A WEIGHTED SEMILINEAR ELLIPTIC EQUATION INVOLVING CRITICAL SOBOLEV EXponents

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Abstract. In this paper we prove the existence of a positive radial solution of the problem

\[-\Delta u = r^{\sigma}|u|^{p-1}u + \lambda r^\alpha u, \quad \text{in } B_R \subset \mathbb{R}^N (r = |x|)\]

for \(\lambda\) in a suitable (and almost optimal) range. Here \(N \geq 3, \sigma, \alpha \geq -2\) and \(p = (N + 2 + 2\sigma)/(N - 2)\) corresponds to the critical Sobolev exponent \(p + 1 = (2N + 2\sigma)/(N - 2)\). Our result extends the previous one due to Brézis and Nirenberg when \(\sigma = \alpha = 0\).

0. Introduction. In a previous paper [8] we considered the problem

\[
\begin{align*}
- &\frac{1}{r^\gamma}(r^\gamma u')' = r^{\sigma}|u|^{q-1}u \quad \text{in } (0, 1) \\
u(1) &= 0, \quad \int_0^1 r^\gamma |u'|^2 dr < \infty \\
u &> 0.
\end{align*}
\tag{0.1}
\]

We recall some of the results we obtained there.

"If \(\gamma > 1\) then the problem has exactly one weak solution for \(1 < q < \frac{\gamma + 3 + 2\sigma}{\gamma - 1}\) and no weak solution for \(q > (\gamma + 3 + 2\sigma)/(\gamma - 1)\)."

In this paper we shall deal exactly with the critical case, namely, \(q = p = (\gamma + 3 + 2\sigma)/(\gamma - 1)\). Instead of (0.1) we consider the more general problem

\[
\begin{align*}
- &\frac{1}{r^\gamma}(r^\gamma u')' = r^{\sigma}|u|^{p-1}u + \lambda r^\alpha u \quad \text{in } (0, 1) \\
u(1) &= 0, \quad \int_0^1 r^\gamma |u'|^2 dr < \infty \\
u &> 0
\end{align*}
\tag{0.2}
\]

where \(\gamma > 1, \sigma, \alpha > -2\).

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In order to describe our results we need to explain some notation. Let us consider the linear eigenvalue problems

$$-\frac{1}{r^\gamma}(r^\gamma u')' = \lambda r^\alpha u, \quad u(1) = 0, \quad \int_0^1 r^\gamma |u'|^2 \, dr < \infty \quad (0.3)$$

and

$$-\frac{1}{r^{2-\gamma}}(r^{2-\gamma} u')' = \mu r^\alpha u, \quad u(1) = 0, \quad \int_0^1 r^{2-\gamma} |u'|^2 \, dr < \infty \quad (0.4)$$

which should be understood in a suitable weak sense.

We denote by $\lambda_1(\alpha, \gamma)$ the least eigenvalue of (0.3) which exists for $\gamma > 1$, $\alpha > -2$ and by $\mu_1(\alpha, \gamma)$ the first eigenvalue of (0.4) which exists for $1 < \gamma < \alpha + 3$. We can now state our main results.

A. There exists no weak solution of (0.2) for $\lambda \leq 0$ or $\lambda \geq \lambda_1(\alpha)$.

B. If $\gamma > \alpha + 3$ then (0.2) has at least a weak solution if and only if $\lambda \in (0, \lambda_1(\alpha))$.

C. If $1 < \gamma < \alpha + 3$ then $\mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma)$ and problem (0.2) has a weak solution for each $\lambda \in (\mu_1(\alpha, \gamma), \lambda_1(\alpha, \gamma))$.

The proof of these results uses a variational method together with the techniques of Brézis-Nirenberg [3] in order to overcome the difficulties raised by the lack of compactness due to the critical Sobolev exponent $p + 1 = (2\gamma + 2 + 2\sigma)/(\gamma - 1)$.

Our results can be directly applied to elliptic PDEs yielding a twofold generalization of some of the results of the Brézis-Nirenberg paper [3]. The paper is divided into three sections. Section 1 deals with weighted Sobolev spaces. In particular, here is proved the existence of a best Sobolev constant in a critical Sobolev imbedding and we compute it explicitly. Section 2 is the core of the paper. Here are stated and proved the existence results for (0.2). Section 3 is devoted to the proof of the inequality $\mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma)$ for $1 < \gamma < \alpha + 3$.

1. Imbedding theorems for weighted Sobolev spaces. Existence of a best constant in the critical case. We first recall some known facts about weighted Sobolev spaces which we have stated and proved in [8] in a form suitable for our purposes. Let $R \in (0, +\infty)$, $E^R_\gamma$ is the closure of the set

$$S = \{u \in C^1[0, R] : u \equiv 0 \text{ in a neighborhood of } R\}$$

in the norm $\| \cdot \|_{\gamma, R}$ defined by

$$\|u\|_{\gamma, R} = \left(\int_0^R r^\gamma |u'|^2 \, dr\right)^{1/2}.$$

When $R = 1$ we write simply $\|u\|_{\gamma, L^s_\theta(0, R)}$ is the weighted $L^s(0, R; r^\theta \, dr)$.

We recall the following results (see [8] for a proof).

Radial Lemma. There exists $C = C(\gamma) > 0$ such that for $u \in E^R_\gamma$

$$|u(r)| \leq \frac{C}{r^{(\gamma - 1)/2}} \|u\|_{\gamma, R}, \quad \forall r \in (0, R).$$
Semilinear Elliptic Equation

Imbedding Lemma. Let $\gamma > 1$, $R < \infty$ and $\theta > \max(-1, \gamma - 2)$. Then $E^R_\gamma \hookrightarrow L^\theta_\phi(0, R)$ continuously if and only if
\[
\frac{\theta + 1}{q} \geq \frac{\gamma - 1}{2}.
\]

Compactness Lemma. Let $\gamma > 1$, $R < \infty$ and $\theta > \max(-1, \gamma - 2)$. The imbedding $E^R_\gamma \hookrightarrow L^\theta_\phi(0, R)$ is compact if
\[
\frac{\theta + 1}{q} > \frac{\gamma - 1}{2}.
\]

We are interested mainly in the critical case; i.e., the situation $(\theta+1)/q = (\gamma-1)/2$ hence
\[
q = \gamma^* = \frac{2(\theta + 1)}{\gamma - 1}. \tag{1.2}
\]
The continuity of the critical imbedding is equivalent to the existence of a constant $K > 0$ such that
\[
\|u\|_{L^{2*}_\gamma(0, R)} \leq K\|u\|_{\gamma, R}. \tag{1.3}
\]
In fact (1.3) holds also with $R = \infty$ (see Maz'ya [7], Sect. 1.3.1). Set
\[
S(\gamma, \theta, R) = \inf\{\|u\|_{\gamma, R}^2 \mid u \in E^R_\gamma, \|u\|_{L^{2*}_\gamma(0, R)}^2 = 1\}. \tag{1.4}
\]
Following the ideas in Aubin [1] we can prove the following result.

Proposition 1.1. Let $\theta > \gamma - 2 > -1$. Then in (1.4) with $R = \infty$ the infimum $S(\gamma, \theta, \infty)$ is achieved by the function
\[
\widetilde{U}(r) = \frac{C}{(1 + r^{2+\sigma})(\gamma - 1)/(2+\sigma)} = C \cdot U(r), \quad \sigma = \theta - \gamma \tag{1.5}
\]
where $C$ is a normalization constant; i.e., such that $\|\widetilde{U}\|_{L^{2*}_\gamma(0, \infty)} = 1$.

Proof: The proof will be carried out in two steps.

