Almost sure Stability of Networked Control Systems under Exponentially Bounded Bursts of Dropouts

Michael Lemmon
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, USA
lemmon@nd.edu

Xiaobo Sharon Hu
Dept. of Computer Science and Engineering
University of Notre Dame
Notre Dame, IN 46556, USA
shu@nd.edu

ABSTRACT

A wireless networked control systems (NCS) is a control system whose feedback path is realized over a wireless communication network. The stability of such systems can be problematic given the random way in which wireless channels drop feedback messages. This paper establishes sufficient conditions for the almost sure stability of NCS under random dropouts. These conditions relate the burstiness in the dropout process to the nominal response of the controlled system. In particular, this means that the burstiness of the dropout process provides a convenient quality-of-service (QoS) constraint on the wireless channel that can be used to adaptively reconfigure the control system in a manner that guarantees the almost sure stability of the NCS. We also show how a probabilistic extension of the network calculus can be used to reconfigure multi-hop communication networks so that this paper's sufficient stability condition is not violated.

Keywords

Wireless, Networked Control Systems, almost sure stability, dropout process, burstiness, network calculus

1. INTRODUCTION

In recent years there has been considerable interest in using wireless communication networks to support the monitoring and management of geographically distributed systems[1]. This interest has been driven by a wireless network's low deployment costs and ease of reconfiguration. Significant concerns arise, however, when wireless networks are suggested for use in time-critical control applications. These concerns stem from the probabilistic nature of message delivery in wireless networks. At any point in time, there is a finite probability that the wireless network will fail to deliver a given packet and this means it is impossible for such networks to meet the *hard* real-time quality-of-service (QoS) constraints usually expected by control applications.

So while wireless communication technologies may be inexpensive and easy to deploy, it is still unclear if this technology can be used for control applications expecting hard real-time guarantees on message delivery.

Control applications have traditionally demanded hard real-time QoS guarantees from network infrastructure. This expectation could be satisfied by wireline networks, but it is an unreasonable expectation for wireless networks. Do these control applications really need to meet these hard real-time deadlines? Are there any control applications that can tolerate firm or event soft real-time guarantees? Firm/soft real-time guarantees may be sufficient if one is willing accept stochastic guarantees on control system stability. Past papers (for example, see [2]) have clearly demonstrated that networked control systems will be mean square stable provided the average rate of dropped feedback data is sufficiently bounded. Hard real-time constraints, therefore, may not be a pre-requisite for all control applications.

The problem with mean square stability, however, is that it only requires that the variance of the state process be asymptotically bounded. This means that sample state trajectories of mean square stable processes have a finite probability of being arbitrarily far from the system's equilibrium point. So while the system may be well-behaved on the average, there is always the possibility of a large transient occurring and for many applications this sort of behavior is simply unacceptable. Demanding applications, therefore, must satisfy a much stronger stability concept than mean square stability. One such stability concept is *almost sure stability*.

Almost sure stability requires the probability of the system's largest excursion from equilibrium for all times $k > T$ go to zero as T goes to infinity. In other words, the further out one goes along the state trajectory, the probability of being arbitrarily far from the equilibrium becomes vanishingly small. This stronger form of stochastic stability is difficult to guarantee in dynamical systems with external disturbances. In fact, it has been shown that if the system disturbance is uniformly bounded, then the resulting system will be almost surely unstable [3]. On the basis of these results, therefore, it may appear highly unlikely that wireless communication should ever be used for highly critical control applications.

Analysis shows that a wireless channel's propensity for generating a long string or *burst* of dropped data packets is the reason for this instability [3]. This observation suggests that if one were able to limit the probability of long bursts of dropouts, it may still be possible to ensure the almost sure stability of the process. This fact was recently exploited in

[4] to show that if no dropout bursts greater than a given length occur, then quantized control systems will be almost surely stable in the presence of uniformly bounded disturbances. While this finding is encouraging, it is of little value in building wireless NCS since there is always a probability that a burst of dropouts may occur.

In wireless NCS it is impossible to require the probability of long dropout bursts to be zero. It may, however be possible to compensate for a long burst by reconfiguring the controller or the network. This is the viewpoint adopted in this paper. In particular, this paper shows that if the dropout process has exponentially bounded burstiness, then one can guarantee the almost sure stability of the control system provided its response to admissible disturbance satisfies certain bounds. In particular, let's characterize a channel's burstiness by a *burst exponent*, γ, that characterizes the probability of a long bursts of dropouts. Let's also characterize a system's disturbance rejection in terms of an exponent, s that characterizes the closed-loop system's disturbance rejection ability. The main finding in this paper shows that provided the product $s\gamma$ is large enough, then we can guarantee the almost sure stability of the closed-loop process. This sufficient condition, therefore, suggests that it may be possible to adaptively reconfigure the controller and network to assure almost-sure stability.

The remainder of this paper is organized as follows. Section 2 introduces the system model under study. The main result of this paper will be found in section 3. This result is the aforementioned sufficient condition for almost sure stability in networked control systems with single hop wireless networks. Experimental results validating this condition will be found in section 4. This experimental section also presents preliminary simulation results concerning the adaptive reconfiguration of the controller in response to changes in the wireless channel's state. Section 5 considers the extension of the main result in section 3 to networked control systems realized over multi-hop wireless networks. The main finding in that section is the statement of a network optimization problem that seeks to minimizes overall network energy consumption subject to a constraint on the burstiness in individual network links. This optimization problems provides the basis for our claim that these concepts can be used to guide the reconfiguration of wireless networks supporting networked control applications. Final remarks will be found in section 6.

2. SYSTEM MODEL

The system under study is shown in figure 1. In this figure, one is interested in stabilizing a discrete-time dynamical system called the *plant*. The plant accepts two real-valued inputs; an external disturbance $\{w_k\}_{k=0}^{\infty}$ and a controlled input, $\{u_k\}_{k=0}^{\infty}$. In response to these inputs, the plant generates a real valued *state*, $\{x_k\}_{k=0}^{\infty}$, that satisfies the following equation,

$$x_{k+1} \quad = \quad \alpha x_k + u_k + w_k \tag{1}$$

for $k = 0, 1, \ldots, \infty$ where $\alpha > 1$ and the initial condition $0 \leq x_0 \in \mathbb{R}$ is given. The only thing known about the disturbance is that it is bounded in a sense to be specified in the next section. The control input at time k is assumed to satisfy the following equation,

$$u_k = (\beta - \alpha)\hat{x}_k \tag{2}$$

for $k = 0, 1, \ldots, \infty$ where $0 < \beta < 1$ and $\hat{x}_k$ is the output of another system called the *channel*.

