Performance of Networked Control Systems under Sporadic Feedback

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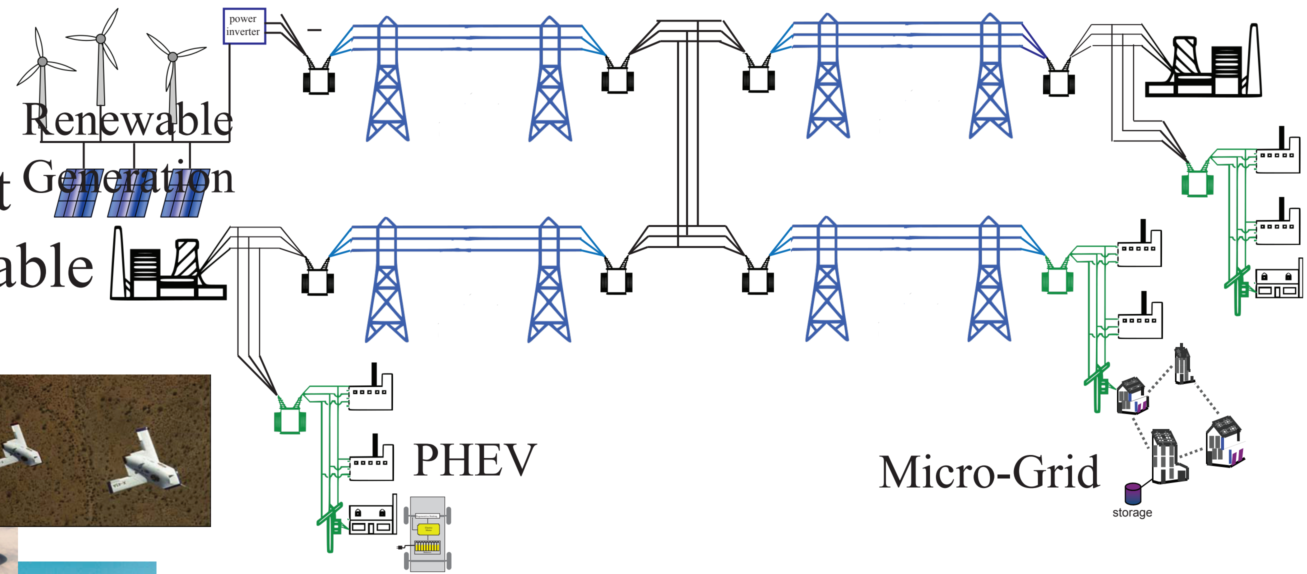
Networked Control Systems

Smart Grid
- Distributed Dispatch
- Demand Management
- Integration of Renewable Generation



Flocks of Autonomous Vehicles
- unmanned or manned aircraft
- in-flight refueling
- situational aware retasking

1) Feedback loops are closed over digital communication networks
2) Variations in network QoS result in sporadic feedback

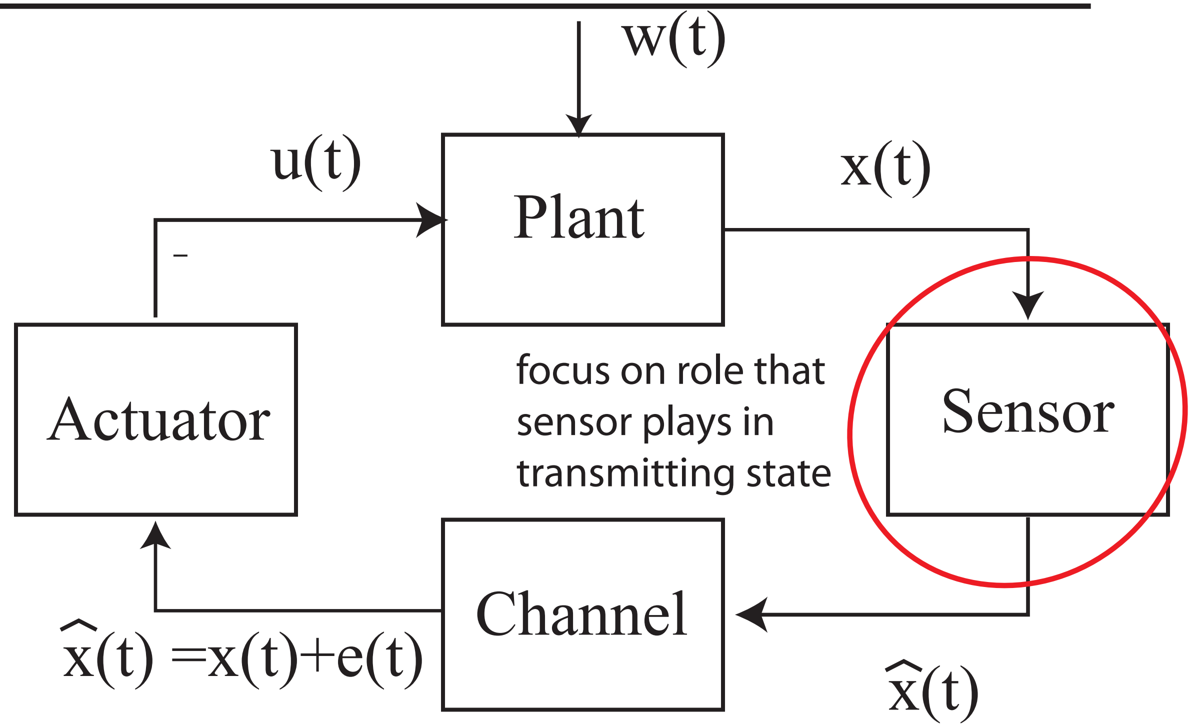
Outline

- Event-triggered Sporadic Feedback
  - ISS Event Triggers and Minimum Attention Control
  - Intersample Time Scaling
  - Controller Design
- Sporadic Feedback over Wireless Channels
  - Almost sure stability
  - Exponential Burstiness
  - Controller Design
- Future Work

