A Random Coding Approach to Gaussian Multiple Access Channels with Finite Blocklength

Ebrahim MolavianJazi and J. Nicholas Laneman
Department of Electrical Engineering
University of Notre Dame
Email: {emolavia,jnl}@nd.edu

***Abstract*— Contrary to the common use of random coding and typicality decoding for the achievability proofs in information theory, the tightest achievable rates for point-to-point Gaussian channels build either on geometric arguments or composite hypothesis testing, for which direct generalization to multi-user settings appears challenging. In this paper, we provide a new perspective on the procedure of handling input cost constraints for tight achievability results. In particular, we show with a proper choice of input distribution and using a change of measure technique, tight bounds can be achieved via the common random coding argument and a modified typicality decoding. It is observed that a codebook generated randomly according to a uniform distribution on the "power shell" is optimal, at least up to the second order. Such insights are then extended to a Gaussian multiple access channel, for which independent uniform distributions on power shells are shown to be very close to optimal, at least up to second order.**

I. Introduction

Random coding and typicality decoding have proven to be powerful tools in information theory, to the point that, these techniques are the standard method for proving most source and channel coding theorems in an intuitive and straightforward manner [1]. They can be used to prove the asymptotic achievability results for point-to-point (P2P) and multi-user memoryless channel models in the discrete and Gaussian memoryless settings [2], as well as those for the most general set of channels, including those with memory or lacking ergodicity [3]. Recently it has been shown that these methods are also capable of achieving the optimal second-order coding rates for P2P discrete memoryless channels in the near-capacity, finite-blocklength regime [4], [5]. That random coding can operate close to capacity in the non-asymptotic case was also shown much earlier in the context of error exponents, although using maximum likelihood (ML) decoding [6].

By contrast, for P2P Gaussian channels subject to a maximal power constraint, the tightest known achievable rates in the non-asymptotic regime do not rely on random coding along with typicality decoding for handling the input cost constraint. The best known achievable rate is due to Shannon [7] who employs random coding but utilizes relatively sophisticated geometric arguments for the analysis of the performance of the optimal maximum likelihood decoder. The next tightest non-asymptotic achievable rate for P2P Gaussian channels is via the recent $\kappa\beta$ bound of Polyanskiy et al. [4], which is similar to the non-random sequential coding of [8], but whose analysis involves a composite hypothesis test to treat the input cost constraint. The main part of the proof in [4] centers around the performance analysis of this composite test. Other bounds involving Gaussian channels are Gallager's random coding error exponent [6] which uses ML decoding, and the cost-constrained version of Feinstein's coding theorem [8], which uses typicality decoding but with a non-random sequential encoding procedure.

The comparison of the choice of input distribution among all these different bounds for P2P Gaussian channels provides an initial interesting observation. Denote the maximal power constraint by P. The loosest bound in the non-asymptotic sense is the cost-constrained version of Feinstein [8] with an independent and identically distributed (i.i.d) Gaussian input distribution $\mathcal{N}(\mathbf{0}, PI_n)$ or $\mathcal{N}(\mathbf{0}, (P-\delta)I_n)$ with an arbitrarily small δ; however, it can be shown that neither result yields the optimal second-order coding rate. Gallager's random coding error exponent [6] achieves relatively better performance by truncating the Gaussian distribution $\mathcal{N}(\mathbf{0}, PI_n)$ and selecting a thin layer as the input distribution such that $P-\delta \leq X^2 \leq P$ for an arbitrary parameter δ. Polyanskiy et al.'s $\kappa\beta$ achievability result [4], which is second-order optimal, is not defined in terms of an input distribution, but the analysis involves codewords which all lie on the n-dimensional "power shell" of radius $\sqrt{nP}$. Finally, Shannon [7] selects the uniform distribution on the aforementioned power shell as the input distribution. These choices suggest that, as the selection of input distribution becomes more refined and the codewords are placed closer to the power shell, they exhibit better non-asymptotic performance.

Motivated by these observations and following Shannon [7], in this paper we focus on random coding with inputs having a *uniform distribution on the power shell* and derive achievable rates resembling those of Polyanskiy et al. [4] in the near-capacity, finite blocklength regime. Unlike [7] and [4], however, we rely on a slightly modified typicality decoding rule, which is more intuitive and relatively less complicated. Our technique appears easier to generalize to multi-user settings, specifically the Gaussian multiple access channel (MAC) for which we provide a second-order achievable region that appears to match a single-user outer bound except for slight gaps at the two corners of the outer bound.

II. Background

A. Random Coding and Typicality Decoding

The basic idea in a random coding and typicality decoding argument can be reviewed most clearly for a P2P channel $P_{Y^n|X^n}$. The channel encoder generates M codewords of the codebook independently at random according to some given n-letter distribution P_{X^n}, where n is the designated blocklength. Observing the output y^n, the decoder then chooses the first codeword $x^n(\hat{m})$ of the codebook which looks "typical" with y^n in a one-sided sense

$$i(x^n(\hat{m}); y^n) > \log \gamma, \tag{1}$$

where γ is a prescribed threshold and $i(x^n(\hat{m}); y^n)$ is the corresponding realization of the *mutual information random variable*

$$i(x^n; y^n) := \log \frac{P_{Y^n|X^n}(y^n|x^n)}{P_{Y^n}(y^n)}. \tag{2}$$

Here, the *reference* distribution P_{Y^n} is the marginal output distribution induced by the input distribution P_{X^n}, i.e.,

$$P_{Y^n}(\cdot) = \sum_{x^n} P_{X^n}(x^n) P_{Y^n|X^n}(\cdot|x^n). \tag{3}$$

Using one realization of such a code $\{x^n(j)\}_{j=1}^M$, the average error probability can be bounded as the sum of an *outage* probability, that the correct codeword does not look typical, and a *confusion* probability, that a preceding codeword incorrectly looks typical, i.e.,[1]

$$\begin{aligned}\epsilon \le \frac{1}{M} \sum_{k=1}^{M} P_{Y^n|X^n=x^n(k)}[i(x^n(k); Y^n) \le \log \gamma] \\ + \frac{1}{M} \sum_{k=1}^{M} P_{Y^n|X^n=x^n(k)} \left[\bigcup_{j=1}^{k-1} i(x^n(j); Y^n) > \log \gamma \right],\end{aligned} \tag{4}$$

and the error probability averaged over all possible realizations of the codebook can be bounded as

$$\begin{aligned}\epsilon \le P_{X^n} P_{Y^n|X^n}[i(X^n; Y^n) \le \log \gamma] \\ + \frac{M-1}{2} P_{X^n} P_{Y^n}[i(X^n; Y^n) > \log \gamma].\end{aligned} \tag{5}$$

The final result is that there exists a deterministic codebook with M codewords whose average error probability ϵ satisfies (5). It is worth mentioning that, in the standard asymptotic analysis of memoryless channels $P_{Y^n|X^n}(y^n|x^n) = \prod_{t=1}^n P_{Y|X}(y_t|x_t)$, the input distribution is selected i.i.d. $P_{X^n}(x^n) = \prod_{t=1}^n P_X(x_t)$, and the threshold is selected as a function of the *average mutual information* $\log \gamma = n\mathbb{I}(X; Y) - o(n) = n\mathbb{E}_{P_X P_{Y|X}}[i(X; Y)] - o(n)$. This leads to the proof of achievability for rates $\frac{\log M}{n} < \mathbb{I}(X; Y)$. In this paper, however, we would like to preserve the general n-letter form of the input distribution.

