Stochastic Reachability of Jump-Diffusion Process Using Sum Of Squares Optimization

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***Abstract*—This note uses sum of squares (SOS) relaxation to solve stochastic reachability problems for jump-diffusion processes. The main result is a polynomial characterization of the infinitesimal generator for the solution of a jump-diffusion process' boundary value problem, thereby enabling one to compute a bound on the probability of reaching a target set in finite time using SOS optimization.**

I. Introduction

Consider a continuous time stochastic process $\{x(t)\}$ whose sample path $x(t) \in \mathcal{X}$ at time $t \in \mathbb{R}_+$ takes values in an open subset $\mathcal{X} \subseteq \mathbb{R}^n$ of Euclidean space. The stochastic reachability analysis computes an upper bound, $\gamma \in [0, 1]$, for the probability that, starting from an initial set $\mathcal{X}_0 \subset \mathcal{X}$, the sample path of $\{x(t)\}$ will reach a given target set $\mathcal{X}_u \subset \mathcal{X}$ in a finite time T, where $T = \inf\{t \in \mathbb{R}_+ \mid x(t) \in \mathcal{X}_u\}$. Formally stated, this problem is to find a constant $\gamma \in [0, 1]$ such that

$$\mathbb{P}\{x(t) \in \mathcal{X}_u \text{ for some } 0 \le t \le T \mid x(0) \in \mathcal{X}_0\} \le \gamma. \quad (1)$$

One approach for computing the bound γ was proposed in [1] for a regular diffusion process $\{x(t)\}$ that satisfies the stochastic differential equation $dx(t) = f(x)dt + \sigma(x)dw(t)$ in which $\{w(t)\}$ is a Wiener process. This approach essentially seeks a "stochastic" Lyapunov function, $V(x(t))$, that generates a supermartingale from which a bound γ in (1) can be deduced. As in standard Lyapunov methods, this approach is hindered by the difficulty of finding such a $V(x(t))$. Recent developments in semidefinite programming and sum of squares (SOS) relaxation methods [2] can now help circumvent this difficulty. As shown recently in [3], the search for function $V(x(t))$ characterizing the probabilitistic bound in (1) can be formulated and solved using SOS optimization methods [4], [5], provided the functions $f(x), \sigma(x)$ are polynomial and the sets $\mathcal{X}, \mathcal{X}_0, \mathcal{X}_u$ are semialgebraic. The success of this computational method has since motivated its use to address related problems in feedback control (e.g. [6]), safety verification (e.g. [7]), modeling and analysis of complex biological networks (e.g. [8]) and model approximation (e.g. [9]).

Many applications, however, require models that also capture the jumps or discontinuous changes on the states due to the presence of extreme or abnormal events. These events are no longer suitable to be described by Wiener processes but are better characterized as stochastic renewal processes. Examples of such applications can be found in models of failure and repair events of a workstation in manufacturing systems (e.g. [10]), abnormal variations on the stock price due to the arrival of important new information about the stocks (e.g. [11]), or the impact of human activities and extreme natural events due to climate change on ecosystem dynamics (e.g. [12], [13]). Stochastic reachability can be used to characterize methods for managing such systems in a safe and sustainable manner and so it is valuable to extend the basic approach in [1], [3] to systems that are described by jump-diffusion processes.

This note extends the methods in [3], [1] to address the stochastic reachability analysis for systems that are described by jump-diffusion processes. To the best of our knowledge, this paper is the first study on using SOS relaxations for stochastic reachability analysis of jump-diffusion processes. As in the case of diffusion processes, the proposed method is also based on searching for a *barrier certificate*, $V(x(t))$, that generates a supermartingale from which the bounds in (1) can be deduced. This note's main contribution is a polynomial characterization of the infinitesimal generator that corresponds to the solution of a jump-diffusion process' boundary value problem (cf. [14]), thereby making it possible to use SOS relaxations to search for the appropriate barrier certificate.

The note is structured as follows. Background on jump-diffusion processes is found in section II. Section III presents the main result on stochastic reachability analysis of jump-diffusion processes. Section IV illustrates an example use of the proposed method in ecosystem management. Section V concludes the note with some remarks.

Notational Conventions: Let $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of positive integers and non-negative real numbers, respectively. Let $\{x(t)\}$ denote a random process whose state $x(t) \in \mathcal{X}$ at time $t \in \mathbb{R}_+$ takes values in an open subset $\mathcal{X} \subseteq \mathbb{R}^n$ of Euclidean space. The total and conditional expected values of a random variable are denoted as $\mathbb{E}\{\cdot\}$ and $\mathbb{E}\{\cdot|\cdot\}$, respectively, and the total and conditional probabilities of an event is denoted as $\mathbb{P}\{\cdot\}$ and $\mathbb{P}\{\cdot|\cdot\}$, repectively. If $\{x(t)\}$ has distribution $F(x)$, then its nth moment is denoted as $\mathbb{M}^n_x = \int x^n dF(x)$.

