Input-to-state stabilizability of quantized linear control systems under feedback dropouts

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***Abstract*— This paper studies the input-to-state stabilizability of quantized linear control systems with external noise under feedback dropouts. A vector of feedback measurements is quantized prior to being transmitted over a communication channel. The transmitted data may be dropped by the channel. The channel dropouts are governed by a stationary model, which is quite general to include many realistic dropout models. This paper derives a lower bound on the constant bit rates which can almost surely stabilize the system in the input-to-state sense under the given dropout condition. A dynamic quantization policy is shown to be able to stabilize the system at that lower rate bound. So the minimum constant stabilizing bit rate has be obtained. The achieved theoretical results are also verified through an example.**

I. Introduction

In recent years there has been increasing interest in implementing the feedback loop of a control system over a nondeterministic digital communication network [1]. This may have many benefits, such as lower cost, higher reliability, and easier maintenance. These advantages are, however, achieved at the cost of loss of perfect feedback information due to packet dropouts and quantization (all data must be quantized into a finite number of bits before transmission). Then the results built upon the perfect feedback assumption have to be re-evaluated. A major concern about such systems is stabilizability, i.e., *whether the originally stabilizable system can still be stabilized under the given network dropout and quantization conditions*. Here stability is measured by input-to-state stability (ISS) in the almost sure sense, which quantitatively characterizes the system's robustness against the input noise and the initial condition[2]. Quantization condition is characterized by the available bit rate (here the constant bit rate case is chosen due to its communication efficiency [3]). We consider a general stationary dropout model, which can cover many realistic dropout conditions. We want to find the minimum constant stabilizing bit rate under the given dropout.

Much research on quantized control systems has been done in the last two decades [4]. Many results on quantized control systems assume that the quantization bits (or symbols) are **errorlessly** (dropout-freely) transmitted. The available quantization policies can be categorized into two groups, static one and dynamic one. *Static quantization policies* take a constant quantization range, and map each bit to a specific subset of that range in a fixed(static) way. Asymptotic stability of noise-free systems is lost under static policies [5]. Compared with static policies, *dynamic quantization policies* may choose a time-varying quantization range and their bit mapping policy can also be time-varying. Although more complicated, the dynamic policies can asymptotically stabilize noise-free linear systems at a finite bit rate[6]. The minimum bit rate to maintain asymptotic stability is given in [7] [8] (for the time-varying bit rate case) and in [9] (for the constant bit rate case). For quantized systems with bounded exogenous noise, bounded-input-bounded-output (BIBO) stability, instead of asymptotic stability, is pursued and the minimum bit rate to achieve such stability is derived [7] [10][11] while the input-to-state stability is investigated in [12]. Due to their efficiency, dynamic quantization polices are chosen in the present paper for stabilization of quantized systems with bounded noise.

Feedback dropout has not got much attention in the previous quantization literature. It seems intuitively pleasing that when the dropout rate is low, the stabilizability of the quantized systems would be preserved under feedback dropouts. In [13], it was asserted that the *almost sure stabilizability* of quantized linear systems with i.i.d. (independent and identically distributed) dropouts can be guaranteed if the average bit rate, $\overline{R}$ satisfies

$$\overline{R} > \sum_{i=1}^{n} max(0, |log_2 \lambda_i|) \tag{1}$$

where λ_i $(i = 1, \cdots, n)$ are the eigenvalues of the discrete-time open-loop system matrix. The above statement is, however, proven to be incorrect in [14]. Furthermore, it is shown that the system state almost surely diverges for any $\overline{R}$ [14]. In order to resolve this diverging issue, one may

- **Choose a weaker notion of stability**, such as mean square stability [15] [11], under the given i.i.d. dropout condition.
- **Put constraints on the dropout sequences**. In [16], the BIBO stabilizability of quantized systems is preserved under some dropout conditions different from the above i.i.d. process.

Because mean square stable systems may still generate sample paths with arbitrarily large state magnitude, the first approach listed above may not be appropriate for real applications. This paper, therefore, mainly focuses on the second approach. In order to come up with appropriate

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dropout constraints, we first have a close look at the i.i.d. dropout model in [13]. Due to the i.i.d. property, it is almost sure that dropout patterns with any number of consecutive dropouts will infinitely often occur. The consecutive dropouts are the main reason to drive the state to diverge[14]. Such arbitrarily long consecutive dropouts are actually prohibited in real networks. Real-time system engineers work hard to avoid consecutive dropouts by proposing and enforcing different constraints on dropouts. Some important constraints include the *skip-over* policy [17] (which places a bounded on the number of consecutive dropouts) and the *(m,k)*-firm guarantee rule [18] (which requires that at least m out of k consecutive attempts succeed). The present paper will give a dropout model (or a constraint on the dropout sequences) more general than the aforementioned ones, and proves the quantized system can be almost surely stabilized under that dropout condition at the minimum constant rate, which extends [12] by explicitly considering feedback dropouts.