Step 1. If the infimum is achieved then it can also be achieved by a positive decreasing function. Let us assume the infimum is achieved. Obviously $\|u\|_{\gamma, \infty} = \|u\|_{\gamma, \infty}$ and $\|u\|_{L^{2*}_\gamma(0, \infty)} = \|u\|_{L^{2*}_\gamma(0, \infty)}$. (The former equality is a consequence of a variant of Stampacchia's lemma; see [6] for a proof.) Hence the infimum can also be reached by positive functions. The functions that realize the infimum satisfy the Euler-Lagrange equation
\[
-\frac{1}{r^{\gamma}}(r^\gamma u')' = \lambda r^\sigma |u|^\gamma - 2 u, \tag{1.6}
\]
where $\lambda \in \mathbb{R}^*$ is a Lagrange multiplier. We may assume $u > 0$. By the Radial Lemma we infer
\[
\lim_{r \to \infty} u(r) = 0. \tag{1.7}
\]
We make the change of variables \( s = r^{-(\gamma-1)} \) and we denote \( v(s) = u(r) \). Then \( v \) satisfies the following equation

\[
v_{ss} + \frac{1}{(\gamma - 1)^2} \frac{\lambda v^{\gamma - 1}}{r^{2+(\gamma - 1)/(\gamma - 1)}} = 0 \quad \text{in} \quad (0, \infty)
\]  

(1.8)

with \( v(0) = 0 \) and \( v_s = dv/ds \). Hence the function \( \lambda v \) is concave. One of the following two situations may occur.

A. \( \lambda < 0 \). Then \( v \) is convex and hence \( v'(0) \) exists and \( v'(0) \geq 0 \). Therefore \( v \) is increasing and obviously \( u \) is decreasing.

B. \( \lambda > 0 \). Then \( v \) is concave and since \( v > 0 \) near \( \infty \) we get \( v_s(\infty) = 0 \) and consequently \( v \) is increasing. Again we get that \( u \) is decreasing.

**Step 2.** The infimum is achieved by the function (1.5). It is easily seen that the function (1.5) satisfies (1.6) with a suitable \( \lambda \). The statement of Step 2 follows from a sharp result due to G.A. Bliss [2]. We state a special case of it.

**Lemma 1.2.** Let \( q > 2 \) and let \( h(x) \geq 0 \) be a measurable real-valued function such that \( \int_0^\infty h^2(x) \, dx \) is finite. Set \( g(x) = \int_0^x g(t) \, dt \). Then

\[
\left( \int_0^\infty g^q(x)x^{q-1} \, dx \right)^{q/2} \leq K \left( \int_0^\infty h^2(x) \, dx \right)^{q/2}
\]  

(1.9)

where \( q = 2\alpha - 2 \) and \( K = 1/(q - \alpha - 1)[(\alpha\Gamma(q/\alpha))/\Gamma(1/\alpha)\Gamma((q - 1)/\alpha)]^\alpha \).

Here \( \Gamma \) is Euler's gamma function. The relation (1.9) holds with equality for every function \( h(x) \) of the form

\[
H_{\alpha}(x) = \frac{C}{(dx^\alpha + 1)^{(\alpha+1)/\alpha}}.
\]  

(1.10)

We see that (1.9) can be restated as

\[
\left( \int_0^\infty g^q(x)x^{q-1} \, dx \right)^{1/q} \leq K^{1/q} \int_0^\infty |g'(x)|^2 \, dx,
\]  

(1.9')

for every increasing function \( g \) such that \( g(0) = 0 \) and \( g' \in L^2(0, \infty) \). If in (1.9') we make the change in variables \( x = 1/r^\nu \), \( u(r) = g(x) \) then we get

\[
\nu^{1/q} \left( \int_0^\infty u^q(r)r^{((\nu+1)/2)-1} \, dr \right)^{1/q} \leq K^{1/q} \nu^{-1} \left( \int_0^\infty |u'(r)|^2 r^{\nu+1} \, dr \right)^{1/2}
\]  

(1.11)

for every positive decreasing function \( u \) such that \( u(r) \to 0 \) as \( r \to \infty \). If in (1.11) we further specialize \( \nu \) and \( q \) such that \( \nu + 1 = \gamma, \frac{\nu}{2} = \theta \) we get \( \nu = \gamma - 1 \) and \( q = (2(\theta+1))/(\gamma - 1) \) and we obtain the critical Sobolev imbedding. Moreover (1.11) becomes equality when \( d/\nu u(r) = H_{\alpha}(x) \), where \( x = r^{-(\gamma - 1)} \). This happens when \( u(r) = U(r) \) when \( U \) is given by (1.5).

**Remark 1.3.** The result above allows one to compute the exact value of \( S(\gamma, \theta, \infty) \).

In fact

\[
S(\gamma, \theta, \infty) = \frac{\|U\|_{H_{\gamma}(0, \infty)}^2}{\|U\|_{L_{\nu}^{\gamma}(0, \infty)}^2}
\]  

(1.12)
\[ \|U\|_{r,\infty}^2 = \int_0^\infty r^\gamma |U'|^2 \, dr, \quad U'(r) = (\gamma - 1) \frac{r^{1+\sigma}}{(1 + r^{2+\sigma})(\gamma+1+\sigma)/(2+\sigma)} \]

\[ \|U\|_{r,\infty}^2 = (\gamma - 1)^2 \int_0^\infty \frac{r^\gamma + 2+2\sigma}{(1 + r^{2+\sigma})(\gamma+1+\sigma)/(2+\sigma)} \, dr \]

\[ = \frac{(\gamma - 1)^2}{2+\sigma} \int_0^\infty \frac{s^{(\gamma+3+2\sigma)/(2+\sigma)-1}}{(1 + s)(\gamma+1+\sigma)/(2+\sigma)} \, ds \]

so that

\[ \|U\|_{r,\infty}^2 = \frac{(\gamma - 1)^2 \Gamma(\frac{\gamma+3+2\sigma}{2+\sigma}) \Gamma(\frac{\gamma-1}{2+\sigma})}{2+\sigma \Gamma(\frac{\gamma+1+\sigma}{2+\sigma})}, \quad (1.13) \]

where we have used the formula

\[ \int_0^\infty \frac{x^{m-1}}{(1 + x)^{m+n}} \, dx = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m + n)}, \quad \forall m, n > 0 \quad (1.14) \]

(cf. Dwight [4]). We now compute in a similar way

\[ \|U\|_{L^\gamma_r(0,\infty)}^2 = \int_0^\infty \frac{r^\theta}{(1 + r^{2+\sigma})^{(\theta+1)/(2+\sigma)}} \, dr \quad (\theta = \gamma + \sigma) \]

\[ = \frac{1}{2+\sigma} \int_0^\infty \frac{s^{(\gamma+\theta+1)/(2+\sigma)-1}}{(1 + s)^{\theta/(2+\sigma)}} \, ds = \frac{1}{2+\sigma} \left[ \frac{\Gamma(\frac{\gamma+\theta+1}{2+\sigma})}{\Gamma(\frac{2+\sigma}{2+\sigma})} \right]^2, \quad (1.15) \]

where again we have used (1.14). It is a straightforward observation that \( S(\gamma, \theta, R) \) is invariant under rescaling so that it does not really depend on \( R \). We set

\[ S(\gamma, \theta) := S(\gamma, \theta, R), \quad R \in (0, \infty). \]

2. Existence of a positive solution. In this section we consider the existence question for the following boundary value problem

\[ \begin{cases} &- \frac{1}{r^\gamma} (r^\gamma u')' = r^\sigma |u|^{p-1} u + \lambda r^\alpha u \quad \text{in} \ (0, 1), \ \lambda \in R, \\ &\alpha > -2, \ \sigma > -2, \ u \in E_\gamma, \ u > 0 \end{cases} \quad (2.1) \]

where \( p = (\gamma + 3 + 2\sigma)/(\gamma - 1), \ \gamma > 1, \ \sigma > -2 \) so that the imbedding \( E_\gamma \hookrightarrow L^{p+1}_\theta \), \( \theta = \gamma + \sigma \) is noncompact.

