THE CO-AREA FORMULA

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1. Statement of the formula

We start with the simplest version.

**Fubini theorem.** Suppose \( \varphi \) is an integrable function on \( \mathbb{R}^{n+k} \). Then

\[
\int_{\mathbb{R}^{n+k}} \varphi(x^1, \ldots, x^{n+k}) |dx^1 \cdots dx^{n+k}|
= \int_{\mathbb{R}^k} \left( \int_{\mathbb{R}^n} \varphi(x^1, \ldots, x^k, x^{k+1}, \ldots, x^{n+k}) dx^{k+1} \cdots dx^{n+k} \right) |dx^1 \cdots dx^k|.
\]

We can reformulate this as follows. Set \( y = (x^1, \ldots, x^k) \), \( x = (x^{k+1}, \ldots, x^{n+k}) \)
and define \( A : \mathbb{R}^{n+k} \to \mathbb{R}^k \), \( (y, x) \mapsto y \). Then

\[
\int_{\mathbb{R}^{n+k}} \varphi(x, y) dV_{n+k}(x, y) = \int_{\mathbb{R}^k} \left( \int_{A^{-1}(y)} \varphi(x, y) dV_n(x) \right) dV_k(y).
\] (1.1)

where \( dV_m \) denotes the \( m \)-dimensional Lebesgue measure.

Consider now a slightly more general case of a linear map

\[
A : \mathbb{R}^{n+k} \to \mathbb{R}^k, \ (x^1, \ldots, x^k, x^{k+1}, \ldots, x^{n+k}) \mapsto (y^1, \ldots, y^k) = (\mu_1 x^1, \ldots, \mu_k x^k),
\] (1.2)

where \( \mu_1, \ldots, \mu_k \) are positive numbers. Applying the Fubini theorem we deduce

\[
\int_{\mathbb{R}^{n+k}} \mu_1 \cdots \mu_k \varphi(x^1, \ldots, x^{n+k}) dV_{n+k}(x^1, \ldots, x^{n+k})
= \int_{\mathbb{R}^{n+k}} \varphi \left( \frac{y^1}{\mu_1}, \ldots, \frac{x^k}{\mu_k}, x^{k+1}, \ldots, x^{n+k} \right) \left| dy^1 \cdots dy^k dx^{k+1} \cdots dx^{n+k} \right|
= \int_{\mathbb{R}^k} \left( \int_{A^{-1}(y)} \varphi(x, y) dV_n(x) \right) dV_k(y).
\] (1.3)


Notes for the “Blue collar seminar on geometric integration theory”.
But for the factor $\mu_1 \cdots \mu_k$, the formulæ (1.1) and (1.3) look similar. To give an invariant meaning to this quantity we need to use the following elementary fact of linear algebra.

**Lemma 1.1.** Suppose that $U$ and $V$ are Euclidean space of dimensions $n+k$ and respectively $k$, $n \geq 0$, and $A : U \to V$ is a surjective linear map. Then there exist Euclidean coordinates $x^1, \ldots, x^{n+k}$ on $U$, Euclidean coordinates $y^1, \ldots, y^k$ on $V$ and positive numbers $\mu_1, \ldots, \mu_k$ such that, in these coordinates the operator $A$ is described by

$$y_j^2 = \mu_j x_j^2, \quad 1 \leq j \leq k.$$  

The numbers $\mu_1^2, \ldots, \mu_k^2$ are the eigenvalues of the positive symmetric operator $AA^* : V \to V$ so that

$$\mu_1 \cdots \mu_k = \sqrt{\det AA^*}.$$  

**Proof.** Let $W$ denote the orthogonal complement of $\ker A$ in $U$. Denote by $A_0$ the restriction of $A$ to $W$ so that $A_0 : W \to V$ is a linear isomorphism. Note that $W$ coincides with the range of the adjoint operator $A^* : V \to U$ so that

$$A_0 A_0^* = AA^*.$$  

We want to find a linear isometry $R : V \to W$ such that the operator

$$B = A_0 R : V \to V$$  

is symmetric. Note that since $R$ is an isometry we have $R^{-1} = R^*$. Moreover we have a commutative diagram

$$\begin{array}{ccc}
W & \xrightarrow{A_0} & V \\
\downarrow R & & \downarrow 1_V \\
V & \xrightarrow{B} & V
\end{array}$$  

Note that $A_0 A^* : V \to V$ is positive and symmetric. We define

$$R := (A_0 A_0^*)^{-1/2} : V \to W.$$  

Let us show that $R$ is indeed an isometry. Indeed, for any $v \in V$ we have

$$(Rv, Rv) = (A_0 A_0^*)^{-1/2} v, (A_0 A_0^*)^{-1/2} v) = (A_0 A_0^*)^{-1/2} v, (A_0 A_0^*)^{-1/2} v) = (v, v).$$  

Clearly $A_0 R = A_0 A_0^* (A_0 A_0^*)^{-1/2} = (A_0 A_0^*)^{1/2}$ is symmetric. Now choose an orthonormal basis that diagonalizes $B$. Transport it via $R$ to an orthonormal basis of $W$. With respect to these bases of $W$ and $V$ the operator $A$ is described by a diagonal matrix with entries consisting of the eigenvalues of $A_0 R = (A_0 A_0^*)^{1/2}$. \hfill \Box

Returning to (1.3) we see that

$$\mu_1 \cdots \mu_k = J_A := \sqrt{\det AA^*}.$$  

The quantity $J_A$ is called the Jacobian of $A$. Thus, we can rewrite (1.3) as

$$\int_{\mathbb{R}^{n+k}} J_A \varphi(x^1, \ldots, x^{n+k}) dV_{n+k}(x^1, \ldots, x^{n+k}) = \int_{\mathbb{R}^k} \left( \int_{A^{-1}(y)} \varphi(x, y) dV_n(x) \right) dV_k(y). \quad (1.4)$$
Lemma 1.1 shows that (1.4) holds for any surjective linear map $\mathbb{R}^{n+k} \to \mathbb{R}^k$.

It is convenient to give a more explicit description of $J_A$. This relies on the concept of Gramm determinant. More precisely, given a collection of vectors $u_1, \ldots, u_k$ in an Euclidean space $U$ we define their Gramm determinant (or Grammian) to be the quantity

$$G(u_1, \ldots, u_k) := \det\left( (u_i, u_j)_U \right)_{1 \leq i, j \leq k},$$

where $(-,-)_U$ denotes the inner product in $U$. Geometrically, $\sqrt{G(u_1, \ldots, u_k)}$ is the $k$-dimensional volume of the parallelipiped spanned by the vectors $u_1, \ldots, u_k$,

$$P(w_1, \ldots, w_k) = \left\{ \sum_{j=1}^k t_j w_j; \ t_j \in [0,1] \right\}.$$  

Note that $G(u_1, \ldots, u_k) = 0$ iff the vectors $u_1, \ldots, u_k$ are linearly dependent and $G(u_1, \ldots, u_k) = 1$ if the vectors $u_1, \ldots, u_k$ form an orthonormal system.