The rest of this paper is organized as follows. Section II presents the mathematical models of the quantized linear system and the feedback dropouts. Under the given dropout condition, we derive a lower bound R_{min} on the constant bit rates to stabilize the quantized system in Section II. That lower bound R_{min} is shown to be achieved by a quantizer in Section III. Simulation results of an example system are also included to demonstrate the correctness of the theoretical results in Section III. Some final remarks are included in Section IV. To improve readability, we move all technical proofs to the appendix, Section V.

II. Mathematical Models

A. Model of the Quantized Linear System

This paper focuses on the system in Fig. 1. In Fig. 1,

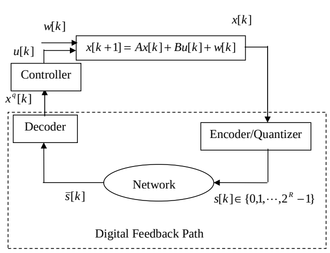

Fig. 1. A quantized linear system

$x[k] \in R^n$ is the state at time instant k $(= 0, 1, 2, \cdots,)$. $x[k]$ is quantized into a R-bit symbols $s[k]$ and sent over the digital communication network. $s[k]$ is either received by the decoder with 1 step delay or dropped. It is assumed that there is reliable ACK to notify the transmitter (encoder/quantizer) regarding dropouts. Define a dropout indicator

$$d[k] = \begin{cases} 1, & \text{Dropout at time } k \\ 0, & \text{Success at time } k \end{cases} \tag{2}$$

$\{d[k]\}$ is referred to as *"dropout sequence"*. The input of the decoder in Fig. 1 is $\overline{s}[k] = \begin{cases} s[k-1], & d[k] = 0 \\ \phi, & d[k] = 1 \end{cases}$ with ϕ representing a dropout. The decoder uses all received symbols $\{\overline{s}[k], \overline{s}[k-1], \cdots, \overline{s}[0]\}$ to generate the state estimate $x^q[k]$, which can also be viewed as a quantized version of $x[k]$. The control input $u[k] \in R^m$ is then constructed from $x^q[k]$. $w[k] \in R^n$ is an exogenous noise signal bounded as

$$\sup_{k \geq 0} \|w[k]\| \leq 0.5W \tag{3}$$

where $\|\cdot\|$ denotes the infinity norm of a vector, and W is known by both the encoder and the decoder. Under a linear controller, the system equation is

$$\begin{cases} x[k+1] &= Ax[k] + Bu[k] + w[k] \\ u[k] &= Gx^q[k] \end{cases} \tag{4}$$

where the matrices A, B and G are of appropriate dimensions. The system is assumed to be stabilizable (under the perfect feedback) and a stabilizing gain G must exist. We consider the input-to-state stability (ISS) of the system [2]

$$\|x[k]\| \leq \beta'(\|x[0]\|, k) + \gamma'(\sup_{j \geq 0} \|w[j]\|), \forall k \geq 0 \tag{5}$$

where $\gamma'(\cdot)$ and $\beta'(\cdot,\cdot)$ are $\mathcal{K}$ and $\mathcal{KL}$ functions [1], respectively.

The quantization error $e[k] = x[k] - x^q[k]$ surely affects stabilizability of the quantized system in eq. 4. It can be shown that the input-to-state stability in eq. 5 is equivalent to the following equation [19]

$$\|e[k]\| \leq \beta(\|e[0]\|, k) + \gamma(\sup_{j \geq 0} \|w[j]\|), \forall k \geq 0 \tag{6}$$

where $\gamma(\cdot)$ and $\beta(\cdot,\cdot)$ are $\mathcal{K}$ and $\mathcal{KL}$ functions, respectively. Therefore this paper establishes the input-to-state stabilizability of the system in eq. 4 through proving eq. 6.

Assumption 1: The system matrix in eq. 4, A, takes a real Jordan canonical form, i.e.,

$$A = diag(J_1, J_2, \cdots, J_P) \tag{7}$$

where J_i is an $n_i \times n_i$ real matrix with a single real eigenvalue λ_i (of the multiplicity of n_i) or a pair of conjugate eigenvalues λ_i and λ_i^* (of the multiplicity of $n_i/2$). $|\lambda_i| \geq 1, \forall i$.