[1] Throughout this paper, we use a non-standard notation of the form $P_X P_Y[f(X, Y) \in \mathcal{A}]$ to explicitly denote the distributions with which the probability $\Pr[f(X, Y) \in \mathcal{A}]$ is calculated when X, Y follow the joint distribution $P_X P_Y$.

One can easily extend the result (5) to input cost constrained settings requiring $X^n \in \mathcal{F}_n$ [4]:

$$\begin{aligned}\epsilon \le P_{X^n} P_{Y^n|X^n}[i(X^n; Y^n) \le \log \gamma] \\ + \frac{M-1}{2} P_{X^n} P_{Y^n}[i(X^n; Y^n) > \log \gamma] + P_{X^n}[\mathcal{F}_n^c].\end{aligned} \tag{6}$$

Considering an i.i.d. Gaussian input $P_{X^n} \sim \mathcal{N}(\mathbf{0}, PI_n)$ and applying the central limit theorem (CLT) in (6) results in the approximate achievability bound only useful for $\epsilon \ge \frac{1}{2}$

$$\frac{\log M}{n} \le C(P) - \frac{\log e}{\sqrt{n}} \sqrt{\frac{P}{1+P}} Q^{-1}\left(\epsilon - \frac{1}{2}\right) + O(1), \tag{7}$$

where[2] $C(P) = \frac{1}{2}\log(1+P)$. With i.i.d. Gaussian input $P_{X^n} \sim \mathcal{N}(\mathbf{0}, (P-\delta)I_n)$, the bound (6) yields the following approximation valid for any $0 \le \epsilon \le 1$:

$$\log M \le nC(P-\delta) - \frac{\log e}{\sqrt{n}} \sqrt{\frac{P-\delta}{1+P-\delta}} Q^{-1}(\epsilon) + O(1), \tag{8}$$

where $\delta > 0$ is an arbitrarily small constant.

B. Polyanskiy et al.'s $\kappa\beta$ Bound

A tighter achievability result for the P2P Gaussian channel is provided in the recent $\kappa\beta$ bound of Polyanskiy et al. [4]. Using a slightly different language from that in [4], this bound fixes an arbitrary *output* distribution Q_{Y^n}, similar to [9], and employs this as the reference distribution for the definition of a *modified* mutual information random variable:

$$\tilde{i}(x^n; y^n) := \log \frac{P_{Y^n|X^n}(y^n|x^n)}{Q_{Y^n}(y^n)}. \tag{9}$$

Building upon the maximal coding idea, the sequential codeword generation process stops after M codewords $\{x^n(j)\}_{j=1}^M$ if the error probability for any choice of the $(M+1)$-th sequence exceeds the target maximal error probability ϵ, i.e.,

$$\begin{aligned}\epsilon < P_{Y^n|X^n=x^n}[\tilde{i}(x^n; Y^n) \le \log \gamma] \\ + P_{Y^n|X^n=x^n} \left[\bigcup_{j=1}^{M} \tilde{i}(x^n(j); Y^n) > \log \gamma \right]\end{aligned} \tag{10}$$

for all sequences $x^n \in \mathcal{F}_n$ where $\mathcal{F}_n$ is the feasible set of codewords according to the input cost constraint. Thinking of the union in the brackets of the second term above as a binary test, one can cast the problem into the framework of the following composite hypothesis test which is used to treat the input cost constraint:

$$\begin{aligned}\kappa_\tau\left(\{P_{Y^n|X^n=x^n}\}_{x^n \in F}, Q_{Y^n}\right) \\ := \min_{Z: P_{Y^n|X^n=x^n}[Z(Y^n)=1] > \tau, \forall x^n \in F} Q_{Y^n}[Z(Y^n) = 1],\end{aligned} \tag{11}$$

where $Z(Y^n)$ is a binary test choosing either the class of conditional channel laws $\{P_{Y^n|X^n=x^n}\}_{x^n \in F}$ if $Z = 1$, or

[2] As usual, $Q^{-1}(\cdot)$ is the inverse of the complementary cumulative distribution function (CDF) of a standard Gaussian distribution $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$.

the unconditional output distribution Q_{Y^n} if $Z = 0$. Notice that rearranging (10) yields

$$P_{Y^n|X^n=x^n}\left[\bigcup_{j=1}^{M}\tilde{i}(x^n(j);Y^n) > \log\gamma\right] > \epsilon - P_{Y^n|X^n=x^n}[\tilde{i}(x^n;Y^n) \le \log\gamma] \ge \tau^*$$

where

$$\tau^* = \epsilon - \sup_{x^n \in F} P_{Y^n|X^n=x^n}[\tilde{i}(x^n;Y^n) \le \log\gamma]. \quad (12)$$

The $\kappa\beta$ bound of [4] for maximal error probability can then be stated as follows:

$$\kappa_{\tau^*}\left(\{P_{Y^n|X^n=x^n}\}_{x^n\in F}, Q_{Y^n}\right) \le Q_{Y^n}\left[\bigcup_{j=1}^{M}\tilde{i}(x^n(j);Y^n) > \log\gamma\right] \quad (13)$$

$$\le M \sup_{x^n\in F} Q_{Y^n}[\tilde{i}(x^n;Y^n) > \log\gamma]. \quad (14)$$

Interpretation of the composite hypothesis test κ_τ and accordingly its evaluation for the P2P Gaussian channel is quite involved. Polyanskiy et al. [4] invoke arguments from abstract algebra to analyze the performance of this test for the feasible set $\mathcal{F}_n = \{x^n \in \mathbb{R}^n : ||x^n|| = \sqrt{nP}\}$ being the "power shell" and the special choice $Q_{Y^n} \sim \mathcal{N}(\mathbf{0}, (1+P)I_n)$ with the selection $\tau^* = 1/\sqrt{n}$, finally concluding that

$$\log\kappa_{\tau^*} \ge \frac{1}{2}\log n + O(1), \quad (15)$$

which with application of the CLT results in the following second-order optimal achievable rate for the P2P Gaussian channel

$$\frac{\log M}{n} \le C(P) - \sqrt{\frac{V(P)}{n}}Q^{-1}(\epsilon) + O(1), \quad (16)$$

where $V(P)$ is the dispersion of the Gaussian P2P channel

$$V(P) = \frac{\log^2 e}{2}\frac{P(P+2)}{(1+P)^2}. \quad (17)$$

Comparing the $\kappa\beta$ bound of [4] with the random coding and typicality decoding method discussed earlier suggests an important insight. Introducing the composite hypothesis bound κ_τ in [4] enables a *change of measure* from $P_{Y^n|X^n=x^n}$ in (10) to Q_{Y^n} in (14) in computing the confusion probability. A similar process occurs in the random coding argument with typicality decoding, as the random generation of the codebook makes it possible to change the measure for computation of the confusion probability from $P_{Y^n|X^n=x^n}$ in (4) to its average P_{Y^n} in (5). We suspect the reason why the composite κ_τ test is introduced in [4] is to enable such a change of measure argument which is required for the evaluation of the confusion probability, but is not directly available in the sequential generation of a maximal coding, which does not incorporate any random generation process. This insight is one of the main ideas we will use in this paper for the analysis of the Gaussian P2P channel and MAC with a random coding and typicality decoding.