System Model - Event-triggered Sampling

$$\dot{x} = f(x, u, w)$$

$$\mathbb{T} = \{\tau_i\}_{i=0}^{\infty}$$



$$\hat{x}(t) = x(\tau_i) \quad \text{for } t \in [\tau_i, \tau_{i+1})$$

$$u(t) = k(\hat{x}(t)) = k(x(t) + e(t))$$

$$\text{Intersample Time} = T_i = \tau_{i+1} - \tau_i$$

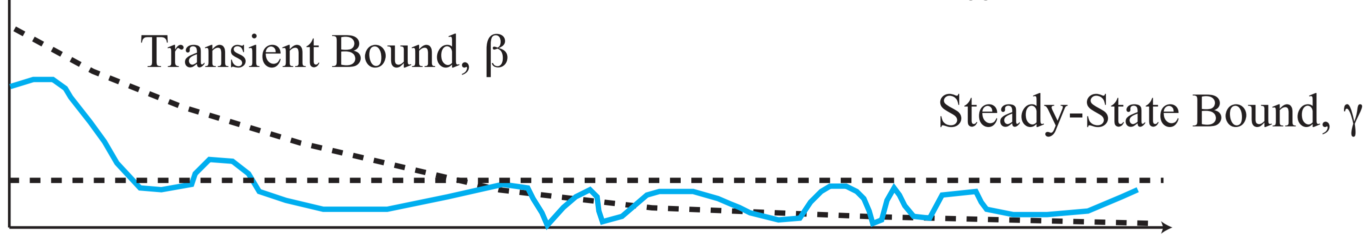
Input to State Stability

- **Process Model:** $\dot{x}(t) = f(x(t), w(t)), \quad x(0) = x_0$

- **Input-to-State Stability (ISS)**

  The system is ISS if there exists $\mathcal{K}L$ function β and class $\mathcal{K}$ function γ such that for any initial condition, $x(0) = x_0$, then the response under any input $u \in \mathcal{L}_\infty$ for all $t \geq 0$ satisfies

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_{\mathcal{L}_\infty})$$



- **ISS-Lyapunov Function**

  C^1 function $V : \Re^n \to \Re$ is **ISS-Lyapunov** function if there exist class $\mathcal{K}$ functions $\underline{\alpha}$, $\overline{\alpha}$, α, and γ such that

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|)$$
$$\dot{V} \leq -\alpha(\|x\|) + \gamma(\|w\|)$$

  If V is an ISS-Lyapunov function, then the system is ISS.

ISS Event-triggering

Consider the system equation

$$\dot{x} = f(x, k(x + e)) = f(x, k(\hat{x}))$$

Given a function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that

INPUT-TO-STATE STABILITY

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)$$
$$\dot{V} \leq -\alpha(|x|) + \gamma(|e|)$$

Select Sampling Instants $\mathbb{T}$ where $0 < \sigma < 1$ so that

EVENT TRIGGER

$$|e(t)| < \gamma^{-1}(\sigma(\alpha(|x|)))$$

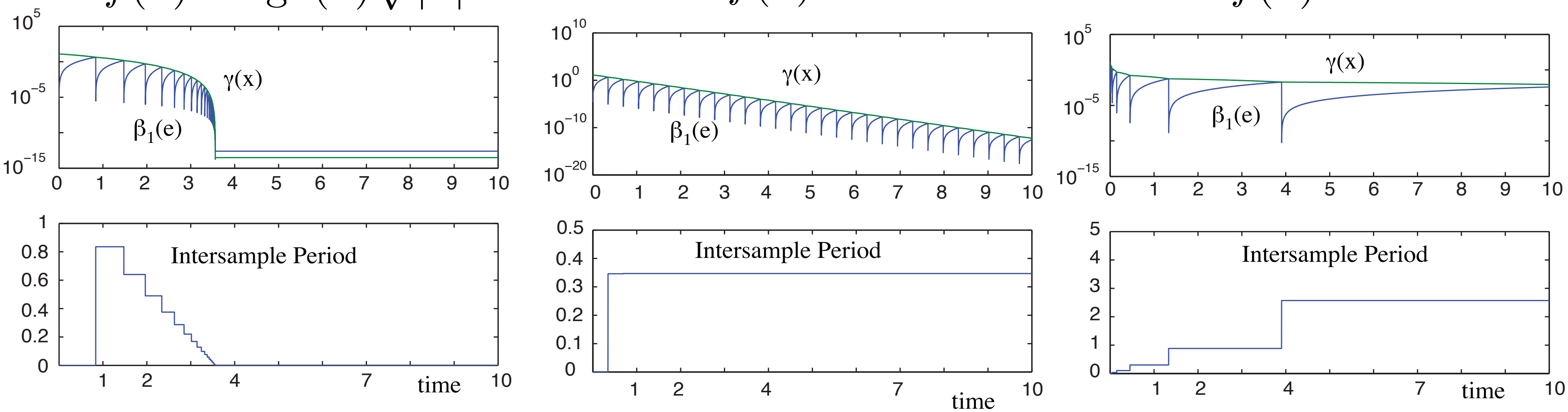
This is sufficient to imply Asymptotic Stability

Event-Trigger and Intersampling Time

- **Process Model:**
$$\begin{aligned}\dot{x}(t) &= f(x(t)) + u(t) \\ u(t) &= -2f(\hat{x}_j(t))\end{aligned}$$

- **Event Trigger:** $\beta_1(\|e_j\|) = e_j^2(t) \geq x^2(t) = \gamma(\|x(t)\|)$

sublinear $f(x) = \mathrm{sgn}(x)\sqrt{|x|}$ | **linear** $f(x) = x$ | **superlinear** $f(x) = x^3$