For notational convenience, we define

$$\alpha(A) = \prod_{i=1}^{P} |\lambda_i|^{n_i} \tag{8}$$

B. Dropout model

For a given dropout sequence $\{d[k]\}$, define the local dropout rate as $\varepsilon_l[k] = \frac{1}{l}\sum_{i=0}^{l-1} d[k+i]$. $0 \leq \varepsilon_l[k] \leq 1$. We can show that the following limit must exist

$$\varepsilon' = \lim_{l_0 \to \infty} \sup_{l \geq l_0} \left(\lim_{k_0 \to \infty} \sup_{k \geq k_0} \varepsilon_l[k] \right) \tag{9}$$

We call ε' in eq. 9 the average dropout rate, which may be different from the ordinary definition of the average dropout rate $\overline{\varepsilon} = \lim_{l \to \infty} \frac{1}{l}\sum_{k=0}^{l-1} d[k]$ For example, $\{d[k]\} = \{101100111000\cdots\}$ gives $\varepsilon' = 1$ v.s. $\overline{\varepsilon} = 0.5$.

[1]A $\mathcal{K}$ function $f(x)$ is continuous, strictly increasing and $f(0) = 0$. A $\mathcal{KL}$ function $g(x, y)$ is a $\mathcal{K}$ function w.r.t. x by fixing y and $\lim_{y \to \infty} g(x, y) = 0$ for any fixed x.

Assumption 2: There exists $0 \le \hat{\varepsilon} < 1$ such that

$$\lim_{l_0 \to \infty} \sup_{l \ge l_0} \left(\lim_{k_0 \to \infty} \sup_{k \ge k_0} \varepsilon_l[k] \right) \le \hat{\varepsilon} \text{ almost surely.} \quad (10)$$

It can be verified that many real-time constraints, such as the *skip-over* policy [17], the *(m,k)*-firm guarantee rule [18], satisfy eq. 10.

Under the dropout condition in eq. 10, *what is the lowest bit rate to stabilize the system?* The following Lemma presents a lower bound on all constant bit rates to stabilize the system in eq. 4. Its proof closely follows that of Proposition 3.2 in [13] and is omitted here.

Lemma 2.1: Under eq. 10, the quantized system in eq. 4 can be almost surely stabilized under at constant bit rate of R only if

$$R \ge R_{min} = \left\lfloor \frac{1}{1-\hat{\varepsilon}} log_2 \left(\alpha(A)\right) \right\rfloor + 1 \quad (11)$$

where $\alpha(A)$ is defined in eq. 8, and $\lfloor \cdot \rfloor$ stands for the flooring operation over a real number.
We will construct a quantizer in Section III, which can achieve the ISS at $R = R_{min}$. So R_{min} in Lemma 2.1 is the minimum stabilizing bit rate.

III. Main results

A. Mathematical preliminaries of quantization policies

The desirable quantizer needs the following preliminaries.

1) Coordinate transformation: For quantized system with complex eigenvalues, a coordinate transformation is defined,

$$z[k] = H^k x[k] \quad (12)$$

where the transformation matrix H is defined as $H = diag(H_1, H_2, \cdots, H_P)$. Each H_i is associated with one of the Jordan blocks J_i in eq. 7. Specifically, $H_i = I_{n_i}$ if λ_i (the eigenvalue of J_i) is real and $H_i = diag\left(r(\theta_i)^{-1}, \cdots, r(\theta_i)^{-1}\right)$ with $r(\theta_i) = \begin{bmatrix} cos(\theta_i) & sin(\theta_i) \\ -sin(\theta_i) & cos(\theta_i) \end{bmatrix}$ if λ_i is complex and $\lambda_i = |\lambda_i| e^{j\theta_i}$. By [7], eq. 12 transforms eq. 4 into

$$z[k+1] = HAz[k] + H^{k+1}Bu[k] + \overline{w}[k] \quad (13)$$

where $\overline{w}[k] = H^{k+1} w[k]$. $\overline{w}[k]$ bounded by $0.5\overline{W} (= 0.5 \times (2W))$. Considering the structure of H, we infer from eq. 12 that $0.5\|x[k]\| \le \|z[k]\| \le 2\|x[k]\|$ for any $k \ge 0$. So the input-to-state stability of eq. 4 (with the noise input of $\{w[k]\}$) is equivalent to that of eq. 13. The present paper, therefore, focuses on the boundedness of $z[k]$. We use $z^q[k]$ to denote the quantized version of $z[k]$ at time k. The quantization error is represented as $e[k] = z^q[k] - z[k]$. As argued in Section II, $\{z[k]\}$ satisfy the ISS requirement in eq. 5 if and only if $\{e[k]\}$ can satisfy eq. 6.