C. Shannon's Geometric Bound

As mentioned before, the best known achievable rate for P2P Gaussian channel is due to Shannon [7] who starts with a random codebook generation according to the uniform distribution on the n-dimensional sphere of radius $\sqrt{nP}$, i.e. the power shell, but follows with the optimal ML decoding method. Since this rule is equivalent to minimum Euclidian distance in $\mathbb{R}^n$, he employs geometric arguments to evaluate and bound the code-ensemble-average probability that the i.i.d. Gaussian channel noise moves the output closer to some incorrect codeword than to the originally transmitted codeword.

A key observation in Shannon's work is his use of the uniform distribution on the power shell, which enables him to develop sharp non-asymptotic bounds. In this paper, we will follow Shannon in this respect, but rely on the more familiar and less complex method of typicality decoding which we show is still capable of achieving sharp non-asymptotic bounds for the Gaussian channel, at least up to the second order.

Having reviewed the basic elements of the different procedures for handling cost constraints, especially in Gaussian settings, we now move on to the formal statement of our problems and results.

III. P2P Gaussian Channel

A general P2P channel with input cost constraint and without feedback consists of an input alphabet $\mathcal{X}$, an output alphabet $\mathcal{Y}$, and an n-letter channel transition probability given by $P_{Y^n|X^n}(y^n|x^n)\colon \mathcal{F}_n \to \mathcal{Y}^n$, where $\mathcal{F}_n \subseteq \mathcal{X}^n$ is the feasible set of n-letter input sequences.

In particular, a P2P memoryless Gaussian channel without feedback consists of an input and an output taking values on the real line $\mathbb{R}$ and a channel transition probability density $P_{Y|X}(y|x)\colon \mathbb{R} \to \mathbb{R}$ whose n-th extension follows $\mathcal{N}(x^n, I_n)$, i.e.,

$$P_{Y^n|X^n}(y^n|x^n) = \prod_{t=1}^{n} P_{Y|X}(y_t|x_t) = (2\pi)^{-n/2}e^{-||y^n-x^n||^2/2}.$$

For such a P2P Gaussian channel, an (n, M, ϵ, P) code is composed of a message set $\mathcal{M} = \{1, ..., M\}$ and a corresponding set of codewords and mutually exclusive decoding regions $\{(x^n(j), D_j)\}$ with $j \in \mathcal{M}$, such that the average error probability satisfies

$$P_e^{(n)} := \frac{1}{M}\sum_{j=1}^{M}\Pr[Y^n \notin D_j | X^n(j) \text{ sent}] \le \epsilon,$$

and each codeword satisfies a maximal power constraint:

$$\frac{1}{n}||x^n(j)||^2 \le P, \qquad \forall j \in \mathcal{M}.$$

Accordingly, a rate $\frac{\log M}{n}$ is *achievable* for the P2P Gaussian channel with finite blocklength n, average error probability ϵ, and maximal power P if such an (n, M, ϵ, P) code exists.

This section summarizes our main results for P2P Gaussian channels. We first state a modified random coding and typicality decoding achievability result for general P2P channels with input cost constraints, which expresses a non-asymptotic achievable rate valid for any blocklength. It basically describes the error probability in terms of the outage, confusion, and constraint-violation probabilities, and is based on the dependence testing (DT) bound of [4].

Theorem 1: For a P2P channel $(\mathcal{X}, P_{Y^n|X^n}(y^n|x^n), \mathcal{Y})$, any input distribution P_{X^n} and any output distribution Q_{Y^n}, there exists an (n, M, ϵ) code satisfying the input cost constraint $\mathcal{F}_n$ with

$$\epsilon \leq P_{X^n}P_{Y^n|X^n}[\tilde{i}(X^n;Y^n) \leq \log\gamma(X^n)] + \frac{M-1}{2}P_{X^n}P_{Y^n}[\tilde{i}(X^n;Y^n) > \log\gamma(X^n)] + P_{X^n}[\mathcal{F}_n^c], \tag{18}$$

where the modified mutual information random variable $\tilde{i}(X^n;Y^n)$ is defined as

$$\tilde{i}(x^n;y^n) := \log\frac{P_{Y^n|X^n}(y^n|x^n)}{Q_{Y^n}(y^n)}, \tag{19}$$

and $\gamma : \mathcal{X}^n \to [0,\infty)$ is an arbitrary measurable function whose optimal choice to give highest rates is $\gamma(x^n) \equiv \frac{M-1}{2}$.

Proof: For brevity, we do not include a complete proof. The main idea is to use the conventional random coding and typicality decoding method, as reviewed in Section II, but instead to define typicality using the modified mutual information random variable. The input cost constraint can be addressed similar to [4] by simply taking the decoding threshold $\gamma = \infty$ for the randomly generated sequences that do not belong to the feasible input set $\mathcal{F}_n$ and remapping all of them to an arbitrary sequence belonging to $\mathcal{F}_n$. ■

If the input distribution is chosen to be i.i.d., then an evaluation of the modified DT bound above would have been straightforward, using a CLT for the first term, the outage probability, and a large deviation bound for the second term, the confusion probability. However, for non-i.i.d. input distributions, such as the uniform distribution on the power shell, the evaluation of the second term, the confusion probability, can be especially challenging due to the potential complicated nature of the actual output distribution P_{Y^n} induced by the input distribution P_{X^n}. We are therefore interested in changing the measure with which the confusion probability is analyzed, as follows:

$$\begin{aligned}
&P_{X^n}P_{Y^n}[\tilde{i}(X^n;Y^n) > \log\gamma(X^n)] \\
&= \mathbb{E}_{P_{X^n}}\left[\mathbb{E}_{P_{Y^n}}\left[1\left\{\tilde{i}(X^n;Y^n) > \log\gamma(X^n)\right\}\right]\right] \\
&= \iint 1\left\{\tilde{i}(x^n;y^n) > \log\gamma(x^n)\right\} dP_{Y^n}(y^n)dP_{X^n}(x^n) \\
&= \iint 1\{\tilde{i}(x^n;y^n) > \log\gamma(x^n)\}\frac{dP_{Y^n}(y^n)}{dQ_{Y^n}(y^n)}dQ_{Y^n}(y^n)dP_{X^n}(x^n) \\
&= \mathbb{E}_{P_{X^n}}\left[\mathbb{E}_{Q_{Y^n}}\left[\frac{dP_{Y^n}(Y^n)}{dQ_{Y^n}(Y^n)}1\{\tilde{i}(X^n;Y^n) > \log\gamma(X^n)\}\right]\right].
\end{aligned} \tag{20}$$