We shall look for a weak solution of (2.1); i.e., a function \( u \in E_\gamma, \ u > 0 \) such that

\[ \int_0^1 r^\gamma u' \varphi' \, dr = \int_0^1 [r^\theta u^p \varphi + \lambda r^{\gamma+\alpha} u \varphi] \, dr, \quad \forall \varphi \in E_\gamma. \quad (2.2) \]

Remark 2.1. (2.1) has no weak solution for \( \lambda \leq 0 \).
This will follow from a Pohozaev-type argument similar to that used in [8]. Assume the contrary, i.e., there exists a weak solution $u$ of (2.1) with $\lambda \leq 0$. In (2.2) we set $\varphi = u$ and we get

$$
\int_0^1 r |u'|^2 \, dr = \int_0^1 [r^\theta u^{p+1} + \lambda r^{\gamma + \alpha} u^2] \, dr.
$$

(2.3)

By standard elliptic regularity, $u \in C^2(0,1)$ so that $u$ is a classical solution of

$$
- (r^\gamma u')' = r^\theta u^p + \lambda r^{\gamma + \alpha} u, \quad \text{in } (0,1), \quad u(1) = 0.
$$

(2.4)

We multiply (2.4) with $ru'$ and we get

$$
\begin{align*}
& r^{\gamma + 1} u'' + \gamma r^{\gamma} u' |u'|^2 + r^{\theta + 1} u^{p+1} u' + \lambda r^{\gamma + \alpha} uu' = 0 \\
& = \frac{1}{2} r^{\gamma + 1} \frac{d}{dr} [r^{\gamma} |u'|^2 + \frac{1}{p+1} r^{\theta + 1} \frac{d}{dr} (u^{p+1})] + \frac{\lambda}{2} r^{\gamma + \alpha} \frac{d}{dr} (u^2) = 0.
\end{align*}
$$

We integrate the last inequality by parts on $(\varepsilon, 1)$. We obtain

$$
\begin{align*}
& \frac{1}{2} |u'(1)|^2 - \frac{1}{2} \varepsilon^{\gamma + 1} |u'(\varepsilon)|^2 - \frac{1}{p+1} \varepsilon^{\theta + 1} u^{p+1}(\varepsilon) - \frac{\lambda}{2} \varepsilon^{\gamma + \alpha + 1} u^2(\varepsilon) \\
& + \frac{\gamma - 1}{2} \int_{\varepsilon}^1 r^{\gamma} |u'(r)|^2 \, dr - \frac{\theta + 1}{p+1} \int_{\varepsilon}^1 r^\theta u^{p+1}(r) \, dr \\
& - \frac{\lambda(\gamma + \alpha + 1)}{2} \int_{\varepsilon}^1 r^{\gamma + \alpha} u^2(r) \, dr = 0.
\end{align*}
$$

(2.5)

Since $u \in E_\gamma \cap L^{p+1}_\theta$ we get that on a subsequence $\varepsilon_k \to 0$

$$
\varepsilon^{\gamma + 1} |u'(\varepsilon_k)|^2 + \varepsilon^{\theta + 1} u^{p+1}(\varepsilon_k) \to 0 \quad \text{as } k \to \infty.
$$

(2.6)

By the Radial Lemma $u^2(\varepsilon) \leq \text{const. } \varepsilon^{-(\gamma - 1)}$ so that

$$
\varepsilon^{(\gamma + \alpha + 1)} u^2(\varepsilon) \leq \text{const. } \varepsilon^{\alpha + 2} = O(1) \quad \text{as } \varepsilon \to 0, \text{ since } \alpha > -2.
$$

(2.7)

If in (2.5) we let $\varepsilon = \varepsilon_k \to 0$ we infer by (2.6), (2.7)

$$
\begin{align*}
& \frac{1}{2} |u'(1)|^2 + \frac{\gamma - 1}{2} \int_0^1 r^{\gamma} |u'(r)|^2 \, dr - \frac{\theta + 1}{p+1} \int_0^1 r^\theta u^{p+1}(r) \, dr \\
& - \frac{\lambda(\gamma + \alpha + 1)}{2} \int_0^1 r^{\gamma + \alpha} u^2(r) \, dr = 0.
\end{align*}
$$

(2.8)

From (2.3) and (2.8) we infer

$$
\begin{align*}
& \frac{1}{2} |u'(r)|^2 + \left( \frac{\gamma - 1}{2} - \frac{\theta + 1}{p+1} \right) \int_0^1 r^{\gamma} |u'(r)|^2 \, dr - \frac{\lambda(\alpha + 2)}{2} \int_0^1 r^{\gamma + \alpha} u^2(r) \, dr = 0.
\end{align*}
$$
and finally since $(\gamma - 1)/2 = (\theta + 1)/(p + 1)$

$$\frac{1}{2} |u'(1)|^2 = \frac{\lambda(\alpha + 2)}{2} \int_0^1 r^{\gamma + \alpha} u^2(r) \, dr \leq 0. \tag{2.9}$$

We get $u'(1) = 0$. We know that also $u(1) = 0$. Therefore it follows—according to the uniqueness in a Lipschitzian Cauchy problem—that $u \equiv 0$.

A special part in our considerations will be played by the following generalized eigenvalue problem

$$\frac{1}{r^\gamma} (r^\gamma u')' = \lambda r^\alpha u, \quad \lambda \in \mathbb{R}, \quad \alpha > -2 \quad u \in E_\gamma, \tag{2.10}$$

which is meant in the following generalized sense

$$\int_0^1 r^\gamma u' \varphi' \, dr = \lambda \int_0^1 r^{\gamma + \alpha} u \varphi \, dr, \quad \forall \varphi \in E_\gamma. \tag{2.11}$$

Since for $\alpha > -2$ the imbedding $E_\gamma \hookrightarrow L^2_{\gamma + \alpha}$ is compact one can prove in a standard manner the following facts:

(F1) The spectrum of (2.10) consists of an unbounded sequence of positive eigenvalues $0 < \lambda_1(\alpha, \gamma) < \lambda_2(\alpha, \gamma) \leq \lambda_3(\alpha, \gamma) < \cdots \to \infty$, each of them having finite multiplicity.

(F2) The eigenvalue $\lambda_1(\alpha, \gamma)$ is simple and the corresponding eigenspace is generated by a positive eigenfunction.

(F3) For every $\gamma > 1$ the mapping $\alpha \to \lambda_1(\alpha, \gamma)$ is decreasing and continuous.

(For a proof of these by now classical statements we refer the reader to the work of D.J. de Figueiredo [5].)

When there is no possibility of confusion we shall write $\lambda_1(\alpha)$ instead of $\lambda_1(\alpha, \gamma)$.

**Remark 2.2.** Problem (2.1) has no weak solution for $\lambda \geq \lambda_1(\alpha, \gamma), \gamma > 1, \alpha > -2$.

Indeed, we set in (2.2) $\varphi = \varphi_1$. We get

$$\int_0^1 r^\gamma u' \varphi_1' \, dr = \int_0^1 [r^\theta u^p + \lambda r^{\gamma + \alpha} u \varphi_1] \, dr.$$ 

If in (2.11) we set $\varphi = u$ and $\lambda = \lambda_1(\alpha)$ we get

$$\int_0^1 r^\gamma u' \varphi_1' \, dr = \lambda_1 \int_0^1 r^{\gamma + \alpha} u \varphi_1 \, dr.$$ 

We infer

$$0 \geq (\lambda - \lambda_1) \int_0^1 r^{\gamma + \alpha} u \varphi_1 \, dr = \int_0^1 r^\theta u^p \varphi_1 \, dr.$$ 

Due to the fact that $\varphi_1 > 0$ it follows $u \equiv 0$.

From the two remarks above we see that a necessary condition for the existence of a solution of (2.1) is that $\lambda \in (0, \lambda_1(\alpha))$. 