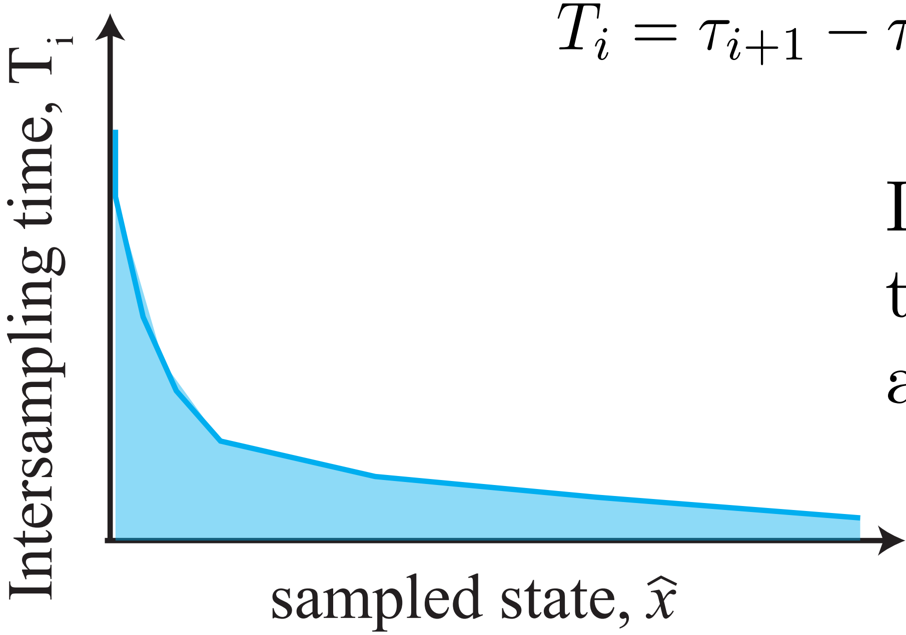
- **Three cases** in which system dynamic is sublinear, linear, or superlinear
  - sublinear dynamics : exhibit ZENO sampling
  - linear dynamics exhibit PERIODIC sampling
  - superlinear dynamics exhibit MINIMUM ATTENTION PROPERTY

Minimum Attention Property

Lower bound on intersampling Time, T_i, as a function of the sampled state, $\hat{x}$.

For all $\epsilon > 0$, there exists $\delta > 0$ and $T \geq 0$ (independent of ϵ) such that for any $|\hat{x}| < \delta$

$$T_i = \tau_{i+1} - \tau_i > T - \epsilon$$



Lower bound on Intersampling time increases to a constant, T, as $|\hat{x}| \to 0$

ISS Event-Trigger with Disturbances

- Consider the plant with disturbance: $\dot{x} = f(x, k(\hat{x}), w)$ and let $\eta : \mathbb{R}^n \to \mathbb{R}^m$ be a **weighting function** with associated error,

$$\tilde{\eta}(x) = \eta(x) - \eta(\hat{x})$$

- Let $V : \mathbb{R}^n \to \mathbb{R}$ be C^1 function such that

$$\begin{aligned} \underline{\alpha}(|x|) &\leq V(x) \leq \overline{\alpha}(|x|) \\ \dot{V} &\leq -\alpha(|x|) + \chi(|x|)\gamma_1(|\tilde{\eta}|) + \gamma_2(|w|) \end{aligned}$$

- The **modified event trigger** renders the sampled data system ISS.

$$|\tilde{\eta}| \leq \gamma_1^{-1}\left(\frac{\sigma\alpha(|x|) + \gamma_3(\overline{w})}{\chi(|x|)}\right) = \theta(|x|)$$

Bounding the Weighted Error

- Rewrite the error's, $e = x - \hat{x}$, time derivative as

$$\frac{d}{dt}|e(t)| = |f(\hat{x} - e, k(\hat{x}), w)|$$

- Under usual Lipschitz assumptions,

$$\frac{d}{dt}|e(t)| < \phi(|\hat{x}|) + L_1|e| + L_2\overline{w}$$

which can be solved to show that the weighted error satisfies

$$|\tilde{\eta}| \leq \frac{\phi(|\hat{x}|) + \delta\overline{w}}{L_1}\left(e^{L_1(t-\tau_i)} - 1\right)$$

Intersampling Time

- Since $|\tilde{\eta}| < \theta(|x|)$, the intersampling time may be bounded as

$$T_i \geq \frac{1}{L_1} \log\left(1 + \frac{\theta(|\hat{x}_{i+1}|)}{\phi(|\hat{x}_i|) + \delta\overline{w}}\right) = T^*(|\hat{x}_i|, |\hat{x}_{i+1}|)$$

  where $\phi(|x|)$ is class $\mathcal{K}$, δ, L_1 are Lipschitz constants, and

$$\theta(|x|) = \gamma_1^{-1}\left(\frac{\sigma\alpha(|x|) + \gamma_3(\overline{w})}{\chi(|x|)}\right)$$

  is the event-triggering threshold

- Minimum attention property requires the term in the parentheses approach a constant strictly greater than 1 as $|\hat{x}_i|$ and $|\hat{x}_{i+1}|$ go to zero.

Minimal Attentive Control under Disturbances

- Intersampling time under bounded ($|w(t)| \leq \overline{w}$) disturbances

$$T_i \geq \frac{1}{L_1} \log \left(1 + \frac{\gamma_1^{-1}\left(\frac{\gamma_3(\overline{w})}{\chi(|\hat{x}_{i+1}|)}\right)}{\phi(|\hat{x}_i|) + \delta\overline{w}} \right)$$

- Since the closed-loop system is ISS under the event-trigger, the numerator is bounded by a positive constant that is independent of $\hat{x}_{i+1}$.
- So as long as χ is a monotone function, this system will be minimally attentive.