The final expression (20) enables us to compute the confusion probability with respect to the more convenient measure Q_{Y^n}, but at the expense of the additional factor $\frac{dP_{Y^n}(Y^n)}{dQ_{Y^n}(Y^n)}$. If we can uniformly bound this factor for all y^n by some positive constant K (or even an slowly growing function K_n independent of y^n), then we will be able to further bound (18) in a very simple way as follows:

$$\epsilon \leq P_{X^n}P_{Y^n|X^n}[\tilde{i}(X^n;Y^n) \leq \log\gamma(X^n)] + K\frac{M-1}{2}P_{X^n}Q_{Y^n}[\tilde{i}(X^n;Y^n) > \log\gamma(X^n)] + P_{X^n}[\mathcal{F}_n^c]. \tag{21}$$

A close examination of [4] shows that the κ_τ performance characteristic in the $\kappa\beta$ bound is also mainly concerned about the factor $\frac{dP_{Y^n}(Y^n)}{dQ_{Y^n}(Y^n)}$ introduced above, and the bound (15) is analogous to the uniform bounding by K in the analysis above. The difference is that our analysis using random coding, typicality decoding, and change of measure is a more transparent procedure and more closely follows conventional lines of argument.

We now specialize the analysis above to the P2P Gaussian channel. First, we choose the input distribution to be the uniform distribution on the power shell

$$P_{X^n}(x^n) = \frac{\delta(||x^n|| = \sqrt{nP})}{S_n(\sqrt{nP})}, \tag{22}$$

where δ is the Dirac delta function and $S_n(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)}r^{n-1}$ is the surface area of an n-dimensional sphere of radius r. Notice that this distribution satisfies the input power constraint with probability one, so that

$$P_{X^n}[\mathcal{F}_n^c] = 0. \tag{23}$$

Moreover, the output distribution induced by this input is

$$P_{Y^n}(y^n) = \frac{1}{2}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right)e^{-||y^n||^2/2}e^{-nP/2}\frac{I_{n/2-1}(||y^n||\sqrt{nP})}{(||y^n||\sqrt{nP})^{n/2-1}}, \tag{24}$$

where $I_v(\cdot)$ is the modified Bessel function of the first kind and v-th order. It is worth mentioning that the general form of the above marginal distribution is obtained in [10] up to its $||y^n||$-independent coefficient. Next, we choose the output distribution Q_{Y^n} to be the capacity-achieving output distribution $\mathcal{N}(\mathbf{0}, (1+P)I_n)$. The following proposition will then bound the divergence term introduced in (20). The reader is referred to [11] for the proof, which is a slight generalization of that in [4, p. 2347].

Proposition 1: Let P_{Y^n} be the distribution (24) induced on the output of the P2P Gaussian channel by the uniform input distribution (22) on the power shell, and let Q_{Y^n} be the capacity-achieving output distribution $\mathcal{N}(\mathbf{0}, (1+P)I_n)$. There exists a positive constant K such that, for sufficiently large n,

$$\frac{dP_{Y^n}(y^n)}{dQ_{Y^n}(y^n)} \leq K, \qquad \forall\, y^n \in \mathbb{R}^n. \tag{25}$$

Remark. Using some more complicated manipulations, this proposition can be shown to be valid for any finite n, but the above statement is enough for our asymptotic analysis.

Combining the modified random coding and typicality decoding bound in Theorem 1, the change of measure argument (20), and the uniform bound of Proposition 1, along with an application of the CLT as in [4], leads to the second-order-optimal achievability bound [4], [5]

$$\frac{\log M}{n} \le C(P) - \sqrt{\frac{V(P)}{n}} Q^{-1}(\epsilon) + O(1),$$

where $C(P)$ and $V(P)$ are the capacity and dispersion of the P2P Gaussian channel, respectively.

IV. Gaussian MAC

A general 2-user multiple access channel (MAC) with input cost constraints and without feedback consists of two input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, an output alphabet $\mathcal{Y}$, and an n-letter channel transition probability given by $P_{Y^n|X_1^nX_2^n}(y^n|x_1^n,x_2^n)\colon \mathcal{F}_{1n} \times \mathcal{F}_{2n} \to \mathcal{Y}^n$, where $\mathcal{F}_{1n} \subseteq \mathcal{X}_1^n$ and $\mathcal{F}_{2n} \subseteq \mathcal{X}_2^n$ are the feasible sets of n-letter input sequences for the two users, respectively.

As a particular example, a memoryless Gaussian MAC without feedback consists of two inputs and an output taking values on the real line $\mathbb{R}$ and a channel transition probability density $P_{Y|X_1X_2}(y|x_1,x_2)\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ whose n-th extension follows $\mathcal{N}(x_1^n + x_2^n, I_n)$, i.e.,

$$\begin{aligned} P_{Y^n|X_1^nX_2^n}(y^n|x_1^n,x_2^n) &= \prod_{t=1}^{n} P_{Y|X_1X_2}(y_t|x_{1t},x_{2t}) \\ &= (2\pi)^{-n/2} e^{-||y^n - x_1^n - x_2^n||^2/2}. \end{aligned}$$

An $(n, M_1, M_2, \epsilon, P_1, P_2)$ code for a Gaussian MAC is composed of two message sets $\mathcal{M}_1 = \{1,...,M_1\}$ and $\mathcal{M}_2 = \{1,...,M_2\}$, and a corresponding set of codeword pairs and mutually exclusive decoding regions $\{(x_1^n(j), x_2^n(k), D_{j,k})\}$, with $j \in \mathcal{M}_1$ and $k \in \mathcal{M}_2$, such that the average error probability satisfies

$$P_e^{(n)} := \frac{1}{M_1M_2}\sum_{j=1}^{M_1}\sum_{k=1}^{M_2} \Pr[Y^n \notin D_{j,k} | X_1^n(j), X_2^n(k) \text{ sent}] \le \epsilon,$$

and each codeword satisfies a maximal power constraint:

$$\begin{aligned} \frac{1}{n}||x_1^n(j)||^2 &\le P_1, \qquad \forall j \in \mathcal{M}_1, \\ \frac{1}{n}||x_2^n(k)||^2 &\le P_2, \qquad \forall k \in \mathcal{M}_2. \end{aligned}$$

Accordingly, an $\left(\frac{\log M_1}{n}, \frac{\log M_2}{n}\right)$ pair is *achievable* for a Gaussian MAC with finite blocklength n, average error probability ϵ, and maximal powers P_1 and P_2 if such an $(n, M_1, M_2, \epsilon, P_1, P_2)$ code exists.

This section summarizes our main results for the Gaussian MAC. We first state a modified random coding and typicality decoding achievability result for a general MAC with input cost constraints, which expresses a non-asymptotic achievable rate valid for any blocklength. It basically describes the error probability in terms of the outage, confusion, and constraint-violation probabilities, and is based on the DT bound for the discrete MAC [12].