Following the ideas of Brézis-Nirenberg [3] we shall consider the minimization problem
\[ S_\lambda = S_\lambda(\gamma, \theta) = \inf\{ \|u\|_\gamma^2 - \lambda\|u\|_{L_{2+\alpha}}^2; \|u\|_{L^{p+1}} = 1 \}. \] (2.12)
If \( u \) is a solution of (2.12) we may assume \( u \geq 0 \) for otherwise we replace \( u \) by \( |u| \).
Then \( u \) satisfies
\[ -\frac{1}{r^\gamma}(r^\gamma u')' - \lambda r^\alpha u = S_\lambda r^\alpha u^p \quad \text{in } (0, 1), \quad u \in E_\gamma \]
in the weak sense (2.2). It follows that if \( S_\lambda > 0 \) then \( ku \) satisfies (2.2) for some appropriate constant \( k > 0 \) (namely \( k = S_\lambda^{1/(p-1)} \)). Thus in order to obtain a solution of (2.1) it is sufficient to check the following conditions.

Problem (2.12) has a solution. \hspace{1cm} (2.13)
\[ S_\lambda > 0. \hspace{1cm} (2.14) \]
Condition (2.14) holds if and only if \( \lambda < \lambda_1(\alpha) \) which is (according to Remark 2.2) a necessary condition for the existence of a solution of (2.1).
A sufficient condition so that (2.13) holds is supplied by the following result.

**Proposition 2.3.** If \( S_\lambda < S \) then the minimization problem (2.12) has a solution.

The proof of this result follows the same lines as the proof of Lemma 1.2 in the paper of Brézis-Nirenberg [3] so we omit it.
Our task is to check when \( S < S_\lambda \). We follow the arguments used in [3] Lemma 1.1 and Lemma 1.3. We treat separately two cases.

**2.1. The case \( \gamma \geq \alpha + 3 \).** The main result of this subsection is the following.

**Proposition 2.4.** Let \( \gamma \geq \alpha + 3 \). Then \( S_\lambda(\gamma, \theta) < S(\gamma, \theta) \) for every \( \lambda > 0 \).

**Proof:** We shall estimate the ratio
\[ Q_\lambda(u_\varepsilon) = \frac{\|u\|_\gamma^2 - \lambda\|u\|_{L_{2+\alpha}}^2}{\|u_\varepsilon\|_{L^{p+1}}^2}, \quad \theta = \gamma + \sigma \]
with
\[ u_\varepsilon(r) = \frac{\varphi(r)}{1 + r^{2+\sigma}} \frac{(\gamma-1)/(2+\sigma)}{r^{\gamma-1)/(2+\sigma)}}, \quad \varepsilon > 0 \] (2.15)
where \( \varphi \in C^\infty[0, 1] \) is a fixed function such that \( \varphi(r) \equiv 1 \) for \( r \) in some neighborhood of 0 and \( \varphi(r) \equiv 0 \) in some neighborhood of 1.
We claim that as \( \varepsilon \to 0 \) we have
\[ \|u_\varepsilon\|_\gamma^2 = \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(1) \] (2.16)
\[ \|u_\varepsilon\|_{L^{p+1}}^2 = \frac{K_2}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(1) \] (2.17)
\[ \|u_\varepsilon\|_{L_{\gamma+\alpha}^{\infty}}^2 = \begin{cases} \frac{K_3}{(\varepsilon^{(\gamma-3+\alpha)/(2+\sigma)})} + O(1), & \gamma > \alpha + 3 \\ K_3 \log \varepsilon, & \gamma = \alpha + 3 \end{cases} \] (2.18)

where \( K_1, K_2, K_3 \) are positive constants such that \( K_1/K_2 = \mathcal{S} \).

**Verification of (2.16).**

\[ u_\varepsilon'(r) = \frac{\varphi'(r)}{(\varepsilon + r^{2+\sigma})(\gamma-1)/(2+\sigma)} - \left(\gamma - 1\right) \frac{r^{1+\sigma}\varphi(r)}{(\varepsilon + r^{2+\sigma})(\gamma+1+\sigma)/(2+\sigma)} \]

Since \( \varphi \equiv 1 \) near 0 it follows that

\[ \|u_\varepsilon\|_{\gamma}^2 = \int_0^1 r\gamma |u_\varepsilon'|^2 \, dr = (\gamma - 1)^2 \int_0^1 \frac{r^{\gamma+2+2\sigma}}{(\varepsilon + r^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, dr + O(1) \]

\[ = \frac{1}{(\varepsilon^{(\gamma-1)/(2+\sigma)})(\gamma - 1)^2} \int_0^{r^{-1/(2+\sigma)}} \frac{s^{\gamma+2+2\sigma}}{(1 + s^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, ds = \frac{K_1}{(\varepsilon^{(\gamma-1)/(2+\sigma)})} + O(1) \]

where

\[ K_1 = (\gamma - 1)^2 \int_0^\infty \frac{s^{\gamma+2+2\sigma}}{(1 + s^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, ds = \|U\|_{L_{\gamma,\infty}}^2 \]

and \( U \) is given by (1.5).

**Verification of (2.17).**

\[ \int_0^1 u_\varepsilon^{p+1,0} \, dr = \int_0^1 \frac{r^{\gamma+\sigma}\varphi^{p+1}(r)}{(\varepsilon + r^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, dr \]

\[ = \int_0^1 \frac{\varphi^{p+1}(r) - 1}{(\varepsilon + r^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, dr + \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, dr \]

\[ = \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, dr + O(1) = \frac{K_2'}{(\varepsilon^{(\gamma+1+\sigma)/(2+\sigma)})} + O(1) \]

where

\[ K_2' = \int_0^\infty \frac{r^{\gamma+\sigma}}{(1 + r^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, dr = \|U\|_{L_{\gamma+1}^{p+1}(0,\infty)}^{p+1} \]

Thus (2.17) follows with \( K_2 = \|U\|_{L_{\gamma+1}^{p+1}(0,\infty)}^2 \) and \( K_1/K_2 = \mathcal{S}(\gamma, \theta) \).

**Verification of (2.18).** We have

\[ \|u_\varepsilon\|_{L_{\gamma+\alpha}^{\infty}}^2 = \int_0^1 \frac{[\varphi^2(r) - 1]r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})(2(\gamma-1))/(2+\sigma)} \, dr + \int_0^1 \frac{r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})(2(\gamma-1))/(2+\sigma)} \, dr \]

\[ = \int_0^1 \frac{r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})(2(\gamma-1))/(2+\sigma)} \, dr + O(1). \]
When $\gamma > \alpha + 3$ we have
\[
\int_0^1 \frac{r^{\gamma + \alpha}}{(\varepsilon + r^{2+\sigma})(2(\gamma -1))/(2+\sigma)} \, dr = \frac{1}{\varepsilon} \int_0^{1/(2+\sigma)} \frac{s^{\gamma + \alpha}}{(1 + s^{2+\sigma})(2(\gamma -1))/(2+\sigma)} \, ds
\]
\[
= \frac{1}{\varepsilon(\gamma -3 -\alpha)/(2+\sigma)} \int_0^\infty \frac{s^{\gamma + \alpha}}{(1 + s^{2+\sigma})(2(\gamma -1))/(2+\sigma)} \, ds + O(1)
\]
and thus (2.18) follows with
\[
K_3 = \int_0^\infty \frac{s^{\gamma + \alpha}}{(1 + s^{2+\sigma})(2(\gamma -1))/(2+\sigma)} \, ds.
\]
When $\gamma = \alpha + 3$ we have
\[
\int_0^1 \frac{r^{2\alpha+3}}{(\varepsilon + r^{2+\sigma})(2(\alpha+2))/(2+\sigma)} \, dr = \frac{1}{2 + \sigma} \int_0^1 \frac{s^{(2(\alpha+2))/(2+\sigma)}}{s} \, ds
\]
\[
= \frac{1}{2 + \sigma} \left[ \log \varepsilon \right] + O(1)
\]
and thus (2.18) follows with $K_3 = 1/(2 + \sigma)$.