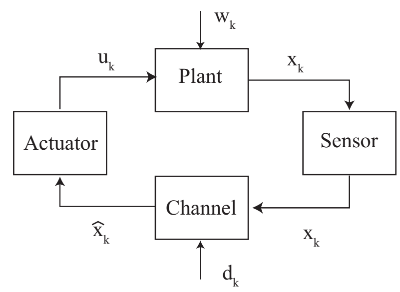

Figure 1: System Model

The channel is a memoryless system that accepts two inputs. Physically this channel is a single-hop wireless communication network. The first input is the plant's state, x_k, measured by a perfect *sensor*. The second input is a binary valued stochastic process, $\{d_k\}_{k=0}^{\infty}$ called the *dropout process*. The relationship between the channel's output, $\hat{x}_k$, and its two inputs is

$$\hat{x}_k \quad = \quad \begin{cases} x_k & \text{if } d_k = 0 \\ 0 & \text{if } d_k = 1 \end{cases} \tag{3}$$

The dropout process, d_k, therefore takes the value of 0 if the channel successfully transport the sensor's measurement, x_k. If $d_k = 1$, then the channel is said to have *dropped* or *erased* the sensor's measurement and it outputs the value 0.

If one combines equations (1)-(3), then the closed-loop system's dynamics are characterized by the following switched difference equation,

$$x_{k+1} = \begin{cases} \alpha x_k + w_k & \text{if } d_k = 1 \\ \beta x_k + w_k & \text{if } d_k = 0 \end{cases} \tag{4}$$

for all $k = 0, 1, \ldots, \infty$. Since $1 < \alpha$, and $0 < \beta < 1$ this means that the undriven system's state is increasing if the sensor data is dropped and it is decreasing asymptotically to zero if there is no dropout. In particular, if we let $x(k; x_0)$ denote the system state at time k assuming initial input x_0, then at time $k \geq 0$ the system state can be shown to be

$$\begin{aligned} x(k; x_0) \quad &= \quad \alpha^{d_{0,k}} \beta^{k-d_{0,k}} x_0 + \sum_{\ell=0}^{k-1} \alpha^{d_{\ell+1,k}} \beta^{k-(\ell+1)-d_{\ell+1,k}} w_\ell \\ &= \quad \alpha^{d_{0,k}} \beta^{k-d_{0,k}} + \sum_{\ell=1}^{k} \alpha^{d_{\ell,k}} \beta^{k-\ell-d_{\ell,k}} w_{\ell-1} \end{aligned} \tag{5}$$

where

$$d_{\ell,k} = \sum_{j=\ell}^{k-1} d_j \tag{6}$$

is the total number of packets that were dropped over the time interval $[\ell, k-1]$ and $d_{k,k} = 0$. We let $\rho_{\ell,k} = \frac{d_{\ell,k}}{k-\ell}$ denote the *local average dropout rate* over time interval $[\ell, k-1]$.

The dropout process $\{d_k\}_{k=0}^{\infty}$ is a stochastic process. The process will be *Bernoulli* with dropout rate $\lambda \geq 0$ if the probability of a dropout at any time $k > 0$ is λ. We will also consider dropout processes that have *exponentially bounded burstiness* or EBB [5]. In particular, given two constants

$\rho, \gamma > 0$, we say the process $\{d_k\}_{k=0}^{\infty}$ is (ρ,γ)-EBB if and only if for all $\sigma > 0$ and all $0 \leq \ell < k$,

$$\Pr\{d_{\ell,k} > \rho(k-\ell) + \sigma\} \leq e^{-\gamma\sigma}. \tag{7}$$

In the above equation, ρ may be viewed as a long term dropout rate and σ may be viewed as the length of a dropout burst (i.e. a dropout burst consists of a several consecutive dropouts). A dropout process is therefore (ρ,γ)-EBB if the probability of the total number of dropouts over an interval Δ being greater than $\rho\Delta + \sigma$ can be exponentially bounded as a function of the *dropout burst length*, σ. Throughout this paper, γ will be referred to as the process' *burst exponent*.

We're interested in studying the *almost sure* stability of the process in equation (4) when the dropouts form a stochastic process. Let the event $A_k^\epsilon(x_0)$ be defined as

$$A_k^\epsilon = \{|x(k; x_0)| > \epsilon\} \tag{8}$$

then the system is *almost sure asymptotically stable* if and only if for all $\epsilon > 0$

$$\Pr\left\{\limsup_k A_k^\epsilon(x_0)\right\} = 0. \tag{9}$$

It is *almost sure practically stable* if there exists $\epsilon > 0$ such that equation (9) holds. Finally the process is said to be *almost sure unstable* if there exists $\epsilon > 0$ such that

$$\Pr\left\{\limsup_k A_k^\epsilon(x_0)\right\} = 1 \tag{10}$$

3. SINGLE-HOP NETWORKS

This paper examines the almost sure stability of control systems assuming the dropout process has exponentially bounded burstiness. The following theorem shows that any Bernoulli process has exponentially bounded burstiness. A version of this theorem was proven in [5]. The following theorem differs form the earlier version in that it explicitly characterizes the burst exponent γ.