Theorem 2: For a MAC $(\mathcal{X}_1, \mathcal{X}_2, P_{Y^n|X_1^nX_2^n}(y^n|x_1^n,x_2^n), \mathcal{Y})$, any pair of independent input distributions $P_{X_1^n}$ and $P_{X_2^n}$, and any triple of (conditional) output distributions $Q^{(1)}_{Y^n|X_2^n}, Q^{(2)}_{Y^n|X_1^n}, Q^{(3)}_{Y^n}$, there exists an (n, M_1, M_2, ϵ) code satisfying input cost constraints $\mathcal{F}_{1n}$ and $\mathcal{F}_{2n}$ with

$$\begin{aligned} \epsilon \le\, & P_{X_1^n}P_{X_2^n}P_{Y^n|X_1^nX_2^n}\left[\tilde{i}(X_1^n;Y^n|X_2^n) \le \log\gamma_1(X_1^n,X_2^n)\right] \\ &+ \frac{M_1-1}{2} P_{X_1^n}P_{X_2^n}P_{Y^n|X_2^n}\left[\tilde{i}(X_1^n;Y^n|X_2^n) > \log\gamma_1(X_1^n,X_2^n)\right] \\ &+ P_{X_1^n}P_{X_2^n}P_{Y^n|X_1^nX_2^n}\left[\tilde{i}(X_2^n;Y^n|X_1^n) \le \log\gamma_2(X_1^n,X_2^n)\right] \\ &+ \frac{M_2-1}{2} P_{X_1^n}P_{X_2^n}P_{Y^n|X_1^n}\left[\tilde{i}(X_2^n;Y^n|X_1^n) > \log\gamma_2(X_1^n,X_2^n)\right] \\ &+ P_{X_1^n}P_{X_2^n}P_{Y^n|X_1^nX_2^n}\left[\tilde{i}(X_1^nX_2^n;Y^n) \le \log\gamma_3(X_1^n,X_2^n)\right] \\ &+ \frac{(M_1-1)(M_2-1)}{2} P_{X_1^n}P_{X_2^n}P_{Y^n}\left[\tilde{i}(X_1^nX_2^n;Y^n) > \log\gamma_3(X_1^n,X_2^n)\right] \\ &+ P_{X_1^n}P_{X_2^n}[\mathcal{F}_{1n}^c \cup \mathcal{F}_{2n}^c], \end{aligned} \tag{26}$$

where the modified mutual information random variables are

$$\tilde{i}(x_1^n;y^n|x_2^n) := \log\frac{P_{Y^n|X_1^nX_2^n}(y^n|x_1^n,x_2^n)}{Q^{(1)}_{Y^n|X_2^n}(y^n|x_2^n)}, \tag{27}$$

$$\tilde{i}(x_2^n;y^n|x_1^n) := \log\frac{P_{Y^n|X_1^nX_2^n}(y^n|x_1^n,x_2^n)}{Q^{(2)}_{Y^n|X_1^n}(y^n|x_1^n)}, \tag{28}$$

$$\tilde{i}(x_1^nx_2^n;y^n) := \log\frac{P_{Y^n|X_1^nX_2^n}(y^n|x_1^n,x_2^n)}{Q^{(3)}_{Y^n}(y^n)}, \tag{29}$$

and where $\gamma_1, \gamma_2, \gamma_3 : \mathcal{X}_1^n \times \mathcal{X}_2^n \to [0,\infty)$ are arbitrary measurable functions whose optimal choices to give the highest rates are as follows:

$$\gamma_1(X_1^n,X_2^n) \equiv \frac{M_1-1}{2}, \qquad \gamma_2(X_1^n,X_2^n) \equiv \frac{M_2-1}{2},$$
$$\gamma_3(X_1^n,X_2^n) \equiv \frac{(M_1-1)(M_2-1)}{2}.$$

The above expression for the random coding and typicality decoding bound is to match our outage-splitting approach later in Theorem 4. However, it is possible to strengthen this bound by focusing on the three outages *jointly*.

Theorem 3: For a MAC $(\mathcal{X}_1, \mathcal{X}_2, P_{Y^n|X_1^nX_2^n}(y^n|x_1^n,x_2^n), \mathcal{Y})$, for any pair of independent input distributions $P_{X_1^n}$ and $P_{X_2^n}$, and any triple of (conditional) output distributions $Q^{(1)}_{Y^n|X_2^n}, Q^{(2)}_{Y^n|X_1^n}, Q^{(3)}_{Y^n}$, there exists an (n, M_1, M_2, ϵ) code satisfying input cost constraints $\mathcal{F}_{1n}$ and $\mathcal{F}_{2n}$ with

$$\begin{aligned} \epsilon \le\, & P_{X_1^n}P_{X_2^n}P_{Y^n|X_1^nX_2^n}\left[\tilde{i}(X_1^n;Y^n|X_2^n) \le \log\gamma_1(X_1^n,X_2^n)\right. \\ &\qquad \cup\ \tilde{i}(X_2^n;Y^n|X_1^n) \le \log\gamma_2(X_1^n,X_2^n) \\ &\qquad \cup\ \left.\tilde{i}(X_1^nX_2^n;Y^n) \le \log\gamma_3(X_1^n,X_2^n)\right] \\ &+ \frac{M_1-1}{2} P_{X_1^n}P_{X_2^n}P_{Y^n|X_2^n}\left[\tilde{i}(X_1^n;Y^n|X_2^n) > \log\gamma_1(X_1^n,X_2^n)\right] \\ &+ \frac{M_2-1}{2} P_{X_1^n}P_{X_2^n}P_{Y^n|X_1^n}\left[\tilde{i}(X_2^n;Y^n|X_1^n) > \log\gamma_2(X_1^n,X_2^n)\right] \\ &+ \frac{(M_1-1)(M_2-1)}{2} P_{X_1^n}P_{X_2^n}P_{Y^n}\left[\tilde{i}(X_1^nX_2^n;Y^n) > \log\gamma_3(X_1^n,X_2^n)\right] \\ &+ P_{X_1^n}P_{X_2^n}[\mathcal{F}_{1n}^c \cup \mathcal{F}_{2n}^c], \end{aligned} \tag{30}$$

where the modified mutual information random variables are defined in (27), (28), and (29) and where $\gamma_1, \gamma_2, \gamma_3 : \mathcal{X}_1^n \times \mathcal{X}_2^n \to [0, \infty)$ are arbitrary measurable functions.