Combining (2.16), (2.17) and (2.18) we get
\[
Q_\lambda(u_\varepsilon) = \begin{cases}
S - \lambda \frac{K_3}{K_2} \varepsilon^{(2+\alpha)/(2+\sigma)} + O(\varepsilon^{(\gamma-1)/(2+\sigma)}), & \gamma > \alpha + 3 \\
S - \lambda \frac{K_3}{K_2} \varepsilon^{(2\alpha+2)/(2+\sigma)} |\log \varepsilon| + O(\varepsilon^{(2\alpha+2)/(2+\sigma)}), & \gamma = \alpha + 3.
\end{cases}
\]
In all cases we deduce that $Q_\lambda(u_\varepsilon) < S$ provided $\varepsilon > 0$ is small enough.

We can state the following result

**Theorem 2.5.** Let $\gamma \geq \alpha + 3$. There exists a solution of (2.1) if and only if $0 < \lambda < \lambda_1(\alpha)$.

**2.2. The case $\gamma < \alpha+3$.** Of special interest to us will be the following eigenvalue problem
\[
-\frac{1}{r^{2-\gamma}}(r^{2-\gamma}u')' = \mu r^\alpha u, \quad u \in E_{2-\gamma}
\]  
(2.19)
which should be understood in the weak sense (2.11).

We shall consider the following more general eigenvalue problem
\[
-\frac{1}{r^\beta}(r^\beta u') = \mu r^\alpha u, \quad u \in E_\beta, \quad \beta < 1
\]  
(2.20)
which is understood in the $E_\beta$-weak sense
\[
\int_0^1 r^\beta u' \varphi' \, dr = \mu \int_0^1 r^{\alpha + \beta} u \varphi \, dr, \quad \forall \varphi \in E_\beta.
\]  
(2.20')

If one makes the change in variables $r = s^{1/(1-\beta)}$, $u(r) = v(s)$, then (2.20) reduces to
\[
v'' = \frac{\mu}{(1-\beta)^2} s^{(\alpha+2)/(1-\beta)-2} v, \quad v \in E_0
\]  
(2.21)
understood in $E_0$-weak sense.

The decisive remark is the following
Lemma 2.6. The space $E_0$ is compactly imbedded in $L^2_0$ for every $\delta > -1$.

Proof: Indeed $E_0$ is compactly imbedded in $L^p_0$ for every $p > 1$ according to Kondrachov's theorem. Now let $\delta > -1$.

\[ \|u\|_{L^2_0}^2 = \int_0^1 r^\delta u^2 \, ds \leq \left( \int_0^1 r^{p\delta} \, dr \right)^{1/p} \left( \int_0^1 u^{2q} \, dr \right)^{1/q} \]

for some $p, q > 1$ such that $p\delta > -1$ and $(1/p) + (1/q) = 1$. Hence $\|u\|_{L^2_0} \leq \text{const.}\|u\|_{L^{p+1}}$ and therefore $E_0$ is compactly imbedded in $L^2_0$. \qed

From the compactness result stated above it follows that if in (2.21) $(\alpha + 2)/(\beta - 1) - 2 > -1$ (or simply $\alpha + \beta + 1 > 0$) then the spectrum is discrete, positive, the first eigenvalue $\mu_1(\alpha, \beta)$ is simple and the corresponding eigenspace is generated by a positive function $\psi_1$.

Lemma 2.7. Let $\psi_1$ be defined as above. Then $\psi_1(r) - \psi_1(0) = O(r^{\alpha+2})$ as $r \to 0$ and $\psi_1(0) > 0$.

Proof: Since $\psi_1 \in E_\beta$ and $\beta < 1$ it follows that $\psi_1 \in L^\infty(0,1)$ and a simple argument shows that $\psi_1 \in C[0,1]$. Using the same arguments as in Proposition 2.4 of [8] it follows that

\[ \psi_1(r) - \psi_1(0) = O(r^{\alpha+2}). \]

Suppose that $\psi_1(0) = 0$. If we make the change of variables $r = s^{1/(1-\beta)}$, $\omega(s) = \psi_1(r)$ then $\omega(s)$ satisfies

\[ -\omega''(s) = \frac{\mu_1}{(1-\beta)^2} s^\delta \omega \quad \text{in } (0,1), \quad \delta = \frac{\alpha + 2}{1-\beta} - 2 \quad \omega \in E_0 \]

in $E_0$-weak sense. By the strong maximum principle we infer $\omega'(0) > 0$ since $\omega(0) = 0$. Invoking once again the arguments of Proposition 2.4 in [8] we infer $\omega(s) = O(s^{\alpha+2})$ as $s \to 0$. Since $\delta > -1$ it follows that $\omega'(0) = 0$. Contradiction! The lemma is proved. \qed

Thus the eigenvalue problem (2.19) leads to a positive discrete spectrum when

\[ 2 - \gamma + \alpha + 1 > 0; \quad \text{i.e., } \gamma < \alpha + 3. \]

We denote the least eigenvalue with $\mu_1(\alpha, \gamma)$ and the corresponding eigenfunction with $\psi_1$ such that $\psi_1 > 0$

\[ \psi_1(0) = 1, \quad \psi_1(r) - 1 = O(r^{\alpha+2}) \text{as } r \to 0. \tag{2.22} \]

Our next task is to estimate

\[ Q_\lambda(u_\varepsilon) = \frac{\|u_\varepsilon\|_{L^\gamma_+}^2 - \lambda\|u_\varepsilon\|_{L^{\lambda+1}_+}^2}{\|u_\varepsilon\|_{R^{\gamma+1}}^2} \]

where

\[ u_\varepsilon(r) = \frac{\psi(r)}{(\varepsilon + r^{2+\sigma})(\gamma-1)/(2+\sigma)}, \quad \psi(r) = \psi_1(r). \]

First we estimate the terms that enter $Q_\lambda(u_\varepsilon)$.
Estimation of $\|u_\varepsilon\|_\gamma^2$.

$$u'(r) = \frac{\psi'(r)}{\varepsilon + r^{2+\sigma}(\gamma-1)/(2+\sigma)} - (\gamma - 1) \frac{r^{1+\sigma} \psi(r)}{\varepsilon + r^{2+\sigma}(\gamma+1)/(2+\sigma)}$$

so that

$$\|u_\varepsilon\|_\gamma^2 = \int_0^1 \left[ \frac{|\psi'(r)|^2 r^\gamma}{\varepsilon + r^{2+\sigma}(2(\gamma-1))/(2+\sigma)} - 2(\gamma - 1) \frac{r^{\gamma+1+\sigma} \psi(r) \psi'(r)}{\varepsilon + r^{2+\sigma}(2\gamma)/(2+\sigma)} + (\gamma - 1)^2 \frac{r^{\gamma+2+2\sigma} \psi^2(r)}{\varepsilon + r^{2+\sigma}(2\gamma+2+2\sigma)/(2+\sigma)} \right] \, dr.$$ 

We integrate by parts the second term above and we get

$$\|u_\varepsilon\|_\gamma^2 = \int_0^1 \frac{|\psi'(r)|^2 r^\gamma}{\varepsilon + r^{2+\sigma}(2(\gamma-1))/(2+\sigma)} \, dr$$

$$+ \varepsilon(\gamma - 1)(\gamma + 1 + \sigma) \int_0^1 \psi^2(r) \frac{r^{\gamma+\sigma}}{\varepsilon + r^{2+\sigma}(2(\gamma+1)/(2+\sigma))} \, dr$$

$$\int_0^1 \psi^2(r) \frac{r^{\gamma+\sigma}}{\varepsilon + r^{2+\sigma}(2(\gamma+1)/(2+\sigma))} \, dr$$

$$= \int_0^1 \frac{r^{\gamma+\sigma}}{\varepsilon + r^{2+\sigma}(2(\gamma+1)/(2+\sigma))} \, dr + \int_0^1 \frac{|\psi^2(r) - 1|}{\varepsilon + r^{2+\sigma}(2(\gamma+1)/(2+\sigma))} \, dr$$

$$= I_1 + I_2$$

$$I_1 = \frac{K_1^\prime}{\varepsilon(\gamma+1)/(2+\sigma)} + O(1)$$

where

$$K_1^\prime = \int_0^\infty \frac{r^{\gamma+\sigma}}{(1 + r^{2+\sigma})/(2+\sigma)} \, dr. \quad (2.23)$$