2) Uncertainty set: Any bounded set in R^N can be over-bounded by a (hyper-)rectangle P with the center of z^P and the side length vector $L = [L_1, L_2, \cdots, L_N]^T$. A rectangle with the center of the origin and the side length vector L is denoted as $rect(L)$ $(= \prod_{i=1}^n [-0.5L_i, 0.5L_i]$ with $\prod$ standing for the Cartesian product). So P can be expressed as $P = z^P + rect(L)$.

Corresponding to the block diagonal structure of A in eq. 7, we relabel L with a 2-dimensional index as $L = [L_{1,1}, \cdots, L_{1,n_1}, \cdots, L_{P,1}, \cdots, L_{P,n_P}]^T$, where $L_{i,j}$ corresponds to the $m-th$ entry of L with $m = \sum_{l=1}^{i-1} n_l + j$.

The decoder does not know the exact value of the $z[k]$, but can know from the received symbols $\{\overline{s}[k]\}$ the set which $z[k]$ lies within. That set is referred to as the *"uncertainty set"*. The uncertainty set is usually a bounded set and can be over-bounded with a rectangle $P[k]$. Without confusion, $P[k]$ is also called the *"uncertainty set"* at time k. By the criterion of minimizing the worst case error, we can estimate $z[k]$ with the center of $P[k]$, $z^q[k]$. The quantization (estimation) error is $e[k] = z^q[k] - z[k] \in rect(L[k])$, where $L[k]$ is the side length vector of $P[k]$. It can be shown that $\{e[k]\}$ satisfies eq. 6 if and only if

$$\|L[k]\| \le \beta_L(\|L[0]\|, k) + \gamma_L(\sup_{j \ge 0} \|w[j]\|), \forall k \ge 0 \quad (14)$$

where $\beta_L(\cdot, \cdot)$ is a $\mathcal{KL}$ function and $\gamma_L(\cdot)$ is a $\mathcal{K}$ function.

3) Evolution of uncertainty sets: As time moves forward, we update $P[k]$, more specifically $z^q[k]$ and $L[k]$, to guarantee that $z[k] \in z^q[k] + rect(L[k]), \forall k$. By eq. 4 and the boundedness of $\{w[k]\}$ in eq. 3, we can update them as

$$z^q[k+1] = HAz^q[k] + H^{k+1}Bu[k] \quad (15)$$

$$L[k+1] = KL[k] + \left[\overline{W}, \overline{W}, \cdots, \overline{W}\right]^T \quad (16)$$

where H is defined in eq. 12, and $K = diag(K_1, K_2, \cdots, K_P)$ with

$$K_i = \begin{bmatrix} |\lambda_i| & 1 & 0 & \cdots & 0 \\ 0 & |\lambda_i| & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & |\lambda_i| \end{bmatrix}_{n_i \times n_i} \text{ when } \lambda_i \text{ is real,}$$

$$K_i = \begin{bmatrix} |\lambda_i|I & E & 0 & \cdots & 0 \\ 0 & |\lambda_i|I & E & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & |\lambda_i|I \end{bmatrix}_{n_i \times n_i} \text{ for complex}$$

λ_i and $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

B. A stabilizing quantizer at $R = R_{min}$

Under the dropout condition in eq. 10, we construct a quantizer at $R = R_{min}$, which can stabilize the system in eq. 4. So the bound R_{min} in Lemma 2.1 is **achievable** and is, therefore, the minimum stabilizing constant bit rate.

Now we start to build the desired quantizer. Let $Q = 2^{R_{min}}$. A positive parameter ρ can be found to satisfy

$$\rho > 1, Q^{1-\hat{\varepsilon}} > \alpha(A) \left(1 + Q\frac{3}{\rho}\right)^n \quad (17)$$

We first assume both the encoder and the decoder agree upon

$$z[0] \in P[0] = z^q[0] + rect(L[0]) \quad (18)$$

The quantizer chooses the "longest" side at $k = 0$ by

$$(I_k, J_k) = arg \max_{i,j} \left(Q^2\rho\right)^j L_{i,j}[k] \quad (19)$$

Partitioning side (I_k, J_k) into Q equal parts, we get a modified side length vector $L^{I_k,J_k}[k]$,

$$L^{I_k,J_k}_{i,j}[k] = \begin{cases} L_{i,j}[k], & (i,j) \neq (I_k, J_k) \\ L_{i,j}[k]/Q, & (i,j) = (I_k, J_k) \end{cases}$$

