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Topological Classification of Morse Functions and Generalisations of Hilbert's 16-th Problem

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Abstract The topological structures of the generic smooth functions on a smooth manifold belong to the small quantity of the most fundamental objects of study both in pure and applied mathematics. The problem of their study has been formulated by A. Cayley in 1868, who required the classification of the possible configurations of the horizontal lines on the topographical maps of mountain regions, and created the first elements of what is called today 'Morse Theory' and 'Catastrophes Theory'. In the paper we describe this problem, and in particular describe the classification of Morse functions on the 2 sphere and on the torus.

Keywords Classification of maps · Morse functions

Mathematics Subject Classifications (2000) 57R99 · 58D15 · 58E05

The topological structures of the generic smooth functions on a smooth manifold belong to the small quantity of the most fundamental objects of study both in pure and applied mathematics. The problem of their study has been formulated by A. Cayley in 1868, who required the classification of the possible configurations of the horizontal lines on the topographical maps of mountain regions, and created the first elements of what is called today 'Morse Theory' and 'Catastrophes Theory'.

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M. Morse has told me, in 1965, that the problem of the description of the possible combinations of several critical points of a smooth function on a manifold looks hopeless to him. L.S. Pontrjagin and H. Whitney were of the same opinion.

I formulate below some recent results in this domain. Thus the classification of the Morse functions on a circle S^1 leads to the Taylor coefficients of the tangent function. On the two-dimensional sphere the number of topologically different Morse functions with T saddle points (that is, having $2T+2$ critical points) grows with T as T^{2T}. The tangent function is replaced in this study by some elliptic integral (discovered by L. Nicolaescu, while he was continuing Arnold's calculation of the number of topologically different Morse functions having 4 saddle points on the sphere S^2—that number is equal to 17 746).

Replacing the 2-sphere by the two-dimensional torus, one obtains an infinite number of topologically different Morse functions with a fixed number of critical points, provided that the topological equivalence is defined by the action of the identity connected component of the group of diffeomorphisms (or homeomorphisms), that is, if the corresponding mappings of the torus are supposed to be homotopic to the identity map.

However, if one accepts mappings permuting the parallels and the meridians of the torus, the classification becomes finite and similar to that of the Morse functions on the 2-sphere.

The topological classification of the functions on the 2-sphere is related to one of the questions of the 16-th Hilbert's problems (on the arrangements of the planar algebraic curves of fixed degree, defined on the real plane $\mathbb{R}^2$). A real polynomial of fixed degree, defined on the real plane $\mathbb{R}^2$, generates a smooth function on S^2 (with one more critical point at infinity), and the topological structure of these functions on the 2-sphere influences the topological properties of the arrangements of the real algebraic curves where the initial polynomials vanish.

It seems that only a small part of the topological equivalence classes of the Morse functions having T saddle points on S^2, contains polynomial representatives of corresponding degree. For instance, in the case of 4 saddles ($T=4$) the polynomials are of degree 4 in two real variables, and provide less that 1000 topological types, from the total number of different topologic types, which is 17 746: the majority of the classes of smooth Morse functions with 4 saddle points have no polynomial representatives.

The classification of the Morse functions on the torus is related similarly to the topological investigation of the trigonometric polynomials. The degree of a polynomial is replaced in the case of the functions on T^2 by the Newton polygon of a trigonometric polynomial (which is the convex hull of the set of the wave-vectors of the harmonics forming the trigonometric polynomial).

When such a convex polygon is fixed, its trigonometric polynomials (formed by the harmonics whose wave-vectors belong to the Newton polygon) realize only a finite subset of the infinite set of the topologically different classes of functions, classified up to those transformations of the torus which are homotopic to the identity map. This finite subset depends on the Newton polygon, and is small when the polygon is small. It is finite for the following reason:

In the torus transformations needed to reduce the initial trigonometric polynomial to the finite set of classes, discussed above, parallels and meridians are sent only to linear combinations of the parallel and meridian classes whose coefficients are not large. The number of the homotopy classes of such transformations of the torus is finite, making finite the resulting set of topological equivalence classes of trigonometric polynomials with fixed Newton polygon (classified up to the torus transformations homotopic to the identity map).