An n-dimensional *multi-index* is an n-tuple $\alpha \equiv (\alpha_1, \cdots, \alpha_n)$ of non-negative integers and its absolute value is defined as $|\alpha| = \Sigma_{i=1}^n \alpha_i$. The sum/difference of two multi-indices is the component-wise sum/difference of the indices. We say that $\alpha \ge \beta$ if and only if $\alpha_i \ge \beta_i$ for $i = 1, \ldots, n$. Let $\alpha! \equiv \alpha_1!\alpha_2! \cdots \alpha_n!$. The binomial coefficient of α and β is defined as $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n} = \alpha!/(\beta!(\alpha-\beta)!)$. For a vector $x \in \mathbb{R}^n$ and an n-dimensional multi-index α, the αth power of x is defined as $x^{[\alpha]} \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. The multi-index binomial theorem states that $(x+y)^{[\alpha]} = \Sigma_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} x^{[\alpha-\beta]} y^{[\beta]}$. It can be shown that

$$\partial^{[\alpha]} x^{[\beta]} = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} x^{[\beta-\alpha]}, & \text{if } \alpha \le \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Given a bounded, real-valued function $V(x) : \mathbb{R}^n \to \mathbb{R}$ and an

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n-dimensional multi-index α, the αth order partial derivative of V is defined as $\partial^{[\alpha]}V = \frac{\partial^{\alpha_1}V}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}V}{\partial x_2^{\alpha_2}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$.

A pth order polynomial may be written in multi-index notation as $V(x) = \Sigma_{|\alpha|\le p}\, c_\alpha x^{[\alpha]}$, where α is a multi-index and c_α is some coefficients associated with the $x^{[\alpha]}$ monomial. The set of all polynomials in variables x with real coefficients is denoted as $\mathbb{R}[x]$. A polynomial $V(x) \in \mathbb{R}[x]$ is said to be positive semidefinite (psd) if $V(x) \ge 0, \forall x \in \mathbb{R}^n$. A necessary condition for $V(x)$ to be psd is that its total degree is even. We say that a polynomial $V(x)$ is SOS if it can be rewritten as $V(x) = \Sigma_{k=1}^{M} q_k^2(x)$ for some set of M polynomials $q_k(x)$ where $k = 1, 2, \ldots, M$. Clearly, a polynomial $V(x)$ being SOS implies $V(x)$ is psd. The set of all SOS polynomials in variables x is denoted as $\Sigma(x)$.

The space of $n\times n$ real symmetric matrices is denoted as $\mathcal{S}^{n\times n}$. A matrix $Q \in \mathcal{S}^{n\times n}$ is positive definite (pd) if $x^TQx > 0$ and is psd if $x^TQx \ge 0$, $\forall x \in \mathbb{R}^n$. We use $\mathcal{S}_{++}^{n\times n}$ and $\mathcal{S}_{+}^{n\times n}$ to denote the space of $n\times n$ symmetric pd and psd matrices, respectively.

II. Jump-Diffusion Processes

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete probability space and let $\{\mathcal{F}_t\}_{t\ge 0}$ be a filtration over $\{\Omega, \mathcal{F}, \mathbb{P}\}$ which satisfies the usual conditions (cf. [15]): (i) $\mathcal{F}_t$ contains the $\mathbb{P}-$negligible sets for all t, (ii) $\mathcal{F}_t$ is right continuous, i.e. $\mathcal{F}_{t^+} = \mathcal{F}_t$, for all t (i.e. the totality of information are observable by time t). Consider a jump-diffusion process (JDP)

$$dx(t) = f(x(t))dt + \sigma(x(t))dw(t) + dJ(t), \quad x(0) = x_0, \tag{2}$$

where $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous functions, $\{x(t)\}$ is a stochastic process, $\{w(t)\}$ is a Wiener process, $\{J(t)\}$ is a shot noise process defined as

$$J(t) = \sum_{\ell=1}^{N(t)} y_\ell e^{-\delta(t-\tau_\ell)}, \quad \ell \in \mathbb{Z}_+. \tag{3}$$

In equation (3), $N(t)$ is a Poisson process with intensity ρ, $\{\tau_\ell\}$ are the event times of a Poisson jump, $\{y_\ell\}$ is an i.i.d. random process with distribution $F(y)$ describing the ℓ-th jump's size, and δ is a real positive constant representing the rate of exponential decay after a jump. The JDP in (2) is understood in Itô's sense and $\{w(t)\}$ is independent of $\{J(t)\}$.

Let $Y(\tau_\ell, y_\ell) = y_\ell e^{\delta\tau_\ell}$, then $J(t)$ in (3) may be written as

$$J(t) = e^{-\delta t}\int_0^t \int_{\mathbb{R}^n} Y(\tau, y) N(d\tau, dy), \tag{4}$$

where $N(d\tau, dy)$ is a Poisson random measure with $\mathbb{E}\{N(dt, dy)\} = \rho dt F(dy)$. We define the increment of $J(t)$ as $dJ(t) = J(t + dt) - J(t)$ where dt is an infinitesimal time increment. Using (4) to expand out $dJ(t)$ and retaining the first order terms in dt, one finds the jump process increment can be written as

$$dJ(t) = -\delta J(t)dt + \int_{\mathbb{R}^n} yN(dt, dy), \tag{5}$$

where the second term in (5) is known as a compound Poisson process. Using the expression for the jump increment in (5), the JDP in (2) can be rewritten as

$$dx(t) = (f(x(t)) - \delta J(t))\, dt + \sigma(x(t))dw(t) + \int_{\mathbb{R}^n} yN(dt, dy), \quad x(0) = x_0. \tag{6}$$

Since $\{J(t)\}$ and $\{w(t)\}$ in (6) are independent Markov processes, one may conclude that the solution of the JDP in (6) is also a Markov process (cf. [16]).