Proof: (Theorems 2 and 3) The complete proof, which again uses random coding and typicality decoding, is similar to that for the discrete MAC [12] and is not included here for brevity. Analogous to the proof of Theorem 1, the modified mutual information random variables are used for defining typicality, and the input cost constraints are included via infinite decoding thresholds. ∎

Similar to the P2P case, an evaluation of the three confusion probabilities in the modified DT bounds of Theorems 2 and 3 with respect to (w.r.t.) a pair of non-i.i.d. input distributions is challenging due to the complicated nature of the induced (conditional) output distributions. Thus, we again appeal to a change of measure argument for computing these confusion probabilities. Analogous to (20), we can show that the following equalities hold.

$$\begin{aligned} &P_{X_1^n} P_{X_2^n} P_{Y^n|X_2^n}\left[\tilde{i}(X_1^n; Y^n|X_2^n) > \log\gamma_1(X_1^n, X_2^n)\right] \\ &= \mathbb{E}_{P_{X^n}P_{X_2^n}}\left[\mathbb{E}_{Q^{(1)}_{Y^n|X_2^n}}\left[\frac{dP_{Y^n|X_2^n}(Y^n|X_2^n)}{dQ^{(1)}_{Y^n|X_2^n}(Y^n|X_2^n)}\right.\right. \\ &\qquad \left.\left. \times 1\left\{\tilde{i}(X_1^n; Y^n|X_2^n) > \log\gamma_1(X_1^n, X_2^n)\right\}\right]\right], \end{aligned} \tag{31}$$

$$\begin{aligned} &P_{X_1^n} P_{X_2^n} P_{Y^n|X_1^n}\left[\tilde{i}(X_2^n; Y^n|X_1^n) > \log\gamma_1(X_1^n, X_2^n)\right] \\ &= \mathbb{E}_{P_{X^n}P_{X_2^n}}\left[\mathbb{E}_{Q^{(2)}_{Y^n|X_1^n}}\left[\frac{dP_{Y^n|X_1^n}(Y^n|X_1^n)}{dQ^{(2)}_{Y^n|X_1^n}(Y^n|X_1^n)}\right.\right. \\ &\qquad \left.\left. \times 1\left\{\tilde{i}(X_2^n; Y^n|X_1^n) > \log\gamma_1(X_1^n, X_2^n)\right\}\right]\right], \end{aligned} \tag{32}$$

$$\begin{aligned} &P_{X_1^n} P_{X_2^n} P_{Y^n}\left[\tilde{i}(X_1^n X_2^n; Y^n) > \log\gamma_1(X_1^n, X_2^n)\right] \\ &= \mathbb{E}_{P_{X^n}P_{X_2^n}}\left[\mathbb{E}_{Q^{(3)}_{Y^n}}\left[\frac{dP_{Y^n}(Y^n)}{dQ^{(3)}_{Y^n}(Y^n)}\right.\right. \\ &\qquad \left.\left. \times 1\left\{\tilde{i}(X_1^n X_2^n; Y^n) > \log\gamma_1(X_1^n, X_2^n)\right\}\right]\right]. \end{aligned} \tag{33}$$

Therefore, we may compute the confusion probability w.r.t. the more convenient measures $Q^{(1)}_{Y^n|X_2^n}$, $Q^{(2)}_{Y^n|X_1^n}$, $Q^{(3)}_{Y^n}$, but at the expense of additional factors. If we can uniformly bound these factors for all x_1^n, x_2^n, y^n by positive constants K_1, K_2, K_3 (or even slowly growing functions K_{1n}, K_{2n} ,K_{3n} independent of x_1^n, x_2^n, y^n), respectively, then we will be able to compute the random coding and typicality decoding bound in a relatively simple form.

Here, we will take the same P2P approach for a Gaussian MAC. First, we choose the pair of input distributions to be independent uniform distributions on the respective power shells

$$P_{X_1^n}(x_1^n) = \frac{\delta(||x_1^n|| = \sqrt{nP_1})}{S_n(\sqrt{nP_1})}, \tag{34}$$

$$P_{X_2^n}(x_2^n) = \frac{\delta(||x_2^n|| = \sqrt{nP_2})}{S_n(\sqrt{nP_2})}. \tag{35}$$

Notice that this pair of distributions satisfies the input power constraint with probability one, that is,

$$P_{X_1^n} P_{X_2^n}[\mathcal{F}_{1n}^c \cup \mathcal{F}_{2n}^c] = 0. \tag{36}$$

Moreover, analogous to the P2P Gaussian channel, the conditional output distributions induced by this input pair are

$$\begin{aligned} P_{Y^n|X_2^n}(y^n|x_2^n) &= \frac{1}{2}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right) e^{-||y^n - x_2^n||^2/2} e^{-nP_1/2} \\ &\quad \times \frac{I_{n/2-1}(||y^n - x_2^n||\sqrt{nP_1})}{(||y^n - x_2^n||\sqrt{nP_1})^{n/2-1}}, \end{aligned} \tag{37}$$

$$\begin{aligned} P_{Y^n|X_1^n}(y^n|x_1^n) &= \frac{1}{2}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right) e^{-||y^n - x_1^n||^2/2} e^{-nP_2/2} \\ &\quad \times \frac{I_{n/2-1}(||y^n - x_1^n||\sqrt{nP_2})}{(||y^n - x_1^n||\sqrt{nP_2})^{n/2-1}}, \end{aligned} \tag{38}$$

where $I_v(\cdot)$ is again the modified Bessel function of the first kind and v-th order. The analysis of the unconditional output distribution P_{Y^n} for such an input pair is more complicated, but results in

$$\begin{aligned} P_{Y^n}(y^n) &= 2^{n/2-2}\pi^{-n/2}\Gamma^2\left(\frac{n}{2}\right) e^{-||y^n||^2/2} e^{-nP_1/2} e^{-nP_2/2} \\ &\times \frac{I_{n/2-1}(||y^n||\sqrt{nP_1})}{(||y^n||\sqrt{nP_1})^{n/2-1}} \frac{I_{n/2-1}(\sqrt{nP_2}(||y^n|| - \sqrt{nP_1}))}{(\sqrt{nP_2}(||y^n|| - \sqrt{nP_1}))^{n/2-1}}. \end{aligned} \tag{39}$$

Next, we choose the triple of (conditional) output distributions to be the capacity-achieving output distributions w.r.t. each case, that is,

$$Q^{(1)}_{Y^n|X_2^n}(\cdot|x_2^n) \sim \mathcal{N}(x_2^n, (1 + P_1)I_n), \tag{40}$$

$$Q^{(2)}_{Y^n|X_1^n}(\cdot|x_1^n) \sim \mathcal{N}(x_1^n, (1 + P_2)I_n), \tag{41}$$

$$Q^{(3)}_{Y^n}(\cdot) \sim \mathcal{N}(\mathbf{0}, (1 + P_1 + P_2)I_n). \tag{42}$$

The following proposition will then bound the factor introduced in (31) and (32) and (33). The reader is referred to [11] for the proof, which is similar to that of Proposition 1.

Proposition 2: Let $P_{Y^n|X_2^n}, P_{Y^n|X_1^n}, P_{Y^n}$ be the (conditional) distributions (37), (38), (39) induced on the output of the Gaussian MAC by a pair of independent uniform input distributions on the respective power shells, and let $Q^{(1)}_{Y^n|X_2^n}, Q^{(2)}_{Y^n|X_1^n}, Q^{(3)}_{Y^n}$ be the (conditional) capacity-achieving output distributions (40), (41), (42). There exists positive constants K_1, K_2, K_3 such that, for any x_1^n and x_2^n on the respective power shells and for sufficiently large n,

$$\frac{dP_{Y^n|X_2^n}(y^n|x_2^n)}{dQ^{(1)}_{Y^n|X_2^n}(y^n|x_2^n)} \le K_1, \qquad \forall\, y^n \in \mathbb{R}^n, \tag{43}$$

$$\frac{dP_{Y^n|X_1^n}(y^n|x_1^n)}{dQ^{(2)}_{Y^n|X_1^n}(y^n|x_1^n)} \le K_2, \qquad \forall\, y^n \in \mathbb{R}^n, \tag{44}$$

$$\frac{dP_{Y^n}(y^n)}{dQ^{(3)}_{Y^n}(y^n)} \le K_3, \qquad \forall\, y^n \in \mathbb{R}^n. \tag{45}$$

Remark. Using some more complicated manipulations, the proposition can be shown to be valid for any finite n, but the above statement is enough for our asymptotic analysis.