Definition 1 The *graph of a Morse function* $f\colon M \to \mathbb{R}$ on a manifold M is the space whose points are the connected components of the level hypersurfaces $f^{-1}(c) \subset M$. This space is a finite complex (at least when M is compact).

Example 1 The twin peak mountain Elbrouz (whose height function f is considered continued to the sphere S^2 with a minimum D at the antipodal point) generates, as the graph of the height function f, the character 'Y' (Fig. 1).

The topographical 'bergshtrechs' at the horizontals map show the antigradient vector directions (followed by the grass of the mountain after the rain).

Example 2 The volcanic Vesuvius mountain has a maximum height point A, a height local minimum point B (crater), and a saddle point C, defining the graph of Fig. 2.

The vertices of the graphs (the end-points A, B, D and the triple point C) represent the critical points of the function f. They form a finite set for a Morse function on a compact manifold M.

We see that the graphs of the two preceding examples are topologically equivalent, while the two corresponding functions are not. Taking this into account, we will distinguish the graphs, ordering the vertices by the critical values of the height.

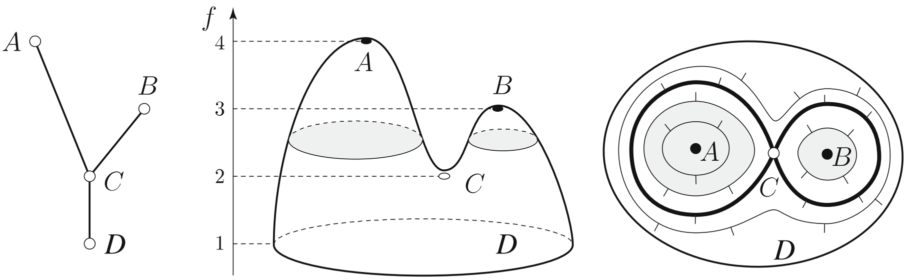


Fig. 1 The twin peak mountain Elbrouz

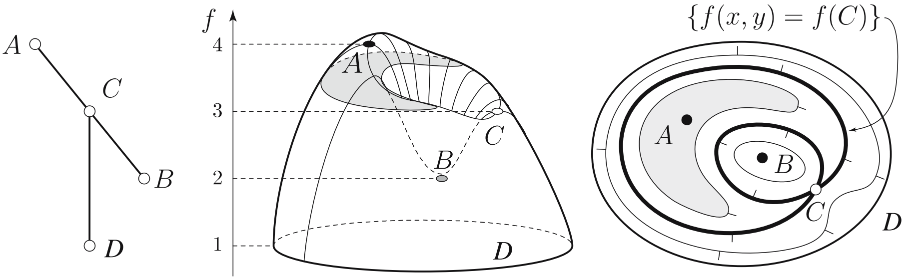


Fig. 2 The volcanic Vesuvius mountain

For instance, for a Morse function with n critical points, we may fix the n critical values to be $1 < 2 < \cdots < n$. Then the first (Elbrouz) graph would obtain the numbering of its vertices

$$\{f(D) = 1,\ f(C) = 2,\ f(B) = 3,\ f(A) = 4\}$$

and the second (Vesuvius) graph's vertex numbering is

$$\{f(D) = 1,\ f(B) = 2,\ f(C) = 3,\ f(A) = 4\}$$

Therefore we will consider the graphs of the Morse functions to be labeled by the above 'height' numbering $(1, 2, \ldots, n)$ *of the vertices* (ordering then maxima, minima, and saddle points).

Examples 1 and 2 show that the heights of the neighbours of a triple vertex can't be all three higher or all three lower than the triple vertex itself (Fig. 3).

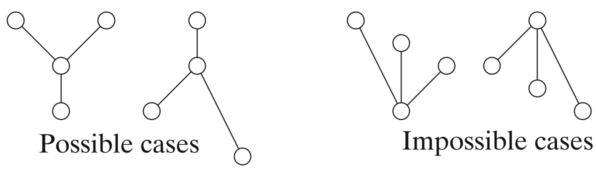
The ordering of the vertices verifying this restriction, will be called *regular*.

