On the Second-Order Cost of TDMA for Gaussian Multiple Access

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***Abstract*—Time division multiple access (TDMA) is widely considered to be a practical multi-access communication scheme that can achieve the highest possible sum-rate with appropriate power allocation. Exploring a Gaussian multiple access channel, we show that this result does not carry over to second-order coding rates. In particular, as the number of users grows, the cost of using TDMA is significant relative to the largest known achievable second-order sum-rate for the Gaussian MAC. The latter sum-rate is established by a random coding argument with non-i.i.d. spherical inputs, which is conveniently analyzed via a central limit theorem (CLT) for functions.**

I. Introduction

Medium access control is a crucial aspect of communication networks, which determines the policy for allocating resources to multiple users in the system. Simplicity of implementation is a critical feature for deciding on the channel access method, which is why time division multiple access (TDMA) or variants thereof are commonly used even in modern wireless networks. TDMA has the advantage that it avoids multi-user interference, but this is usually at the expense of lower communication rates due to the shorter time frames that each user can utilize. An interesting observation, however, is that TDMA achieves the optimal sum-rate in the regime of asymptotically long channel codes [1]. This observation has encouraged the application of TDMA as a simple, yet powerful channel access policy.

In this paper, motivated by applications in machine-to-machine (M2M) communications and the Internet of things (IoT) in which channel coding may be limited to finite blocklength, we revisit the TDMA method form the perspective of second-order channel coding rates. We consider the K-user Gaussian MAC modeled as

$$Y^n = \sum_{u=1}^{K} X_u^n + Z^n, \tag{1}$$

where $X_u^n := (X_{u1}, \cdots, X_{un})$ is the n-letter input for the user $u \in \{1, \cdots, K\}$ satisfying power constraint $||X_u^n||^2 \leq nP_u$ almost surely, and $Y^n := (Y_1, \cdots, Y_n)$ is the n-letter output resulting from the multi-user interference and the additive white Gaussian noise $Z^n \sim \mathcal{N}(\mathbf{0}, I_n)$. To highlight the key finding of this paper, let us consider the symmetric case, in which all users $u \in \{1, \cdots, K\}$ have maximal power constraints equal to P. To maximize the sum-rate of TDMA, the first-order asymptotic analysis allocates a fraction n/K of the time to each user, in which all the other users remain silent, and guarantees the optimal sum-rate $C(KP)$ using simple single-user encoding and decoding methods, where

$$C(P) := \frac{1}{2}\log(1+P) \tag{2}$$

is the capacity of the point-to-point Gaussian channel with signal-to-noise ratio P. In this paper, however, we will show that the maximum second-order sum-rate achieved by TDMA (using any power control strategy) with moderately short coding blocklength n and average block error probability ϵ is

$$R^*_{\text{TD}} = C\left(KP\right) - \sqrt{\frac{K}{n} V\left(KP\right)} Q^{-1}\left(1 - \sqrt[K]{1-\epsilon}\right) + O\left(\frac{\log n}{n}\right) \tag{3}$$

which can be well approximated for small $\epsilon \leq 0.01$ by

$$R^*_{\text{TD}} \approx C\left(KP\right) - \sqrt{\frac{K}{n} V\left(KP\right)} Q^{-1}\left(\frac{\epsilon}{K}\right) + O\left(\frac{\log n}{n}\right). \tag{4}$$

Here, $Q^{-1}(\cdot)$ is the inverse of the complementary CDF of a standard Gaussian distribution, $Q(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$, and $V(P)$ is the dispersion of the Gaussian channel [2], [3]

$$V(P) := \frac{\log^2 e}{2} \frac{P(P+2)}{(1+P)^2}. \tag{5}$$

A larger sum-rate can be achieved by allowing all the users to access the channel all the time, but with proper encoding at the transmitter and joint decoding at the receiver. In particular, it is straightforward to show [4] that random coding with independent and identically distributed (i.i.d.) Gaussian inputs $X_u^n \sim \mathcal{N}(\mathbf{0}, \bar{P} I_n)$ achieves

$$R^*_{\text{G}} = C\left(K\bar{P}\right) - \sqrt{\frac{V_{\text{G}}\left(K\bar{P}\right)}{n}} Q^{-1}\left(\epsilon\right) + O\left(\frac{1}{n}\right), \tag{6}$$

where $\bar{P} := P - \delta$ with δ a small constant or even decaying but slower than $1/\sqrt{n}$, and $V_{\text{G}}(P)$ is the Gaussian-induced dispersion [5]

$$V_{\text{G}}(P) := \log^2 e \, \frac{P}{1+P}. \tag{7}$$

Following Shannon [6], a still larger sum-rate can be achieved via non-i.i.d. inputs that are uniformly distributed over the n-dimensional *power shell* $||x_u^n|| = \sqrt{nP}$. This ensures that each