Combining the modified outage-splitting random coding and typicality decoding bound in Theorem 2, the change of measure arguments (31), (32), (33), and the uniform bounds of Proposition 2, along with an application of the CLT leads to the following second-order achievable region for the Gaussian MAC. Details of the analysis of the outage and confusion probabilities are provided in [11].

Theorem 4: An achievable region for the Gaussian MAC with maximal power constraints P_1 and P_2 is given by the union of all $(\frac{\log M_1}{n}, \frac{\log M_2}{n})$ pairs satisfying

$$\begin{aligned}\frac{\log M_1}{n} &< C(P_1) - \sqrt{\frac{V(P_1)}{n}}Q^{-1}(\lambda_1\epsilon)+O(1),\\ \frac{\log M_2}{n} &< C(P_2) - \sqrt{\frac{V(P_2)}{n}}Q^{-1}(\lambda_2\epsilon)+O(1),\\ \frac{\log M_1}{n}+\frac{\log M_2}{n} &< C(P_1+P_2)-\sqrt{\frac{V(P_1+P_2)}{n}}Q^{-1}(\lambda_3\epsilon)+O(1),\end{aligned} \tag{46}$$

for some choice of positive constants $\lambda_1, \lambda_2, \lambda_3$ satisfying $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

In the above theorem, the average error probability ϵ is basically split among the three outage events of a 2-user Gaussian MAC according to some $(\lambda_1, \lambda_2, \lambda_3)$ partitioning. Instead, we can assign essentially all the average error probability ϵ to the joint outage event. Therefore, upon repeating the same procedure as in Theorem 4, but with the joint-outage bound in Theorem 3 and the multi-dimensional CLT, we obtain the following achievable region for the Gaussian MAC, which is similar to the results of [13] and [14] for the discrete MAC.

Theorem 5: An achievable region for the Gaussian MAC with maximal power constraints P_1 and P_2 is given by the union of all $(\frac{\log M_1}{n}, \frac{\log M_2}{n})$ pairs satisfying

$$\begin{bmatrix}\frac{\log M_1}{n}\\ \frac{\log M_2}{n}\\ \frac{\log M_1}{n}+\frac{\log M_2}{n}\end{bmatrix} \in \mathbf{C}(P_1,P_2)-\frac{1}{\sqrt{n}}Q^{-1}(\epsilon;\mathbf{V}(P_1,P_2))+O(1), \tag{47}$$

where $Q^{-1}(\epsilon;\mathbf{\Sigma})$ is the inverse complementary CDF of a 3-dimensional Gaussian random variable defined as

$$Q^{-1}(\epsilon;\mathbf{\Sigma}):=\left\{\mathbf{z}\in\mathbb{R}^3 : \Pr\left(\mathcal{N}(\mathbf{0},\mathbf{\Sigma})\le\mathbf{z}\right)\ge 1-\epsilon\right\}, \tag{48}$$

and where the shorthand notations $\mathbf{C}(P_1, P_2)$ and $\mathbf{V}(P_1, P_2)$ are defined as

$$\mathbf{C}(P_1,P_2) = \begin{bmatrix} C(P_1)\\ C(P_2)\\ C(P_1+P_2)\end{bmatrix}, \tag{49}$$

$$\mathbf{V}(P_1,P_2)=\begin{bmatrix} V(P_1) & V_{1,2}(P_1,P_2) & V_{1,3}(P_1,P_2)\\ V_{1,2}(P_1,P_2) & V(P_2) & V_{2,3}(P_1,P_2)\\ V_{1,3}(P_1,P_2) & V_{2,3}(P_1,P_2) & V(P_1+P_2)\end{bmatrix}, \tag{50}$$

where with shorthand $u = 1, 2$, we have

$$V_{1,2}(P_1,P_2) = \frac{\log^2 e}{2}\frac{P_1P_2}{(1+P_1)(1+P_2)}, \tag{51}$$

$$V_{u,3}(P_1,P_2) = \frac{\log^2 e}{2}\frac{P_u(2+P_1+P_2)}{(1+P_u)(1+P_1+P_2)}. \tag{52}$$

Both achievable regions in Theorems 4 and 5 suggest that taking finite blocklength into account introduces a rate penalty (for the interesting case of $\epsilon < \frac{1}{2}$) that depends on blocklength, error probability and Gaussian MAC dispersions.

V. Numerical Evaluation

In this section, we numerically evaluate the performance of our achievable rate region for a symmetric Gaussian MAC with finite blocklength $n = 500$ in the (noise-limited) low-power $P_1 = P_2 = 0$ dB regime. For comparison, we also evaluate and depict the following: the conventional asymptotic capacity [1]; the simple single-user finite-blocklength outer bounds [13]; the finite-blocklength achievable region via a random Gaussian codebook, and that via TDMA.

To explore how tight the two achievable regions we have characterized in our Theorems 4 and 5 are, we consider a simple outer bound that can be developed using single-user results as follows. The achievable rate for each user cannot exceed that when the other user is silent. In addition, one can combine the two users into a super-user transmitting a super-message consisting of the two messages of the users over a P2P Gaussian channel using the sum power $P_1 + P_2$. For each case, the total error probability ϵ is assigned to only one of the outage events. Combining these arguments, we obtain the following simple outer bound:

$$\begin{aligned}\frac{\log M_1}{n} &< C(P_1) - \sqrt{\frac{V(P_1)}{n}}Q^{-1}(\epsilon)+O(1),\\ \frac{\log M_2}{n} &< C(P_2) - \sqrt{\frac{V(P_2)}{n}}Q^{-1}(\epsilon)+O(1),\\ \frac{\log M_1}{n}+\frac{\log M_2}{n} &< C(P_1+P_2)-\sqrt{\frac{V(P_1+P_2)}{n}}Q^{-1}(\epsilon)+O(1).\end{aligned} \tag{53}$$

To illustrate the tightness of the random codebooks with power shell input distribution, we compare our achievable region with the region achieved by a pair of random codebooks which are, as usual [1], generated according to independent Gaussian distributions. One can easily show an extension of (8) to a Gaussian MAC such that

$$\begin{aligned}\frac{\log M_1}{n} &< C(P_1-\delta_1)-\frac{\log e}{\sqrt{n}}\sqrt{\frac{P_1-\delta_1}{1+P_1-\delta_1}}Q^{-1}(\lambda_1\epsilon)+O(1),\\ \frac{\log M_2}{n} &< C(P_2-\delta_2)-\frac{\log e}{\sqrt{n}}\sqrt{\frac{P_2-\delta_2}{1+P_2-\delta_2}}Q^{-1}(\lambda_2\epsilon)+O(1),\\ \frac{\log M_1}{n}+\frac{\log M_2}{n} &< C(P_1+P_2-\delta 1-\delta_2)\\ &\quad -\frac{\log e}{\sqrt{n}}\sqrt{\frac{P_1+P_2-\delta_1-\delta_2}{1+P_1+P_2-\delta_1-\delta_2}}Q^{-1}(\lambda_3\epsilon)+O(1),\end{aligned} \tag{54}$$

for an arbitrarily small constant $\delta > 0$ and any choice of constants satisfying $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