The regularly ordered graph is a topological invariant of a Morse function. For Morse functions f on the spheres ($M = S^m, m > 1,\ f\colon\ M \to \mathbb{R}$) the graphs are trees. For Morse functions on a surface of genus g, the graph has g independent cycles (1 for the torus surface $\mathbb{T}^2$, two for the sphere with two handles, three for the bretzel surface and so on).

In the case of the sphere of dimension 2 this regularly ordered graph is *the only topological invariant* of a Morse function $f\colon S^2 \to \mathbb{R}$ (up to diffeomorphisms or up to homeomorphisms of S^2, if we fix the critical values to be $\{1, 2, \ldots, n\}$). All regular orderings (of any tree) with n vertices are realized as the graphs of such Morse functions.

Counting (in 2005) the number $\varphi(T)$ of such regularly ordered trees with T triple points (and of diffeomorphism classes of Morse functions having

Fig. 3 Possible, impossible cases



values $\{1, 2, \ldots, n\}$ at their $2T + 2$ critical points, including the T saddle points) Arnold proved the following:

Theorem 1 *There exists positive constants a and b such that for any T the inequalities $aT^T < \varphi(T) < b\,T^{2T}$ hold.*

The article [1] claims that the upper bound of Theorem 1 is closer to the genuine asymptotics of the function φ, mentioning some arguments for this conjecture, based on (unproved) ergodic properties of random graph ordering.

This conjecture was then proved by L. Nicolaescu [6] who replaced the recurrent inequality used by Arnold to prove his upper bound theorem by an exact nonlinear recurrent relation (similar to the Givental's mirror symmetry proof in quantum field theory).

Nicolaescu's recurrent relation involves 43 terms of which Arnold had used only the first leading term to prove his inequality. Using the computer, Nicolaescu solved this nonlinear recurrence explicitly, representing φ in terms of the coefficients of the power series expansion of some elliptic integral.

Studying the behaviour in the complex domain of these integrals, Nicolaescu succeeded to prove the Arnold conjecture on the growth of φ (leaving unproved however, the conjecture on the ergodic theory of random graphs which had led Arnold to his conjecture on the asymptotics of φ).

The first values of the function φ (calculated in Arnold [1]) are

T	1	2	3	4
$\varphi(T)$	2	19	428	17 746

Already the easy computation of $\varphi(2) = 19$ provides some of the ideas of the general structure theory.

According to Nicolaescu's computer $\varphi(5) = 1178792$. Arnold was unable to draw all the shapes of functions with $T = 5$ saddles, drawing only those with $T = 4$.

In the case $T = 5$ the upper bound $T^{2T} = \frac{10}{2}^{10} \approx 10^{10-3}$ is only 10 times higher than the exact value provided by Nicolaescu.

A Morse polynomial f of degree m in two variables has at most $(m-1)^2$ critical points. Let $m = 2k$ be even. Suppose that the leading form at infinity (of degree m) is positive. In this case the polynomial provides a Morse function $\widetilde{f}\colon S^2 \to \mathbb{R}$ having exactly $4k^2 - 4k + 2$ critical points (taking that at infinity into account).

The relation $2T + 2 = 4k^2 - 4k + 2$ provides the value $T = 2k(k-1)$ for the number T of the saddles. For the polynomials of degree $m = 4$ we find $k = 2$ and $T = 4$. Therefore, the topological types of these 4-th degree polynomials (with 9 critical points in $\mathbb{R}^2$) are included in the set of 17 746 classes of topologically equivalent smooth Morse functions on the sphere S^2, with $T = 4$ saddles.

The topological equivalence is defined here by the actions of the homeomorphisms (or diffeomorphisms) both of the sphere, where the functions are defined, and the orientation preserving homeomorphisms of the axis of values. One also needs the mappings of the axis of values, if the critical values are not fixed e.g. as $\{1, 2, \ldots, 9\}$, but may move.