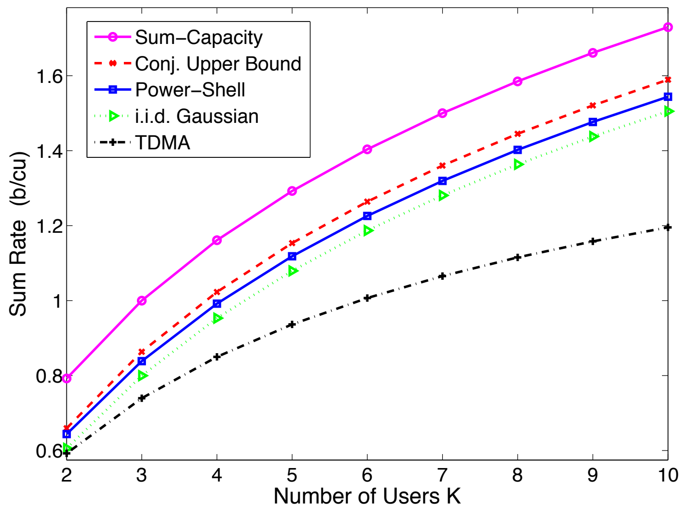

Fig. 1. Sum-rate, up to second-order, in bits per channel use (b/cu) of a symmetric Gaussian MAC as a function of the number K of users. The setting has blocklength $n = 500$, average block error probability $\epsilon = 10^{-3}$, and power constraint $P_u = 0$ dB for all users $u \in \{1, \cdots, K\}$. The power margin for the i.i.d. Gaussian input is $\delta = 0.052$.

codeword utilizes all the power budget. We will show that this scheme achieves the following second-order sum-rate:

$$R^*_{\text{PS}} = C\,(KP) - \sqrt{\frac{V\,(KP) + V_c\,(K,P)}{n}}\,Q^{-1}\,(\epsilon) + O\left(\frac{1}{n}\right), \tag{8}$$

where $V_c(K,P)$ is a cross dispersion

$$V_c(K,P) := \frac{\log^2 e}{2}\,\frac{K(K-1)P^2}{(1+KP)^2}, \tag{9}$$

arising from the inner-product of independent power shell inputs. In the last section of this paper, we conjecture that dropping this cross dispersion leads to a second-order upper bound on the sum-rate of the form

$$R^*_{\text{UB}} = C\,(KP) - \sqrt{\frac{V\,(KP)}{n}}\,Q^{-1}\,(\epsilon) + O\left(\frac{\log n}{n}\right). \tag{10}$$

Figure 1 compares, up to second order, the sum rate achievable via TDMA with that achievable using the Gaussian and power shell coding schemes, all as functions of the number of users K. For reference, the asymptotically optimal sum-rate $C(KP)$ and a conjectured upper bound [7] are also depicted. One observes that TDMA is not second-order sum-rate optimal. In fact, it has a considerably high back-off from the first-order asymptotic sum-capacity $C(KP)$, and this gap increases as the number of users grows. The power-shell and the Gaussian coding schemes, however, very closely follow the conjectured upper bound and the asymptotic sum-rate.

It is worth mentioning that in our setup, the number of users can be large, but still needs to satisfy $K = O(1)$ with respect to the blocklength. A first-order study of the case in which K grows as a function of blocklength n is developed in [8].

II. TDMA Analysis

In this section, we present achievability and converse arguments that characterize the second-order performance of the time-sharing scheme for the general, potentially asymmetric case.

Theorem 1: The second-order characterization of the TDMA with power control scheme for the K-user Gaussian MAC with power constraint P_u for user $u \in \{1, \cdots, K\}$ is the set of all tuples $(R_1, \cdots, R_K)$ satisfying

$$R_u < \alpha_u C\left(\frac{P_u}{\alpha_u}\right) - \sqrt{\frac{\alpha_u}{n} V\left(\frac{P_u}{\alpha_u}\right)}\,Q^{-1}(e_u) + O\left(\frac{\log n}{n}\right), \tag{11}$$

for some $0 \le \alpha_u \le 1$ and $0 \le e_u \le \epsilon$ for all $u \in \{1, \cdots, K\}$; $\sum_{u=1}^K \alpha_u = 1$; and

$$\sum_{u=1}^{K} (-1)^{u+1} \sum_{1 \le l_1 < \cdots < l_u \le K} e_{l_1} \cdots e_{l_u} = \epsilon. \tag{12}$$

