

Forecasting the Resilience of Networked Dynamical Systems Under Environmental Perturbation

[Extended Abstract]

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1. INTRODUCTION

Many real life systems can be viewed as networked systems that are composed by interconnected compartments which exchange mass or energy between each other and with their environment through fluxes. Such interaction with the environment make these systems subject to external perturbations that cause systems parameters to vary away from the nominal values. For nonlinear networked systems, such parameter variations can change the qualitative behaviors of the system (i.e. phase portrait or stability) through a bifurcation [6]. These changes may result in a *regime shifts* [8] in which the system "flips" from a nominal operating state to an alternative state. Regime-shifts can be catastrophic for users who have grown accustomed to the quality of services provided by the system prior to the shift. Examples of this can be found in the eutrophication of shallow lakes as a result of human-induced nutrient enrichment or the decline of fisheries due to overfishing practices [8]. Another prime example occurs when voltage collapses cascade through the electric power network [4]. Each of these shifts has the potential to disrupt the services that these systems provide to the society. Forecasting the resilience of these networked systems to parameter variations is therefore crucial for managing their security and sustainability [1, 5].

Consider networked systems $\dot{x} = f(x, \mu)$ whose equilibrium x^* depend on parameter μ . The resilience of a system under parameter variation can be measured by the distance $\gamma = |\mu^* - \mu^0|$ between the nominal parameter μ^0 and the closest critical parameter μ^* at which a bifurcation occur. The quantity γ , often called *distance to closest bifurcation*,

is an indicator of how close the system is to a collapse. The computation of γ is generally difficult since the *bifurcation set* containing μ^* is usually unknown. Prior works [2, 4] have used numerical optimization techniques to search for the minimum γ subject to the constraints that a bifurcation occurs at μ^* . These methods, however, are computationally demanding since the search for minimum γ requires the computation of equilibrium x^* at every iteration.

This paper uses sum-of-square (SOS) relaxation [7] to obtain a lower bound on the global minimum of the "distance-to-bifurcation", γ , in networked dynamical system. Our approach uses algebraic geometry methods [3] to reduce the size of the constraints and to avoid the computation of equilibrium x^* at every iteration of the optimization. We have recently applied this method to a class of non-negative systems that has a kinetic realization [9]. This paper extends the method in [9] for dynamical systems which may neither non-negative nor has a kinetic realization. The method is illustrated to study voltage collapse in power network [2].

2. NECESSARY BIFURCATION CONDITIONS

Consider an n -dimensional *polynomial system* whose state trajectories $x(t, x_0)$ satisfy

$$\dot{x}(t) = f(x(t), \mu), \quad x(0) = x_0, \quad (1)$$

for all $t \geq 0$ in which $f(x, \mu) \in \mathbb{R}^n(\mu)[x]$ is polynomial in the unknown $x \in \mathbb{R}^n$ with coefficient $\mu \in \mathbb{R}^p$. The system's *equilibria* are vectors in $\mathbb{Q}^n(\mu)$ that are zeros of the right hand side of (1). In other words, $x^*(\mu)$ is an equilibrium if

$$x^*(\mu) \in \{x \in \mathbb{Q}^n(\mu) : f(x, \mu) = 0\}. \quad (2)$$

Computing the algebraic expression for $x^*(\mu)$ in high dimensional systems usually requires the use of symbolic methods. These methods are based on the fact that a set of polynomials generate an *ideal* in the polynomial ring and that the zeros of the system of polynomials are equivalent to the zeros of any Gröbner basis of that ideal [3].

Let $J = \frac{\partial f(x, \mu)}{\partial x} |_{x^*}$ be the Jacobian matrix of (1) with characteristic polynomial $p(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$, where the coefficients $a_i(\mu)$ are function of the parameter, μ . For $z = 1, \dots, n$, let Δ_z denotes the z th Hurwitz determinant of $p(s)$. A local bifurcation can be characterized in term of the *eigenvalue condition* of J with some additional *transversality conditions* [6]. Let Ω^{SN} and Ω^H be the set of parameters for which a saddle node (also pitchfork

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and transcritical) and hopf bifurcations occur. As shown in [9], these sets are semi-algebraic sets characterized by $a_i(\mu)$ and Δ_z . One may then denote the parameter set for which at least one type of bifurcation occurs as $\Omega = \Omega^{SN} \cup \Omega^H$. Thus, system (1) will not have a bifurcation if Ω is empty.

When system (1) is *non-negative* and has a *kinetic realization*, the bifurcation set can be expressed only in terms of the parameters, rather than the parameters and the equilibria. As a result, this parameterization can simplify the computation of the distance-to-bifurcation [9].