It is also interesting to compare the achievable rate region with that of time-division multiple access (TDMA). For TDMA with power control, the two users can share the n channel uses, use single-user coding strategies, and average the error probability ϵ. Specifically, user 1 transmits in the first αn channel uses with power P_1/α and rate such that an average error probability $\beta\epsilon$ is achieved, and user 2 transmits in the remaining $\bar{\alpha}n := (1-\alpha)n$ channel uses with power $P_2/\bar{\alpha}$ and rate such that an average error probability $\tilde{\beta}\epsilon$ is achieved. Since the average error probability of this scheme can be characterized as $\epsilon = \beta\epsilon + \tilde{\beta}\epsilon - \beta\tilde{\beta}\epsilon^2$, we choose $\tilde{\beta} = (1-\beta)/(1-\beta\epsilon)$. Using the power shell uniform input distribution for each user and relying on the Gaussian P2P results [4], [5], the TDMA strategy achieves the following set of rate pairs:

$$\frac{\log M_1}{n} < \alpha C\left(\frac{P_1}{\alpha}\right) - \sqrt{\frac{\alpha}{n} V\left(\frac{P_1}{\alpha}\right)} Q^{-1}(\beta\epsilon) + O(1),$$

$$\frac{\log M_2}{n} < \bar{\alpha} C\left(\frac{P_2}{\bar{\alpha}}\right) - \sqrt{\frac{\bar{\alpha}}{n} V\left(\frac{P_2}{\bar{\alpha}}\right)} Q^{-1}\left(\frac{(1-\beta)\epsilon}{1-\beta\epsilon}\right) + O(1),$$

for some $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$.

Figures 1 illustrates the comparison between our two (splitting and joint) achievable regions for the Gaussian MAC using independent power shell inputs, the achievable region using Gaussian inputs, the achievable region using TDMA with power control, the simple single-user outer bound, and the asymptotic capacity region. We observe that coding with power shell inputs outperforms that using a Gaussian distribution and TDMA. Specifically, Gaussian random codebooks, while optimal for achieving capacity, are not second-order optimal and their finite blocklength achievable rate region falls inside that of power shell inputs. The independent power shell inputs are also seen to closely approach the simple single-user outer bounds everywhere, except for a slight gap at the two corners that shrinks with increasing blocklength. It is also interesting to notice that, contrary to the infinite blocklength case, the TDMA strategy with power control is not even sum-rate optimal.

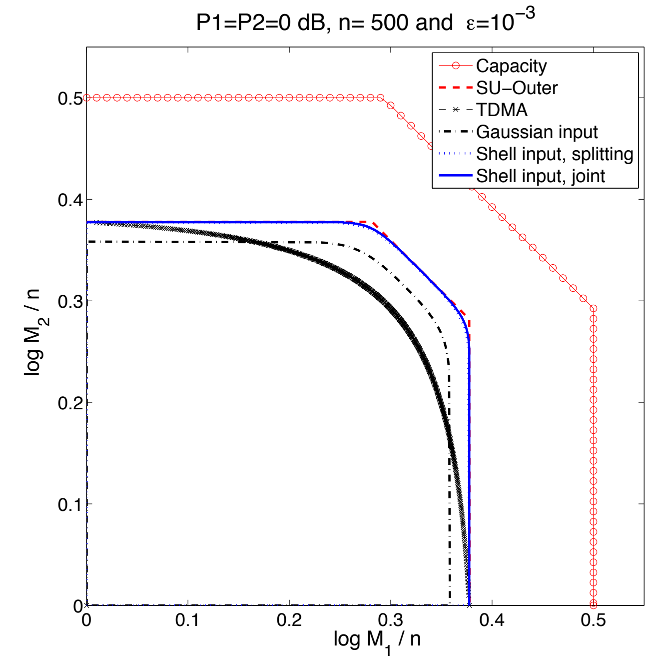

Fig. 1. Symmetric Gaussian MAC in the low-power regime

References

[1] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley, 1991.

[2] A. El Gamal and Y.-H. Kim, *Network Information Theory*, Cambridge University Press, UK, 2012.

[3] S. Verd and T.-S. Han, "A General Formula for Channel Capacity," *IEEE Transactions on Information Theory*, vol. 40, no. 4, pp. 1147–1157, Jul. 1994.

[4] Y. Polyanskiy, H. V. Poor and S. Verd, "Channel Coding Rate in the Finite Blocklength Regime," *IEEE Transactions on Information Theory*, vol. 56, no. 5, pp. 2307–2359, May 2010.

[5] M. Hayashi, "Information Spectrum Approach to Second-Order Coding Rate in Channel Coding," *IEEE Transactions on Information Theory*, vol. 55, no. 11, pp. 4947–4966, Nov. 2009.

[6] R. G. Gallager, *Information Theory and Reliable Communication*, New York: Wiley, 1968.

[7] C. E. Shannon, "Probability of Error for Optimal Codes in a Gaussian Channel," *Bell Systems Technical Journal*, vol. 38, pp. 611–656, 1959.

[8] A. Feinstein, "A New Basic Theorem of Information Theory," *IRE Transactions on Information Theory*, vol. 4, no. 4, pp. 2–22, 1954.

[9] J. Wolfowitz, "Notes on a General Strong Converse," *Information and Control*, vol. 12, pp. 1–4, 1968.

[10] C. Robert, "Modified Bessel Functions and Their Applications in Probability and Statistics," *Statistics & Probability Letters*, vol. 9, no. 2, pp. 155–161, 1990.

[11] E. MolavianJazi and J. N. Laneman, "A Finite-Blocklength Perspective on Multi-Access Channels," *IEEE Transactions on Information Theory*, to be submitted, 2012.

[12] E. MolavianJazi and J. N. Laneman, "Simpler Achievable Rate Regions for Multiaccess with Finite Blocklength," in *Proc. IEEE International Symposium on Information Theory*, pp. 36–40, Boston, MA, 2012

[13] V. Y. F. Tan and O. Kosut, "On the Dispersion of Three Network Information Theory Problems," submitted to *IEEE Transactions on Information Theory*, Available at http://arxiv.org/abs/1201.3901, 2012.

[14] Y. W. Huang and P. Moulin, "Finite Blocklength Coding for Multiple Access Channels," in *Proc. IEEE International Symposium on Information Theory*, pp. 831–835, Boston, MA, 2012.

[15] R. G. Gallager, "A Perspective on Multi-Access Channels," *IEEE Transactions on Information Theory*, vol. 31, no. 2, pp. 124–142, 1985.