Now consider a Markov process $\{x(t)\}$ with right continuous sample path and consider any function $V(x(t)) : \mathbb{R}^n \to \mathbb{R}$. The (infinitesimal) generator of $\{x(t)\}$ is an operator, $\mathcal{L}$, whose action on $V(x(t))$ is defined by

$$\mathcal{L}V(x) = \lim_{h\searrow 0_+} \frac{\mathbb{E}\{V(x(h))|V(x_0)\} - V(x_0)}{h} \text{ (if the limit exists)},$$

where $\searrow$ means that the limit is taken from the right. For the jump process in (5) and a function $V(x(t)) \in C^2(\mathbb{R}^n)$ that is twice continuously differentiable and bounded for all $x \in \mathbb{R}^n$ (denote this class of functions as $C^2(\mathbb{R}^n)$), one can show that its generator, $\mathcal{L}_{JP}$, is (cf. [15])

$$\mathcal{L}_{JP}V(x(t)) = \rho\int_0^\infty (V(x+y) - V(x))\, dF(y) - \frac{\partial V(x)}{\partial x}\delta J(t).$$

Combining the generator of the above jump process with the generator of diffusion process $dx(t) = f(x)dt + \sigma(x)dw(t)$ (cf. [17]), one may conclude that the generator, $\mathcal{L}$, of the JDP in (6) is given by

$$\begin{aligned}\mathcal{L}V(x(t)) = &\frac{\partial V(x(t))}{\partial x}(f(x(t)) - \delta J(t)) \\ &+ \frac{1}{2}\mathrm{Tr}\left(\sigma^T(x(t))\frac{\partial^2 V(x(t))}{\partial x^2}\sigma(x(t))\right) \\ &+ \rho\int_0^\infty (V(x+y) - V(x))dF(y).\end{aligned} \tag{7}$$

Dynkin's formula for JDP in (6) which can be used to characterize a supermartingale $V(x(t))$ is stated below.

Lemma 2.1 ([14]): Consider the JDP in (6) defined on a bounded open set $\mathcal{X} \subseteq \mathbb{R}^n$ with smooth boundary $\partial\mathcal{X}$. Let $\tau < \infty$ with $\tau \le \tau_{\mathcal{X}} := \inf\{t \in \mathbb{R}_+ \,|\, x(t) \in \partial\mathcal{X}\}$ be a stopping time. For $V(x(t)) \in C^2(\mathbb{R}^n)$, suppose $\mathbb{E}\left\{|V(x(\tau))| + \int_0^\tau |\mathcal{L}V(x(s))|ds\right\} < \infty$. Then

$$V(x(\tau)) = V(x_0) + \int_0^\tau \mathcal{L}V(s, x(s))ds. \tag{8}$$

Now recall that a process $\{V(x(t))\}$ is said to be a supermartingale with respect to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ generated by the process $\{x(t)\}$ if: (i) $\forall t \ge 0$, $V(x(t))$ is $\mathcal{F}_t$-measurable, (ii) $\mathbb{E}\{|V(x(t))|\} < \infty$, and (iii) $\mathbb{E}\{V(x(t_2))|V(x(t_1))\} \le V(x(t_1))$ for all $0 \le t_1 \le t_2 \le \tau$ (cf. [17]). By the choice of $V(x(t)) \in C^2(\mathbb{R}^n)$ in (7) and the boundedness of $x \in \mathcal{X}$, it is known that $V(x(t))$ will always satisfy conditions (i) and (ii), respectively (cf. [14]). If $V(x(t))$ also satisfies $\mathcal{L}V(x(t)) \le 0, \forall x \in \mathcal{X}$ with $\mathcal{L}V(x(t))$ as defined in (7), then Dynkin's formula in (8) implies that condition (iii) will also be satistied. One may then conclude that $V(x(t)) \in C^2(\mathbb{R}^n)$ with $\mathcal{L}V(x(t)) \le 0$, $\forall x \in \mathcal{X}$ is a supermartingale with respect to $\{x(t)\}$. In this paper, we will consider nonnegative supermartingale, i.e. $V(x(t)) \ge 0$, $\forall x \in \mathcal{X}$, for which the following inequality holds.

Lemma 2.2 ([1]): Let $\{V(x(t))\}$ be a supermartingale with respect to the process $\{x(t)\}$ where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ and $0 \le t \le \tau := \inf\{t : x(t) \notin \mathcal{X}\}$. Let $V(x(t))$ be nonnegative in $\mathcal{X}$. Then for any constant $\theta > 0$ and any $x(0) = x_0 \in \mathcal{X}$,

$$\mathbb{P}\left\{\sup_{0\le t\le\tau} V(x(t)) \ge \theta \,\Big|\, x(0) = x_0\right\} \le \frac{V(x_0)}{\theta}. \tag{9}$$

III. Main Results

As discussed in section I, provided that the functions $f(x), \sigma(x)$ and the jump term in (6) are polynomials and the sets $\mathcal{X}, \mathcal{X}_0, \mathcal{X}_u$ in (1) are semialgebraic, the search for a barrier certificate, $V(x(t))$, can be formulated as an SOS optimization. In this SOS optimization, $V(x(t))$ is a polynomial function whose coefficients are the decision variables that will be determined in the optimization task. Thus, our goal is to formulate a polynomial representation for the conditions which guarantee $\{V(x(t))\}$ to be a supermartingale (cf. section II). One issue in formulating such a representation comes from the integral term in the generator of the JDP in (7). The following proposition shows how to address this issue.