Equivalently

$$G(u_1, \ldots, u_k) = (u_1 \wedge \cdots \wedge u_k, u_1 \wedge \cdots \wedge u_k)_{\Lambda^k U}$$

where $(-,-)_{\Lambda^k U}$ denotes the inner product on $\Lambda^k U$ induced by the inner product in $U$.

**Lemma 1.2.** Let $A : U \to V$ be as in Lemma 1.1. Fix a basis $f_{k+1}, \ldots, f_{n+k}$ of $U_0 := \ker A$ and vectors $u_1, \ldots, u_k$ such that $Au_1, \ldots, Au_k$ span $V$. Then

$$J_A^2 = \frac{G(Au_1, \ldots, Au_k)G(f_{k+1}, \ldots, f_{n+k})}{G(u_1, \ldots, u_k, f_{k+1}, \ldots, f_{n+k})}. \tag{1.5}$$

**Proof.** We first prove the result when $\dim U = \dim V$. In this case the collection $u_1, \ldots, u_k$ is a basis of $U$. Fix an orthonormal basis $e_1, \ldots, e_k$ of $U$ denote by $T : U \to U$ the linear operator

$$e_j \mapsto u_j.$$  

Then

$$G(u_1, \ldots, u_k) = \det T^* T,$$

$$G(Au_1, \ldots, Au_k) = \det((AT)^*(AT)) = |\det T^*| |\det AA^*| |\det T| = J_A^2 \det TT^*.$$  

To deal with the general case, we denote by $P_0$ the orthogonal projection onto $U_0$. Now define

$$\hat{\Lambda} : U \to \hat{V} := V \oplus U_0, \quad u \mapsto Au \oplus P_0 u.$$  

we equip $\hat{V}$ with the product Euclidean structure.

Let us observe that

$$J_A = J_{\hat{\Lambda}}.$$  

Indeed, with respect to the direct sum decomposition $\hat{V} = V \oplus U_0$ the operator $\hat{\Lambda} \hat{\Lambda}^*$ has the block decomposition

$$\hat{\Lambda} \hat{\Lambda}^* = \begin{bmatrix} AA^* & 0 \\ 0 & 1_{U_0} \end{bmatrix}$$

so that

$$\det \hat{\Lambda} \hat{\Lambda}^* = \det AA^*.$$  

Observe that in $\Lambda^{k+n}(V \oplus U_0)$ we have the equality

$$\hat{\Lambda} u_1 \wedge \cdots \hat{\Lambda} u_k \wedge f_{k+1} \wedge \cdots \wedge f_{k+n} = Au_1 \wedge \cdots Au_k \wedge f_{k+1} \wedge \cdots \wedge f_{k+n}.$$
Proof. We consider first the case when $g_X$ is surjective. We denote by $J$ densities induced by the Riemann metric on $X$ and respectively $g_N$.

Suppose now that $X$ and $Y$ are $C^1$ manifolds of dimensions $n + k$ and respectively $k$, $n \geq 0$ equipped with Riemann metrics $g_X$ and $g_Y$. We denote by $|dV_X|$ and $|dV_Y|$ the volume densities induced by $g_M$ and respectively $g_N$.

Suppose that $F : X \to Y$ is a $C^1$-map such that for any $p \in M$ the differential $D_pF : T_pX \to T_{F(p)}Y$ is surjective. We denote by $J_F(p)$ the Jacobian of this map.

**Theorem 1.3 (The co-area formula: version 1).** For any nonnegative function $\varphi : X \to \mathbb{R}$ which is measurable with respect to the measure defined by $|dV_X|$ we have

$$\int_X J_F(p)\varphi(p)|dV_X(p)| = \int_Y \left( \int_{F^{-1}(q)} \varphi(p)|dV_{F^{-1}(q)}(p)| \right) |dV_Y(q)|,$$

where $|dV_{F^{-1}(q)}|$ denotes the volume density on the fiber $F^{-1}(q)$ induced by the restriction of $g_X$ to $F^{-1}(q)$.

*Proof.* We consider first the case when $X$ is an open subset of $\mathbb{R}^{n+k}$ with coordinates $(x^1, \ldots, x^{n+k})$ equipped with a $C^1$-metric $g_X$, $Y$ is an open subset of $\mathbb{R}^k$ with coordinates $(y^1, \ldots, y^k)$ equipped with a metric $g_Y$ and the map $F$ is given by

$$y^j = x^j, \ j = 1, \ldots, k.$$

We have

$$|dV_X| = \sqrt{G(\partial_{x^1}, \ldots, \partial_{x^{n+k}})}|dx^1 \cdots dx^{n+k}|$$

$$= \sqrt{G(\partial_{x^1}, \ldots, \partial_{x^{n+k}})}|dy^1 \cdots dy^k dx^k+1 \cdots dx^{n+k}|,$$

$$= : \rho_X$$

$$|dV_{F^{-1}(q)}| = \sqrt{G(\partial_{x^1}, \ldots, \partial_{x^{n+k}})}|dx^1 \cdots dx^{n+k}|,$$

$$= : \rho_{F^{-1}(q)}$$

where the subscript $X$ indicates that the inner product in the definition of the above Gramm determinants is the one determined by the Riemann metric on $X$. Similarly

$$|dV_Y| = \sqrt{G_Y(\partial_{y^1}, \ldots, \partial_{y^k})}|dy^1 \cdots dy^k| = \sqrt{G_Y(DF\partial_{x^1}, \ldots, DF\partial_{x^k})}|dy^1 \cdots dy^k|.$$

$$= : \rho_Y$$
Using the Fubini theorem we deduce that for any nonnegative, measurable function \( \phi : X \to \mathbb{R} \) we have

\[
\int_X \rho_Y \phi_X |dy^1 \cdots dy^k dx^{k+1} \cdots dx^{n+k}| = \int_Y \left( \int_{F^{-1}(y)} \rho_X \phi |dx^{k+1} \cdots dx^{n+k}| \right) \rho_Y |dy^1 \cdots dy^k|
\]

\[
= \int_Y \left( \int_{F^{-1}(y)} \rho_X \phi_F |dx^{k+1} \cdots dx^{n+k}| \right) |dV_Y(y)| = \int_Y \left( \int_{F^{-1}(y)} \rho_X \phi |dV_{F^{-1}(y)}| \right) |dV_Y(y)|.
\]

