in sensor subsystem $S_s$ defined in equation (13) and the triggering set in controller subsystem $S_c$ defined in equation (20), the average cost $J(S_s, S_c)$ given by the two weakly coupled triggering sets is bounded from above by $\overline{J}(S_s, S_c) = \overline{J}_s(S_s) + \overline{J}_c(S_c, S_s)$, where $J_s(S_s)$ and $J_c(S_c, S_s)$ are defined in equation (15) and (24), respectively.*

## 7 Simulation Results

In this section, an example is used to demonstrate Theorem 7. We first calculate the quadratic triggering sets $S_s$ and $S_c$ according to equation (13) and (20), and search for the controller gain $K$ such that inequality (22) is satisfied. The system, then, is run with the calculated controller gain $K$, and the transmission is triggered with the computed triggering sets. Next, the average cost given by simulation is compared with the upper bound given in Theorem 7 to demonstrate Theorem 7. Finally, we show the number of transmission times in sensor subsystem, the number of transmission times in controller subsystem, and the number of times when both sensor and controller transmit (concurrent transmission times) to illustrate that transmissions in the sensor subsystem don't necessarily trigger transmissions in the controller subsystem, or vice versa.

Let's consider the system with $A$ to be $\begin{bmatrix} 0.4 & 0 \\ 0 & 1.01 \end{bmatrix}$, $B$ to be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $C$ to be $\begin{bmatrix} 0.1 & 1 \end{bmatrix}$. The variances of the system noises are $W = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$, and $V = 0.3$. The weight matrix $Z$ is chosen to be an identity matrix.

Given $\delta_s^2 = 1.5$, $\lambda_s = 3$, $\delta_c^2 = 1.02$ and $\rho_c = 0.3$, we can obtain the triggering set in sensor subsystem $S_s$ as

$$S_s = \left\{ e^-_{KF,RO} : e^{-T}_{KF,RO} \begin{bmatrix} 2.5641 & 0 \\ 0 & 4.0543 \end{bmatrix} e^-_{KF,RO} \leq 0.8414 \right\},$$

the triggering set in controller subsystem $S_c$ as

$$S_c = \left\{ \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} : \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix}^T \begin{bmatrix} 1.3315 & -0.2836 & -0.3512 \\ -0.2836 & 3.6377 & 2.6808 \\ -0.3512 & 2.6808 & 13.7606 \end{bmatrix} \begin{bmatrix} \overline{x}_{RO}(k) \\ u_a(k-1) \end{bmatrix} \right.$$
$$\left. \leq 0.9008\lambda_c \right\},$$

and the controller gain $K = [-0.1967 \ -0.3133]$. The controller gain $K$ must be chosen such that equation (22) is satisfied. The closed loop system is run for 3000 steps with $\lambda_c$ varying from 0 to 8.

Figure 7 shows that the average cost given by simulation ($J$) is always bounded from above by the upper bound given by Theorem 7 ($J_{up}$). The $x$-axis of this plot indicates the communication price in controller subsystem $\lambda_c$, and the $y$-axis is the average cost. We can see that for any $\lambda_c$, the average cost $J$ (stars) is always bounded from above by the upper bound given by Theorem 7 (crosses), which is consistent with Theorem 7.

Figure 2 shows that transmissions in the sensor subsystem don't always trigger the transmissions in the controller subsystem, or vice versa. The $x$-axis of this plot is the communication price $\lambda_c$ in the controller subsystem, and the $y$-axis indicates the number of transmissions. We can see that the number of concurrent transmissions(circles) is always less or equal to both the numbers of transmission times in sensor and controller subsystems, which indicates that