Remark 1. The parameters $\{\alpha_u\}_{u=1}^K$ and $\{e_u\}_{u=1}^K$ denote the fraction of blocklength allocated to and the average block error probability achieved by each user in its TDMA time slot, respectively. To clarify, (12) for the example of $K = 3$ reads:

$$e_1 + e_2 + e_3 - e_1e_2 - e_1e_3 - e_2e_3 + e_1e_2e_3 = \epsilon. \tag{13}$$

Before giving the proof of Theorem 1, let us show how it implies the sum-rate (3) for the symmetric case. First note that, the selections $\alpha_u = 1/K$ and $e_u = e := 1 - \sqrt[K]{1-\epsilon}$ for all $u \in \{1, \cdots, K\}$ satisfy the conditions of Theorem 1 and recover (3). In particular,

$$\sum_{u=1}^{K} (-1)^{u+1} \binom{K}{u} e^u = 1 - (1-e)^K = \epsilon. \tag{14}$$

To prove that these are the optimal selections, note that any pair of $\alpha := (\alpha_1, \cdots, \alpha_K)$ and $\mathbf{e} := (e_1, \cdots, e_K)$ tuples achieve the following sum-rate:

$$R^*_{\text{TS}}(\alpha, \mathbf{e}) = \left(\sum_{u=1}^{K} \alpha_u C\left(\frac{P}{\alpha_u}\right)\right) + \frac{1}{\sqrt{n}}\left(-\sum_{u=1}^{K} \sqrt{\alpha_u V\left(\frac{P}{\alpha_u}\right)}\,Q^{-1}(e_u)\right) + O\left(\frac{\log n}{n}\right), \tag{15}$$

To maximize $R^*_{\text{TS}}(\alpha, \mathbf{e})$ first over the choice of α, we rely on a vector-extension of [2, Lemma 63] which states the maximum occurs by choosing the α that individually maximizes the second-order term, only among those α that individually maximize the first-order term; moreover, the residual term is $O\left(\frac{1}{n}\right)$ if the two functions in the first- and second-order terms satisfy differentiability conditions; cf. [2, Lemma 64]. In our case, the only α that maximizes the first order term is the well-known choice of equal fractions $\alpha_u = \frac{1}{K}$ for all $u \in \{1, \cdots, K\}$. Thus, it is the global optimizer for the function $R^*_{\text{TS}}(\alpha, \mathbf{e})$ with $O\left(\frac{1}{n}\right)$ as the residual, since regularity conditions are satisfied. To maximize the resulting function, one performs

$$\min_{\mathbf{e}} \sum_{u=1}^{K} Q^{-1}(e_u) \tag{16}$$

$$\text{subject to} \sum_{u=1}^{K} (-1)^{u+1} \sum_{1 \le l_1 < \cdots < l_u \le K} e_{l_1} \cdots e_{l_u} = \epsilon. \tag{17}$$

One can verify that the solution of this minimization problem is symmetric, i.e. $e_1 = \cdots = e_k := e$, thus satisfying (14).

Now, we turn to the proof of Theorem 1.

Proof: Since each user in its turn observes an interference-free P2P Gaussian channel, both achievability and converse parts follow directly from the achievability and converse results of the P2P Gaussian channel [2], [3], respectively, upon recalling that (i) the total fraction of users' turns adds up to one, and (ii) the error probability analysis follows from

$$\Pr\left[\bigcup_{u=1}^{K} \mathcal{E}_u\right] = \epsilon, \tag{18}$$

where $\mathcal{E}_u$, with $\Pr[\mathcal{E}_u] = e_u$, denotes the individual error event of user $u \in \{1, \cdots, K\}$ on a P2P Gaussian channel. Due to the independence of the error events, (18) simplifies to (12) via the inclusion-exclusion principle, thus concluding the proof. ■

III. Achievable Region with Power Shell Input

In this section, we state and prove the second-order achievable rate region using the power shell input for the general, potentially asymmetric Gaussian MAC. As we mentioned in Section I, this gives the largest region among all the regions we analyze in this paper.

Theorem 2: A second-order achievable rate region with power shell inputs for the K-user Gaussian MAC with power constraint P_u for user $u \in \{1, \cdots, K\}$ is the set of all rate tuples $(R_1, \cdots, R_K)$ satisfying

$$\mathbf{R} \in \mathbf{C}(\mathbf{P}) - \frac{1}{\sqrt{n}} Q^{-1}(\epsilon; \mathbf{V}(\mathbf{P})) + O\left(\frac{1}{n}\right)\mathbf{1}, \tag{19}$$

where: $\mathbf{1}$ is the all-one vector of length $2^K - 1$; $\mathbf{R}$ denotes the $2^K - 1$ tuple with elements $R_{\mathcal{S}} = \sum_{u \in \mathcal{S}} R_u$ with $\mathcal{S} \subseteq \{1, \cdots, K\}$; $\mathbf{P}$ denotes the $2^K - 1$ tuple with elements