Minimal Attentive Control without Disturbances

- Intersampling time when $w(t) = 0$ is

$$T_i \geq \frac{1}{L_1} \log\left(1 + \frac{\theta(|\hat{x}_{i+1}|)}{\phi(|\hat{x}_i|)}\right)$$

where

$$\theta(|\hat{x}_{i+1}|) = \gamma_1^{-1}\left(\frac{\sigma\alpha(|\hat{x}_{i+1}|)}{\chi(|\hat{x}_{i+1}|)}\right)$$

- The system is *strictly* minimally attentive (i.e. $T^* \to \infty$ as $\hat{x} \to 0$ when

$$\lim_{|\hat{x}_i| \to 0} \frac{\phi(|\hat{x}_i|)}{\theta(|\hat{x}_{i+1}|)} = 0$$

Controller Specifications for Strict Minimum Attentiveness

- Note that without disturbance the system is asymptotically stable so that $|\hat{x}_i| \to 0$ and $|\hat{x}_{i+1}| \to 0$.

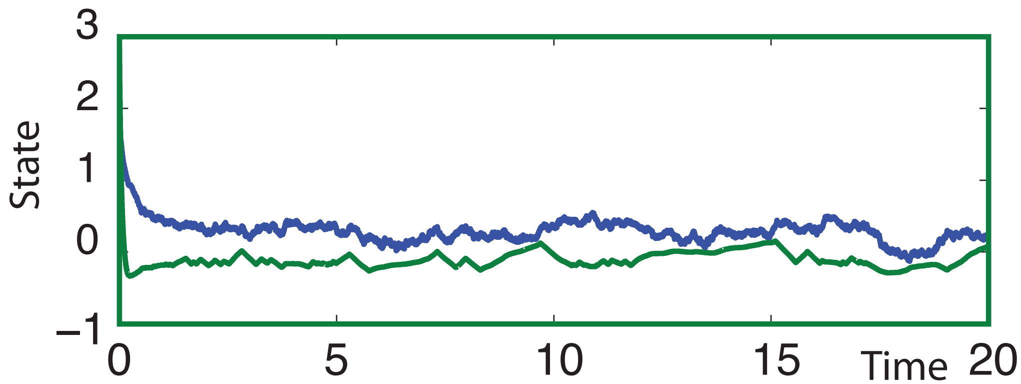
- System is strictly minimally attentive if

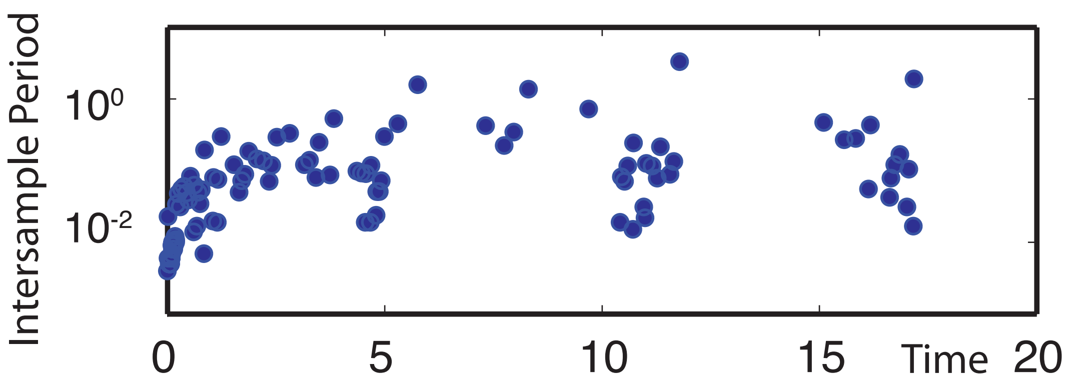
$$\lim_{s\to 0} \frac{\phi(s)}{s} = 0 \quad f(x, k(x), 0) \text{ has order greater than linear}$$
$$\lim_{s\to 0} \frac{\phi(s)}{\theta(s)} = 0 \quad \text{order of } \gamma_1 \text{ greater than order of } \alpha(s)/\chi(s)$$
$$\lim_{s\to 0} \theta(s) = 0 \quad \text{Order of } \alpha \text{ greater than order of } \chi.$$

- These are constraints on the class $\mathcal{K}$ functions bounding $\dot{V}$. In particular, the second condition places a constraint on the *error rejection* ability of the controller.

- The controller must be appropriately selected to ensure the strict minimum attention property.

Example - Controller Design





- System Equations:

$$\dot{x}_1 = -2x_1^3 + x_2^3 + \frac{w}{\sqrt{(2x_1 + x_2)^2 + 1}}$$
$$\dot{x}_2 = x_1(5e^{x_1} - 5 + u) - 2x_2^3 + x_1^3$$