By (2.22) we get

$$|I_2| \leq \text{const.} \int_0^1 \frac{r^{\gamma+\sigma+\alpha+2}}{\varepsilon + r^{2+\sigma}(2(\gamma+1)/(2+\sigma))} = O(f(\varepsilon))$$

where

$$f(\varepsilon) = \begin{cases} \varepsilon^{-(\gamma+\sigma-\alpha-1)/(2+\sigma)}, & \gamma + \sigma > \alpha + 1 \\ |\log \varepsilon|, & \gamma + \sigma = \alpha + 1 \\ 1, & \gamma + \sigma < \alpha + 1. \end{cases}$$

We infer that

$$\|u_\varepsilon\|_\gamma^2 = \int_0^1 \frac{|\psi^1(r)|^2 r^\gamma}{\varepsilon + r^{2+\sigma}(2(\gamma-1))/(2+\sigma)} \, dr + \frac{1}{\varepsilon(\gamma-1)/(2+\sigma)} + O(\varepsilon) + O(\varepsilon(f(\varepsilon))).$$
\( \varepsilon \to 0. \)

A simple computation shows that

\[
\|u_\varepsilon\|_{L^p}^2 = \int_0^1 \frac{|\psi'(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})(2(\gamma-1))/(2+\sigma)} \, dr + \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(\varepsilon f(\varepsilon)) 
\]

(2.24)

as \( \varepsilon \to 0 \), where

\[
K_1 = (\gamma - 1)(\gamma + 1 + \sigma)K_1' 
\]

(2.25)

and

\[
\varepsilon f(\varepsilon) = \begin{cases} 
\varepsilon^{(\alpha+3-\gamma)/(2+\sigma)}, & \gamma + \sigma > \alpha + 1 \\
\varepsilon |\log \varepsilon|, & \gamma + \sigma = \alpha + 1 \\
\varepsilon, & \gamma + \sigma < \alpha + 1.
\end{cases}
\]

**Estimation of \( \|u_\varepsilon\|_{L^{p+1}}^2 \).**

\[
\|u_\varepsilon\|_{L^{p+1}}^{p+1} = \int_0^1 \frac{\psi^{p+1}(r) r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, dr = \int_0^1 \frac{r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, dr \]

\[
= \int_0^1 \frac{[\psi^{p+1}(r) - 1] r^{\gamma+\sigma}}{(\varepsilon + r^{2+\sigma})(2(\gamma+1+\sigma))/(2+\sigma)} \, dr = I_1 + I_2.
\]

As above we infer

\[
\|u_\varepsilon\|_{L^{p+1}}^{p+1} = \frac{K_1'}{\varepsilon^{(\gamma+1+\sigma)/(2+\sigma)}} + O(g(\varepsilon))
\]

where

\[
g(\varepsilon) = \begin{cases} 
\varepsilon^{-(\gamma+\sigma-\alpha-1)/(2+\sigma)}, & \gamma + \sigma > \alpha + 1 \\
|\log \varepsilon|, & \gamma + \sigma = \alpha + 1 \\
1, & \gamma + \sigma < \alpha + 1.
\end{cases}
\]

Hence

\[
\|u_\varepsilon\|_{L^{p+1}}^2 = \frac{K_2}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + O(g(\varepsilon)) 
\]

(2.26)

with

\[
K_2 = [K_1']^{2/(p+1)}.
\]

**Estimation of \( \|u_\varepsilon\|_\gamma^2 - \lambda \|u_\varepsilon\|_{L^{2+\alpha}_\gamma}^2 \).** By (2.24) we get

\[
\|u_\varepsilon\|_\gamma^2 - \lambda \|u_\varepsilon\|_{L^{2+\alpha}_\gamma}^2 = \frac{K_1}{\varepsilon^{(\gamma-1)/(2+\sigma)}} + \int_0^1 \frac{|\psi'(r)|^2 r^\gamma}{(\varepsilon + r^{2+\sigma})(2(\gamma-1))/(2+\sigma)} \, dr
\]

\[
- \lambda \int_0^1 \frac{\psi^2(r) r^{\gamma+\alpha}}{(\varepsilon + r^{2+\sigma})(2(\gamma-1))/(2+\sigma)} + O(\varepsilon f(\varepsilon))
\]
\[
\int_0^1 \frac{|\psi'(r)|^2 r}{(e + r^2 + \sigma)(e + (\gamma - 1) + (2 + \sigma))} \, dr - \lambda \int_0^1 \frac{\psi^2(r)r^{\gamma + \alpha}}{(e + r^2 + \sigma)(e + (\gamma - 1) + (2 + \sigma))} \, dr
\]
\[
= \int_0^1 \left[ (r^2\psi'(r))^2 r^{\gamma - 2} - \lambda r^{\alpha + 2 - \gamma}\psi^2(r) \right] \frac{r^{\gamma - 2}}{(e + r^2 + \sigma)(e + (\gamma - 1) + (2 + \sigma))} \, dr
\]
\[
= \int_0^1 R(\psi)N(\varepsilon, r) \, dr + \int_0^1 R(\psi)[N(\varepsilon, r) - 1] \, dr
\]
\[
= \int_0^1 R(\psi) \, dr + O(\varepsilon).
\]

We infer
\[
\|u_\varepsilon\|_\gamma^2 - \lambda \|u_\varepsilon\|_{L^2_{\gamma + \alpha}}^2 = K_1 \int_0^1 R(\psi) \, dr + O(\varepsilon f(\varepsilon)) \quad \text{as } \varepsilon \to 0 \tag{2.28}
\]
where again we have used the fact that \(O(\varepsilon) + O(\varepsilon f(\varepsilon)) = O(\varepsilon f(\varepsilon)) \) as \(\varepsilon \searrow 0\). By (2.26) and (2.28) we infer
\[
Q_\lambda(u_\varepsilon) = \frac{K_1}{K_2} + \varepsilon^{(\gamma - 1)/(2 + \sigma)} \int_0^1 R(\psi) \, dr + O(\varepsilon^{(\gamma - 1)/(2 + \sigma)} f(\varepsilon)).
\]
From \(\gamma < \alpha + 3\) we get \(\varepsilon f(\varepsilon) = o(1)\) and therefore
\[
= \frac{K_1}{K_2} + \varepsilon^{(\gamma - 1)/(2 + \sigma)} \int_0^1 R(\psi) \, dr + o(\varepsilon^{(\gamma - 1)/(2 + \alpha)}) \quad \text{as } \varepsilon \searrow 0. \tag{2.29}
\]
If we recall our definition of \(\psi\) we get
\[
\int_0^1 R(\psi) \, dr = \int_0^1 \left[ r^{2 - \gamma}|\psi'(r)|^2 - \lambda r^{\gamma + \alpha - \gamma}\psi^2(r) \right] \, dr
\]
\[
= (\mu_1(\alpha) - \lambda) \int_0^1 r^{\alpha + 2 - \gamma}\psi^2(r) \, dr = (\mu_1(\alpha) - \lambda)K_3,
\]
where \(K_3\) is a positive constant. This yields
\[
Q_\lambda(u_\varepsilon) = \frac{K_1}{K_2} + K_3(\mu_1(\alpha) - \lambda)\varepsilon^{(\gamma - 1)/(2 + \sigma)} + o(\varepsilon^{(\gamma - 1)/(2 + \sigma)}) \quad \text{as } \varepsilon \searrow 0. \tag{2.30}
\]
Now we claim that \(K_1/K_2 = S\). Indeed \(S = \|U\|_{L^2_{\gamma + \alpha}}^2\). By (2.23), (2.27) we see that \(K_2 = \|U\|_{L^2_{\gamma + 1}}^2\). We have to check that \(K_1 = \|U\|_{L^2_{\gamma + 1}}^2\). From (1.13) we get
\[
\|U\|_{L^2_{\gamma, \infty}}^2 = \frac{(\gamma - 1)^2}{2 + \sigma} \frac{\Gamma(\frac{2 + \gamma}{2 + \sigma})}{\Gamma(\frac{2(\gamma + 1 + \sigma)}{2 + \sigma})}.
\]
\(K_1\) can be easily computed using (1.14) and we get
\[
K_1 = \frac{(\gamma - 1)(\gamma + 1 - \sigma)}{(2 + \sigma)^2} \frac{\Gamma(\frac{\gamma + 1 + \sigma}{2 + \sigma})^2}{\Gamma(\frac{2(\gamma + 1 + \sigma)}{2 + \sigma})}.
\]
So that we actually have to check

\[(\gamma + 1 + \sigma)\left[\Gamma\left(\frac{\gamma + 1 + \sigma}{2 + \sigma}\right)\right]^2 = (\gamma - 1)\Gamma\left(\frac{\gamma + 3 + 2\sigma}{2 + \sigma}\right)\Gamma\left(\frac{\gamma - 1}{2 + \sigma}\right). \tag{2.31}\]