Suppose that above \( \rho_Y \phi = J_F \varphi \), i.e., \( \phi = \frac{J_F}{\rho_Y} \varphi \). Then the above equality can be rewritten

\[
\int_X J_F(x) \varphi(x) |dV_X(x)| = \int_Y \left( \int_{F^{-1}(y)} \rho_X J_F |dV_{F^{-1}(y)}| \right) |dV_Y(y)|.
\]

The co-area formula is proved once we show that

\[
\frac{\rho_X J_F}{\rho_F \rho_Y} = 1, \quad \text{i.e.,} \quad J_F = \frac{\rho_Y \rho_F}{\rho_X}.
\]

The last equality follows from (1.5).

The general case of the co-area formula can be reduced to the special case via partition of unity and the implicit function theorem. \( \square \)

**Corollary 1.4.** Let \( X, Y \) and \( F : X \to Y \) be as in Theorem 1.3. Then for any measurable function \( \phi : X \to \mathbb{R} \) we have

\[
\int_X \phi(p) |dV_X(p)| = \int_Y \left( \int_{F^{-1}(q)} \frac{\phi(p)}{J_F(p)} |dV_{F^{-1}(q)}(p)| \right) |dV_Y(q), \quad (1.7)
\]

**Proof.** Apply (1.6) to \( \varphi = \frac{\phi}{J_F} \). \( \square \)

**Corollary 1.5.** Suppose \( X \) is a \( C^1 \) manifold equipped with a \( C^1 \)-metric \( g_X \), and \( f : X \to \mathbb{R} \) is a \( C^1 \) function with no critical points. Then for any measurable function \( \phi : X \to \mathbb{R} \) we have

\[
\int_X \phi(p) |dV_X(p)| = \int_\mathbb{R} \left( \int_{\{f=t\}} \frac{\phi(p)}{|\nabla f(p)|} |dV_{f^{-1}(t)}(p)| \right) dt. \quad (1.8)
\]

In particular, by setting \( \phi = 1 \) we deduce

\[
\text{vol}(X) = \int_\mathbb{R} \left( \int_{\{f=t\}} \frac{1}{|\nabla f(p)|} |dV_{f^{-1}(t)}(p)| \right) dt. \quad (1.9)
\]

**Example 1.6.** We want to show how to use (1.9) to compute \( \sigma_n \), the “area” of the unit sphere

\[
S^n = \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=0}^n x_j^2 = 1 \right\}.
\]

Let \( S^n_\ast \) denote the unit sphere with the poles \( x_0 = \pm 1 \) removed. Then \( \sigma_n = \text{vol}(S^n_\ast) \).
Consider \( f : S^n_\ast \to \mathbb{R}, f(x_0, \ldots, x_n) = x_0 \). This function has no critical points on \( S^n_\ast \). Let \( p \in S^n_\ast \) such that \( f(p) = x_0(p) = t \). Denote by \( \varphi \) the angle between the radius \( Op \) and the \( x_0 \)-axis. Note that

\[
\cos \varphi = x_0 = t.
\]

The gradient of \( f \) is the projection of \( \partial x_0 \) on the tangent plane \( T_pS^n \). We deduce that

\[
|\nabla f(p)| = |\partial x_0| \sin \varphi = (1 - t^2)^{1/2}.
\]

The level set \( \{ f = t \} \) is an \((n-1)\)-dimensional sphere of radius \((1 - t^2)^{1/2}\) and we deduce

\[
\int_{\{ f = t \}} \frac{1}{|\nabla f(p)|} |dV_{f^{-1}(t)}(p)| = (1 - t^2)^{-1/2} \text{vol} (f = t) = \sigma_{n-1}(1 - t^2)^{\frac{n-2}{2}}.
\]

Hence

\[
\sigma_n = \sigma_{n-1} \int_{-1}^{1} (1 - t^2)^{\frac{n-2}{2}} dt = 2\sigma_{n-1} \int_{0}^{1} (1 - t^2)^{\frac{n-2}{2}} dt
\]

\((t = \sqrt{s})\)

\[= \sigma_{n-1} \int_{0}^{1} (1 - s)^{\frac{n}{2}-1} s^{\frac{1}{2}-1} ds =: B\left(\frac{n}{2}, \frac{1}{2}\right).\]

The integral \( B(p, q) \) was computed by Euler who showed that

\[
B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.
\]

Hence

\[
\frac{\sigma_n}{\sigma_{n-1}} = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.
\]

Using the equalities \( \sigma_0 = 2 \) and \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \) we deduce

\[
\sigma_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.
\]

We can obtain easily \( \omega_n \), the volume of the unit \( n \)-dimensional ball,

\[
\omega_n = \frac{1}{n} \sigma_{n-1} = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2} \Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}.
\]

(1.10)

\[\square\]

2. The Hausdorff Measure

Suppose \((X,d)\) is a separable metric space. Fix a nonnegative real number \( r \). For any positive number \( \delta \) and any set \( S \subset X \) we set

\[
H^r_\delta(S) := \frac{\omega^r}{2r} \inf \left\{ \sum_{j \geq 1} (\text{diam } B_j)^r ; \ S \subset \bigcup_{j \geq 1} B_j, \ \text{diam } B_j < \delta \right\}.
\]

Note that if

\[0 < \delta_0 < \delta_1 \Rightarrow H^r_{\delta_0}(S) \geq H^r_{\delta_1}(S)\]

Thus the limit

\[
\lim_{\delta \to 0} H^r_\delta(S)
\]
exists and we denote it by $H^r$. The correspondence $S \mapsto H^r(S)$ is an outer measure satisfying the Caratheodory condition, [6, Chap.12]
\[
\text{dist } (S_1, S_2) > 0 \Rightarrow H^r(S_1 \cup S_2) = H^r(S_1) + H^r(S_2).
\]
This implies, [6, Chap. 5], that any Borel set $B$ is measurable with respect to $H^r_+$, i.e.,
\[
H^r_+(Y) = H^r_+(Y \cap B) + H^r_+(Y \setminus B), \forall Y \subset X.
\]
We denote $\sigma^r_+(X)$ the set of $H^r_+$-measurable subset of $X$ and by $H^r_-$ the restriction of $H^r_+$ to $\sigma^r_+(X)$. The measure $H^r_-$ is called the $r$-the Hausdorff measure.