According to my calculations, among the 17 746 topologically different classes of the Morse functions on S^2 with 4 saddles at most, one thousand of classes have a polynomial representative of degree 4.

I have no conjecture on the growth rate of the number of classes realizable by polynomials of the corresponding degree, for a growing value of the number T of saddles.

A trigonometric Morse polynomial of degree n has at most $2n$ critical points on the circle S^1. For the case where all the $2n$ critical values are different, the table (from 'serpent' theory [4]) provides the following numbers φ of the topologically different functions $f : S^1 \to \mathbb{R}$ with $2n$ critical points (considered up to the orientation preserving homeomorphisms or diffeomorphisms of S^1 and of $\mathbb{R}$)

$2n$	2	4	6	8	10
φ	1	2	16	272	7 936

Every such class of smooth functions includes some trigonometric polynomial of degree n.

Theorem 2 *The numbers $\varphi(n)$ of classes of the oriented topological equivalence of Morse functions with $2n$ critical points and $2n$ critical values on the circle, provide the Taylor series of the tangent function (as of the exponential generating function):*

$$\tan t = \frac{1}{1!}t + \frac{2}{3!}t^3 + \frac{16}{5!}t^5 + \frac{272}{7!}t^7 + \frac{7936}{9!}t^9 + \ldots$$

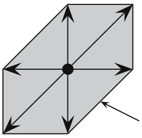
We consider next the *trigonometric polynomials of two variables* as some smooth functions on the 2-torus, $f\colon \mathbb{T}^2 \to \mathbb{R}$.

Example 3 The $\widetilde{A}_2$ affine Coxeter group of trigonometric polynomials of degree 1 form the following 6-parameter vector space of trigonometric polynomials in two variables x and y:

$$(*) \quad f = a\cos x + b\sin x + c\cos y + d\sin y + p\cos(x+y) + q\sin(x+y).$$

This family of trigonometric polynomials has been studied in the articles by Arnold [2, 3]. The number of critical points of such Morse polynomials $(*)$ on $\mathbb{T}^2$ does not exceed 6. In the general case of the arbitrary trigonometric polynomials the upper bound of the number of critical points on $\mathbb{T}^2$ is provided by the doubled area of the Newton polygon ($n!$ the volume of the Newton polyhedron for the n-torus case $\mathbb{T}^n$) (Fig. 4).

Fig. 4 Newton polygon



Definition 2 The *Diff-equivalence* of smooth functions on $\mathbb{T}^2$ is the belonging to the same orbit of the natural action on functions on the torus of the group *Diff* ($\mathbb{T}^2$) of the diffeomorphisms of the torus, accompanied by the orientation preserving diffeomorphisms of the axis of the values.

The *Diff*$_0$*-equivalence* is defined similarly, replacing, however, the group *Diff*($\mathbb{T}^2$) by its connected subgroup *Diff*$_0$($\mathbb{T}^2$), formed by the diffeomorphisms homotopic to the identity, which is the connected component of the identity in *Diff*($\mathbb{T}^2$) (whose elements may interchange the meridian and parallel classes).

The articles by Arnold [2, 3] contain the proof of

Theorem 3 *The number of equivalence classes of the* C^∞ *Morse functions with 6 critical points on the two-dimensional torus* $\mathbb{T}^2$ *and of the trigonometric polynomials* $(*)$ *of the affine Coxeter group* $\widetilde{A}_2$ *have the following values:*

	Diff	*Diff*$_0$
$\mathscr{C}^\infty$	16	∞
$\widetilde{A}_2$	2	6

The 16 topologically different types of the smooth Morse functions with 6 critical points and values on the torus are described in Arnold [3] by the 16 regularly ordered graphs having 3 triple points (□), three end points (◯) and one cycle (for each graph). These 16 graphs, ordered by the natural height function, are shown in Fig. 5.

Among all these topological equivalence classes only two classes (marked by the sign $(*)$) contain some of the trigonometric polynomials.