From the well-known relation $\Gamma(x + 1) = x\Gamma(x)$ we infer

\[\Gamma\left(\frac{\gamma + 1 + \sigma}{2 + \sigma}\right) = \frac{\gamma - 1}{2 + \sigma}\Gamma\left(\frac{\gamma - 1}{2 + \sigma}\right)\]

and

\[\Gamma\left(\frac{\gamma + 3 + 2\sigma}{2 + \sigma}\right) = \frac{\gamma + 1 + \sigma}{2 + \sigma}\Gamma\left(\frac{\gamma + 1 + \sigma}{2 + \sigma}\right).\]

(2.31) follows from the equalities above. Hence

\[\frac{K_1}{K_2} = S. \tag{2.32}\]

By (2.30) and (2.32) we get

If $\lambda > \mu_1(\alpha)$ then $S_\lambda < S. \tag{2.33}$

Now we can state

**Proposition 2.8.** If $\mu_1(\alpha) < \lambda_1(\alpha)$ ($\alpha > -2, 1 < \gamma < \alpha + 3$) then the problem possesses at least a solution for each $\lambda \in (\mu_1(\alpha), \lambda_1(\alpha))$.

There is still a question we must answer, namely when does the following spectral equality hold?

\[\mu_1(\alpha) < \lambda_1(\alpha). \tag{2.34}\]

We recall that $\mu_1(\alpha) = \mu_1(\alpha, \gamma)$ and $\lambda_1(\alpha) = \lambda_1(\alpha, \gamma)$, are the least eigenvalues of the following eigenvalue problems

\[-\frac{1}{r^{2-\gamma}}(r^{2-\gamma}\psi')' = \mu r^\alpha \psi, \quad \psi \in E_{2-\gamma}, \tag{2.35}\]

and respectively

\[-\frac{1}{r^{\gamma}}(r^\gamma \varphi')' = \lambda r^\alpha \varphi, \quad \varphi \in E_\gamma \tag{2.36}\]

where $\alpha > -2, 1 < \gamma < \alpha + 3$ and these eigenvalue problems are understood in the weak sense (2.20') and (2.11).

The aim of our next section is to answer this question.

**3. Proof of the spectral inequality.** The main result of this section is the following.
Proposition 3.1. \( \mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma) \) for all \( \alpha > -2, 1 < \gamma < \alpha + 3 \).

The proof will be carried out in several steps. Let us first denote by \( D \) the set
\[
D = \{(\alpha, \gamma) | 1 < \gamma < \alpha + 3\}.
\]
In the \((\alpha, \gamma)\)-plane \( D \) looks like Figure 1.

Let us denote by \( S \) the set
\[
S = \{(\alpha, \gamma) \in D | \mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma)\}.
\]
We also consider the following semigroup of transformations of the \((\alpha, \gamma)\)-plane \((H_\beta)_{\beta > 0}\) where the \( H_\beta \) are given by the law
\[
(\alpha, \gamma) \xmapsto{H_\beta} (\alpha_\beta, \gamma_\beta) = \beta(\alpha + 2, \gamma - 1) + (-2, 1). \tag{3.1}
\]
\((H_\beta)_{\beta > 0}\) is a semigroup of homoteties of pole \( C(-2, 1) \). Clearly, \( H_\beta(D) \subset D \forall \beta > 0 \).

Proof of Proposition 3.1.

Step 1. \( H_\beta(S) \subset S, \forall \beta > 0 \). We have to prove that if \( \mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma) \) then also
\[
\mu_1(\alpha_\beta, \gamma_\beta) < \lambda_1(\alpha_\beta, \gamma_\beta).
\]
Indeed, let us make the change of variables \( r = s^\beta \) in the equations (2.35) and (2.36). We get
\[
-\frac{1}{s^{\gamma_\beta}}(s^{\gamma_\beta} \psi')' = \beta^2 s^\alpha s \psi, \quad \psi \in E_{2-\gamma_\beta}, \tag{3.2}
\]
and respectively
\[
-\frac{1}{s^{\gamma_\beta}}(s^{\gamma_\beta} \varphi')' = \beta^2 s^\alpha \varphi, \quad \varphi \in E_{\gamma_\beta} \tag{3.3}
\]
where \( \alpha_\beta \) and \( \gamma_\beta \) are given by (3.1). This yields \( \mu_1(\alpha_\beta, \gamma_\beta) = \beta^2 \mu_1(\alpha, \beta) \) and \( \lambda_1(\alpha_\beta, \gamma_\beta) = \beta^2 \lambda_1(\alpha, \gamma) \) and thus \( \mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma) \) if and only if \( \mu_1(\alpha_\beta, \gamma_\beta) < \lambda_1(\alpha_\beta, \gamma_\beta) \).

Step 2. \( \mu_1(\alpha, 2) < \lambda_1(\alpha, 2) \), for all \( \alpha > -1 \). We denote by \( \psi \) and respectively \( \varphi \) eigenfunctions corresponding to \( \mu_1 \) and \( \lambda_1 \) such that \( \psi, \varphi > 0 \) in \((0,1)\). Then \( \psi \) and \( \varphi \) satisfy
\[
-\psi'' = \mu_1 r^\alpha \psi, \quad \varphi \in E_0 \tag{3.4}
\]
and respectively

\[ -\frac{1}{r^2}(r^2 \varphi')' = \lambda_1 r^\alpha \varphi, \quad \psi \in E_2. \quad (3.5) \]

As in Lemma 2.7 we can prove that

\[ \varphi(0) > 0 \quad (3.6) \]

\[ \varphi(r) - \varphi(0) = O(r^{\alpha+2}) \quad \text{as} \ r \searrow 0. \quad (3.7) \]

Hence \( \varphi'(0) = 0 \) and \( \varphi \in E_0 \). From (3.5) we infer

\[ -\varphi'' = \lambda_1 r^\alpha + \frac{2}{r} \varphi'. \quad (3.8) \]

Multiplying (3.8) with \( \varphi \) and then integrating on \((0,1)\) we get

\[ \int_0^1 |\varphi'|^2 \, dr = \lambda_1 \int_0^1 r^\alpha \varphi^2 \, dr + \int_0^1 \frac{2\varphi \varphi'}{r} \, dr. \]

Since

\[ \int_0^1 |\varphi'|^2 \, dr \geq \mu_1 \int_0^1 r^\alpha \varphi^2 \, dr \]

it follows

\[ \mu_1 \int_0^1 r^\alpha \varphi^2 \, dr \leq \lambda_1 \int_0^1 r^\alpha \varphi^2 \, dr + \int_0^1 \frac{2\varphi \varphi'}{r} \, dr. \]

It is a routine exercise using (3.5) to prove that \( \varphi' < 0 \). Thus

\[ \mu_1 \int_0^1 r^\alpha \varphi^2 \, dr < \lambda_1 \int_0^1 r^\alpha \varphi^2 \, dr \]

hence \( \mu_1 < \lambda_1 \).