Now $P[k] = z^q[k] + rect(L[k])$ is partitioned into Q smaller subsets $P_s[k]$

$$P_s[k] = z^q_s[k] + rect(L^{(I_k,J_k)}[k]), s = 0, \cdots, Q-1,$$

where $z^q_s[k] = z^q[k] + z_s^{(I_k,J_k)}$ and $z_s^{(I_k,J_k)}$ is an n-dimensional vector with the (I_k, J_k)-th element equal to $\frac{-Q+(2s+1)}{2Q} L_{I_k,J_k}[k]$ and other elements of 0. $z[k]$ must lie within one of the subsets, to say, $P_{s_0}[k]$. Set $s[k] = s_0$, code $s[k]$ into R_{min} bits and send them to the decoder. Upon receiving $s[k]$, decoder sends ACK back to the encoder to confirm the receipt of $s[k]$. Due to ACK, the encoder and the decoder always agree upon the information of $z[k]$: either $z[k] \in z^q[k] + rect(L[k])$ (when $s[k]$ is dropped) or $z[k] \in z^q_{s[k]}[k] + rect(L^{(I_k,J_k)}[k])$ (when $s[k]$ is successfully transmitted). Based on the system equation 15, the encoder and decoder update the state set, $P[k+1]$ $(= z^q[k+1] + rect(L[k+1]))$, as

$$\begin{cases} \text{When } d[k] = 1: \\ \begin{cases} L[k+1] = KL[k] + [\overline{W}, \cdots, \overline{W}]^T \\ z^q[k+1] = HAz^q[k] + H^{k+1}Bu[k] \end{cases}, \\ \text{When } d[k] = 0: \\ \begin{cases} L[k+1] = & KL^{I_k,J_k}[k] + [\overline{W}, \cdots, \overline{W}]^T \\ z^q[k+1] = & HAz^q[k] + H^{k+1}Bu[k] \\ & +HAz^{(I_k,J_k)}_{s[k]} \end{cases}. \end{cases} \quad (20)$$

where the control variable is computed as

$$u[k] = G\left(H^{-k} z^q[k]\right). \quad (21)$$

The above quantization policy is summarized into

Algorithm 1: **Encoder/Decoder initialization:** Initialize $z^q[0]$ and $L[0]$ so that $z[0] \in z^q[0] + rect(L[0])$ and set $k = 0$.

Encoder Algorithm:

1) **Select** the indices (I_k, J_k) by eq. 19.
2) **Quantize** the state $z[k]$ by setting $s[k] = s$ if $z[k] \in z^q[k] + z_s^{(I_k,J_k)} + rect(L^{(I_k,J_k)}[k])$.
3) **Transmit** $s[k]$ and wait for ACK. If ACK is received before time $k+1$, $d[k] = 0$; otherwise, $d[k] = 1$.
4) **Update** $z^q[k+1]$ and $L[k+1]$ by eq. 20 immediately before time $k+1$. Update time index, $k = k+1$ and return to step 1.

Decoder Algorithm:

1) **Compute** control for time k by eq. 21.
2) **Wait** for the quantized data, $s[k]$, from the encoder. If $s[k]$ is received before time k, send ACK to decoder and set $d[k] = 0$; otherwise, set $d[k] = 1$.
3) **Update** $z^q[k+1]$ and $L[k+1]$ by eq. 20 immediately before time $k+1$. Update time index, $k = k+1$ and return to step 1.

Under the quantizer in Algorithm 1, the quantized system in eq. 4 is input-to-state stable in the almost sure sense. That result is formally presented by Theorem 3.1. Its proof is moved to Section V to improve readability.

Theorem 3.1: Let $R_{min} = \left\lfloor \frac{1}{1-\varepsilon} log_2\left(\alpha(A)\right) \right\rfloor + 1$ and $Q = 2^{R_{min}}$. The dropout model in eq. 10 is assumed. The quantized linear system in eq. 4 is almost surely input-to-state stable under the quantizer in Algorithm 1.

Remark: input-to-state stability describes more precisely the dependence The input-to-state stability in Theorem 3.1 unifies the asymptotic stability of noise-free quantized systems [9] and the BIBO stability of quantized systems with bounded noise [16] at the minimum bit rate. Compared with the results in [12], Theorem 3.1 explicitly takes the dropouts into account.

C. Simulation results

We consider an example with $A = \begin{bmatrix} 1.1 & 1 & 0 \\ 0 & 1.1 & 1 \\ 0 & 0 & 1.1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $G = [-1.29, -3.56, -3.27]$. The dropout sequence is governed by a $(2,3)$-firm model, i.e., among any 3 consecutive packets, at least 2 ones are transmitted successfully. So $\hat{\varepsilon} = 1/3$, $R_{min} = 1$ and $Q = 2$. According to eq. 17, choose $\rho = 109.1$. Initial conditions are $L[0] = [1,1,1]^T$, $x[0] = [0,0,0]^T$, $x^q[0] = [0,0,0]^T$. The simulation results for $W = 1$ and $W = 0$ are shown in Fig. 2. Note that the zoom-in versions of the two figures are also shown. It confirms Theorem 3.1 by showing that $\|L[k]\|$ and $\|x[k]\|$ are bounded for $W = 1$ and asymptotically converge to 0 for $W = 0$.