All the other 14 cases are eliminated by the algebraic geometry of the elliptic level curves of functions $(*)$, as is explained in articles by Arnold [2, 3].

This finite set of the 16 *Diff*-equivalence classes provides an infinity of different *Diff*$_0$-classes for the following reason.

A generic point P on the cycle of the graph does not separate the graph. Hence the level line component, represented by P, does not separate the torus $\mathbb{T}^2$ into two parts, and therefore this closed curve is not homologous to zero on the torus.

Its homology class does not depend on the choice of the point P on the cycle, since any pair of such points $\{P, P\}'$ separates the graph into two parts, and hence the cycle $P - P'$ is the boundary of a domain on the torus.

This 1-dimensional homology class is an invariant of the *Diff*$_0$-equivalence (at least up to the choice of its sign).

Triple points (□), end points (○)

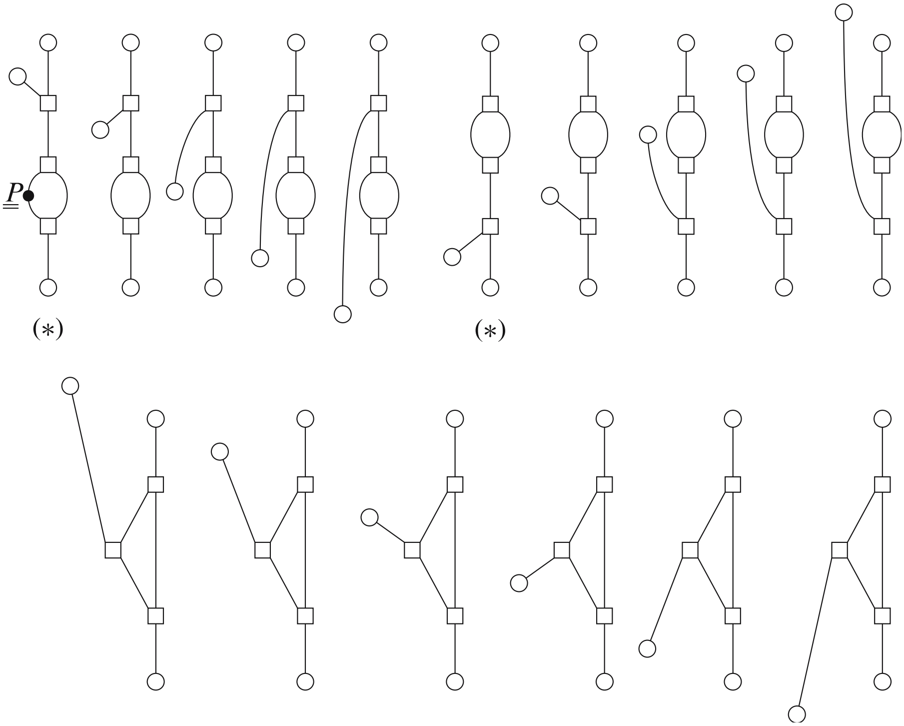


Fig. 5 The 16 graphes

For suitable choices of Morse functions with 6 critical points and values on the torus, it can take an infinity of different values. All these 1-homology classes are *Diff*-equivalent, each of them being represented by a non-bounding, non-self-intersecting closed curve on the torus.

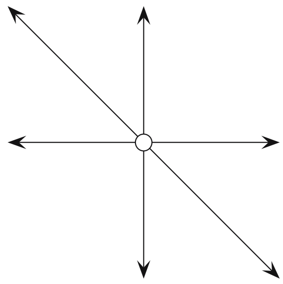
Thus we obtain an infinite set of the *Diff*$_0$-equivalence classes of the Morse functions having 6 critical points and 6 critical values on the 2-torus $\mathbb{T}^2$.

However the trigonometric polynomials (∗) provide only a finite part of this infinite set of the *Diff*$_0$-equivalence classes. Namely, in Arnold [2] it is proved that the algebraic geometry of real elliptic curves restricts the homology class of the level line component P on the torus: it may attain only 3 values (or 6 if we take the orientation into account).

