Challenging problems

Combinatorics

1. Suppose \( m, n, p \) are positive integers. Show that a \( m \times n \) chess table can be covered using \( 1 \times p \) pieces if and only if \( p \) divides \( m \) or \( n \).

2. Suppose we are given 1999 points such that no three are collinear and no four of them lie on the same circle. Show that among these 1999 points there exist three such that the circle they determine contains exactly 998 points inside and 998 points outside.

Geometry

1. \( ABCD \) was a square. Peter marked points \( M, N, P, Q \) on the sides of this square and then erased the square. Can you reconstruct the square using a ruler and compass?

2. Prove that in any triangle \( ABC \) the internal bisector of the angle \( \angle BAC \) is also the internal bisector of the angle \( \angle HAO \) where \( H \) is the orthocenter of \( ABC \) and \( O \) is its circumcenter.

Number theory

1. (a) Find all the triples of positive integers \( a, b, c \) such that

\[
\frac{1}{4} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.
\]

(b) Prove that for any integer \( n \geq 6 \) we can find positive integers \( x_1, \ldots, x_n \) such that

\[
\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = 1.
\]