**Example 2.1.** (a) If $M$ is a $C^1$-manifold of dimension $m$ equipped with a $C^0$-Riemann metric $g$ that induces a metric space structure on $M$, then for any Borel set $B \subset M$ we have
\[
H^m_M(B) = \text{vol}_g(B).
\]
In particular, $H^m_M$ coincides with the measure induced by the volume density determined by $g$.
(b) If $M$ is a $C^1$-submanifold of dimension $k$ of $C^1$ Riemann manifold $X$ of dimension $n$, then
\[
H^k_X(M) = \text{vol}_M(M),
\]
where $\text{vol}_M(M)$ denotes the volume of $M$ with respect to the Riemann metric induced by the Riemann metric on $X$.
(c) If $X, Y$ are locally compact metric spaces, $F : X \to Y$ is a Lipschitz map with Lipschitz constant $\leq L$, and $B \subset X$ is a Borel set, then $F(B)$ is $H^r_Y$-measurable and
\[
H^r_Y(F(B)) \leq L^r H^r_X(B).
\]
For proofs of the above statements (a), (b), (c) we refer to [6, Chap 12].

We have the following density result concerning Hausdorff measurable functions. For a proof we refer to [3, §4.3] or [5, §3].

**Theorem 2.2.** Suppose that $X$ a separable metric space $S \subset X$ is a $H^m$-measurable set such that $H^m(S) < \infty$. Then for for $H^m$-almost any $x \in X \setminus S$ we have
\[
\limsup_{r \downarrow 0} \frac{H^m(B_r(x) \cap S)}{\omega_m r^m} = 0.
\]

**Corollary 2.3.** Suppose $X$ is an $n$-dimensional Riemann manifold and $S \subset X$ is a $C^1$ submanifold of dimension $m$. There exists a subset $S^* \subset S$ such that the following hold.

- $H^m(S \setminus S^*) = 0$.
- For any $x \in S^*$ we have
\[
\lim_{r \downarrow 0} \frac{H^m(S \cap B_r(x))}{H^m(X \cap B_r(x))} = 1.
\]

**Proof.** We have
\[
H^m(S \cap B_r(x)) = H^m(X \cap B_r(x)) - H^m(S^c \cap B_r(x)).
\]
From Theorem 2.2 we deduce that there exits a subset \( S^* \subset S \) such that \( \mathcal{H}^m(S \setminus S^*) = 0 \) and for any \( x \in S^* \) we have
\[
\Theta^m(S^*, x) = 0, \quad \text{i.e.} \quad \limsup_{r \searrow 0} \frac{\mathcal{H}^m(S^* \cap B_r(x))}{\omega_m r^m} = 0.
\]
We deduce
\[
\liminf_{r \searrow 0} \frac{\mathcal{H}^m(S \cap B_r(x))}{\omega_m r^m} = \liminf_{r \searrow 0} \frac{\mathcal{H}^m(X \cap B_r(x))}{\omega_m r^m} - \limsup_{r \searrow 0} \frac{\mathcal{H}^m(S^* \cap B_r(x))}{\omega_m r^m} = \liminf_{r \searrow 0} \frac{\mathcal{H}^m(X \cap B_r(x))}{\omega_m r^m}.
\]
The desired conclusion follows by observing that
\[
\lim_{r \searrow 0} \frac{\omega_m r^m}{\mathcal{H}^m(X \cap B_r(x))} = 1 \geq \liminf_{r \searrow 0} \frac{\mathcal{H}^m(S \cap B_r(x))}{\omega_m r^m}.
\]
The result is now obvious.

**Theorem 2.4** (Eilenberg inequality). Suppose \((X, d_X)\) is a separable metric space and \( Y \) is a \( C^1 \)-manifold of dimension \( k \) equipped with a \( C^0 \)-Riemann metric \( g \). Denote by \( d_Y : Y \times Y \to \mathbb{R} \) the metric on \( Y \) induced by \( g \). Let \( F : X \to Y \) be a map satisfying the Lipschitz condition
\[
d_Y(F(x_1), F(x_2)) \leq Ld_X(x_1, x_2), \quad \forall x_1, x_2 \in X.
\]
Then for any \( m \geq k \) there exists a constant\(^1\) \( C(m, k) > 0 \) such that for any Borel set \( A \subset X \) we have
\[
\int_Y \mathcal{H}^{m-k}_X(A \cap F^{-1}(y)) d\mathcal{H}^k_Y(y) \leq C(m, k)L^k \mathcal{H}^m(A),
\]
where \( \int^* \) denotes the upper Lebesgue integral.

For a proof of this inequality we refer to [1, §13.3] or [3, §5.2.1]. The strategy behind the proof is identical to the strategy behind the proof of Lemma 2.7 described a bit later. As explained in [3, §5.2.1], this inequality implies that the following technical result.

**Corollary 2.5.** Let \( F : X \to Y \) be as in Theorem 2.4. Then for any \( m \geq k \) and any Borel subset \( A \subset X \) the map
\[
Y \ni y \mapsto \mathcal{H}^{m-k}_X(A \cap F^{-1}(y)) \in [0, \infty]
\]
is \( \mathcal{H}^k_Y \)-measurable.

**Theorem 2.6** (The co-area formula: version 2). Suppose \( X \) and \( Y \) are connected, Riemann \( C^1 \)-manifolds of dimensions \( n + k \) and respectively \( k, n \geq 0 \). If \( F : X \to Y \) is a \( C^1 \)-map satisfying the Lipschitz condition
\[
d_Y(F(x_1), F(x_2)) \leq Ld_X(x_1, x_2), \quad \forall x_1, x_2 \in X,
\]
then, for any \( \mathcal{H}^{n+k}_X \)-measurable subset \( A \subset X \) we have
\[
\int_A J_F(x)d\mathcal{H}^{n+k}_X(x) = \int_Y \mathcal{H}^n_M(A \cap F^{-1}(y)) d\mathcal{H}^k_Y(y) \tag{2.1}
\]
\[^1\]We can choose \( C(m, k) = \frac{\omega_{m-k} \omega_k}{\omega_m} \)
Proof. We follow closely the strategy in [1, §13.4]. We prove (2.1) in several several gradually more general cases.

**Step 1.** We prove that $I(A) = J(A)$ if $A$ is compact and $F$ has no critical points on $A$. Choose a small open neighborhood $O$ of $A$ in $M$ such that $F$ has no critical points in $O$. The equality $I(A) = J(A)$ follows from Theorem 1.3 applied to the map $F: O \to Y$ and the function $\varphi = 1_A$, the indicator function of $A$.