Proposition 3.1: Let $y \in \mathbb{R}^n$ be an n-dimensional independent random variable with distribution $F(y)$. Let $V(x) = \sum_{|\alpha|\le p} c_\alpha x^{[\alpha]}$ be a multi-index representation of a polynomial function. Then

$$\int (V(x+y) - V(x))\, dF(y) = \sum_{1\le|\beta|\le p} \frac{1}{\beta!}\partial^{[\beta]}[V(x)]\,\mathbb{M}^{|\beta|}, \tag{10}$$

and the generator in (7) can be rewritten as

$$\tilde{\mathcal{L}}V(x(t)) = \sum_{1\le|\beta|\le p} \frac{1}{\beta!}\partial^{[\beta]}[V(x)] + \frac{\partial V(x(t))}{\partial x}(f(x(t)) - \delta J(t)) + \frac{1}{2}\mathrm{Tr}\left(\sigma^T(x(t))\frac{\partial^2 V(x(t))}{\partial x^2}\sigma(x(t))\right) \tag{11}$$

Proof: We only need to show that (10) holds since its substitution into the integral term in (7) gives the generator in (11). Let us write

$$V(x+y) = \sum_{|\alpha|\le p} c_\alpha (x+y)^{[\alpha]} = \sum_{|\alpha|\le p} c_\alpha \sum_{0\le|\beta|,\beta\le\alpha} \binom{\alpha}{\beta} x^{[\alpha-\beta]} y^{[\beta]},$$
$$= \sum_{|\alpha|\le p} c_\alpha \left[x^{[\alpha]} + \sum_{1\le|\beta|,\beta\le\alpha} \binom{\alpha}{\beta} x^{[\alpha-\beta]} y^{[\beta]}\right].$$

For notational convenience, let us denote the difference $V(x+y) - V(x)$ as $\Delta V(x,y)$. Using the above sum, one can write this difference as

$$\Delta V(x,y) = \sum_{|\alpha|\le p} c_\alpha \sum_{1\le|\beta|,\beta\le\alpha} \binom{\alpha}{\beta} x^{[\alpha-\beta]} y^{[\beta]},$$

and since

$$\partial^{[\beta]}\left[x^{[\alpha]}\right] = \begin{cases} \frac{\alpha!}{(\alpha-\beta)!}x^{[\alpha-\beta]} & \text{if } \beta\le\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

the expression for $\Delta V(x,y)$ can be rewritten as

$$\Delta V(x,y) = \sum_{|\alpha|\le p} c_\alpha \sum_{1\le|\beta|,\beta\le\alpha} \frac{1}{\beta!}\partial^{[\beta]}\left[x^{[\alpha]}\right] y^{[\beta]}.$$

Expand out the first summation to obtain

$$\begin{aligned}\Delta V(x,y) = &\sum_{|\alpha|=1} c_\alpha \sum_{|\beta|=1} \frac{1}{\beta!}\partial^{[\beta]}\left[x^{[\alpha]}\right] y^{[\beta]} \\ &+ \sum_{|\alpha|=2} c_\alpha \sum_{1\le|\beta|\le 2} \frac{1}{\beta!}\partial^{[\beta]}\left[x^{[\alpha]}\right] y^{[\beta]} + \cdots \\ &+ \sum_{|\alpha|=p} c_\alpha \sum_{1\le|\beta|\le p} \frac{1}{\beta!}\partial^{[\beta]}\left[x^{[\alpha]}\right] y^{[\beta]}.\end{aligned}$$

The order of the summations can now be interchanged since α and β are no longer directly coupled to yield

$$\begin{aligned}\Delta V(x,y) = &\sum_{|\beta|=1} \frac{1}{\beta!}\left[\sum_{|\alpha|=1} c_\alpha \partial^{[\beta]}\left[x^{[\alpha]}\right]\right] y^{[\beta]} + \cdots \\ &+ \sum_{1\le|\beta|\le p} \frac{1}{\beta!}\left[\sum_{|\alpha|=p} c_\alpha \partial^{[\beta]}\left[x^{[\alpha]}\right]\right] y^{[\beta]}.\end{aligned}$$

Reordering the terms in the first summation yields,

$$\begin{aligned}\Delta V(x,y) = &\sum_{|\beta|=1} \frac{1}{\beta!}\left[\sum_{1\le|\alpha|\le p} c_\alpha \partial^{[\beta]}\left[x^{[\alpha]}\right]\right] y^{[\beta]} + \cdots \\ &+ \sum_{|\beta|=p} \frac{1}{\beta!}\left[\sum_{|\alpha|=p} c_\alpha \partial^{[\beta]}\left[x^{[\alpha]}\right]\right] y^{[\beta]}.\end{aligned}$$

Because $\partial^{[\beta]}\left[x^{[\alpha]}\right] = 0$ when $\alpha \le \beta$, the summation limits of the inner sums can be extended from 1 to p thereby yielding

$$\begin{aligned}\Delta V(x,y) = &\sum_{|\beta|=1} \frac{1}{\beta!}\left[\sum_{1\le|\alpha|\le p} c_\alpha \partial^{[\beta]}\left[x^{[\alpha]}\right]\right] y^{[\beta]} + \cdots \\ &+ \sum_{|\beta|=p} \frac{1}{\beta!}\left[\sum_{1\le|\alpha|\le p} c_\alpha \partial^{[\beta]}\left[x^{[\alpha]}\right]\right] y^{[\beta]}. \end{aligned}\tag{12}$$

Now note that

$$\partial^{[\beta]}V(x) = \partial^{[\beta]}\left[\sum_{|\alpha|\le p} c_\alpha x^{[\alpha]}\right] = \sum_{1\le|\alpha|\le p} c_\alpha \partial^{[\beta]}\left[x^{[\alpha]}\right],$$

which is the inner sum in (12) and so the difference becomes

$$\Delta V(x,y) = \sum_{1\le|\beta|\le p} \frac{1}{\beta!}\partial^{[\beta]}[V(x)]\,y^{[\beta]}.$$