THEOREM 3.1. *Let $\{d_k\}_{k=0}^{\infty}$ be a Bernoulli process with parameter λ. For any $\rho > \lambda$ there exists a constant $\gamma > 0$ such that*

$$\Pr\{d_{\ell,k} \geq \rho(k-\ell) + \sigma\} \leq e^{-\gamma\sigma} \tag{11}$$

for all $k > \ell \geq 0$ where

$$\gamma = \sup\left\{z \in \mathbb{R}^+ : \lambda e^z + 1 - \lambda \leq e^{\rho z}\right\} \tag{12}$$

PROOF. The Markov inequality implies that

$$\begin{aligned}\Pr\{d_{\ell,k} \geq \rho(k-\ell)+\sigma\} &= \Pr\left\{e^{zd_{\ell,k}} \geq e^{z(\rho(k-\ell)+\sigma)}\right\} \\ &\leq \mathbb{E}\left[e^{zd_{\ell,k}}\right] e^{-z(\rho(k-\ell)+\sigma)}\end{aligned}$$

for any $z > 0$. $\mathbb{E}[e^{zd_{\ell,k}}]$ is the moment generating function for the random variable $d_{\ell,k}$. Since d_k is an independent and identically distributed process we can see that

$$\mathbb{E}[e^{zd_{\ell,k}}] = (\lambda e^z + 1 - \lambda)^{k-\ell}.$$

The probability in equation (11) may therefore be bounded as

$$\Pr\{d_{\ell,k} \geq \rho(k-\ell)+\sigma\} \leq f(z)^{k-\ell} g(-z)^{k-\ell} e^{-z\sigma} \tag{13}$$

where

$$\begin{aligned} f(z) &= (\lambda e^z + 1 - \lambda) \\ g(z) &= e^{\rho z}\end{aligned}$$

Note that

$$\begin{aligned} f(0) &= g(0) = 1 \\ f'(0) = \lambda &< \rho = g'(0)\end{aligned}$$

This means that $1 \leq f(z) \leq g(z)$ for all $z \in [0,\gamma]$ for the γ given in the theorem's statement. We can use this fact in equation (13) to obtain

$$\begin{aligned}\Pr\{d_{\ell,k} \geq \rho(k-\ell)+\sigma\} &\leq g(z)^{k-\ell} g(-z)^{k-\ell} e^{-z\sigma} \\ &\leq e^{-z\sigma}\end{aligned}$$

for any $z \in [0,\gamma]$ which completes the proof. $\square$

A number of papers (see [2] for instance) characterize the mean square stability of a linear process under Bernoulli dropouts. The following theorem establishes similar results for homogeneous (i.e. no input) versions of the linear systems in equation (4) under dropout processes that have exponentially bounded burstiness.

THEOREM 3.2. *Assume the droput process is (ρ,γ)-EBB where*

$$0 < \rho < -\frac{\log\beta}{\log\alpha - \log\beta} < 1 \tag{14}$$

If the input $w_k = 0$ for all $k \geq 0$, then the system in equation (4) is almost sure asymptotically stable.

PROOF. Under the assumption that $w_k = 0$ for all k, equation (5) reduces to

$$x_k = \alpha^{d_{0,k}} \beta^{k-d_{0,k}} x_0. \tag{15}$$

Now assume that a particular instance of the dropout process satisfies the inequality

$$d_{\ell,k} \leq \rho(k-\ell) + \sigma \tag{16}$$

for $0 < \ell < k$ and for a given $\sigma > 0$. With this particular dropout process the above equation (15) can be bounded as

$$x_k \leq \mu^k \left(\frac{\alpha}{\beta}\right)^\sigma x_0$$

for all $k \geq 0$ where $\mu = \alpha^\rho \beta^{1-\rho}$. The right hand side of the above inequality can be bounded by a polynomial function of k. In particular, for any $\gamma > 0$ there exists a positive $C > 0$ such that

$$\mu^k < Ck^{-\frac{2}{\gamma}\log(\alpha/\beta)}$$

for all $k \geq 1$. This allows us to bound x_k with a polynomial function of k,

$$x_k \leq Ck^{-\frac{2}{\gamma}\log(\alpha/\beta)} \left(\frac{\alpha}{\beta}\right)^\sigma x_0 \tag{17}$$

So we know that if the dropout sequence d_k satisfies equation (16) then the system state is bounded above as in equation (17).

So let's consider the event $A_k^\epsilon(x_0)$. From equation (17) we know that if $x_k > \epsilon$ and if $d_{\ell,k} \leq \rho(k-\ell)+\sigma$ for some choice of σ, then

$$\epsilon < Ck^{-\frac{2}{\gamma}\log(\alpha/\beta)}(\alpha/\beta)^\sigma x_0.$$

Taking the log of both sides and solving for σ provides a bound on σ of the form,

$$\sigma > \frac{\log(\epsilon/Ck^{-\frac{2}{\gamma}\log(\alpha/\beta)}x_0)}{\log(\alpha/\beta)} = \sigma^*(\epsilon) \tag{18}$$

The right hand side of equation (18) is a lower bound on the burst length σ giving rise to the system state, $x_k > \epsilon$. In other words, if $x_k > \epsilon$, then the dropout process must have had burst length $\sigma > \sigma^*(\epsilon)$. Since we assumed the process is (ρ, γ)-EBB, the probability of this event occurring must be less than $e^{-\gamma\sigma^*}$. We may, therefore, bound the probability of $A_k^\epsilon(x_0)$ as

$$\begin{aligned}\Pr\{A_k^\epsilon(x_0)\} &\leq \Pr\{d_{\ell,k} > \rho(k-\ell) + \sigma^*(\epsilon)\} \\ &\leq \exp\left(-\gamma\left(\frac{\log(\epsilon/Ck^{-\frac{2}{\gamma}\log(\alpha/\beta}x_0)}{\log(\alpha/\beta)}\right)\right) \\ &= C_1 k^{-2}\end{aligned}$$

where $C_1 = \left(\frac{Cx_0}{\epsilon}\right)^{\frac{\gamma}{\log(\alpha/\beta)}}$.