$$P_{\mathcal{S}} = \sum_{u \in \mathcal{S}} P_u\,; \tag{20}$$

$\mathbf{C}(\mathbf{P})$ denotes the $2^K - 1$ tuple with elements $C_{\mathcal{S}} := C(P_{\mathcal{S}})$; $\mathbf{V}(\mathbf{P})$ is the $(2^K - 1) \times (2^K - 1)$ dispersion matrix with elements

$$\begin{aligned} V_{\mathcal{S},\mathcal{S}'} &= V(P_{\mathcal{S}}, P_{\mathcal{S}'}) \\ &:= \frac{\log^2 e}{2} \frac{P_{\mathcal{S}} P_{\mathcal{S}'} + 2P_{\mathcal{S}\cap\mathcal{S}'} + (P_{\mathcal{S}\cap\mathcal{S}'})^2 - \sum_{u \in \mathcal{S}\cap\mathcal{S}'} P_u^2}{(1 + P_{\mathcal{S}})(1 + P_{\mathcal{S}'})}; \end{aligned} \tag{21}$$

and $Q^{-1}(\epsilon;\mathbf{\Sigma})$ is the inverse complementary CDF of a $(2^K–1)$-dimensional Gaussian random variable defined as the set

$$Q^{-1}(\epsilon; \mathbf{\Sigma}) := \left\{\mathbf{z} \in \mathbb{R}^{2^K-1} : \Pr\left(\mathcal{N}(\mathbf{0}, \mathbf{\Sigma}) \le \mathbf{z}\right) \ge 1 - \epsilon\right\}. \tag{22}$$

Since the evaluation of the rate region in Theorem 2 is cumbersome for large K, we provide a simpler rate region via the outage-splitting idea [7].

Theorem 3: A second-order achievable rate region with power shell inputs for the K-user Gaussian MAC with power constraint P_u for user $u \in \{1, \cdots, K\}$ is the set of all rate tuples $(R_1, \cdots, R_K)$ satisfying

$$R_{\mathcal{S}} \le C(P_{\mathcal{S}}) - \sqrt{\frac{V(P_{\mathcal{S}}) + V_c(P_{\mathcal{S}})}{n}} Q^{-1}(\lambda_{\mathcal{S}}\epsilon) + O\left(\frac{1}{n}\right), \tag{23}$$

for all subsets of the users $\mathcal{S} \subseteq \{1, \cdots, K\}$ and any choice of non-negative coefficients $\lambda_{\mathcal{S}}$ that sum to one, $\sum_{\mathcal{S} \subseteq \{1,\cdots,K\}} \lambda_{\mathcal{S}} = 1$, where $R_{\mathcal{S}}$ and $P_{\mathcal{S}}$ are defined as in Theorem 2, $V(P)$ is the dispersion (5) of the P2P Gaussian channel, and $V_c(P_{\mathcal{S}})$ is a cross dispersion

$$V_c(P_{\mathcal{S}}) := \frac{\log^2 e}{2} \frac{(P_{\mathcal{S}})^2 - \sum_{u \in \mathcal{S}} P_u^2}{(1 + P_{\mathcal{S}})^2}. \tag{24}$$

The sum-rate (8) for the symmetric Gaussian MAC is an immediate consequence of Theorem 3 upon noting that the dominant inequality for the sum-rate is the one corresponding to the set of all users $\mathcal{S} = \{1, \cdots, K\}$ with $P_{\mathcal{S}} = KP$. In particular, the error probabilities of all other subsets of users are in the large deviation regime, so that $\lambda_{\mathcal{S}}\epsilon$ captures all but a vanishing part of the error probability ϵ. This vanishing perturbation can then be dropped via an application of Taylor's theorem, cf. [9] for related details.

The key tool for our analysis of power shell inputs is a powerful, yet convenient, version of the central limit theorem (CLT), called the *CLT for functions*, which handles sums of *dependent* random variables that can be expressed as *functions of sums of independent* random variables. In particular, let $\{\mathbf{W}_t := (W_{1t}, ..., W_{Kt})\}_{t=1}^{\infty}$ be i.i.d. random vectors with positive variance and finite third moment, and let $\mathbf{f}(\mathbf{w}) = (f_1(\mathbf{w}), ..., f_L(\mathbf{w}))$ be an L-component vector-function with continuous second-order partial derivatives in a neighborhood of $\mathbf{w} = \mathbb{E}[\mathbf{W}_1]$. Then, for any convex Borel-measurable set $\mathcal{D}$ in $\mathbb{R}^L$, there exists a finite positive constant B such that [10], [7, Prop. 1]