**Step 2.** We prove that $I(A) = J(A)$ if $F$ has no critical points on $A$. We choose a family of compact sets $C_\varepsilon \subset A$, $\varepsilon > 0$ such that

$$\mathfrak{H}_{X}^{n+k}(A \setminus C_\varepsilon) \leq \varepsilon, \quad C_\varepsilon \subset C_\varepsilon', \quad \forall \varepsilon \geq \varepsilon' > 0.$$ 

From the monotone convergence theorem we deduce that

$$\lim_{\varepsilon \to 0} I(C_\varepsilon) = I(A). \quad (2.2)$$

From Step 1 we deduce

$$J(A) = J(C_\varepsilon) + J(A \setminus C_\varepsilon) = I(C_\varepsilon) + J(A \setminus C_\varepsilon). \quad (2.3)$$

From the Eilenberg inequality we deduce

$$J(A \setminus C_\varepsilon) \leq C(m,k)L^k\varepsilon;$$

so that

$$\lim_{\varepsilon \to 0} J(A \setminus C_\varepsilon) = 0. \quad (2.4)$$

We obtain (2.1) by letting $\varepsilon \to 0$ in (2.3) and then invoking (2.2) and (2.4).

**Step 3.** We prove that $I(A) = J(A)$ for any $A$. Choose a compact set $C_\varepsilon \subset A$ such that

$$\mathfrak{H}_{X}^{n+k}(A \setminus C_\varepsilon) < \varepsilon.$$ 

Define

$$C^0_\varepsilon := \{ x \in C_\varepsilon; \quad J_F(x) = 0 \}.$$ 

Then

$$J(A) - I(A) = J(A \setminus C_\varepsilon) - I(A \setminus C_\varepsilon) + J(C_\varepsilon \setminus C^0_\varepsilon) - I(C_\varepsilon \setminus C^0_\varepsilon) + J(C^0_\varepsilon) - I(C^0_\varepsilon)$$

$$= J(A \setminus C_\varepsilon) - I(A \setminus C_\varepsilon) + J(C_\varepsilon \setminus C^0_\varepsilon) - I(C_\varepsilon \setminus C^0_\varepsilon) + J(C^0_\varepsilon).$$

From Step 2 we know that $J(C_\varepsilon \setminus C^0_\varepsilon) - I(C_\varepsilon \setminus C^0_\varepsilon) = 0$ and the proof of Step 2 shows that

$$\lim_{\varepsilon \to 0} \left( J(A \setminus C_\varepsilon) - I(A \setminus C_\varepsilon) \right) = 0.$$

Hence

$$J(A) - I(A) = \lim_{\varepsilon \to 0} J(C^0_\varepsilon).$$

The equality (2.1) now follows from the following Sard-like result.

**Lemma 2.7.** If $C$ is a compact subset of $X$ such that $J_F(x) = 0$, for any $x \in C$ then

$$\int_Y \mathfrak{H}_X^n(C \cap F^{-1}(y)) \, d\mathfrak{H}_Y^n(y) = 0.$$

\qed
Proof of Lemma 2.7. Let us first observe that for any $p \in C$ and any $\varepsilon > 0$ there exists $r_\varepsilon = r_\varepsilon(p)$ such that for any $0 < r < r_\varepsilon(x)$ we have

$$\mathcal{H}^k(F(B(p, r))) \leq \varepsilon L^{k-1} r^k.$$  \hfill (2.5)

Indeed, we have $\text{rank} \, D_p F \leq k - 1$. The definition of the differential of $F$ at $x$ implies that, given a choice of coordinates $x$ near $p$ such that $x(p) = 0$ we have

$$F(x) = F(0) + A_p x + o(|x|), \quad A_p := D_p F.$$ 

Hence, for any $\varepsilon > 0$, the set $F(B(p, r))$ is contained in a $k$-dimensional polydisk of the form $\mathbb{D}^{k-1}(F(p), L r) \times [-\varepsilon r, \varepsilon r]$ if $r$ is sufficiently small, $r < r_\varepsilon(p)$. Above, $\mathbb{D}^{k-1}(y, R)$ indicates a $(k - 1)$-disk of center $y$ and radius $R$. Since $C$ is compact we can assume that

$$r_\varepsilon := \inf_{p \in C} r_\varepsilon(p) > 0.$$ 

We deduce that

$$\mathcal{H}^k(F(S \cap C)) \leq \varepsilon L^{k-1} \text{diam}(S)^k, \quad \forall S \subset X, \quad \text{diam} \, S < \frac{1}{2} r_\varepsilon.$$ \hfill (2.6)

For any $s > 0$ we can find a countable cover of $C$ in $X$ by measurable sets $(X^s_i)_{i \geq 1}$ such that

$$\text{diam}(X^s_i) < \frac{1}{s} \quad \text{and} \quad \mathcal{H}^{n+k}(C) \geq \frac{\omega_{n+k}}{2^{n+k}} \sum_{i \geq 1} (\text{diam} \, X^s_i)^{n+k} - \frac{1}{s}.$$ \hfill (2.7)

By definition

$$\mathcal{H}^n(C \cap f^{-1}(y)) \leq \frac{\omega_{n+k}}{2^{n+k}} \liminf_{s \to \infty} \sum_{i \geq 1} (\text{diam} \, X^s_i \cap f^{-1}(y))^n.$$ 

For any set $E \subset X$ we denote by $\varphi_E$ the characteristic function of the closure of $F(E)$. We can then rewrite the above equality as

$$\mathcal{H}^n(C \cap f^{-1}(y)) \leq \frac{\omega_{n+k}}{2^{n+k}} \liminf_{s \to \infty} \sum_{i \geq 1} (\text{diam} \, X^s_i)^n \varphi_{X^s_i}(y).$$

The Fatou lemma then implies

$$\int_Y \mathcal{H}^n(C \cap f^{-1}(y)) \, d\mathcal{H}^k_Y \leq \frac{\omega_{n+k}}{2^{n+k}} \liminf_{s \to \infty} \sum_{i \geq 1} (\text{diam} \, X^s_i)^n \int_Y \varphi_{X^s_i}(y) d\mathcal{H}^k_Y.$$

Fix $\varepsilon > 0$. We deduce from (2.6) that for $s$ sufficiently large, $s > s_\varepsilon$ we have

$$\int_Y \varphi_{X^s_i}(y) d\mathcal{H}^k_Y \leq \varepsilon L^{k-1} (\text{diam} \, X^s_i)^k.$$ 