If we now sum these probabilities over all $k \geq 1$, we obtain

$$\sum_{k=1}^{\infty} \Pr\{A_k^\epsilon(x_0)\} \leq C_1 \sum_{k=1}^{\infty} k^{-2} = \frac{C_1\pi^2}{6}$$

This sum is clearly bounded for any finite ϵ so by the first Borel-Cantelli lemma we can conclude that

$$\Pr\left\{\limsup_k A_k^\epsilon(x_0)\right\} = 0$$

for any $\epsilon > 0$ which means the system is almost sure asymptotically stable. $\square$

It is well known that homogeneous systems under Bernoulli dropouts are almost sure stable [2] [6]. The result in theorem 3.2 extends that prior result to the larger class of dropout processes with exponentially bounded burstiness. We'd now like to extend these results to inhomogeneous systems. In general, it is well known that the system in equation (4) is almost surely unstable when driven by a *uniformly bounded* disturbance [3]. This is because at any time k, there is a finite probability of having a dropout burst whose length is great enough to force x_k to exceed ϵ. One way to get around this issue is to simply require that no burst occurs with length beyond the critical threshold of σ^*. This approach was adopted in [4] for quantized feedback control systems with random dropouts. The following theorem adopts a less heavy-handed approach. In particular, we assume the dropout process is (ρ, γ)-EBB and identify those class of inputs for which the system is almost sure asymptotically stable.

The following theorem assumes that the dropout process is (ρ, γ)-EBB with a given ρ that is less than the bound ρ^* found in equation (14) of theorem 3.2. The other parameter, γ, of course is the *burst exponent* of the dropout process. Since we already know that this system is almost sure unstable under uniformly bounded dropouts, we relax the uniform bound and require that the response of an "averaged" closed-loop system is bounded above Ck^{-s} where $s > 0$. We refer to s as the system's *response exponent*. The larger s is, the faster the system rejects the input disturbance w. The main result in the following theorem shows that if the product $s\gamma$ is greater than $\log\alpha - \log\beta$, then the closed-loop system will be almost surely asymptotically stable.

THEOREM 3.3. *Assume the dropout process is (ρ, γ)-EBB where*

$$\rho < -\frac{\log\beta}{\log\alpha - \log\beta} < 1 \tag{19}$$

Assume that the system input, w is such that there exist positive real constants C and s such that

$$\mu^k x_0 + \sum_{j=0}^{k-1} \mu^j w_{k-j-1} \leq Ck^{-s} \tag{20}$$

where $\mu = \alpha^\rho\beta^{1-\rho}$. If s and γ satisfy,

$$s\gamma > \log\alpha - \log\beta \tag{21}$$

then the driven system in equation (4) is almost sure asymptotically stable.

PROOF. From equation (5) we know

$$x_k \;=\; \alpha^{d_{0,k}}\beta^{k-d_{0,k}}x_0 + \sum_{\ell=1}^{k} \alpha^{d_{\ell,k}}\beta^{k-\ell-d_{\ell,k}}w_{\ell-1}$$

Let's consider a dropout sequence that satisfies the bound,

$$d_{\ell,k} \leq \rho(k-\ell) + \sigma \tag{22}$$

for all $0 < \ell < k$ where $\sigma > 0$ is given. This implies that

$$\begin{aligned} x_k &\leq \alpha^{\rho k+\sigma}\beta^{(1-\rho)k-\sigma}x_0 \qquad (23) \\ &\quad + \sum_{\ell=1}^{k} \alpha^{\rho(k-\ell)+\sigma}\beta^{(1-\rho)(k-\ell)-\sigma}w_{\ell-1} \\ &= \left(\mu^k x_0 + \sum_{j=0}^{k-1}\mu^j w_{k-j-1}\right)\left(\frac{\alpha}{\beta}\right)^\sigma \qquad (24)\end{aligned}$$

where $\mu = \alpha^\rho\beta^{1-\rho}$. Under the assumption in equation (19), it can be readily shown that $0 < \mu < 1$. Moreover, the other assumption in equation (20) and the above relation in equation (24) imply that

$$x_k \leq Ck^{-s}\left(\frac{\alpha}{\beta}\right)^\sigma \tag{25}$$

So if the dropout sequence, d_k, satisfies equation (22), then the system state x_k must be bounded as shown in equation (25).

Now let's consider the event $A_k^\epsilon(x_0)$ so that $x_k > \epsilon$ at time instant k. If the dropout sequence also satisfies equation (22) then we can use equation (25) to infer that

$$\epsilon \leq Ck^{-s}\left(\frac{\alpha}{\beta}\right)^\sigma$$

taking the log of both sides and solving for σ yields the bound

$$\sigma > \frac{\log(\epsilon/Ck^{-s})}{\log(\alpha/\beta)} = \sigma^*(\epsilon) \tag{26}$$

The right hand side of equation (26) is a lower bound on the burst length, σ giving rise to $x_k > \epsilon$. Since the dropout process was assumed to be (ρ, γ)-EBB, the probability of this event occurring is $e^{-\gamma\sigma^*}$. We may therefore bound the probability of event $A_k^\epsilon(x_0)$ as

$$\begin{aligned}\Pr\{A_k^\epsilon(x_0)\} &\leq \Pr\{d_{\ell,k} > \rho(k-\ell)+\sigma^*(\epsilon)\}\\ &\leq \exp\left(-\gamma\left(\frac{\log(\epsilon/Ck^{-s})}{\log(\alpha/\beta)}\right)\right)\\ &= C_1 k^{-\frac{s\gamma}{\log(\alpha/\beta)}}\end{aligned}$$

where $C_1 = \left(\frac{C}{\epsilon}\right)^{\frac{\gamma}{\log(\alpha/\beta)}}$.

So as we did in the proof for theorem 3.1, we sum these probabilities over $k = 1$ to ∞ to obtain

$$\sum_{k=1}^{\infty}\Pr\{A_k^\epsilon(x_0)\} \leq C_1\sum_{k=1}^{\infty} k^{-\frac{\gamma}{\log(\alpha/\beta)}}$$

This sum is convergent if $\frac{s\gamma}{\log(\alpha/\beta)}$ is greater than one and that sum is given by the Riemann zeta function, $\zeta(s)$. This is precisely the condition assumed in equation (21). We can therefore conclude that the sum of these probabilities is bounded for any choice of ϵ we can make. By the first Borel-Cantelli lemma this implies that $\Pr\{\limsup_k A_k^\epsilon\} = 0$ which means the system is almost sure asymptotically stable. $\square$

Unlike the earlier result in [3] where the disturbance was uniformly bounded, theorem 3.3 restricts the input to eventually go to zero so that the bound in equation (20) holds. We may take, however, the polynomial exponent, s, on the right hand side of equation (20) to be arbitrarily close to zero so that in the limiting case we approach the uniformly bounded case. To guarantee almost sure asymptotic stability as $s \to 0$, one would then need to have the burst exponent γ go to infinity as well, which we mean that the probability of any burst essentially goes to zero. If, however, we are only interested in assuring almost sure practical stability (for a specified ϵ), then these limiting conditions require that the probability of a burst of length greater than $\sigma^*(\epsilon)$ goes to zero. This finding is consistent with recent results in [4].