- Candidate ISS-Lyapunov Function

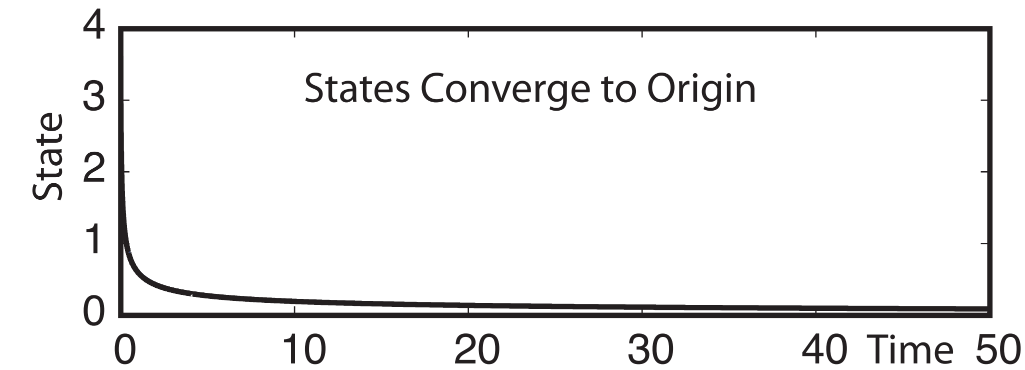
$$\dot{V} = x_1^2 + x_1 x_2 + x_2^2$$

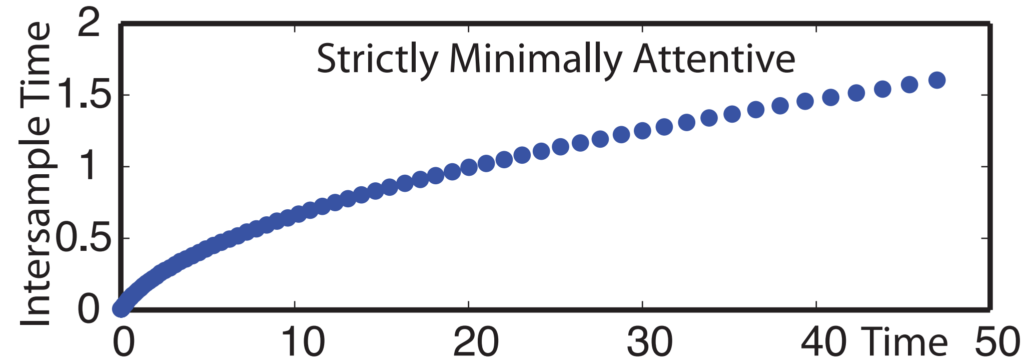
- ISS Dissipative Inequality

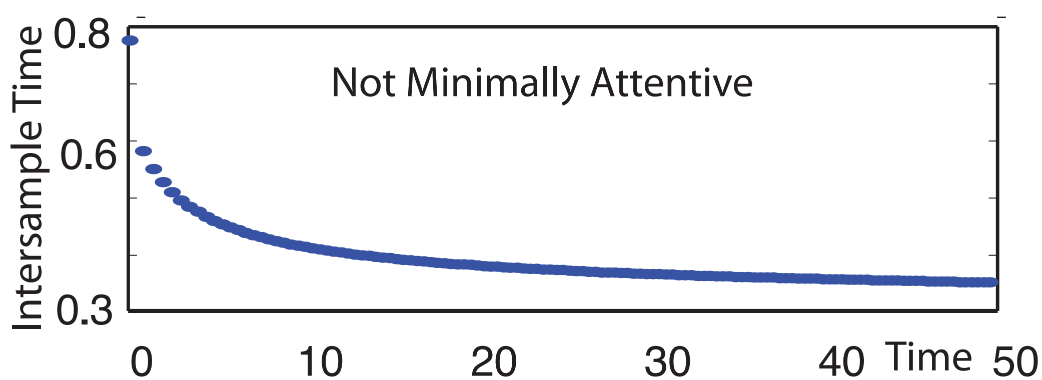
$$\begin{aligned}\dot{V} &\leq -\alpha(|x|) + \chi(|x|)\gamma_1(|\tilde{u}|) + \gamma_2(|w|) \\ &= -3|x|^4 + \sqrt{5}|x|^2|\tilde{u}| + |w|\end{aligned}$$

- Universal formula used to construct ISS controller k.

Error Rejection and Minimum Attention Property







- System with no disturbance, $w = 0$. System Equations:

$$\begin{aligned} \dot{x}_1 &= -2x_1^3 + x_2^3 \\ \dot{x}_2 &= x_1(5e^{x_1} - 5 + u) - 2x_2^3 + x_1^3 \end{aligned}$$

Use two different controllers/event-triggers.

- Strictly Minimally Attentive under event-triggering threshold

$$\theta(r) = \gamma_1^{-1}\left(\frac{\sigma\alpha(r)}{\chi(r)}\right) = \frac{1.2r^2}{5\sqrt{5}}$$

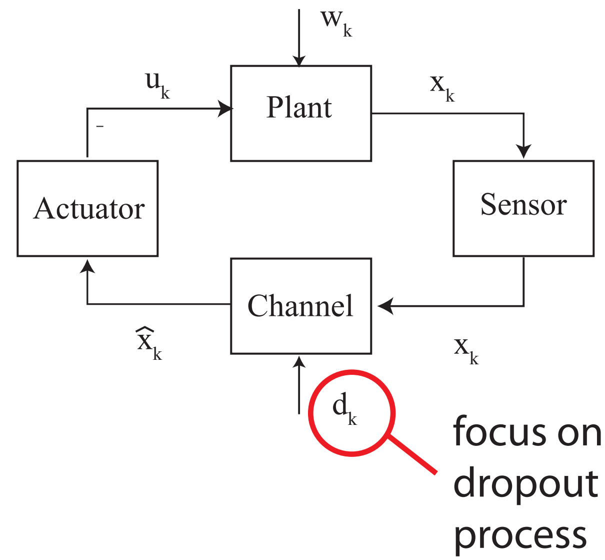
- Not Minimally Attentive under event-triggering threshold

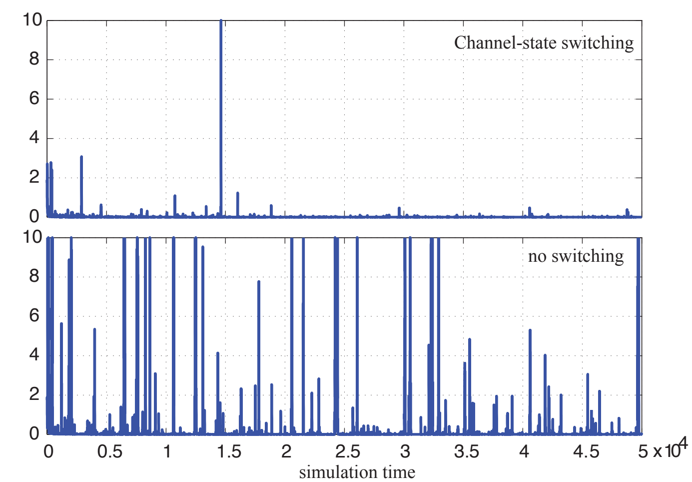
$$\theta(r) = \frac{1.2r^3}{5\sqrt{5}}$$

$\theta(r)$ is same order as f.