Integrating both sides with respect to $F(y)$, and since each component of y is independent, gives

$$\begin{aligned}\int \Delta V(x,y)dF(y) &= \sum_{1\le|\beta|\le p} \frac{1}{\beta!}\partial^{[\beta]}[V(x)]\int y^{|\beta|}dF(y) \\ &= \sum_{1\le|\beta|\le p} \frac{1}{\beta!}\partial^{[\beta]}[V(x)]\,\mathbb{M}^{|\beta|},\end{aligned}$$

where we have noted that the integral $\int y^{|\beta|}dF(y) = \mathbb{M}^{|\beta|}$ is the $|\beta|$-th moment of y. Substitution of the above expression into (7) gives the JDP generator in (11). ■

Using the generator in (11), the supermartingale inequality in (9), and Dynkin's formula in (8), then proposition 3.2 below can be used to characterize a function $V(x(t))$ that bounds the probability in (1) for the JDP in (6). The proof of this proposition is given in the appendix and is based on the proof in [1, Theorem 1] except that we use the generator in (11).

Proposition 3.2: For a constant $\gamma > 0$ and a function $V(x(t)) \in C^2(\mathbb{R}^n)$, consider the open set $\Omega_{V,\gamma} = \{x \in \mathcal{X} \,|\, V(x) < \gamma\}$. Let $\{x(t)\}$ be a right continuous JDP in (6) defined on $\Omega_{V,\gamma}$ until at least some time $\tau > \tau_\gamma \doteq \inf\{t \in \mathbb{R}_+ \,|\, x(t) \notin \Omega_{V,\gamma}\}$. Let $\tilde{\mathcal{L}}V(x(t))$ be the JDP's generator defined in (11) and let $V(x(t))$ be in the domain of $\tilde{\mathcal{L}}V(x(t))$. For a constant $\alpha > 0$ and a finite time interval $t \in [0,T]$, assume the following condition holds in $\Omega_{V,\gamma}$.

$$\tilde{\mathcal{L}}V(x(t)) \le -\alpha V(x(t)) + \beta(t), \tag{13}$$

where $\beta(t)$ is a continuous, strictly positive function on $[0,T]$. In the interval $t \in [0,T]$, define

$$W(x(t)) = e^{\frac{\alpha t}{\beta(t)}} V(x(t)) + \frac{\beta(t)}{\alpha}\left(e^{\frac{\alpha \mathcal{B}(T)}{\beta(t)}} - e^{\frac{\alpha \mathcal{B}(t)}{\beta(t)}}\right), \quad (14)$$

with $\mathcal{B}(t) = \int_0^t \beta(s)ds$. Let $\lambda > 0$ be such that if $V(x(t)) > \gamma$, then $W(x(t)) \geq \lambda$. Then,

$$\mathbb{P}\{\sup_{0\leq t\leq T} V(x(t)) \geq \gamma \mid V(x_0)\} \leq \frac{V(x_0) + \frac{1}{\theta}(e^{\theta\mathcal{B}(T)} - 1)}{\gamma e^{\theta\mathcal{B}(t)} + \frac{1}{\theta}(e^{\theta\mathcal{B}(T)} - e^{\theta\mathcal{B}(t)})}. \quad (15)$$

Remark 3.3: Note that the condition in (13) which allows the generator $\tilde{\mathcal{L}}V(x)$ to be positive is less restrictive than the requirement $\tilde{\mathcal{L}}V(x) \leq 0$ used in [3]. This comparison therefore suggests that a better estimate of the probability bound in (15) can be obtained by choosing $\beta(t)$ in (13) to be a function with a small maxima and whose value decreases as time increases. One choice of such function is illustrated in the next section.

A. SOS Optimization

This section presents a two-step SOS optimization method to compute the probability bound in proposition 3.2. The first step in this method searches for a Lyapunov function $V(x)$ and the set $\Omega_{V,\gamma} \equiv \{x \in \mathcal{X}, \gamma > 0 \,|\, V(x) < \gamma\}$ for the deterministic part of the JDP in equation (6). Note that the set $\Omega_{V,\gamma}$ is the region of attraction (ROA) that corresponds to Lyapunov function $V(x)$. Using the obtained $V(x)$ and $\Omega_{V,\gamma}$, the second step estimates a constant $\alpha > 0$ and a function $\beta(t)$ which satisfy the condition in (13). This two-step optimization method is illustrated in algorithm 1 and each of these steps is discussed as follows.

1) Computation of $V(x)$ and $\Omega_{V,\gamma}$: Consider a polynomial system $\dot{x}(t) = f(x)$ with $f(x) \in \mathbb{R}[x]$, $x(0) = x_0$ and consider its linearization in the neighborhood of an equilibrium point x^* defined by $\dot{\tilde{x}}(t) = A\tilde{x}(t)$ where $A = \frac{\partial f(x)}{\partial x}|_{x^*}$. Without loss of generality, we assume x^* is the origin. The linearized system is said to be asymptotically stable if and only if there exist matrices $P \in \mathcal{S}_{++}^{n\times n}$ and $Q \in \mathcal{S}_{+}^{n\times n}$ which satisfy the Lyapunov equation $A^T P + PA = -Q$. Let $V_0(x) = x^T Px$ be the Lyapunov function for the linearized system. Then for a constant $\gamma > 0$ and a function $\sigma_1(x) \in \mathbb{R}[x]$ with $\sigma_1(x) \geq 0$ and $\sigma_1(0) = 0$, the set

$$\Omega_{V_0,\gamma}(x) = \{x \mid V_0(x) \leq \gamma\} \subseteq \{x \mid \nabla V_0(x) \cdot f(x) + \sigma_1(x) < 0\},$$

is the ROA of the origin such that for all $x_0 \in \Omega_{V_0,\gamma}$ then $\lim_{t\to\infty} x(t) = 0$. Using the generalized $\mathcal{S}-$procedure (cf. [18]), the ROA $\Omega_{V_0,\gamma}$ can be enlarged by searching a function $V(x)$ and a constant $\gamma > 0$ using the SOS optimization below.