**Step 3.** \( S = D \). We have proved so far that the half line \( (T) : \alpha > -1, \gamma = 2 \) lies in \( S \) (see Figure 1). Hence by Step 1 we get \( \bigcup_{\beta \geq 0} H_\beta(T) \subset S \). It is an easy geometric remark that

\[ \bigcup_{\beta > 0} H_\beta(T) = D \quad (\text{see Figure 1}). \]

Proposition 3.1 is proved. \( \blacksquare \)

Now we can state the following result.

**Theorem 3.2.** a) For \( \gamma \geq \alpha + 3, \alpha > -2 \) problem (2.1) has a solution if and only if \( \lambda \in (0, \lambda_1(\alpha, \gamma)) \).

b) For \( 1 < \gamma < \alpha + 3, \alpha > -2 \) we have \( \mu_1(\alpha, \gamma) < \lambda_1(\alpha, \gamma) \) and problem (2.1) has a solution if \( \lambda \in (\mu_1(\alpha, \beta), \lambda_1(\alpha, \gamma)) \) and no solution if \( \lambda \leq 0 \) or \( \lambda \geq \lambda_1 \).

**Remark 3.3.** In the paper [3] of Brézis-Nirenberg it was proved that if \( \gamma = 2, \alpha = 0 \) then problem (2.1) has a solution if and only if \( \lambda \in (\pi^2/4, \pi^2) \). An easy
computation shows that \( \mu_1(0,2) = \pi^2/4 \) and \( \lambda_1(0,2) = \pi^2 \) and thus our result seems optimal. It is therefore natural to ask the following question: Is it true that in the case \( 1 < \gamma < \alpha + 3 \) problem (2.1) has no solution for \( \lambda < \mu_1(\alpha, \gamma) \)?

**Remark 3.4.** We consider the problem

\[
-\Delta u = r^\sigma |u|^{p-1} u + \lambda r^\sigma u \quad \text{in } B_R \subset \mathbb{R}^N \quad (r = |x|)
\]

\[
u = 0 \quad \text{on } \partial B_R
\]

\[
u > 0 \quad \alpha, \sigma \geq -2, \quad p + 1 = (2N + 2\sigma)/(N - 2).
\]

By a weak solution of (3.8) we mean a function \( u \in H^1_0(B_R) \) \( u > 0 \) satisfying

\[
\int_{B_R} \nabla u \nabla \varphi = \int_{B_R} r^\sigma |u|^{p-1} u \varphi dx + \lambda \int_{B_R} r^\sigma u \varphi dx, \quad \forall \varphi \in H^1_0(B_R).
\]  

(3.10)

It is easily seen that at least at a formal level the radial solutions of (3.9) satisfy

\[
-\frac{1}{r^{N-1}} (r^{N-1} u') = r^\sigma |u|^{p-1} u + \lambda r^\sigma u \quad \text{in } (0,R), \quad u' = du/dr, \quad u > 0, \quad u \in E_{N-1}.
\]  

(3.11)

We claim that weak solutions of (3.11) in the sense of (2.2) satisfy (3.10). Indeed, if \( u \) satisfies (3.11), then \( u \) satisfies

\[
-\Delta u = r^\sigma |u|^{p-1} u + \lambda r^\sigma u \quad \text{in } B_R \setminus \{0\}
\]  

(3.12)

by standard elliptic regularity. Let \( \varphi \in H^1_0(\Omega) \). We multiply (3.12) with \( \varphi \) and then we integrated on \( D_\varepsilon = \{ x \in \mathbb{R}^N : \varepsilon \leq |x| \leq R \} \) using Green's formula

\[
\int_{D_\varepsilon} \nabla u \nabla \varphi dx - \int_{D_\varepsilon} r^\sigma |u|^{p-1} u \varphi dx - \lambda \int_{D_\varepsilon} r^\sigma u \varphi dx + \int_{|x|=\varepsilon} u'(\varepsilon) \varphi(x) dS = 0.
\]

Here \( dS \) is the surface element on \( |x| = \varepsilon \), \( dS = \sum_{N-1} \varepsilon^{N-1} d\theta; \ d\theta \) is the surface element on \( |x| = 1 \) and \( \sum_{N-1} \) is the \((N-1)\)-dimensional measure of the sphere \( |x| = 1 \). We get

\[
\int_{D_\varepsilon} \nabla u \nabla \varphi dx - \int_{D_\varepsilon} r^\sigma |u|^{p-1} u \varphi dx - \lambda \int_{D_\varepsilon} r^\sigma u \varphi dx + \sum_{N-1} \int_{|x|=\varepsilon} \varepsilon^{N-1} u'(\varepsilon) \varphi(\varepsilon, \theta) d\theta = 0.
\]  

(3.13)

Since \( u \in E_{N-1} \) we deduce that \( \varepsilon^N u'(\varepsilon)^2 \to 0 \) on a subsequence \( \varepsilon = \varepsilon_k \to 0 \). Hence \( \varepsilon_k^{N-1} u'(\varepsilon_k) \to 0 \) since \( N > 1 \). In (3.13) we let \( \varepsilon = \varepsilon_k \to 0 \). This yields

\[
\int_{B_R} \nabla u \cdot \nabla \varphi dx - \int_{B_R} r^\sigma |u|^{p-1} u \varphi dx - \lambda \int_{B_R} r^\sigma u \varphi dx = 0;
\]

i.e., exactly (3.10). Now it is easy to formulate existence results for the problem (3.9). Let \( \lambda_1(\alpha) \) be the first eigenvalue of the following eigenvalue problem

\[
-\Delta u = \lambda r^\sigma u \quad \text{in } B_R, \quad u = 0 \quad \text{on } \partial B_R.
\]  

(3.14)

As in Figueiredo [5] we infer that \( \lambda_1(\alpha) \) is simple and the corresponding eigenfunction may be chosen positive. We deduce that in fact \( \lambda_1(\alpha) = \lambda_1(\alpha, N - 1) \) where \( \lambda_1(\alpha, N - 1) \) was defined at (2.11). Let \( \mu_1(\alpha) \) be the first eigenvalue of the eigenvalue problem (2.35) with \( \gamma = N - 1 \).

We can now state
Theorem 3.5. a) For $N \geq \alpha + 4$, $\alpha \geq -2$ problem (3.9) has a radial weak solution if and only if $\lambda \in (0, \lambda_1(\alpha))$.

b) For $2 < N < \alpha + 4$, $\alpha > -2$, we have $\mu_1(\alpha) < \lambda_1(\alpha)$ and if $\lambda \in (\mu_1(\alpha), \lambda_1(\alpha))$ problem (3.9) has a radial weak solution. It has no radial weak solution if $\lambda \leq 0$, $\lambda \geq \lambda_1$.

Concerning the regularity of radial weak solutions it can be easily proved that these are classical solutions of (3.9). Indeed $u \in E_{N-1}$. Let $\theta = (N-1) + (\sigma + 1)/2$, $q = (p - 1)N/2$. Then, a simple computation shows that $(N-1)\frac{q}{2} = \frac{q+1}{q}$ so that by the imbedding lemma we get that $u \in L^q_\theta(0, R)$; i.e., $r^{(N-1)\frac{q}{2}(\sigma + 1)/2}|u|^{(p-1)N/2} \in L^1(0, R)$ which can be restated as $r^\theta|u|^{p-1} \in L^{N/2}(B_R)$. Hence if we set $a(r) = r^\theta|u|^{p-1}$ then $u$ is a weak solution of

$$-\Delta u = a(r)u + \lambda r^\alpha u \text{ in } H^1_0(B_R)$$

with $a \in L^{N/2}(B_R)$.

We can now proceed as in Brézis-Nirenberg [3] to infer that $u \in L^{\infty}(B_R)$ and hence by standard elliptic regularity $u \in C^2(B_R)$.

REFERENCES