Hence

$$\int_Y \mathcal{H}^n(C \cap f^{-1}(y)) \, d\mathcal{H}^k_Y \leq \varepsilon L^{k-1} \frac{\omega_{n+k}}{2^{n+k}} \liminf_{s \to \infty} \sum_{i \geq 1} (\text{diam} \, X^s_i)^{n+k}$$

$$\leq \varepsilon L^{k-1} \left( \mathcal{H}^{n+k}(C) + \frac{1}{s_\varepsilon} \right).$$ \hfill (2.7)

Now let $\varepsilon \to 0.$ \hfill \(\square\)
Remark 2.8. The proof of the Eilenberg inequality follows an identical strategy with the inequality (2.5) replaced by the inequality
\[ \mathcal{H}^k(F(S)) \leq C(m,k)(\dim S)^k \]
for any Borel set \( S \subseteq X \) with sufficiently small diameter. □

Corollary 2.9. Let \( F : X \to Y \) be as in Theorem 2.6. Then for any nonnegative measurable function \( \varphi : X \to \mathbb{R} \) we have
\[
\int_X \varphi(x) J_F(x) d\mathcal{H}^{n+k}_X(x) = \int_Y \left( \int_{F^{-1}(y)} \varphi(x) d\mathcal{H}^n_X(x) \right) d\mathcal{H}^k_Y(y).
\] (2.8)

Proof. By Theorem 2.6 the equality (2.8) is true when \( \varphi \) is the characteristic function of a measurable subset of \( X \). By linearity, (2.8) is true for linear combinations of such functions. We now observe that for any measurable nonnegative function \( \varphi \) we can find a sequence of simple functions \( (\varphi_\nu)_{\nu \geq 1} \) that converges increasingly and almost everywhere to \( \varphi \). □

Corollary 2.10. Let \( F : X \to Y \) be as in Theorem 2.6. Then for any compactly supported continuous function \( \varphi : X \to \mathbb{R} \) we have
\[
\int_X \varphi(x) J_F(x) d\mathcal{H}^{n+k}_X(x) = \int_Y \left( \int_{F^{-1}(y)} \varphi(x) d\mathcal{H}^n_X(x) \right) d\mathcal{H}^k_Y(y),
\] (2.9)
and both sides are finite. □

Corollary 2.11. Suppose that \( F : X \to Y \) is as in Theorem 2.6 and additionally, \( X \) and \( Y \) are oriented. Denote by \( Y^* \) the set of regular values of \( Y \). When \( y \in Y^* \) we orient the fiber \( F^{-1}(y) \) using the fiber first convention

orientation \((X) = \text{orientation} \ F^{-1}(y) \land \text{orientation} \ (Y)\).

Then for any compactly supported \( C^1 \)-form \( \eta \in \Omega^n(X) \) the map
\[
Y^* \ni y \mapsto \int_{F^{-1}(y)} \eta \in \mathbb{R}
\]
is measurable and
\[
\int_Y \left( \int_{F^{-1}(y)} \eta \right) dV_Y(y) = \int_X (\eta \land F^* dV_Y)(x).
\] (2.10)

More generally, if \( \alpha \in \Omega^{k+n}(X) \) is a compactly supported \( C^1 \)-form then
\[
\int_X \alpha = \int_Y \left( \int_{F^{-1}(y)} \frac{\alpha}{F^* dV_Y} \right) dV_Y(y),
\] (2.11)
where, along a regular fiber \( F^{-1}(y) \), the Gelfand-Leray residue \( \frac{\alpha}{F^* dV_Y} \) is defined by the equality
\[
\frac{\alpha}{F^* dV_Y} = \eta|_{F^{-1}(y)}, \ \forall \eta \text{ such that } \eta \land F^* dV_Y = \alpha.
\]
Proof. We prove (2.10) first. Observe that there exists a unique, compactly supported continuous function $\varphi : X \to \mathbb{R}$ such that

$$\eta \wedge F^*dV_Y = \varphi dV_X.$$ 

Corollary 2.10 implies that

$$\int_X \eta \wedge F^*dV_Y = \int_X \varphi dV_X = \int_Y \left( \int_{F^{-1}(y)} \frac{\varphi}{J_F} dV_{F^{-1}(y)} \right) dV_Y.$$ 

To complete the proof we need to show that if $y_0$ is a regular value of $F$, then

$$\frac{\varphi}{J_F} \mid_{F^{-1}(y_0)} dV_{F^{-1}(y_0)} = \eta \mid_{F^{-1}(y_0)}.$$ 

We rely on the same arguments used in the proof of Theorem 1.3. Fix $x_0 \in F^{-1}(y_0)$. Fe can find local coordinates $y^1, \ldots, y^k$ near $y_0$ in $Y$ and coordinates $(x^1, \ldots, x^k, x^{k+1}, \ldots, x^{k+n})$ near $x_0$ in $X$ such that in these coordinates $F$ is the linear projection $y^j = x^j, \ j = 1, \ldots, k$.

We write

$$dx' = dx^{k+1} \wedge \cdots \wedge dx^{k+n}, \ dx'' = dx^1 \wedge \cdots \wedge dx^k, \ dy = dy^1 \wedge \cdots \wedge dy^k.$$ 

We assume that the coordinates are ordered so that

$$dV_X = \rho_X dx' \wedge dx'', \ dV_Y = \rho_Y dy, \ dV_{F^{-1}(y_0)} = \rho_F dx'.$$

As in the proof of Theorem 1.3 we have

$$J_F = \frac{\rho_Y}{\rho_F} \rho_X.$$ 

In the coordinates $(x', x'')$ we can write

$$\eta = \eta' dx' + \text{other terms}$$

where $\eta' = \eta'(x', x'')$ is a locally defined $C^1$-function. Note that

$$\eta \mid_{F^{-1}(y_0)} = \eta' dx'.$$

We deduce that

$$\eta \wedge F^*dV_Y = \eta \wedge (\rho_Y dx'') = \eta' \rho_Y dx' \wedge dx'' = \frac{\eta' \rho_Y}{\rho_X} dV_X.$$ 

Hence, in the coordinates $(x', x'')$ we have

$$\varphi = \frac{\eta' \rho_Y}{\rho_X}.$$ 

We conclude that

$$\frac{\varphi}{J_F} dV_{F^{-1}(y_0)} = \frac{\varphi}{J_F} \rho_F dx' = \frac{\eta' \rho_Y \rho_F}{\rho_X J_F} dx' = \eta' dx' = \eta \mid_{F^{-1}(y_0)}.$$ 

Observe that (2.11) follows from (2.10). With $y_0$ a regular value of $F$ as before and $x_0 \in F^{-1}(y_0)$, we write $\alpha$ locally near $x_0$ as a product

$$\alpha = \eta \wedge F^*dV_Y.$$ 

The form $\eta$ is not unique, but its restriction to $F^{-1}(y_0)$ is. Then, by definition,

$$\eta \mid_{F^{-1}(y_0)} = \frac{\alpha}{F^*dV_Y}.$$
3. Lipschitz maps

To formulate our last and most general version of the co-area formula we need to recall a few facts about Lipschitz maps between locally Euclidean sets.