Sporadic Feedback due to Dropouts

- Sporadic feedback also occurs when feedback packets are *dropped* by the communication link.
- Prior work has examined the impact such dropouts have on *mean square stability*
- We'll examine a stronger notion of stability (almost sure) under dropouts that are *exponentially bursty.*
- The result will highlight the relation between disturbance rejection, stability, and sporadic feedback.



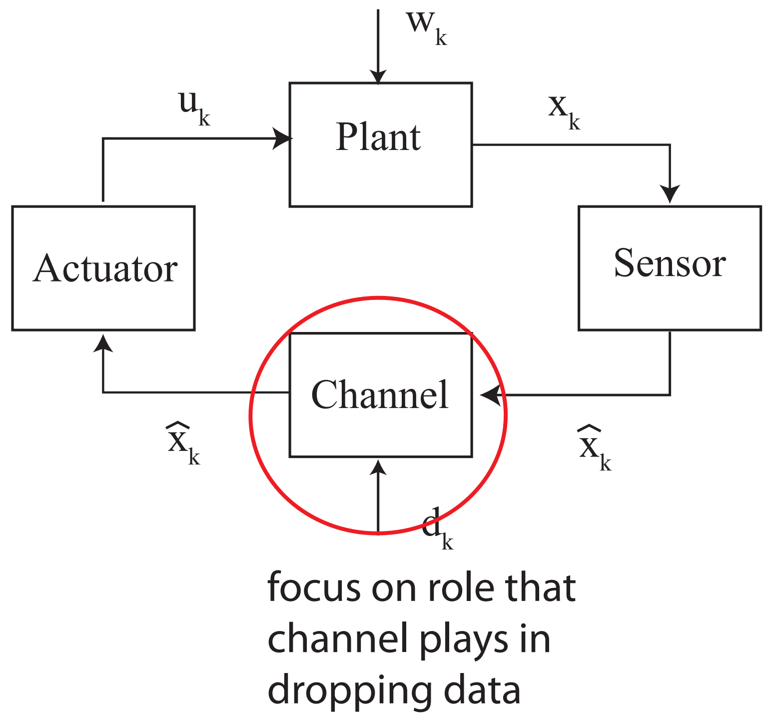


Two different mean square stable systems

System Model - Channel Dropouts

- Scalar Positive System:

$$x_{k+1} = \begin{cases} \alpha x_k + w_k & d_k = 1 \\ \beta x_k + w_k & d_k = 0 \end{cases}$$



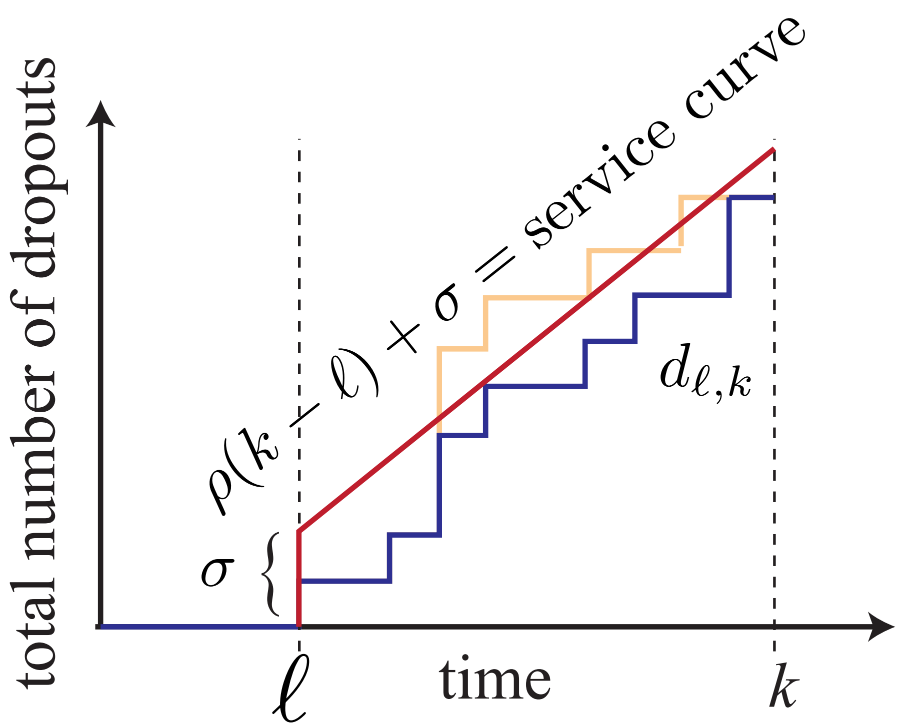
- $\{d_k\}_{k=0}^{\infty}$ is a dropout process.
- Cumulative dropout process is

$$d_{\ell,k} = \sum_{j=\ell}^{k-1} d_j$$

- The system state is

$$x(k;x_0) = \alpha^{d_{0,k}} \beta^{k-d_{0,k}} + \sum_{\ell=1}^{k} \alpha^{d_{\ell,k}} \beta^{k-\ell-d_{\ell,k}} w_{\ell-1}$$

Exponentially Bounded Bursty Dropouts



- The dropout process $\{d_k\}_{k=0}^{\infty}$ is (ρ,σ)-**exponentially bounded bursty (EBB)** if there exists $\gamma > 0$ such that

$$\Pr\{d_{\ell,k} > \rho(k-\ell)+\sigma\} \leq e^{-\gamma\sigma}$$

- The parameter ρ is an "average dropout rate"
- The parameter σ is the "size" of a dropout "burst" (set of consecutive dropouts).
- γ is called the *burst exponent*