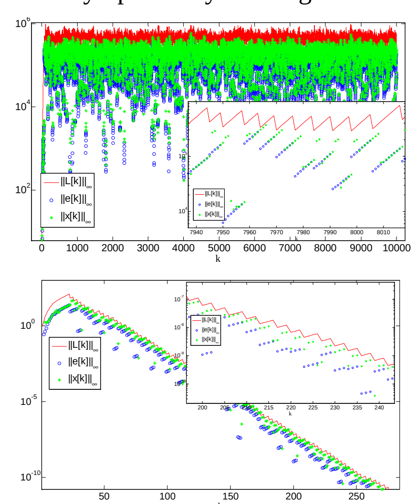

Fig. 2. $\|L[k]\|$, $\|e[k]\|$ and $\|x[k]\|$ with : (top) $W = 1$; (bottom) $W = 0$.

IV. Conclusion

This paper studies the effects of the quantization bit rate and the feedback dropouts on the input-to-state stability of linear systems with bounded noise. Under a general dropout model, it derives the minimum stabilizing constant bit rate and shows its achievability through constructing a quantizer.

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V. APPENDIX:PROOF OF THEOREM 3.1

Theorem 3.1 is proven through showing $L[k]$

$$\|L[k]\| \leq c_1\|L[0]\|\eta^k + c_2 W, \forall k \tag{22}$$

where c_1, c_2 are η are constants to be determined.

By eq. 10 and 11, we can bound$\varepsilon_l[k]$ as

Lemma 5.1: There exists $\delta > 0$, $N \in \mathcal{N}$ and $k_1 \in \mathcal{N}$ to almost surely guarantee that, for $\forall l \geq N, \forall k \geq k_1$,

$$\begin{cases} \varepsilon_l[k] \leq \hat{\varepsilon} + \delta \\ Q^{1-\hat{\varepsilon}-\delta} \geq \alpha(A)\left(1+\frac{3Q}{\rho}\right)^n \end{cases} \tag{23}$$

$$\eta = \sqrt[n]{\frac{\alpha(A)\left(1+\frac{3Q}{\rho}\right)^n}{Q^{1-\hat{\varepsilon}-\delta}}} < 1. \tag{24}$$

By comparing $\overline{W}(=2W)$ and $\|L[k_1]\|$, we see there are two cases: (i). $\overline{W} \geq \|L[k_1]\|$; (ii). $\overline{W} < \|L[k_1]\|$. We will find upper bounds on $\|L[k]\|$ ($k \geq k_1$) for both cases, respectively. By combining these bounds, together with a bound on $\|L[k]\|$ for $k < k_1$, we will get eq. 22.

A. When $\overline{W} \geq \|L[k_1]\|$

Define

$$r_{i,j}[k] = \begin{cases} \max\left(L_{i,j}[k], \rho\overline{W}\right), & j = n_i \\ \max\left(L_{i,j}[k], \rho r_{i,j+1}[k]\right), & j < n_i \end{cases} \tag{25}$$

$$p[k] = \prod_{i=1}^{P}\prod_{j=1}^{n_i} r_{i,j}[k] \tag{26}$$

It is straightforward to get

$$\begin{cases} r_{i,j}[k] \geq L_{i,j}[k] \\ r_{i,j}[k] \geq \rho^{n_i-j+1}\overline{W} \geq \rho\overline{W} \geq \overline{W} \end{cases}. \tag{27}$$

where the parameter ρ is defined in eq. 17. There are two bounds on the growth rate of $r_{i,j}[k]$.

Lemma 5.2: For $\forall k, \forall i = 1, \cdots, P; j = 1, \cdots, n_i$,

$$\frac{r_{i,j}[k+1]}{r_{i,j}[k]} \leq |\lambda_i|\left(1+\frac{3Q}{\rho}\right). \tag{28}$$

Lemma 5.3: Suppose side (I_k, J_k) is the longest at time k according to the criterion in Algorithm 1. When $d[k] = 0$ and $L_{I_k,J_k}[k] \geq Q^2\rho^{n_{I_k}-J_k+1}\overline{W}$,

$$\frac{r_{I_k,J_k}[k+1]}{r_{I_k,J_k}[k]} \leq \frac{|\lambda_i|}{Q}\left(1+\frac{3Q}{\rho}\right). \tag{29}$$

The proofs to Lemmas 5.2 and 5.3 comes from eq. 19 and 20 and the definitions of $r_{i,j}[k]$. Due to space limitation, they are omitted.