**Theorem 3.1** (Rademacher). Suppose $U_k \subset \mathbb{R}^{n_k}, k = 0, 1$ are open sets and $F : U_0 \to U_1$ is a Lipschitz map. Then the map $F$ is almost everywhere differentiable and the differential is a measurable map $U_0 \to \text{Hom}(\mathbb{R}^{n_0}, \mathbb{R}^{n_1})$. Moreover, for any $\varepsilon > 0$ there exists a $C^1$ map $F_\varepsilon : U_0 \to \mathbb{R}^{n_1}$ such that

$$\text{vol}\left(\{x \in U_0; \ F(x) \neq F_\varepsilon(x)\}\right) + \text{vol}\left(\{x \in U_0; \ DF(x) \neq DF_\varepsilon(x)\}\right) < \varepsilon$$

For a proof we refer to [3, §5.1].

**Theorem 3.2** (Extension theorem). Suppose that $S \subset \mathbb{R}^n$ is a closed subset and $F : S \to \mathbb{R}$ is a Lipschitz function. Then $f$ admits an extension to a Lipschitz function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ that has the same Lipschitz constant as $f$. □

For a proof we refer to [3, Thm. 5.1.12].

**Theorem 3.3** (The co-area formula: version 3). Suppose $X$ and $Y$ are $C^1$ Riemann manifolds of dimensions $n+k$ and respectively $k, n \geq 0$. If $F : M \to N$ is a map satisfying the Lipschitz condition

$$d_Y(F(x_1), F(x_2)) \leq Ld_X(x_1, x_2), \ \forall x_1, x_2 \in X,$$

then, for any $\mathcal{H}^{n+k}_X$-measurable subset $A \subset X$ we have

$$\int_A J_F(x) d\mathcal{H}^{n+k}_X(x) = \int_Y \mathcal{H}^n_M(\ A \cap F^{-1}(y)) d\mathcal{H}^k_Y(y). \ \ (3.1)$$

**Proof.** Clearly, it suffices to prove the theorem for sets $A$ with the following property: $A$ is contained in a coordinate neighborhood $U_0 \subset X$ and $F(U_0)$ is contained in a coordinate neighborhood $U_1 \subset Y$ such that $U_0$ is bi-Lipschitz homeomorphic to a bounded open subset in $\mathbb{R}^{n+k}$ and $U_1$ is bi-Lipschitz homeomorphic to a bounded open set in $\mathbb{R}^k$. For any $\varepsilon > 0$ we can find a compact subset $C_\varepsilon \subset U_0$ and a $C^1$-map $F_\varepsilon : U_0 \to \mathbb{R}^k$ such that

$$\mathcal{H}^{n+k}_X(U_0 \setminus C_\varepsilon) < \varepsilon, \ F|_{C_\varepsilon} = F_\varepsilon|_{C_\varepsilon}, \ J_F|_{C_\varepsilon} = J_{F_\varepsilon}|_{C_\varepsilon}.$$ 

Then

$$I(A) - J(A) = I(A \cap C_\varepsilon) - J(A \cap C_\varepsilon) + I(A \setminus C_\varepsilon) - J(A \setminus C_\varepsilon).$$

The monotone convergence theorem implies that

$$\lim_{\varepsilon \searrow 0} I(A \setminus C_\varepsilon) = 0$$

while the Eilenberg inequality implies that

$$\lim_{\varepsilon \searrow 0} J(A \setminus C_\varepsilon) = 0.$$
On the other hand, there exists an open neighborhood \( V_0 \) of \( C_\varepsilon \) in \( U_0 \) such that \( F_\varepsilon(V_0) \subset U_1 \). Applying Theorem 2.6 to the \( C^1 \)-map \( F_\varepsilon : V_0 \to U_1 \) we deduce that

\[
I(C_\varepsilon) = \int_{C_\varepsilon} J_F(x) d\mathcal{H}_{X}^{n+k}(x) = \int_{C_\varepsilon} J_{F_\varepsilon}(x) d\mathcal{H}_{X}^{n+k}(x)
\]

\[
= \int_{Y} \mathcal{H}_M^n(C_\varepsilon \cap F_{\varepsilon}^{-1}(y)) d\mathcal{H}_{Y}^{k}(y) = \int_{Y} \mathcal{H}_M^n(C_\varepsilon \cap F^{-1}(y)) d\mathcal{H}_{Y}^{k}(y) = J(C_\varepsilon).
\]

\( \square \)

**Corollary 3.4** (Area formula). Let \( X, Y \) be two \( n \)-dimensional \( C^1 \)-manifolds equipped with \( C^0 \)-Riemann metrics and \( F : X \to Y \) a Lipschitz map. Then

\[
\int_{Y} \#F^{-1}(y) d\mathcal{H}^n(y) = \int_{X} J_F(x) d\mathcal{H}^n(x).
\]

\( \square \)

### 4. Rectifiable sets

A set \( S \subset \mathbb{R}^n \) is said to be *countably \( m \)-rectifiable* if it is \( \mathcal{H}^m \)-measurable and

\[
S \subset S_0 \cup \left( \bigcup_{j \geq 1} F_j(\mathbb{R}^m) \right),
\]

where

- \( \mathcal{H}^m(S_0) = 0 \);
- the functions \( F_j : \mathbb{R}^m \to \mathbb{R}^n \) are Lipshitz, \( \forall j \geq 1 \).

We have the following result, [3, §5.4].

**Proposition 4.1.** Suppose that \( S \subset \mathbb{R}^n \) is \( \mathcal{H}^m \)-measurable and countably \( m \)-rectifiable. Then

\[
S = \bigcup_{j=0}^{\infty} S_j,
\]

where

- \( \mathcal{H}^m(X_0) = 0 \);
- \( S_i \cap S_j = \emptyset \) if \( i \neq j \);
- for \( j \geq 1 \) there exists an \( m \)-dimensional \( C^1 \)-submanifold \( X_j \subset \mathbb{R}^n \) such that \( S_j \subset X_j \).