3. DISTANCE-TO-BIFURCATION

The previous section suggests that the non-existence of a particular bifurcation is equivalent to the emptiness of the corresponding bifurcation set. In general, checking the emptiness of Ω^{SN} can be difficult. In recent years, however, it has proven fruitful to consider convex relaxations of the problem [7] in which one checks for the emptiness of the set $\tilde{\Omega}(\beta) \cap \Omega^{SN}$, where $\tilde{\Omega}(\beta)$ is a semi-algebraic set defined by a positive semi-definite *certificate* function $V(\mu)$. In particular, let β be a real constant and let $\alpha(|\mu - \mu_0|)$ be a class \mathcal{K} function in which μ is the parameter set and μ_0 is a known initial parameter. The *certificate* set is defined as

$$\tilde{\Omega}(\beta) = \{\mu \in \mathbb{R}^p : \alpha(|\mu - \mu_0|) \leq \beta\}. \quad (3)$$

Given a specific $\beta > 0$, if the intersection $\tilde{\Omega}(\beta) \cap \Omega^{SN} = \emptyset$, then the distance to a saddle node bifurcation cannot be less than $\alpha^{-1}(\beta)$. The key point is that verifying whether the set $\tilde{\Omega}(\beta) \cap \Omega^{SN}$ is empty or not can be done using SOS program. By defining the certificate function as $V(\mu) = \alpha(|\mu - \mu_0|)$, a bound $\bar{\beta}$ for the minimum distance to a saddle node bifurcation can be obtained by the following SOS program [9].

$$\begin{aligned} & \text{maximize } \bar{\beta} \\ & \text{such that } a_{n-1}^2(\mu)(V(\mu) - \bar{\beta}) + r(\mu)a_n(\mu) \text{ is SOS.} \end{aligned}$$

A similar result would also hold for the Hopf bifurcation.

4. THE VOLTAGE COLLAPSE PROBLEM

Consider a simple power network [2] in figure 1 whose underlying differential equations are given by

$$\begin{aligned} \dot{\omega} &= \frac{1}{M}[P_m - P_{e1}(\delta, V) - D_G\omega] \\ \dot{\delta} &= \omega - \frac{1}{D_L}[P_{e2}(\delta, V) - P_d] \\ \dot{V} &= \frac{1}{\tau}[Q_e(\delta, V) - Q_d] \end{aligned} \quad (4)$$

where $\delta = \delta_1 - \delta_2$, $P_{e1} = G - V(G \cos \delta - B \sin \delta)$, $P_{e2} = -V^2G + V(G \cos \delta + B \sin \delta)$, $Q_e = -V^2(B - B_c) - V(G \sin \delta - B \cos \delta)$, $G = R/[R^2 + (X_L - X_x)^2]$, $B = (X_L - X_x)/[R^2 + (X_L - X_x)^2]$. We will use the methods in previous section to

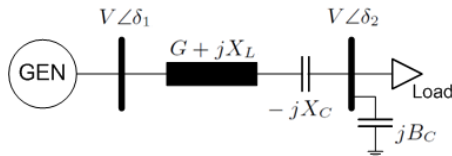


Figure 1: A simple power network [2].

determine the distance to saddle node bifurcation (i.e. voltage collapse). We pose the problem in term of parameters $k = (P_d, Q_d)$ (the load powers) and compute the minimum $\beta = |k^* - k^0|$ such that the equilibrium and the bifurcation conditions are satisfied. Assuming $\omega = 0$, $X_c = B_c = 0$, $R = 0.1$, $X_L = 0.5$ and letting $x = \sin \delta$ and $y = \cos \delta$, one can verify that these conditions satisfy the following equations.

$$\begin{aligned} 0 &= -V^2G + VGy + VBx - P_d, \\ 0 &= -V^2B - VGx + VB_y - Q_d, \\ 0 &= B^2 + G^2 - 2B^2Vy - 2G^2Vy \\ 0 &= x^2 + y^2 - 1. \end{aligned}$$

For simplicity, let's assume a constant power factor of $Q_d = 0.25P_d$. Computing a Gröbner basis, \mathbb{G} , for the ideal generated by the above polynomials yield a single polynomial

$$\begin{aligned} \mathbb{G} &= B^4 - B^3P_d + 2B^2G^2 - 4B^2GP_d - 4B^2P_d^2 - BG^2P_d \\ &\quad + 2BGP_d^2 + G^4 - 4G^3P_d - 0.25G^2P_d^2, \end{aligned}$$

and the minimum distance to bifurcation can be recasted as

$$\begin{aligned} & \text{maximize: } \beta \\ & \text{such that: } (P_d^* - P_d^0)^2 + (Q_d^* - Q_d^0)^2 - \beta + r(\mu)\mathbb{G} \text{ is SOS.} \end{aligned}$$

The SOSTOOLS [7] were used to solve the above SOS program and found a minimum $\beta = 0.443$ which corresponds to $\mu^* = (0.6661, 0.1665)$. These are the same results obtained in [2], but the underlying optimization problem is much simpler than that used in [2] as it only uses a single constraint.

5. ACKNOWLEDGMENTS

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