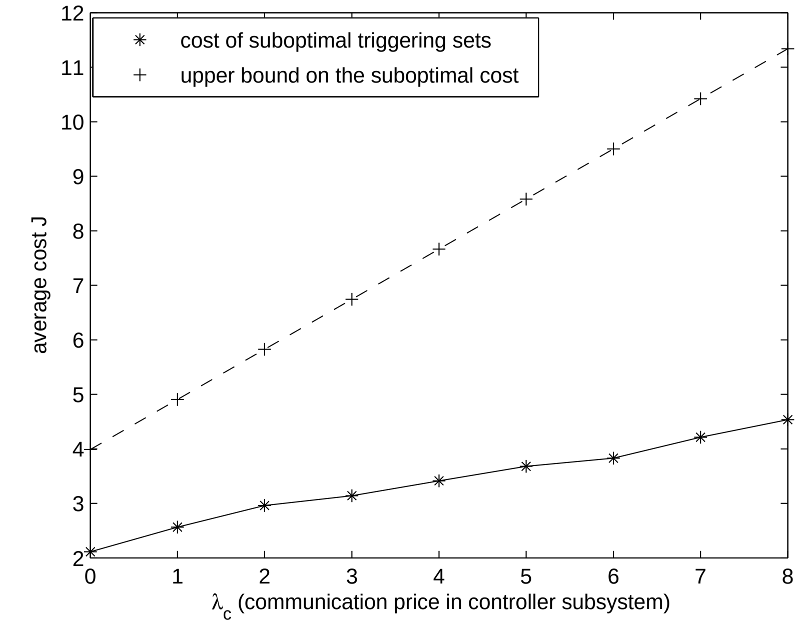


**Fig. 2** average cost

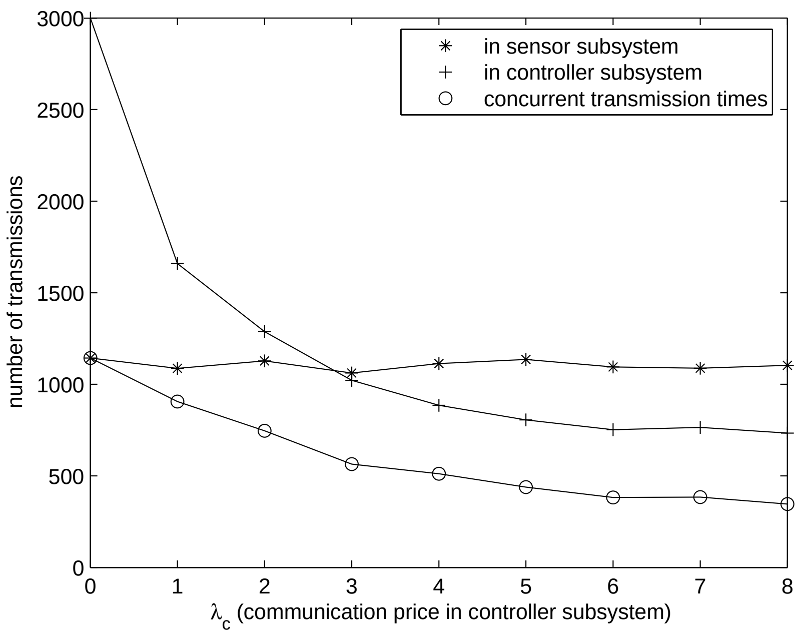


**Fig. 3** number of transmissions

the transmission in sensor subsystem doesn't always trigger transmissions in the controller subsystem, or vice versa.

## 8 Conclusion

This paper shows how to weaken coupling between sensor events and controller events in event triggered output feedback system. By 'weakly coupled', we mean that the triggering events in both sensor and controller only use local information to decide when to transmit data, and the transmission in one link doesn't necessarily trigger the transmission in the other link. These results extend earlier work in event triggered output feedback control [1] by removing the need for strong coupling. We also show that with the triggering events and controller we designed, the cost of the closed loop system is bounded from above, and an explicit upper bound on the cost is obtained. Our simulation results demonstrate the proposed weakly coupled triggering events and the upper bound on the cost of the closed loop system.

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