The results in this section confined their attention to linear scalar systems in which the state x_k is always going to be positive. This may, at first, appear to be a significant limitation. We view the system in equation (4), however, as a Lyapunov comparison system. In other words x_k corresponds to the Lyapunov function of the system at time k. In this regard, we believe these results can be extended to any nonlinear systems that are input-to-state stable (ISS) when there are no dropouts.

4. EXPERIMENTAL RESULTS

This section presents experimental results from preliminary Monte Carlo simulations examining how tight the sufficient condition in theorem 3.3 might be. We consider the system in equation (4) with a Bernoulli dropouts having parameter λ. The closed-loop dynamic is characterized by the parameter $\beta = .2$. The open-loop dynamic constant α takes values of 5, 2, and 1.25. These are the three system examined in these simulation studies.

We first study these systems by fixing the input to the system and then varying the dropout parameter λ. We select a parameter $\rho < \rho^*$ where

$$\rho^* = -\frac{\log\beta}{\log\alpha - \log\beta},$$

see equation (14) in theorem 3.2. We selected $\rho = 0.8\rho^*$. We then drove the system in equation (4) with an input function $w_k = \frac{1}{\sqrt{k}}$ for $k \geq 1$ and determine the exponent $\overline{s}$ which must closely overbounds the system's actual response x_k with the function $Ck^{-\overline{s}}$. The value of $\overline{s}$ was then used in equation (21) to determine an upper bound, γ^*, on the burst exponent γ such that the overall system would be almost surely asymptotically stable. From this γ^*, we then computed the dropout parameter λ^* using the relation

$$\lambda^* e^{\gamma^*} + 1 - \lambda^* = e^{\rho\gamma^*}$$

in equation (12). The dropout parameter λ^* computed above represents a threshold level above which we can expect the driven system to be almost surely unstable.

To test this hypothesis, we simulated the driven system with the given input where the dropout rate λ was varied between 0.01 and 0.5. For each λ, the system was simulated 10 times over the time interval from 0 to 10000. The maximum excursion of the state after time instant 5000 was recorded for each simulation run. The mean, maximum, and minimum value of each collection was recorded and then plotted. Figure 2 shows this plot for the three systems in which α = 5, 2, or 1.25. The x-axis in the plot represents the dropout parameter λ on a log scale. The y-axis represents the mean, maximum, and minimum values that x_k achieved over the 10 simulation runs at the specified dropout parameter. The dark blue line shows the mean value. The max and min values are marked by the dashed error bars. For each simulation, the computed threshold, λ^*, is marked by the black vertical line.

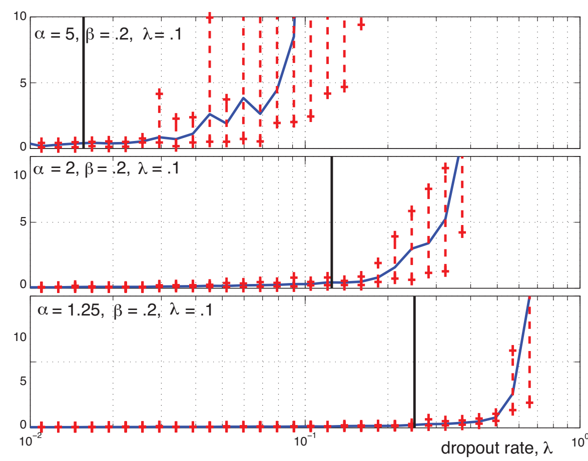

Figure 2: Simulation results where system input was held constant and dropout process parameter, λ, was varied between 0.01 **to** 0.5

The parameters of these three systems are shown in table 1. The threshold dropout rates for the three systems are λ^* = 0.015, 0.125, and 0.25, respectively. These thresholds are marked by the dark vertical line in each plot in figure 2. The plots show that to the right of the threshold, the variation in the state, x_k, increases dramatically. The left of this threshold, the state remains close to zero for all times after 5000. This is precisely the behavior one would expect if the systems were almost sure asymptotically stable for $\lambda < \lambda^*$. These results, therefore, seem to confirm the findings in theorem 3.3.

We also studied this system from the standpoint of varying the response exponent s in equation (20). For these

	α	β	ρ^*	γ^*	λ^*
system 1	5	.2	.5	6.19	0.015
system 2	2	.2	.69	4.42	0.125
system 3	1.25	.2	.88	3.52	0.25

Table 1: Parameter Values for Simulation Experiment Varying Dropout Rate λ

	α	β	λ	γ^*	s^*
system 1	5	.2	.1	3.4	-0.95
system 2	2	.2	.1	5.1	-0.45
system 3	1.25	.2	.1	7.7	-0.24

Table 2: Parameters for simulation where system responsiveness was varied

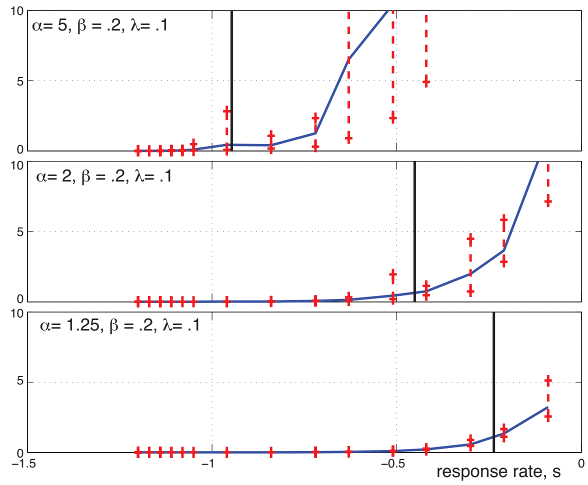

Figure 3: Simulation results where the dropout rate was held to $\lambda = 0.1$ and the response exponent, s, for the system was varied between -1.5 and $-.1$.