By eq. 26, $p[k]$ is just the product of all $r_{i,j}[k]$. Combining Lemmas 5.2 and 5.3, we get

Lemma 5.4:

$$p[k+1] \leq \alpha(A)\left(1+\frac{3Q}{\rho}\right)^n p[k] < Qp[k], \forall k. \tag{30}$$

When $d[k] = 0$ and $p[k] \geq \prod_{i=1}^{P}\prod_{j=1}^{n_i}\left(Q^2\rho^{n_i-j+1}\overline{W}\right)$,

$$p[k+1] \leq \frac{1}{Q}\alpha(A)\left(1+\frac{3Q}{\rho}\right)^n p[k] \tag{31}$$

Now we partition the time instants into windows with the duration of N (see Lemma 5.1 for the definition of N). Lemma 5.4 yields an upper bound on $p[mN+k_1]$ ($m = 0, 1, \cdots$).

Lemma 5.5: It is almost sure that

$$p[mN+k_1] \leq Q^N\prod_{i=1}^{P}\prod_{j=1}^{n_i}\left(Q^2\rho^{n_i-j+1}\overline{W}\right), \forall m \geq 0. \tag{32}$$

For $mN+k_1 \leq k < (m+1)N+k_1$, we implement eq. 30 from $mN+k_1$ to k, together with Lemma 5.5, to reach

Corollary 5.6: It is almost sure that

$$p[k] \leq Q^{2N}\prod_{i=1}^{P}\prod_{j=1}^{n_i}\left(Q^2\rho^{n_i-j+1}\overline{W}\right), \forall k \geq k_1 \tag{33}$$

$p[k]$ is the product of n terms, $r_{i',j'}[k](i' = 1, \cdots, P; j' = 1, \cdots, n_{i'})$. Among these terms, we consider a particular one with $i' = i, j' = j$. With the lower bounds of $r_{i',j'}[k](i' \neq$

i or $j' \neq j$) in eq. 27 and the upper bound of $p[k]$ in Corollary 5.6, we obtain

Proposition 5.7: For $\forall k \geq k_1$,

$$L_{i,j}[k] \leq r_{i,j}[k] \leq Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \right) \right) \overline{W}. \quad (34)$$

B. When $\overline{W} < \|L[k_1]\|$

There exist k_2 ($k_2 > k_1$) such that $\|L[k_1]\|\eta^{k_2-k_1} \geq \overline{W}$ and $\|L[k_1]\|\eta^{k2-k_1+1} < \overline{W}$, where η is defined in eq. 24.

1) Under the condition $k \leq k_2$,: we redefine $r_{i,j}[k]$ and $p[k]$ into $r'_{i,j}[k]$ and $p'[k]$ as

$$\begin{cases} r'_{i,j}[k] = \begin{cases} \max(L_{i,n_i}[k], \rho\eta^{k-k_1}\|L[k_1]\|), & j = n_i \\ \max(L_{i,j}[k], \rho r_{i,j+1}[k]), & j < n_i \end{cases} \\ p'[k] = \prod_{i=1}^{P} \prod_{j=1}^{n_i} r'_{i,j}[k] \end{cases}$$

Similar to Lemmas 5.2 and 5.3, we can get

Lemma 5.8: For $\forall k \in \{k_1, k_1+1, \cdots, k_2\}$,

$$\begin{cases} \frac{r'_{i,j}[k+1]}{r'_{i,j}[k]} \leq |\lambda_i| \left(1 + \frac{3Q}{\rho}\right) \\ p'[k+1] \leq \alpha(A) \left(1 + \frac{3Q}{\rho}\right)^n p[k] < Qp'[k] \end{cases}. \quad (35)$$

When $L_{I_k,J_k}[k] \geq Q^2 \rho^{n_{I_k} - J_k + 1} \eta^{k-k_1} \|L[k_1]\|$ and $d[k] = 0$,

$$\frac{r'_{I_k,J_k}[k+1]}{r'_{I_k,J_k}[k]} \leq \frac{|\lambda_i|}{Q} \left(1 + \frac{3Q}{\rho}\right). \quad (36)$$

When $p'[k] \geq \prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \eta^{k-k_1} \|L[k_1]\| \right)$ and $d[k] = 0$,

$$p'[k+1] \leq \frac{1}{Q} \alpha(A) \left(1 + \frac{3Q}{\rho}\right)^n p'[k] \quad (37)$$

Under the condition in eq. 23,

$$p'[k] \leq Q^{2N} \prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \eta^{k-k_1} \|L[k_1]\| \right) \quad (38)$$