$$\begin{aligned} \max \quad & \gamma, \\ \text{s.t.} \quad & (V(x) - \sigma_2(x)) \in \Sigma(x), \\ & -\left(\frac{\partial V(x)}{\partial x} \cdot f(x) + \sigma_1(x) + s(x)(\gamma - V(x))\right) \in \Sigma(x), \end{aligned} \quad (16)$$

where $s(x) \in \Sigma(x)$ and $\sigma_2(x) \in \mathbb{R}[x]$ with $\sigma_2(x) \geq 0$ and $\sigma_2(0) = 0$. For a feasible solution of the above SOS optimization, the set $\Omega_{V,\gamma}(x) = \{x \in \mathcal{X} \mid V(x) \leq \gamma\}$ is a subset of the maximal ROA of the origin. Note that the second constraint in the SOS optimization (16) is bilinear in the unknowns γ and $V(x)$ and so one has to iterate between γ and $V(x)$ by using $V_0(x)$ for initialization (see e.g. [18], [19] for the detail of such iteration).

Algorithm 1 Two-step SOS optimization.

Step 1 - Computation of $V(x)$ and $\Omega_{V,\gamma}$

Require: $f(x)$ and d (prespecified order of $V(x)$)
procedure $[V, \Omega_{V,\gamma}]$ = ROA$(f(x), d)$
  solve SOS optimization (16)
end procedure

Step 2 - Computation of α and c

Require: $V(x)$, $\Omega_{V,\gamma}$, $\epsilon_0 > \underline{\epsilon}$
procedure $[\alpha, c]$ = BOUND $(V, \Omega_{V,\gamma}, \epsilon_0)$
  set $\underline{\epsilon} > 0$, $\epsilon \leftarrow \epsilon_0$
  while $\epsilon \geq \underline{\epsilon}$ **do**
    max α, s.t. $[-\tilde{\mathcal{L}}V - \alpha V + \epsilon] \in \Sigma(x)$
    while α exists **do**
      max c, s.t. $[t^{q-1} - (c + t^{2q})(\tilde{\mathcal{L}}V + \alpha V)] \in \Sigma(x)$
    end while
    $\epsilon \leftarrow \epsilon/2$ ▷ Bisection on ϵ
  end while
  return α **and** c ▷ The optimal α and c
end procedure

2) Computation of α and $\beta(t)$: For given Lyapunov function $V(x)$ and the corresponding ROA $\Omega_{V,\gamma}$, the second optimization step searches for a constant $\alpha > 0$ and a function $\beta(t)$ which satisfy equation (13). In this paper, we consider the function $\beta(t)$ to be a rational function of time of the form $\beta(t) = t^{q-1}/(c + t^{2q})$ for $q = 1, 2, \dots$ and $q \in \mathbb{Z}_+$ which satisfies the condition discussed in remark 3.3. With this choice of $\beta(t)$, the estimates for constants α and c can be computed using the following SOS optimization.

$$\begin{aligned} \max \quad & \alpha,\, c, \\ \text{s.t.} \quad & \left(-(c + t^{2q})(\tilde{\mathcal{L}}V(x) + \alpha V(x)) + t^{q-1}\right) \in \Sigma(x), \end{aligned} \quad (17)$$

where the constraint of the optimization in (17) is the SOS relaxation of the condition in (13). Since the constraint in (17) is bilinear in the decision variables, one has to solve it iteratively between α and c as illustrated in step 2 of algorithm 1. The specified constant $\underline{\epsilon} > 0$ in algorithm 1 can be used as a tightness criteria of the bound in proposition 3.2. In particular, remark 3.3 suggests that a small value of $\underline{\epsilon}$ will gives a better estimate of the bound in equation (15).

Remark 3.4: The feasibility of α and c in step 2 of algorithm 1 depends on the existence of function $V(x)$ (of order d) and the set $\Omega_{V,\gamma}$ which satisfy condition (13). If the solution for α for a given $V(x)$ of order d and fixed $\underline{\epsilon}$ is not feasible, one may repeat step 1 to search for higher order $V(x)$.

IV. Example

This section illustrates an example use of the method presented in the previous sections in ecosystems management. In particular, we consider the problem of choosing a harvesting strategy to manage the bass-crayfish population in freshwater lakes. Bass-crayfish interaction is an intraguild predation system in which both species compete for the same resource while also predate one another. The model used in the example has two equilibria; one in which the bass dominate the ecosystem and the other in which the crayfish dominate the ecosystem. An outbreak of crayfish is undesirable as it can suppress the bass population. If such an outbreak occurs, management strategies are needed to shift the crayfish-dominated equilibrium point to the bass-dominated equilibrium point. One such strategy is to permit the harvesting of crayfish by anglers. In general, this harvesting process can be modeled as a jump process in which the size and the intensity of harvesting events are variables that the ecosystem manager needs to set.