experiments, we used the same three systems and we fixed the dropout rate $\lambda = 0.1$. Under these assumptions we can then compute ρ^* as before and selected $\rho = 0.8\rho^*$. For this value of ρ we computed the bound, γ^*, on the burst exponent using equation (12). This value of γ^* was then used in equation (21) to determine the threshold s^* for the system's response exponent. For systems whose actual response exponent, $\overline{s} > s^*$, we would expect the system to no longer be almost surely asymptotically stable. The computed values for γ^*, and s^* are shown in table 2

The results from this simulation experiment are plotted in figure 3 for all three systems. The x-axis is the response exponent ,s, that was actually seen in the simulation. As before in figure 2, the y-axis is the mean, max, and min values achieved by the system state after time instant 5000 over 10 independent simulation runs. The dark vertical line in each plot marks the threshold, s^*, computed for table 2. For simulation runs to the right of these vertical lines, we expect a large increase in the variation of the state (as shown by the larger error bars and larger mean values). To the left of this line, the system states are relatively small. We take this behavior as indicative of the dividing line between almost sure stability and instability as suggested in theorem 3.3. Once again, therefore, these simulation results seem to be able to predict over what range of burst exponents, γ, and response exponents, s, we can expect these systems to be almost sure asymptotically stable.

The simulation results in figures 2 and 3 support theorem 3.3's assertion regarding a tradeoff between the system's response exponent, s, and the dropout process' burst exponent, γ. This suggests a strategy for reconfiguring the control law in wireless networked control systems. It is well known that radio communication channels with Ralyeigh fading can be modeled as two state Markov chains whose two states represent the channel state [7]. The channel state can be detected quickly by monitoring the signal to noise ratio (SNR) at the receiver. From the SNR, one may compute the bit error rate and thereby predict the dropout rate expected at the receiver. What this means is that the channel state and the channel's dropout parameter can be observed and may be used by the control application to reconfigure its controller and thereby guarantee the almost sure stability of the process, even if the channel state randomly changes according to a Markov chain.

A simple simulation was devised to test this idea. We assume a dropout process whose dropout parameter λ switches between 0.01 and 0.1 according to a two-state Markov chain. The Markov chain state, q is either $q_1 = \texttt{GOOD}$ with $\lambda = 0.01$ and $q_2 = \texttt{BAD}$ with $\lambda = 0.1$. The transition probability $\Pr(q_i : q_j)$ is 0.99 if $i = j$ and is 0.01 if $i \neq j$. The system under study has a state x_k that satisfies the following difference equation

$$x_{k+1} = \begin{cases} 5x_k + w_k & \text{if } d_k = 1 \text{ (dropout)} \\ 0.75x_k + w_k & \text{if } d_k = 0 \text{ and } q = \texttt{GOOD} \\ 0.5x_k + w_k & \text{if } d_k = 0 \text{ and } q = \texttt{BAD} \end{cases}$$

where $w_k = \frac{1}{k^{0.8}}$. In this case, the control system's gain is a function of the channel state. The top plot in figure 4 shows a common state trajectory for this system. This system is almost surely asymptotically stable as one can see the pulses due to packet dropouts grow less frequent as time increases. As a point of comparison, the second plot in figure 4 shows the state trajectory of a system in which the control gain is independent of channel state. In particular, we let $x_k = 0.75x_k + w_k$ regardless of the channel condition. For this case, one sees very large pulses (some on the order of 10^4) arbitrarily far out in time. So without this channel aware switching the system loses almost sure asymptotic stability.

5. MULTI-HOP NETWORKS

The results in section 3 pertain to a networked control system in which the feedback communication channel is essentially a single hop. The last section suggested that if the burst exponent of this single hop is too small, then we can modify the controller to still assure closed-loop stability. We may, however, also consider this from the standpoint of controlling the network. In other words, for a given response exponent, how might we reconfigure the network to enforce the sufficient stability condition in theorem 3.3? This section examines that question with regard to a multi-hop communication network (rather than single-hop). In particular, we use a probabilistic extension of the network

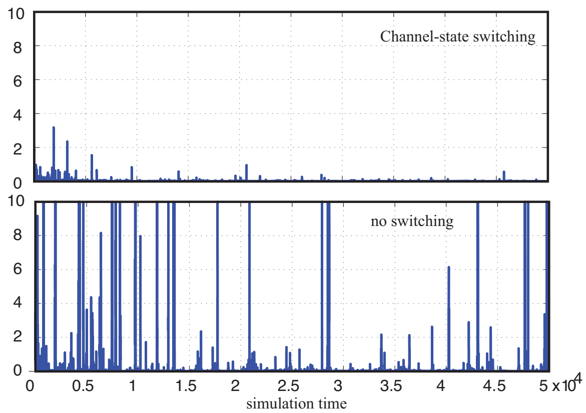

Figure 4: Simulation history of a wireless networked control system whose dropouts following a two-state Markov chain. The top plot shows the response of a channel-aware switching controller and the bottom plot shows the response of a controller that does not switch on channel state.

calculus to identify an optimization problem whose solution yields the burst exponents of individual network links whose end-to-end quality of service enforces the almost sure stability condition in theorem 3.3.

Let's consider a networked control system whose feedback information is transported over the network shown in figure 5. This network consists of N wireless nodes connected in series. The first node receives sensor data and transports this over a multi-hop network to a destination node that is connected to the control system's actuator.

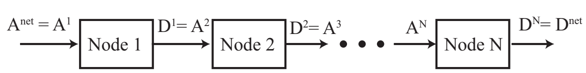

Figure 5: Traffic flow through N forwarding nodes

Rather than assuming that each link's dropout process is Bernoulli, we adopt the viewpoint used in probabilistic extensions of the network calculus [5] where each link provides its arriving packets with a *statistical service curve*. Using recent results in [11], we use these link service curves to identify a *network* statistical service curve and then show that this network service curve results in an end-to-end dropout process that has exponentially bounded burstiness. The main finding is that the network's burst exponent can be related back to burst exponents for each link in the network. This relationship takes the form of a constraint on the link burst exponents which can then be used to guide the reconfiguration of the overall network. The following subsections first review some basic results in [11] regarding statistical service curves. We then go on to use these results to characterize the network's burst exponent.