$$\begin{aligned} &\Bigg| \Pr\left[\mathbf{f}\left(\frac{1}{n}\sum_{t=1}^{n} \mathbf{W}_t\right) \in \mathcal{D}\right] \\ &\quad - \Pr\left[\mathcal{N}\left(\mathbf{f}\left(\mathbb{E}[\mathbf{W}_1]\right), \frac{1}{n}\mathbf{J}\mathrm{Cov}(\mathbf{W}_1)\mathbf{J}^T\right) \in \mathcal{D}\right]\Bigg| \le \frac{B}{\sqrt{n}}, \end{aligned} \tag{25}$$

where $\mathbf{J}$ is the Jacobian matrix of $\mathbf{f}(\mathbf{w})$ at $\mathbf{w} = \mathbb{E}[\mathbf{W}_1]$ consisting of the following first-order partial derivatives

$$J_{lk} := \left.\frac{\partial f_l(\mathbf{w})}{\partial w_k}\right|_{\mathbf{u}=\mathbb{E}[\mathbf{W}_1]} \quad l = 1, ..., L, \quad k = 1, ..., K. \tag{26}$$

We are now ready to present the proof of Theorems 2 and 3.

Proof: The proof builds on random coding and modified typicality decoding. Each user $u \in \{1, \cdots, K\}$ independently generates M_u independent codewords $\{x_u^n(j_u)\}_{j_u=1}^{M_u}$ according to the n-letter input distribution $P_{X_u^n}(x_u^n)$. To communicate a message tuple $(j_1, \cdots, j_K)$, each user u sends $x_u^n(j_u)$. Upon reception of y^n, the decoder searches for the first codeword tuple $(x_1^n(\hat{j}_1), \cdots, x_K^n(\hat{j}_K))$ that looks "typical" with y^n in the following one-sided sense:

$$\tilde{\imath}\left(x_{\mathcal{S}}^n(\hat{j}_{\mathcal{S}}); y^n \middle| x_{\mathcal{S}^c}^n(\hat{j}_{\mathcal{S}^c})\right) > \log(\kappa_{\mathcal{S}} M_{\mathcal{S}}), \tag{27}$$

for all subsets $\mathcal{S} \subseteq \{1, \cdots, K\}$, where $j_{\mathcal{S}}$ and $x_{\mathcal{S}}^n(j_{\mathcal{S}})$ are the $|\mathcal{S}|$-tuples consisting of the messages j_u and the codewords $x_u^n(j_u)$, respectively, for all $u \in \mathcal{S}$. Moreover, we define $M_{\mathcal{S}} := \prod_{u \in \mathcal{S}} M_u$, the $\kappa_{\mathcal{S}}$ coefficients

$$\kappa_{\mathcal{S}} := \sup_{x_{\mathcal{S}^c}^n \in \mathcal{X}_{\mathcal{S}^c}^n,\, y^n \in \mathcal{Y}^n} \frac{dP_{Y^n|X_{\mathcal{S}^c}^n}(y^n|x_{\mathcal{S}^c}^n)}{dQ^{(\mathcal{S})}_{Y^n|X_{\mathcal{S}^c}^n}(y^n|x_{\mathcal{S}^c}^n)}, \tag{28}$$

and $\tilde{\imath}$ the modified mutual information random variable

$$\tilde{\imath}\left(x_{\mathcal{S}}^n; y^n \middle| x_{\mathcal{S}^c}^n\right) := \log \frac{P_{Y^n|X_{\mathcal{S}}^n X_{\mathcal{S}^c}^n}(y^n|x_{\mathcal{S}}^n, x_{\mathcal{S}^c}^n)}{Q^{(\mathcal{S})}_{Y^n|X_{\mathcal{S}^c}^n}(y^n|x_{\mathcal{S}^c}^n)}, \tag{29}$$

with any reference (conditional) output distribution $Q^{(\mathcal{S})}_{Y^n|X_{\mathcal{S}^c}^n}$.

Following standard arguments [1], [2], [7], one can prove the existence of a code with blocklength n, average error probability ϵ, and satisfying the power constraints such that[1]

$$\begin{aligned}\epsilon \le\; & P_{\mathbf{X}^n} P_{Y^n|\mathbf{X}^n}\left[\bigcup_{\mathcal{S} \subseteq \{1,\cdots,K\}} \tilde{\imath}(X_{\mathcal{S}}^n; Y^n|X_{\mathcal{S}^c}^n) \le \log(\kappa_{\mathcal{S}} M_{\mathcal{S}})\right] \\ & + \sum_{\mathcal{S} \subseteq \{1,\cdots,K\}} \kappa_{\mathcal{S}} M_{\mathcal{S}} P_{\mathbf{X}^n} Q^{(\mathcal{S})}_{Y^n|X_{\mathcal{S}^c}^n}\left[\tilde{\imath}(X_{\mathcal{S}}^n; Y^n|X_{\mathcal{S}^c}^n) > \log(\kappa_{\mathcal{S}} M_{\mathcal{S}})\right] \\ & + P_{\mathbf{X}^n}\left[\bigcup_{u \in \{1,\cdots,K\}} ||X_u^n||^2 > nP_u\right],\end{aligned} \tag{30}$$

where $P_{\mathbf{X}^n} := \prod_{u=1}^K P_{X_u^n}$.