**Definition 4.2.** If \( S \) is a \( \mathcal{H}^m \)-measurable subset of \( \mathbb{R}^n \), then we say that an \( m \)-dimensional vector subspace \( W \subset \mathbb{R}^n \) is the approximate tangent space for \( S \) at \( x \in \mathbb{R}^n \) if

\[
\lim_{r \to 0} \frac{1}{r} \int_{r^{-1}(S-x)} f(y) d\mathcal{H}^m(y) = \int_{W} f(y) d\mathcal{H}^m(y), \quad \forall f \in C^0_{\text{cpt}}(\mathbb{R}^n).
\]

**Proposition 4.3.** Suppose that \( S \subset \mathbb{R}^n \) is a countably \( m \)-rectifiable set such that \( \mathcal{H}^m(S \cap K) < \infty \) for any compact subset \( K \subset \mathbb{R}^m \). Then there exists a subset \( S_{\text{sing}} \subset S \) such that

- \( \mathcal{H}^m(S_{\text{sing}}) = 0 \) and
- for any \( x \in S \setminus S_{\text{sing}} \) there exist an approximate tangent space to \( S \) at \( x \).
Proof. We write \( S \) as in Proposition 4.1

\[
S = \bigcup_{j=0}^{\infty} S_j
\]

where \( S_j \) is contained in a \( C^1, m \)-dimensional submanifold \( X_j \subset \mathbb{R}^n, S_i \cap S_j = \emptyset, \forall i \neq j \), \( \mathcal{H}^m(S_0) = 0 \). For \( j > 0 \) we denote by \( S_j^* \) the set of points \( x \in S_j \) such that

\[
\lim_{r \to 0} \frac{\mathcal{H}^m((S - S_j) \cap B_r(x))}{r^m} = \lim_{r \to 0} \frac{\mathcal{H}^m((X_j - S_j) \cap B_r(x))}{r^m} = 0.
\]

By Theorem 2.2 we have \( \mathcal{H}^m(S_j \setminus S_j^*) = 0 \). We will show that \( S \) admits an approximate tangent space at any point \( x \in S_j^* \). Indeed, suppose \( f \in C_0^0(\mathbb{R}^n) \). For simplicity assume that \( \text{supp} \, f \subset B_1(0) \), and \( f \geq 0 \). Then using the change in variables \( y = \frac{1}{r}(z - x) \)

\[
\int_{\frac{1}{r}(S-x)} f(y) d\mathcal{H}^m(y) = \frac{1}{r^m} \int_S f \left( \frac{1}{r}(z - x) \right) d\mathcal{H}^m(z)
\]

Now observe that

\[
\frac{1}{r^m} \left| \int_{B_r(x) \cap S} f \left( \frac{1}{r}(z - x) \right) d\mathcal{H}^m(z) - \int_{B_r(x) \cap S_j} f \left( \frac{1}{r}(z - x) \right) d\mathcal{H}^m(z) \right| 
\leq \sup_f \frac{\mathcal{H}^m(B_r(x) \cap (S_j \setminus S))}{r^m} \to 0,
\]

and

\[
\frac{1}{r^m} \left| \int_{B_r(x) \cap S_j} f \left( \frac{1}{r}(z - x) \right) d\mathcal{H}^m(z) - \int_{B_r(x) \cap X_j} f \left( \frac{1}{r}(z - x) \right) d\mathcal{H}^m(z) \right| 
\leq \sup_f \frac{\mathcal{H}^m(B_r(x) \cap (X_j \setminus S_j))}{r^m} \to 0.
\]

Hence

\[
\lim_{r \to 0} \left( \int_{\frac{1}{r}(S-x)} f(y) d\mathcal{H}^m(y) - \int_{\frac{1}{r}(X_j-x)} f(y) d\mathcal{H}^m(y) \right) = 0.
\]

Suppose \( X \subset \mathbb{R}^n \) is a \( C^1 \) \( m \)-dimensional manifold of finite (induced) volume and \( S \subset X \) is a \( \mathcal{H}^m \)-measurable subset. We set \( S^c := X \setminus S \).

Suppose that \( S \subset \mathbb{R}^N \) is a countably \((n+k)\)-rectifiable subset of \( \mathbb{R}^N \). We can then express \( S \) as a disjoint union

\[
S = \bigcup_{j=0}^{\infty} S_j
\]

where \( \mathcal{H}^{n+k}(S_0) = 0 \), and \( S_j \) is contained in a \( C^1 \)-submanifold \( X_j \subset \mathbb{R}^N, \dim X_j = n + k, \forall j \geq 1 \).

A Lipschitz map on \( f : S \to \mathbb{R}^M \) admits a Lipschitz extension to the closure of \( S \) and thus a Lipschitz extension to a map \( F : \mathbb{R}^N \to \mathbb{R}^M \). The restriction of \( F \) to each \( X_j \) is \( \mathcal{H}^{n+k} \) a.e. differentiable and for \( \mathcal{H}^{n+k} \) a.e. point \( p \in S_j \) we have a differential
\[ D_p F : T_p S \rightarrow \mathbb{R}^n. \]

One can show that if \( F(S) \) is contained in a countably \( k \)-rectifiable subset \( Z \subset \mathbb{R}^n \), then for \( \mathcal{H}^{n+k} \)-a.e. point \( p \in S \) the set \( Z \) admits a tangent space at \( q = F(p) \) and moreover

\[ D_p F(T_p S) \subset T_q Z. \]

We denote by \( J_F(p) \) the Jacobian of this map. We can now state the final version of the coarea formula.

**Theorem 4.4** (The co-area formula: final version). Suppose that \( S \subset \mathbb{R}^N \) is \((n+k)\)-rectifiable, \( Z \subset \mathbb{R}^M \) is \( k \)-rectifiable and \( F : S \rightarrow Z \) is a Lipschitz map. Then, for any \( \mathcal{H}^{n+k} \)-measurable subset \( A \subset S \) we have\(^2\)

\[
\int_A J_F(p) \, d\mathcal{H}^{n+k}(p) = \int_Z \mathcal{H}^n \left( F^{-1}(z) \right) \cap A \, d\mathcal{H}^k(z). \tag{4.1}
\]

The proof is obtained by putting together all the facts we have gathered so far. For details we refer to [2, Thm. 3.2.22] or [5, §12]

**References**


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\(^2\)Implicit in the statement of (4.1) is the fact that the various integrands are measurable with respect to the appropriate measures.