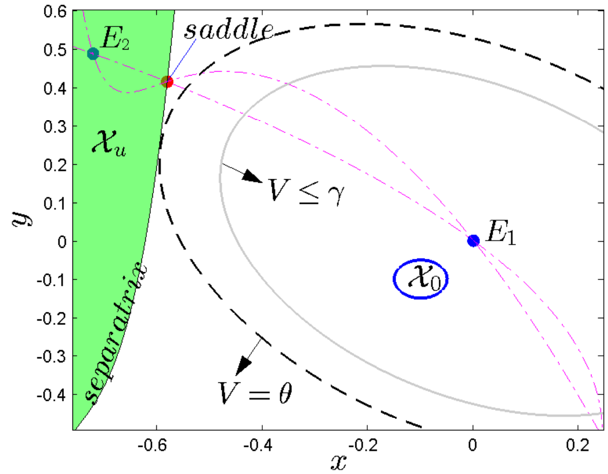
A normalized model of crayfish (x) and bass (y) interaction under harvesting events is given by [20]

$$\begin{aligned} dx(t) &= x(1 - x^2 - 0.7y) - \frac{0.08yx^2}{0.01 + x^2} + \sigma dw_1(t) \\ &\quad - \sum_{i=1}^{N(t)} z_i \delta(t - \tau_i), \qquad (18) \\ dy(t) &= 1.5y(1 - y^2 - 0.9x) + \frac{0.01yx^2}{0.01 + x^2} + \sigma dw_2(t). \end{aligned}$$

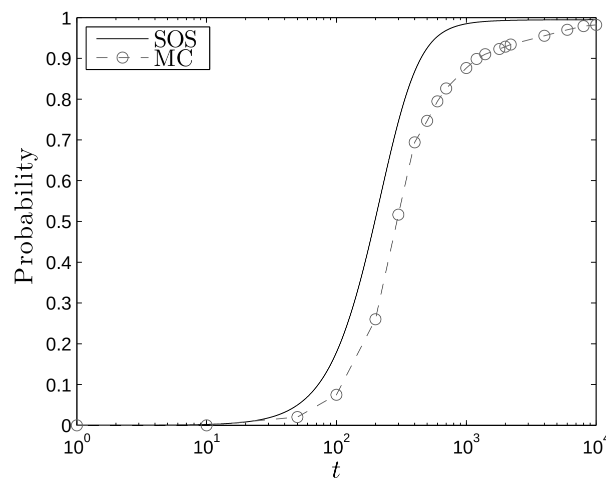
In model (18), the Wiener processes $\{w_i(t)\}$, $(i = 1, 2)$ describe small fluctuations in each population due to variations in growth rate or other environmental conditions. The last term of the state x's dynamics in (18) models the harvest of crayfish as a compound Poisson process in which the harvest size $\{z_i\}_{i=1}^{N_t}$ and the harvest times $\{\tau_i\}_{i=1}^{N_t}$ are i.i.d. with exponential distribution of intensity μ and λ, respectively, and $N(t)$ is the number of harvest events in the interval $[0, t]$. In the absence of stochastic processes $\{w(t)\}$ and $\{N(t)\}$, model (18) has three equilibria (two stable and one unstable) in $\mathbb{R}^2_+$. Figure 1a plots the isoclines, identifies the two stable equilibria (E_1, E_2) and their ROA, and marks the separatrix between the two ROAs. Note that the plot in figure 1a is obtained after shifting one of the equilibria (i.e. E_1) to the origin.

Assuming the system's current state lies in the ROA of equilibrium E_1, we are interested in bounding the probability that the sample path of (18) crosses the separatrix for a given jump intensity. We consider the set $\mathcal{X} = \{(x, y) \in \mathbb{R}^2 \,|\, -0.7 \leq x \leq 0.25,\ -0.5 \leq y \leq 0.6\}$ and define the initial set $\mathcal{X}_0$ as a circle of radius 0.05 centered at $(x, y) = (-0.1, -0.1)$. We aim to bound the probability that the sample paths of (18) reach the target set $\mathcal{X}_u$ defined by the bass-dominated region (shaded area in figure 1a). To compute this bound, we use SOSOPT [21] and SOSTOOLS [4] combined with Sedumi [5] to solve the SOS optimizations in (16)-(17).

We first solve the SOS optimization in (16) and found a second order Lyapunov function $V(x, y) = 0.6092x^2 + 0.4606xy + 0.6805y^2$ whose ROA $\Omega_{V,\gamma} = \{(x, y) \in \mathcal{X} \,|\, V \leq \gamma\}$ with $\gamma = 0.122$ is plotted in figure 1a. The closest intersection between the level set of $V(x, y)$ and the separatrix occurs at $V(x, y) = \theta = 0.19$ and so the probability that the JDP's sample paths started in $\mathcal{X}_0$ reach $\mathcal{X}_u$ in a finite time T is defined as $\mathbb{P}\{\sup_{0 \leq t \leq T} V(x, y) \geq \theta \,|\, V(x_0, y_0) \in \mathcal{X}_0\}$ (cf. figure 1a). We choose $\beta(t) = t/(c + t^4)$ and set $\sigma = 0.05$, $\mu = 0.075$, $\lambda = 0.2$ for the parameters of the JDP in (18). Using the obtained $V(x, y)$ and setting $\underline{\epsilon} = 10^{-4}$ with $\epsilon_0 = 0.1$, we then solve the SOS optimization in (17) and found $\alpha = 0.0887$ and $c = 2.56 \times 10^5$. The obtained functions $V(x, y)$ and $\beta(t)$ and constant α can then be used to compute the bound in (15). Our simulation results show that both steps achieve optimal solutions as indicated by the reports from Sedumi (feasibility ratio ≈ 1 with duality gaps of order 10^{-10}, cf. [5]).