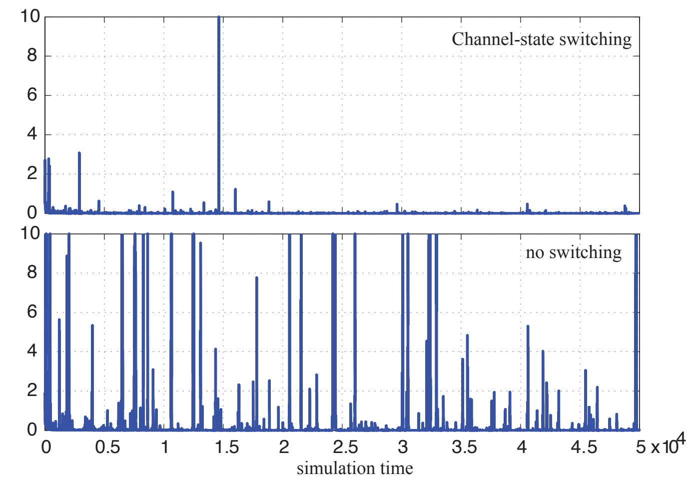
Almost Sure Stability

- Mean square stability

$$\lim_{k\to\infty} \mathbb{E}[x_k^T x_k] \to c$$

There is a finite probability of being arbitrarily far away from origin.

Two different mean square stable systems



- Almost sure stability requires that the probability of being arbitrarily far from the origin goes to zero as $k \to \infty$.

$$\Pr\{\limsup_k A_k^\epsilon\} = 0 \text{ where } A_k^\epsilon = \text{event } |x_k| > \epsilon$$

- Almost sure stability is a *stronger* stability concept than mean square stability.

- More useful for safety-critical systems.

Almost Sure Stability without Disturbances

- Assume that $d_{\ell,k} \leq \rho(k-\ell)+\sigma$ where $\rho < -\frac{\log\beta}{\log\alpha - \log\beta}$, We can then bound the state as

$$x_k \leq \mu^k \left(\frac{\alpha}{\beta}\right)^{\sigma} x_0$$

  where $\mu = \alpha^{\rho}\beta^{1-\rho}$.

- We can use this bound on x_k along with the definition of exponential burstiness to show that

$$\sum_{k=1}^{\infty} \Pr\left\{A_k^{\epsilon}(x_0)\right\} \leq C_1 \sum_{k=1}^{\infty} k^{-2} = \frac{C_1 \pi^2}{6}$$

- From the Borel-Cantelli lemma

$$\textstyle\sum_{k=1}^{\infty} \Pr\left\{A_k^{\epsilon}\right\} < \infty \Rightarrow \Pr\left\{\limsup_k A_k^{\epsilon}\right\} = 0 \Leftrightarrow \text{A.S. stability}$$

Almost Sure Stability with Disturbances

If we now consider disturbances where $\rho < -\frac{\log\beta}{\log\alpha - \log\beta}$, we can use the same techniques to show that the disturbed system is almost sure stable provided

- Disturbance is asymptotically rejected in the sense that there exists $s > 0$ (**response exponent**) such that

$$\mu^k x_0 + \sum_{j=0}^{k-1} \mu^j w_{k-j-1} \leq C k^{-s}$$

- The response exponent, s, and burst exponent, γ satisfy

$$s\gamma > \log\alpha - \log\beta$$

Disturbance Rejection and Dropout Sensitivity

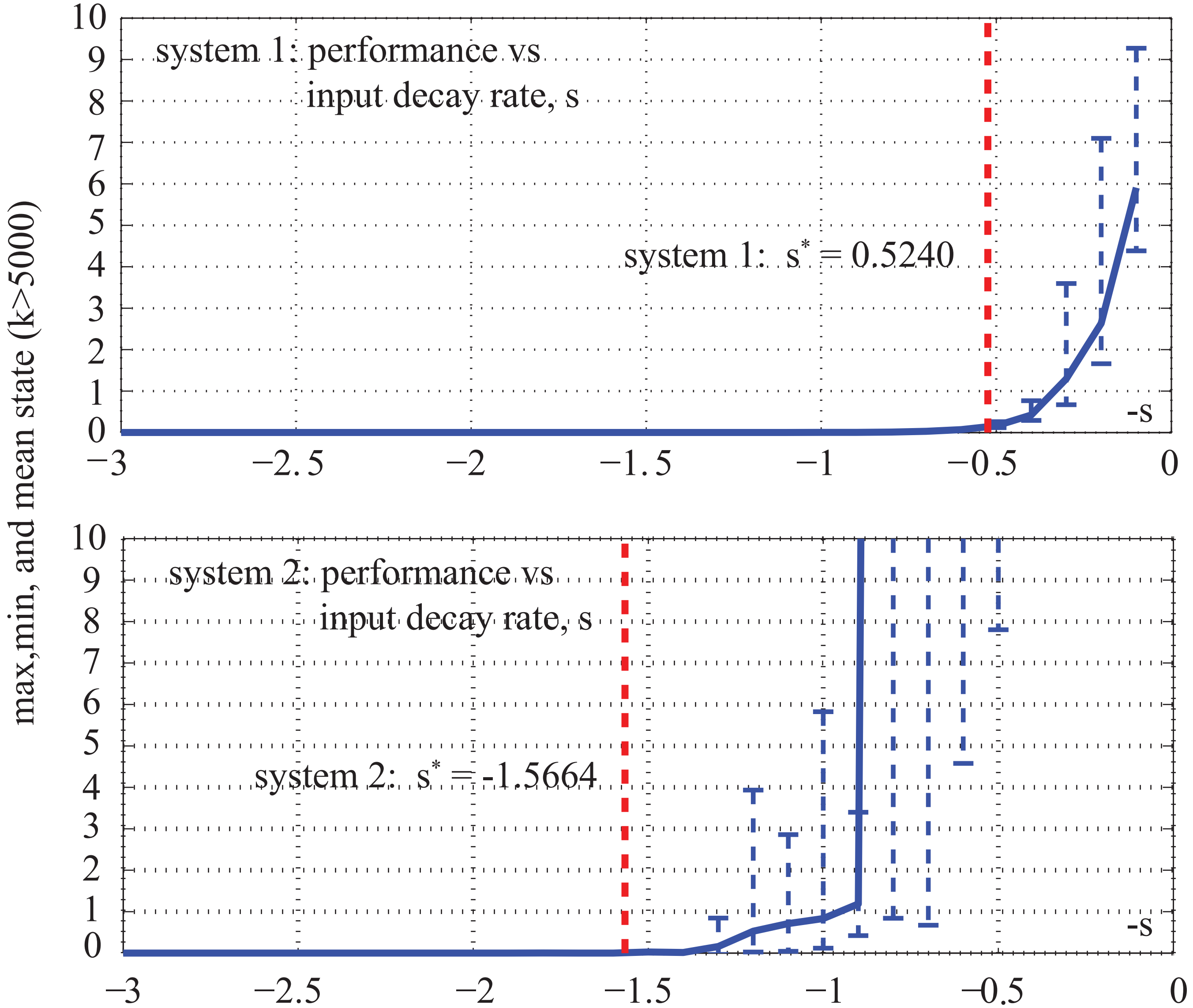
- If the response to the input is uniformly bounded ($s = 0$), then we know the system is *almost sure unstable*.