Probabilistic Network Calculus: The network calculus [8, 9, 10] uses a min-plus algebra to relate the end-to-end quality of service (latency) in deterministic networks to the QoS of each link. Our problem needs a probabilistic extension of the network calculus that relates the burstiness of each link to the network's end-to-end burstiness [5]. If one uses the techniques in [5] to bound end-to-end network latency, one finds that this bound grows as $O(N^3)$ where N is the number of network nodes. A recent alternative approach [11] allows one to obtain an upper bound on a network's latency that grows as $O(N \log N)$. The following development uses this later method to bound the network's burst exponent. The method makes use of *statistical service curves*

To define a statistical service curve, it will be useful to introduce some notational conventions. In particular, we let $(x)_+ = \max(0, x)$ where x is any real number. We let $x \wedge y = \min\{x, y\}$ for any real x and y. Given a function $S(\cdot) : \mathbb{Z} \to \mathbb{R}$, we let S_δ be the function where $S_\delta(k) = S(k) + \delta k$. Given two functions $A(\cdot) : \mathbb{Z} \to \mathbb{R}$ and $S(\cdot) : \mathbb{Z} \to \mathbb{R}$, we define the min-plus convolution of A and S as the function $A * S$ that takes values

$$(A * S)(k) = \inf_{0 \leq \ell \leq k} \{A(k - \ell) + S(\ell)\} \tag{27}$$

for all $k \geq 0$. With these notational conventions, we can now introduce the concept of a statistical service curve.

Consider a network node whose input is a stochastic process $A = \{A(k)\}_{k=0}^{\infty}$ called the *arrival process*. $A(k)$ denotes the total number of packets received by the node over the time interval $[0, k]$. We let the output of the node be a stochastic process $D = \{D(k)\}_{k=0}^{\infty}$ called the *departure process*. $D(k)$ represents the total number of packets that have departed the node over the interval $[0, k]$. Given a function $S(\cdot) : \mathbb{Z} \to \mathbb{R}$, we say that S is a *statistical service curve* for the node provided for any real $\sigma > 0$,

$$\Pr\{D(k) < (A * (S - \sigma)_+)(k)\} < \epsilon(\sigma) \tag{28}$$

where $\epsilon(\cdot) : \mathbb{R} \to \mathbb{R}$ is a non-increasing function called the *error function*.

Theorem 1 in [11] characterizes the network service curve for the network shown in figure 5. Assuming that node i for $i = 1, 2, \ldots, N$ provides a statistical service curve $S^i(\cdot) : \mathbb{Z} \to \mathbb{R}$ with error function $\epsilon^i(\cdot)$, then for any $\delta > 0$, the function

$$S^{\text{net}} = S^1 * S^2_{-\delta} * S^3_{-2\delta} * \cdots * S^N_{(N-1)\delta} \tag{29}$$

is a statistical service curve for the network with an error function

$$\epsilon^{\text{net}}(\sigma) = \inf_{\sigma_1 + \cdots + \sigma_N = \sigma} \left[\epsilon^N(\sigma_N) + \sum_{j=1}^{N-1} \int_{\sigma_j}^{\infty} \epsilon^j(u) du \right] \tag{30}$$

The proof for the above result will be found in [11].

Network Burst Exponent: We now use the result in equations (29) and (30) to bound the network's end-to-end burst exponent. We start with the following theorem that asserts a single node with a statistical service curve will have a dropout process that has exponentially bounded burstiness. The proof of this theorem requires the following technical lemma.

LEMMA 5.1. *if $S(k) = ((1 - \rho)k)_+$ for all k and $A(k)$ is such that $A(k + \ell) - A(k) < \ell$, then for all k*

$$(A * (S - \sigma)_+)(k) \geq (A(k) - (\rho k + \sigma))_+ \tag{31}$$

PROOF. We can use the bound $(A(k + \ell) - A(k) < \ell)$ to bound the min-plus convolution,

$$\begin{array}{ll} & (A*(S-\sigma)_+)(k) \\ = & \inf_{0\leq\ell\leq k}\{A(k-\ell)+((1-\rho)\ell-\sigma)_+\} \\ = & \inf_{0\leq\ell\leq k}\{A(k)+(A(k-\ell)-A(k))+((1-\rho)\ell-\sigma)_+\} \\ \geq & \inf_{0\leq\ell\leq k}\{A(k)-\ell+((1-\rho)\ell-\sigma)_+\} \\ \geq & \inf_{0\leq\ell\leq k}\left\{(A(k)-(\rho\ell+\sigma))_+\right\} \\ = & (A(k)-(\rho k+\sigma))_+ \end{array}$$

The lemma follows since this holds for all $k \geq 0$. □

We use lemma 5.1 to establish the following theorem. This theorem asserts under relatively mild assumptions that if the wireless node provides a statistical service curve with error function $e^{-\gamma\sigma}$, then the dropout process has exponentially bounded burstiness with the same error function.

THEOREM 5.2. *Consider a node with arrival process $A(k)$ and departure process $D(k)$. Suppose there exist constants $0 < \rho < 1$ and $\gamma > 0$ such that for all $\sigma > 0$, the node provides a statistical service curve $S(k) = ((1-\rho)k)_+$ to the arrival process with error function $\epsilon(\sigma) = e^{-\gamma\sigma}$. Then the dropout process $d_{0,k} = A(k) - D(k)$ is (ρ, γ)-EBB.*

PROOF. Under the assumptions we know that

$$\Pr\{D(k) < (A*(S-\sigma)_+)(k)\} < e^{-\gamma\sigma}$$

Equation (31) implies that the following events satisfy

$$\{D(k) < (A(k)-(\rho k+\sigma))_+\} \subset \{D(k) < (A*(S-\sigma)_+)(k)\}$$

So the probability of the left hand event must be less than the probability of the right hand event which is, in turn, less than $e^{-\gamma\sigma}$, thereby completing the proof. □