These 6 realizable classes form in the one-dimensional homology group of the torus the Dynkin diagram of the Coxeter group A_2 (whence the name $\widetilde{A}_2$ of the class (∗) of trigonometric polynomials) (Fig. 6).

For the other classes (say defined by arbitrary Newton polygons) the number of *Diff*-equivalence and *Diff*$_0$-equivalence classes are unknown. I do not know how fast the number of *Diff*-equivalence classes grows when the number T of saddle points is large (neither for the smooth Morse functions, nor for the trigonometric polynomials having a fixed Newton polygon). Even

Fig. 6 $H_1(\mathbb{T}^2, \mathbb{Z})$



the finiteness proof for the number of *Diff*-equivalence classes (for each value of T) is not published.

This abundance of unsolved problems was the reason to choose these questions of real algebraic geometry and of Morse functions statistics for this paper: I hope that the readers will go further than me in this rapidly evolving domain at the intersection of all the branches of mathematics.

It is rather strange that the computer contribution to real algebraic geometry is still almost negligible while theoretical mathematicians have made a lot.

The only real contribution that I know, is the recent result of a former student of the Université Paris-Jussieu, Adriana Ortiz-Rodriguez, working at Mexico (and having started these works in Paris). This result describes the topology of the parabolic curves on algebraic surfaces of the projective space, and belongs to the intersection of real algebraic geometry and symplectic geometry.

The graph $\{z = f(x, y)\}$ of a polynomial f in two real variables x and y is a smooth surface in $\mathbb{R}^3$. The parabolic lines of this surface are projected to the $\{(x, y)\}$-plane as algebraic curves of degree $2n - 4$.

The question studied by A. Ortiz-Rodriguez is to evaluate the possible numbers of closed parabolic curves for polynomials of a given degree: how large may this number of parabolic curves be? And how large may be the number of connected components of the corresponding algebraic curve of degree $2n - 4$ in the real projective plane?

The classical Harnack theorem of real algebraic geometry implies that for the polynomial f of degree $n = 4$ the number M of closed parabolic curves cannot exceed 4.

It is not too difficult to construct examples of polynomials f of degree 4, for which the number of closed parabolic curves attains the value $M = 3$. But the problem of wether the case of 4 closed parabolic curves is realizable by some polynomial of degree 4 resisted to all the attempts of mathematicians, and only the computer helped to solve it.

Namely, in a year of uninterrupted calculations, the Mexico computer of A. Ortiz-Rodriguez has studied 50 millions of polynomials of degree 4.

Among all these polynomials, just 3 polynomials have, each of them, 4 closed parabolic curves on their graphs.

For the polynomials f of degree n, A. Ortiz-Rodriguez has proved (in her thesis, preceding the experiment described above) the inequalities

$$an^2 \leqslant M \leqslant bn^2$$

for the maximal number $M(n)$ of closed parabolic curves on the graph of a polynomial of degree n: for any such polynomial, the number of parabolic curves is at most bn^2, and there exist polynomials of degree n for which the parabolic curve number is at least an^2.

Unfortunately, b is higher than a, and thus the question of the genuine asymptotics of $M(n)$ is waiting for the courageous researchers (and computer experiments).

The situation is even more difficult with another similar problem (also studied by A. Ortiz-Rodriguez). Consider in the 3-dimensional projective space $\mathbb{R}P^3$ a smooth algebraic surface of degree n. How large may be the number of its closed parabolic curves?

Here the Ortiz-Rodriguez boundaries are

$$An^3 \leqslant M \leqslant Bn^3,$$

but the coefficient B is a dozen of times higher than A and the genuine asymptotic behaviour of the maximal number $M(n)$ of the connected components of the parabolic curves on smooth real projective hypersurfaces of degree n remains unknown.

These real algebraic geometry versions of the higher dimensional Plücker formulae remain a challenge for modern algebraic geometry, which seems, however, to be unable to study the real things. Of course the celebrated Tarski-Seidenberg theorem implies the existence of an algorithm providing the needed answers, but the required time for the real application is usually much larger than the whole life span of the universe.

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