Now, we choose the input distribution $P_{X_u^n}$ to be the uniform distribution on the power shell $||x_u^n|| = \sqrt{nP_u}$, for which the constraint-violation probability in the third term of (30) is zero. Moreover, we select the reference distribution as $Q^{(\mathcal{S})}_{Y^n|X_{\mathcal{S}^c}^n}(y^n|x_{\mathcal{S}^c}^n) \sim \mathcal{N}(y^n; x_{\mathcal{S}^c}^n, (1+P_{\mathcal{S}})I_n)$, which can be considered as the output of the channel $Y^n = X_{\mathcal{S}}^n + x_{\mathcal{S}^c}^n + Z^n$ with the input $X_{\mathcal{S}}^n$ following $Q^{(\mathcal{S})}_{X_{\mathcal{S}}^n} \sim \mathcal{N}(\mathbf{0}, P_{\mathcal{S}} I_n)$.

A critical step in the analysis is to show that

$$\kappa_{\mathcal{S}} = O(1), \qquad \forall\, \mathcal{S} \subseteq \{1, \cdots, K\}. \tag{31}$$

If $|\mathcal{S}| = 1$, (31) immediately follows from [7, Prop. 2]. If $|\mathcal{S}| \ge 2$, (31) is proved by extending our idea in [7, Prop. 3], that the probability density function (pdf) of the superposition of any number of independent uniform-on-the-shell *input* distributions is bounded by a constant times the pdf of a Gaussian *input* distribution with an equal sum-power:

$$\sup_{x_{\mathcal{S}}^n \in \mathcal{X}_{\mathcal{S}}^n} \frac{dP_{X_{\mathcal{S}}^n}(x_{\mathcal{S}}^n)}{dQ^{(\mathcal{S})}_{X_{\mathcal{S}}^n}(x_{\mathcal{S}}^n)} \le c_{\mathcal{S}} = O(1), \tag{32}$$

which is proved in the Appendix. Since $\kappa_{\mathcal{S}}$ is the ratio of the corresponding outputs through the same channel $Y^n = X_{\mathcal{S}}^n + x_{\mathcal{S}^c}^n + Z^n$, we conclude (31). The confusion probability in the second term of (30), $P_{\text{conf}}(\mathcal{S}) := \kappa_{\mathcal{S}} M_{\mathcal{S}} P_{\mathbf{X}^n} Q^{(\mathcal{S})}_{Y^n|X_{\mathcal{S}^c}^n}\left[\tilde{\imath}(X_{\mathcal{S}}^n; Y^n|X_{\mathcal{S}^c}^n) > \log(\kappa_{\mathcal{S}} M_{\mathcal{S}})\right]$, can now be analyzed using the strong large deviation result of [2, Lemma 47] to conclude

$$\sum_{\mathcal{S} \subseteq \{1,\cdots,K\}} P_{\text{conf}}(\mathcal{S}) \le O\left(\frac{1}{\sqrt{n}}\right), \tag{33}$$

since the number of users is constant w.r.t. the blocklength.

Hence, it only remains to analyze the outage probability in the first term of (30). We have

$$\begin{aligned}\tilde{\imath}\left(x_{\mathcal{S}}^n; y^n \middle| x_{\mathcal{S}^c}^n\right) = nC(P_{\mathcal{S}}) + \frac{\log e}{2(1+P_{\mathcal{S}})}\Big[P_{\mathcal{S}}(n - ||Z^n||^2) \\ + 2\langle X_{\mathcal{S}}^n, Z^n\rangle + \sum_{\substack{u,\bar{u} \in \mathcal{S} \\ u < \bar{u}}} 2\langle X_u^n, X_{\bar{u}}^n\rangle\Big].\end{aligned} \tag{34}$$