Theorem 5.2 can now be used to establish the main result of this section. Let's return to the network shown in figure 5 and let's assume that this is the feedback channel used by the networked control system in equation (4). Let's further assume that the ith node in this network ($i = 1, 2, \ldots, N$) provides a statistical service curve $S^i(k) = ((1-\rho^i)k)_+$ with error function $\epsilon^i(\sigma) = e^{-\gamma^i\sigma}$ for all $\sigma > 0$, some $\gamma^i > 0$ and some $\rho^i > -\frac{\log\beta}{\log\alpha - \log\beta}$. From theorem 1 in [11], we know that the network service curve, S^{net}, in equation (29) has the error function

$$\epsilon^{\text{net}}(\sigma) = \inf_{\sigma_1+\cdots+\sigma_N=\sigma}\left[e^{-\gamma^N\sigma_N} + \sum_{j=1}^{N-1}\frac{1}{\delta\gamma^j}e^{-\gamma_j\sigma_j}\right]$$

for any $\delta > 0$. We can select a specific partition of the delays $\sigma_i = \frac{\sigma}{N}$ so that

$$\epsilon^{\text{net}}(\sigma) \leq \left[e^{-\gamma^N\sigma/N} + \sum_{j=1}^{N-1}\frac{1}{\delta\gamma^j}e^{-\gamma_j\sigma/N}\right]$$

Now from theorem 3.3, for a given response exponent, s, we know that the end-to-end process burst exponent, γ^{net}, should be greater than

$$\gamma^* = \frac{\log\alpha - \log\beta}{s}$$

We need to select the individual link exponents, γ^* for $i = 1, 2, \ldots, N$ such that the error function $\epsilon^{\text{net}}(\sigma)$ is less than the burst error function $e^{-\gamma^*\sigma}$. If this is done we can expect the networked control system to remain almost sure stable. In other words, we need to select the link exponents γ^i so that

$$\epsilon^{\text{net}}(\sigma) \leq \left[e^{-\gamma^N\sigma/N} + \sum_{j=1}^{N-1}\frac{1}{\delta\gamma^j}e^{-\gamma^j\sigma/N}\right] \leq e^{-\gamma^*\sigma} \tag{32}$$

In light of theorem 3.3, if the end-to-end ρ of the network's service curve still satisfies the required bound in equation (14), then equation (32) represents an inequality constraint on the burst exponents of each link, that must be satisfied to assure the almost sure stability of the networked control system.

We suggest that the inequality in equation (32) can be used to adaptively reconfigure the wireless nodes in the network. Let's consider a wireless radio node in which a single message packet consists of M information bits. For this packet to be successfully received, all M information bits must be received. Let's assume that the node transmits $L > M$ bits at R bits/second with power w. If the bit error rate is known then we can easily compute the probability that M information bits will be received within a specified deadline $D > LR$. If this probability is too small we can take steps to decrease the bit error rate (increase broadcast power) or we can increase the bit transmission rate R. In either case one may, within realistic limits, formulate an optimization problem whose solution would generate a set of link burst exponents that satisfy the inequality constraint in equation (32). The decision variable in this problem would be either the link's bit rate or transmission power.

For example, let's consider a scenario in which the radio nodes adjust their transmitted bit rate, R_i, for $i = 1, 2, \ldots, N$. Let $E(R_i)$ denote the energy each node expends in transmitting a packet at this bit rate. We may then pose the following problem that seeks to minimize the summed energy of all network nodes subject to the end-to-end burstiness constraint required to achieve almost sure stability. This optimization problem could take the form,

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} E(R_i) \\ \text{with respect to:} & R_1, R_2, \cdots, R_N \\ \text{subject to:} & R_i < \overline{R}_i, \quad (i = 1, 2, \ldots, N) \\ & \left[e^{-\gamma^N\sigma/N} + \sum_{j=1}^{N-1}\frac{1}{\delta\gamma^j}e^{-\gamma^j\sigma/N}\right] \leq e^{-\gamma^*\sigma} \end{array}$$

where γ^i (the link's burst exponent) is a function of the node's transmission rate R_i, $\overline{R}_i$ is an upper bound on the ith node's maximum allowable transmission rate, and δ is a tuning parameter.

This section has suggested how probabilistic extensions of the network calculus might be used in conjunction with theorem 3.3 to adaptively reconfigure wireless networks to ensure almost sure stability in networked control systems. This work is still in progress and simulated verification is still being done. The basic approach involves using the network calculus to form an optimization problem whose solution minimizes overall network energy consumption while ensuring the link transmission rates satisfy a bound on the end-to-end burstiness of the network. The resulting optimization problem appears to be separable and convex so that any one of a number of distributed optimization algorithms might be used to solve this problem [12, 13, 14]. We hope to have some examples of this approach in time for the conference.

6. CONCLUDING REMARKS

This paper studied the almost sure stability of discrete-time linear systems under dropout processes that have exponentially bounded burstiness. The main finding is theorem 3.3 which provides a sufficient characterization for almost sure stability in systems whose driven response decays to zero at an arbitrarily slow rate. The sufficient condition establishes a tradeoff between the system's rate of decay and the dropout process' burst exponent. Preliminary simulation experiments suggest that the bound in theorem 3.3 is reasonably tight.

Theorem 3.3 suggests a method by which one can adaptively reconfigure a networked control system to maintain almost sure stability over wireless networks with random dropouts. If, for instance, one has no control over the wireless channel, then it may be possible to adjust the controller to increase the system's rate of decay for a bounded class of inputs. This may be used to compensate for temporary increases in dropout burstiness. Another approach for reconfiguring the system focuses on the links in a multi-hop network. Using results from the probabilistic network calculus we identified a constraint on the network's burst exponents that could be used to guide the adaptation of link bit rates to try and meet the almost sure stability conditions presented in theorem 3.3.

The results in this paper, therefore, suggest a promising direction for adaptively reconfiguring a wireless sensor-actuator network to guarantee the almost sure stability of the controlled process. The results in theorem 3.3 provide guidance on how to adjust either the control application or the network's communication infrastructure to achieve these goals. Future work will continue to explore the directions suggested by these results. In particular, we are developing co-design methods that simultaneously adjust controller and network structures to ensure overall application performance. These methods, in our opinion, provide a realistic way of achieving performance levels in wireless networked control systems that are often only seen in wired hard real-time systems.

7. ACKNOWLEDGEMENT

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