- If the response exponent can be taken arbitrarily close to zero, then the system is A.S. stable provided the burst exponent is sufficiently large.

- In fact if we can guarantee that the probability of a burst greater than $\sigma^*(\epsilon)$ can never occur then we can guarantee A.S. stability.

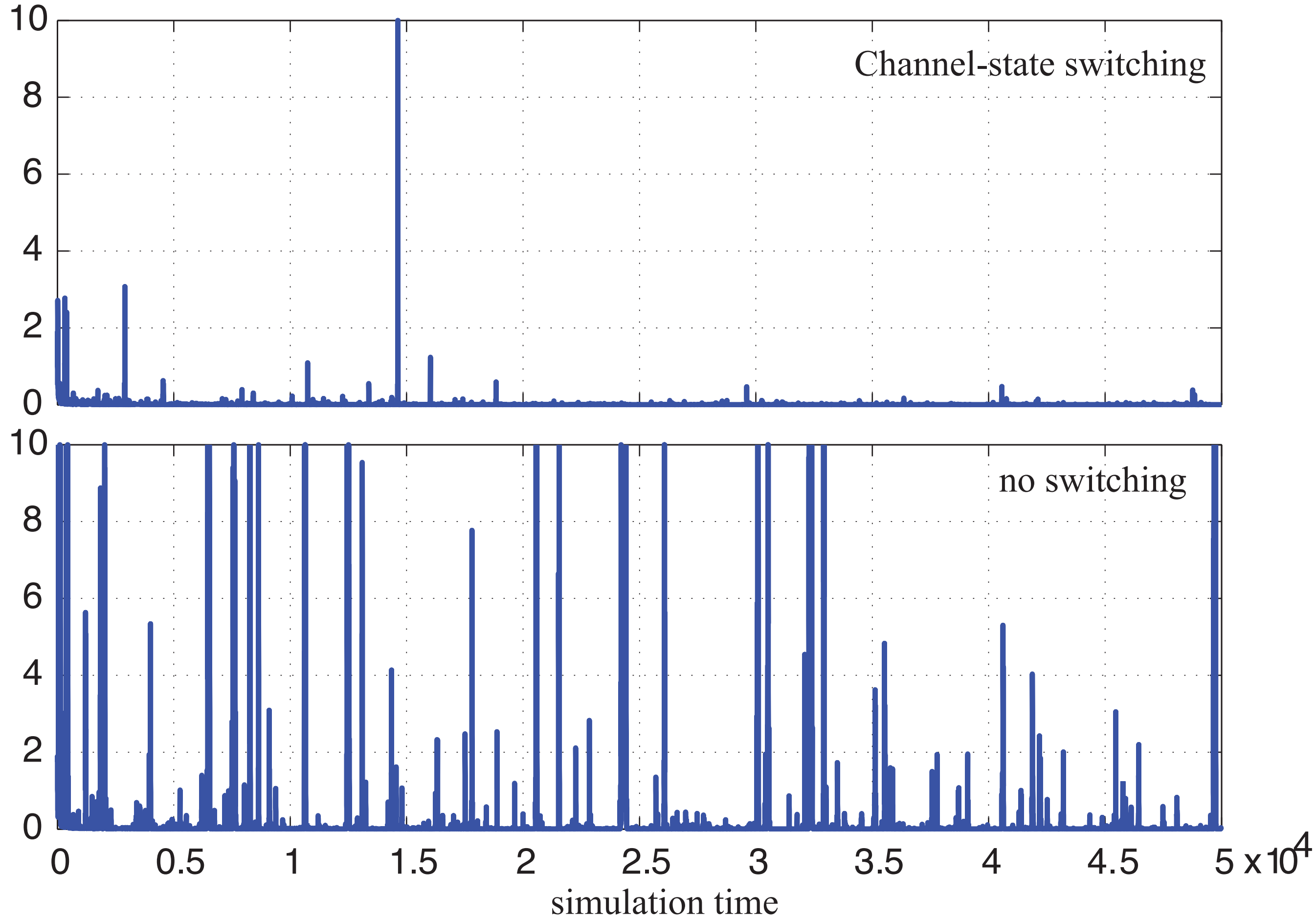
- This shows a fundamental tradeoff exists between a system's sensitivity to dropout bursts and its disturbance rejection ability.

Simulation Experiments Bernoulli Channel



Simulation Experiments - Gilbert-Elliott Channel

Switching controller on detected channel state



Final Remarks

- Impact of Sporadic Feedback on Control System Performance/Stability
- Sporadic Feedback due to
  - event-triggering (choice)
  - channel burstiness (dropouts)
- In both cases, a useful strategy involves changing/switching controller's disturbance rejection ability to provide control performance assurances.
- Application to real-time control over wireless communication links