Note that, due to the non-i.i.d. structure of the power shell input, $\tilde{\imath}\left(x_{\mathcal{S}}^n; y^n \middle| x_{\mathcal{S}^c}^n\right)$ is a sum of *dependent* random variables. However, recall that X_u^n is distributed uniformly on the power shell $||x_u^n|| = \sqrt{nP_u}$ if and only if

$$X_{ut} = \sqrt{nP_u}\frac{G_{ut}}{||G_u^n||}, \tag{35}$$

where $G_u^n \sim \mathcal{N}(\mathbf{0}, I_n)$. Therefore, $\tilde{\imath}\left(x_{\mathcal{S}}^n; y^n \middle| x_{\mathcal{S}^c}^n\right)$ can be expressed as a *function* of independent random variables, and analyzed via our CLT of functions in (25). In particular, $\frac{1}{n}\tilde{\imath}\left(x_{\mathcal{S}}^n; y^n \middle| x_{\mathcal{S}^c}^n\right) = f_{\mathcal{S}}\left(\frac{1}{n}\sum_{t=1}^n \mathbf{W}_t\right)$, where the function $f_{\mathcal{S}}(\mathbf{w})$ with $1 + 2K + \binom{K}{2}$ input variables is defined as

$$\begin{aligned}f_{\mathcal{S}}(\mathbf{w}) = C(P_{\mathcal{S}}) + \frac{\log e}{2(1+P_{\mathcal{S}})}\Bigg[P_{\mathcal{S}} w_0 + \sum_{u \in \mathcal{S}} \frac{2w_u}{\sqrt{1 + w_{(K+u)}}} \\ + \sum_{\substack{u,\bar{u} \in \mathcal{S} \\ u < \bar{u}}} \frac{2w_{u,\bar{u}}}{\sqrt{1 + W_{(K+u)}}\sqrt{1 + W_{(K+\bar{u})}}}\Bigg],\end{aligned} \tag{36}$$

and $\mathbf{W}_t = (W_{0t}; \{W_{ut}\}_{u=1}^K; \{W_{(K+u)t}\}_{u=1}^K; \{W_{(u,\bar{u})t}\}_{\substack{u,\bar{u}=1 \\ u<\bar{u}}}^K)$ is the set of random variables

$$W_{0t} = 1 - Z_t^2, \qquad W_{ut} = \sqrt{P_u} G_{ut} Z_t, \tag{37}$$

$$W_{(K+u)t} = G_{u,t}^2 - 1, \quad W_{(u,\bar{u})t} = \sqrt{P_u P_{\bar{u}}} G_{ut} G_{\bar{u}t}, \tag{38}$$

for all $u, \bar{u} \in \{1, \cdots, K\}, u < \bar{u}$.

Since $\mathbf{W}_t$ is zero mean, we obtain $f_{\mathcal{S}}(\mathbb{E}[\mathbf{W}_1]) = C(P_{\mathcal{S}})$. Moreover, it is straightforward to verify that $\mathbf{W}_t$ has finite third moment and a diagonal covariance matrix as:

$$\text{Cov}(\mathbf{W}_1) = \text{Diag}[2; P_1 \cdots P_K; 2 \cdots 2; (P_1 P_2) \cdots (P_{K-1} P_K)],$$

[1] We use $P_X P_Y P_{Z|X}[f(X,Y,Z) \in \mathcal{A}]$ to indicate that (X,Y,Z) follow the joint distribution $P_{XYZ}(x,y,z) = P_X(x) P_Y(y) P_{Z|X}(z|x)$ in determining the probability $\Pr[f(X,Y,Z) \in \mathcal{A}]$.

and that $f_{\mathcal{S}}(\mathbf{w})$ has a Jacobian matrix $\mathbf{J}$ whose row corresponding to $\mathcal{S}$ is $\frac{\log e}{2}$ times

$$\Bigg[\underbrace{\frac{P_{\mathcal{S}}}{1+P_{\mathcal{S}}}}_{\text{size } 1};\underbrace{\frac{2}{1+P_{\mathcal{S}}}1\{u\in\mathcal{S}\}}_{\text{size } K};\underbrace{\mathbf{0}}_{\text{size } K};\underbrace{\frac{2}{1+P_{\mathcal{S}}}1\{u<u'\in\mathcal{S}\}}_{\text{size } \binom{K}{2}}\Bigg].$$

Therefore, the $(\mathcal{S},\mathcal{S}')$-entry in $\mathbf{J}\mathrm{Cov}(\mathbf{W}_1)\mathbf{J}^T$ is computed as

$$\left(\frac{\log e}{2}\right)^2\Bigg[\frac{2P_{\mathcal{S}}P_{\mathcal{S}'}}{(1+P_{\mathcal{S}})(1+P_{\mathcal{S}'})} + \sum_{1\le u\le K}\frac{2P_u\cdot 1\{u\in\mathcal{S}\}}{1+P_{\mathcal{S}}}\frac{2\cdot 1\{u\in\mathcal{S}'\}}{1+P'_{\mathcal{S}}} + \sum_{1\le u,\bar{u}\le K}\frac{2P_uP_{\bar{u}}\cdot 1\{u<\bar{u}\in\mathcal{S}\}}{1+P_{\mathcal{S}}}\frac{2\cdot 1\{u<\bar{u}\in\mathcal{S}'\}}{1+P'_{\mathcal{S}}}\Bigg]$$

which simplifies to (21). Combining the CLT for functions (25) with (30) and (33), we obtain

$$\Pr\left[\mathcal{N}\left(\mathbf{C}(\mathbf{P}),\frac{\mathbf{V}(\mathbf{P})}{n}\right)>\mathbf{R}+O\left(\frac{1}{n}\right)\mathbf{1}\right]\ge 1-\left(\epsilon-O\left(\frac{1}{\sqrt{n}}\right)\right).$$

The proof of Theorem 2 concludes by recalling the symmetry property and the inverse complementary CDF definition (22) of the multi-dimensional Gaussian RV.