Similar to eq. 27, we get, for $\forall k \in \{k_1, k_1+1, \cdots, k_2\}$,

$$\begin{cases} r'_{i,j}[k] \geq L_{i,j}[k] \\ r'_{i,j}[k] \geq \rho^{n_i-j+1} \eta^{k-k_1} \|L[k_1]\| > \eta^{k-k_1} \|L[k_1]\| \end{cases}. \quad (39)$$

Considering the definition of $p'[k]$ and applying eq. 39 to eq. 38, we get when $k_1 \leq k \leq k_2$,

$$L_{i,j}[k] \leq Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \right) \right) \eta^{k-k_1} \|L[k_1]\|. \quad (40)$$

2) Under the condition $k \geq k_2 + 1$: Starting from $k \geq k_1$, the time instants are grouped into epoches with the duration of N. Let $m_0 = \lfloor (k_2 - k_1)/N \rfloor$. Because $\|L[k_1]\|\eta^{k_2+1-k_1} \leq \overline{W}$ and $k_2 + 1 - k_1 \leq (m_0+1)N$, we know

$$\eta^{m_0 N} \|L[k_1]\| \leq \frac{1}{\eta^N} \overline{W}$$

Because $m_0 N + k_1 \leq k_2$, eq. 40 is applicable and yields

$$\begin{aligned} & L_{i,j}[m_0 N + k_1] \\ \leq \quad & Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \right) \right) \eta^{m_0 N} \|L[k_1]\| \quad (41) \\ \leq \quad & Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \right) \right) \frac{\overline{W}}{\eta^N} \quad (42) \end{aligned}$$

Define $\overline{W}' = \frac{Q^{2N}}{\eta^N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \right) \right) \overline{W}$. Note that $\overline{W}' \geq \overline{W}$. Similar to $r_{i,j}[k]$ and $p[k]$, we define, for $k \geq m_0 N + k_1$,

$$r''_{i,j}[k] = \begin{cases} \max\left(L_{i,j}[k], \rho\overline{W}'\right), & j = n_i \\ \max\left(L_{i,j}[k], \rho r''_{i,j+1}[k]\right), & j < n_i \end{cases} \quad (43)$$

$$p''[k] = \prod_{i=1}^{P} \prod_{j=1}^{n_i} r''_{i,j}[k] \quad (44)$$

So we can repeat the previous procedure for the case of $\|L[k_1]\| \leq \overline{W}$ to get a result similar to eq. 34

$$\begin{aligned} & L_{i,j}[k] \\ \leq \quad & Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \right) \right) \overline{W}' \\ = \quad & \left(Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \right) \right) \right)^2 \frac{1}{\eta^N} \overline{W} \quad (45) \end{aligned}$$

for $\forall k \geq m_0 N + k_1$. Of course, the above inequality holds for $k \geq k_2$ due to $k_2 \geq m_0 N + k_1$.

C. Final proof to Theorem 3.1

From time 0 to k_1, we can easily deduce the following inequality on $\|L[k]\|$ by the updating rule of $L[k]$

$$\|L[k+1]\| \leq (\alpha(A) + 2) \|L[k]\| + \overline{W}$$

So it is straightforward to reach

$$\|L[k]\| \leq (\alpha(A)+2)^{k_1} \|L[0]\| + (\alpha(A)+2)^{k_1} \overline{W}, \quad (46)$$

for $\forall k \in \{0, 1, \cdots, k_1\}$. Eq. 34, 40, 45 and 46 provide 4 upper bounds on $L_{i,j}[k]$ under 4 different conditions. These 4 bounds can be bounded by $c_1\|L[0]\|\eta^k + \frac{1}{2}c_2\overline{W}$ from above for $\forall k \geq 0$ with

$$\begin{cases} c_1 = & Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \right) \right) \eta^{-k_1} (\alpha(A)+2)^{k_1} \\ c_2 = & 2 \left(Q^{2N} \left(\prod_{i=1}^{P} \prod_{j=1}^{n_i} \left(Q^2 \rho^{n_i - j+1} \right) \right) \right)^2 \\ & \times \eta^{-k_1 - N} (\alpha(A)+2)^{k_1} \end{cases}$$

Then it is almost sure that

$$\begin{aligned} \|L[k]\| \quad & \leq \quad c_1 \|L[0]\| \eta^k + \frac{1}{2} c_2 \overline{W} \\ & \leq \quad c_1 \|L[0]\| \eta^k + c_2 W, \quad \forall k \geq 0 \end{aligned}$$

where the relationship $\overline{W} = 2W$ is utilized. The proof of Theorem 3.1 has been completed. $\diamondsuit$