(a) Phase portrait and ROA.



(b) Probability bound.

Fig. 1: Phase portrait, ROA and probability bound.

Figure 1b (solid curve) shows the probability bound obtained using the proposed SOS optimization method for a finite time $T = 10^4$. One may then conclude that the transition from the crayfish-dominated to the bass-dominated lakes for the specified harvesting parameters can be expected to occur after $T \geq 10^3$. Although higher harvesting intensities can be chosen to ensure a shorter transition period, one should realize that it may also drive the crayfish population toward extinction. As discussed in [22], a good harvesting parameter should be chosen to ensure that both the deadline for a transition and the desired level of ecosystem biodiversity are achieved.

Figure 1b also plots an estimate of the probability bound obtained from 500 realizations of a Monte Carlo (MC) simulation (dashed circle curve). One may see that the SOS optimization result upper bounds the MC simulation result with an average difference of 0.1 over the specified period T. The main advantage of the SOS optimization method can be seen in term of the computation time. For example, the MC simulation takes about one minute to compute the bound for given $T = 10^3$ and this computation time increases linearly as

T increases. On the other hand, the SOS optimization method only requires an average of 30 seconds to compute the bound for any values of T. This illustrates the effectiveness of the SOS optimization method to prove reachability of a process without having to rely on excessive simulations of the process.

V. Concluding Remarks

This note presented an extension of the SOS optimization method in [3] to solve stochastic reachability problems for jump-diffusion processes. To the best of our knowledge, this paper is the first study on using SOS relaxation techniques for stochastic reachability analysis of jump-diffusion processes. The extension is achieved by identifying a polynomial expression for the JDP's generator from which a two-step SOS optimization method can be formulated to compute probability bound in equation (1). The paper illustrated the value of the approach on an example drawn from ecosystem management.

Appendix

Proof of proposition 3.2

Proof: Let $T > 0$ and consider the set

$$\Omega_{W,\lambda} \triangleq \{x \in \mathbb{R}^n, t \in \mathbb{R}_+ \; : \; W(x(t)) < \lambda, \; t < T\}. \quad (19)$$

Let $\tau \doteq \inf_t\{t \in \mathbb{R}_+ \,|\, x(t) \notin \Omega_{W,\lambda}\}$ and consider the process $\{x(s)\} = \{x(t \wedge \tau)\}$ taking values on the set $\Omega_{W,\lambda}$, where $t \wedge \tau = \min(t, \tau)$. Let $\hat{\mathcal{L}}$ be the infinitesimal generator of $\{x(s)\}$ acting on the function $W(x(s))$ defined on $\Omega_{W,\lambda}$. From the definition of $W(x(s))$ in (14), we have

$$\begin{aligned}\hat{\mathcal{L}}W(x(s)) &= \theta\beta(s)e^{\theta\mathcal{B}(s)}V(x(s)) + e^{\theta\mathcal{B}(s)}\tilde{\mathcal{L}}V(x(s)) - \beta(s)e^{\theta\mathcal{B}(s)}, \\ &= e^{\theta\mathcal{B}(s)}\left(\theta\beta(s)V(x(s)) + \tilde{\mathcal{L}}V(x(s)) - \beta(s)\right).\end{aligned}$$

Since $\theta\beta(s) = \frac{\alpha\beta(s)}{\max_{0\le t\le s}\beta(t)} = \alpha$ and by the condition on $\tilde{\mathcal{L}}V(x(t))$ in (13), then

$$\hat{\mathcal{L}}W(x(s)) \le e^{\theta\mathcal{B}(s)}\left(\alpha V(x(s)) - \alpha V(x(s)) + \beta(s) - \beta(s)\right) \le 0.$$

Applying Dynkin's formula to function $W(x(t))$ gives

$$\mathbb{E}\{W(x(t))\} = \mathbb{E}\{W(x(0))\} + \mathbb{E}\left\{\int_0^s \hat{\mathcal{L}}W(x(t))\,dt\right\} \le W(x(0)),$$

which implies that $W(x(t))$ is a supermartingale with respect to process $\{x(s)\}$. Thus, for $\lambda \ge 0$

$$\begin{aligned}\mathbb{P}\left\{\sup_{0\le t\le T} W(x(t)) \ge \lambda \,\middle|\, W(x(0))\right\} &\le \frac{W(x(0))}{\lambda} \\ &\le \frac{V(x_0) + (e^{\theta\mathcal{B}(T)} - 1)/\theta}{\lambda}.\end{aligned} \quad (20)$$

Now since $W(x(t)) \ge \lambda$ implies $V(x(t)) \ge \gamma$, then

$$W(x(t)) = e^{\theta\mathcal{B}(t)}V(x(t)) + \frac{1}{\theta}\left(e^{\theta\mathcal{B}(T)} - e^{\theta\mathcal{B}(t)}\right) \ge \lambda$$

can be rearranged to obtain

$$V(x(t)) \ge e^{-\theta\mathcal{B}(t)}\left(\lambda - \frac{1}{\theta}\left(e^{\theta\mathcal{B}(T)} - e^{\theta\mathcal{B}(t)}\right)\right).$$

Thus, the condition that $V(x(t)) \ge \gamma$ implies

$$\gamma = e^{-\theta\mathcal{B}(t)}\left(\lambda - \frac{1}{\theta}\left(e^{\theta\mathcal{B}(T)} - e^{\theta\mathcal{B}(t)}\right)\right).$$

Solving the above equation for λ and substituting back into (20) gives the probability bound in equation (15). ■

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