For the proof of Theorem 3, we instead use the union bound to split the joint-outage event as in [7] and assign a portion $\lambda_{\mathcal{S}}\epsilon$ of the error probability to the individual outage events of each subset $\mathcal{S}$ of users. The analysis would then rely solely on the regular one-dimensional CLT and only deals with the diagonal entries of the dispersion matrix $\mathbf{V}(\mathbf{P})$, i.e., only the variances of the corresponding mutual information random variables. ■

IV. A Conjectured Outer Bound

A set of straightforward outer bounds for the second-order capacity region of the K-user Gaussian MAC can be obtained via single-user bounds that assume a genie has provided the receiver with the messages of all users, except one of them. We suspect an extension of this idea to all subsets $\mathcal{S}\subseteq\{1,\cdots,K\}$ holds, which implies our conjectured outer bound in (10) for the symmetric case.

Conjecture 1: An outer bound on the second-order rate region of the K-user Gaussian MAC with power constraint P_u for the user $u\in\{1,\cdots,K\}$ is the set of all rate tuples $(R_1,\cdots,R_K)$ satisfying

$$R_{\mathcal{S}}\le C(P_{\mathcal{S}})-\sqrt{\frac{V(P_{\mathcal{S}})}{n}}Q^{-1}(\epsilon)+O\left(\frac{\log n}{n}\right), \tag{39}$$

for all subsets of the users $\mathcal{S}\subseteq\{1,\cdots,K\}$, where all notations are defined as in Theorem 2.

Appendix

In this appendix, we sketch the proof of inequality (32) in the confusion probability analysis. The main idea is to first derive the pdf of the super-imposed spherical distributions and then prove that it is bounded w.r.t. the corresponding Gaussian pdf. For brevity, we focus on the case of symmetric Gaussian MAC with equal powers P across users; the proof for the asymmetric case generalizes similar arguments.

Lemma 1: The pdf of the superposition of $k=2,3,\cdots$ power shell input distributions is given by

$$P_{U_k^n}(u_k^n)=c_k\frac{(S_{n-1}(1))^{k-1}}{(S_n(\sqrt{nP}))^k}(\sqrt{2\pi P})^{k-2}(nP) \times\frac{(k^2nP-||u_k^n||^2)^{k-2}}{(k(k-2)nP+||u_k^n||^2)^{\frac{k-1}{2}}} \times\left(\frac{(k^2nP-||u_2^n||^2)^{k-1}}{(k-1)^{k-1}k^k}\right)^{\frac{n-3}{2}}, \tag{40}$$

where $0<||u_k^n||<k\sqrt{nP}$ and $c_k<1$ is a normalizing constant, independent of n and P, that depends only on the number of layers k.

The proof of this lemma is given elsewhere [4], but basically relies on a recursive formula for the pdf of interest along with the Laplace method for integration [11].

To prove inequality (32), let $d(u_k^n):=\frac{P_{U_k^n}(u_k^n)}{Q_{U_k^n}(u_k^n)}$ where $Q_{U_k^n}(u_k^n)=(2\pi)^{-n/2}(kP)^{-n/2}e^{-||u_k^n||^2/(2kP)}$ is the pdf of the superposition of k layers of i.i.d. Gaussian inputs. Letting $||u_k^n||=\sqrt{nt}$ with $t\in(0,k^2P)$, we obtain $\ln d(u_k^n)=a(t)+\frac{n}{2}f_{k,P}(t)$ where

$$a(t):=c+(k-2)\ln(k^2P-t)-\frac{k-1}{2}\ln(k(k-2)P+t)$$

with c a constant only depending on k and P, and

$$f_{k,P}(t):=-1-k\ln P+\ln(kP)-(k-1)\ln(k-1) -k\ln k+(k-1)\ln(k^2P-t)+\frac{t}{kP},$$

for sufficiently large n. It is then straightforward to verify that $f_{k,P}(t)$ achieves its maximum at $t=kP$ where its value is $f_{k,P}(kP)=0$. Therefore, $f_{k,P}(t)\le 0$ for all $t\in(0,k^2P)$ that implies $d(u_k^n)\le\exp(\max_t a(t))=O(1)$, since $a(t)$ has a finite maximum that occurs at $t=0$.

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