THE CROFTON FORMULA

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ABSTRACT. I discuss the classical Crofton formula for curves in the plane.

1. AFFINE LINES IN THE PLANE

Denote by $\mathcal{Graff}_1(\mathbb{R}^2)$ the set of affine lines in the plane. For any $L \in \mathcal{Graff}_1$ denote by $[L] \perp$ the line through the origin perpendicular to $L$. The resulting map $\mathcal{Graff}_1(\mathbb{R}^2) \ni L \mapsto [L] \perp \in \mathbb{RP}^1$ defines a structure of real line bundle $\mathcal{Graff}_1(\mathbb{R}^2) \rightarrow \mathbb{RP}^1$ canonically isomorphic to the tautological line bundle $\mathcal{U}_1 \rightarrow \mathbb{RP}^1$. More precisely, the isomorphism associates to a line $\ell \in \mathbb{R}^1$ and a point $p \in \ell$ the affine line $L$ through $p$ and perpendicular to $\ell$.

We can identify $\mathcal{Graff}_1(\mathbb{R}^2)$ with the Möbius band $\mathbb{R}^2/\sim$, where $(\varphi, t) \sim (\varphi + n\pi, (-1)^n t), \ n \in \mathbb{Z}$.

More precisely, to the equivalence class $[\varphi, t]$ we associate the line $L_{[\varphi, t]} = \{(x, y) \in \mathbb{R}^2; \ t = x \cos \varphi + y \sin \varphi \}$.

We have a metric $g = d\varphi^2 + dt^2$ on $\mathcal{Graff}_1(\mathbb{R})$ with volume density $|dV_g| = |d\varphi \wedge dt|$. This density is invariant with respect to the group of affine isometries of $\mathbb{R}^2$ which acts in the obvious way on $\mathcal{Graff}_1(\mathbb{R})$.

2. THE CROFTON FORMULA

Suppose that $C$ is simple closed $C^2$-curve in $\mathbb{R}^2$ parametrized by arclength $[0, S] \ni s \mapsto (x(s), y(s)) \in \mathbb{R}^2, \ |x'(s)|^2 + |y'(s)|^2 = 1, \ S := \text{length}(C)$.

For any affine line $L \in \mathcal{Graff}_1(\mathbb{R}^2)$ we denote by $|C \cap L| \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ the cardinality of the intersection $L \cap C$.

**Theorem 2.1** (Crofton). The function $\mathcal{Graff}_1(\mathbb{R}^2) \ni L \mapsto |L \cap C| \in \mathbb{R}$ is measurable and

$$\int_{\mathcal{Graff}_1(\mathbb{R}^2)} |L \cap C| \ |dV_g(L)| = 2\text{length}(C).$$

Proof. Consider the incidence set
\[ \mathcal{J}(C) = \{ (\mathbf{p}, L) \in C \times \text{Graff}_1(\mathbb{R}^2); \quad \mathbf{p} \in L \} = \{ (s, \varphi, t) \in [0, S] \times [0, \pi) \times \mathbb{R}; \quad t = x(s) \cos \varphi + y(s) \sin \varphi \}. \]
We can regard \( \mathcal{J}(C) \) as the graph of the map
\[ [0, S] \times [0, \pi) \to \mathbb{R}, \quad (s, \varphi) \mapsto t(s, \varphi) = x(s) \cos \varphi + y(s) \sin \varphi. \]
Viewed as a submanifold of \( \mathbb{R}^3 \) is equipped with the induced metric
\[ \hat{g} = dt^2 + ds^2 + d\varphi^2 = (dt(s, \varphi))^2 + ds^2 + d\varphi^2. \]
We use \((s, \varphi)\) as coordinates on \( \mathcal{J}(C) \) so that the vector fields \( \partial_s, \partial_\varphi \) can be identified with the 3-dimensional vector fields
\[ \partial_s = (1, 0, \partial_s t), \quad \partial_\varphi = (0, 1, \partial_\varphi t). \]
Consider the maps
\[
\begin{array}{ccc}
\alpha & : & \mathcal{J}(C) \\
\beta & : & \text{Graff}_1(\mathbb{R}^2), \\
C & \mapsto & \text{Graff}_1(\mathbb{R}^2),
\end{array}
\]
In coordinates we have
\[ \alpha(s, \varphi) = s, \quad \beta(s, \varphi) = (\varphi, t(s, \varphi)). \tag{2.1} \]
Note that the fiber \( \beta^{-1}(L) \) can be identified with the set \( L \cap C \). The fiber of \( \beta \) over \( \mathbf{p} \in C \) can be identified with \( \mathbb{RP}^1(\mathbf{p}) \subset \text{Graff}_1(\mathbb{R}^2) \), the space of affine lines through \( \mathbf{p} \).
Using the coarea formula [1] we deduce
\[ \int_{\text{Graff}_1(\mathbb{R}^2)} |L \cap C| dV_\hat{g} = \int_{\mathcal{J}(C)} J_\beta |dV_\hat{g}| = \int_C \left( \int_{\mathbb{RP}^1(\mathbf{p})} \frac{J_\beta}{J_\alpha} d\sigma_\mathbf{p} \right) ds, \tag{2.2} \]
where \( J_\alpha, J_\beta \) are the Jacobians of \( \alpha \) and respectively \( \beta \), and \( d\sigma_\mathbf{p} \) denotes the arclength along \( \mathbb{RP}^1(\mathbf{p}) \) with respect to the metric induced by the metric \( \hat{g} \) on \( \mathcal{J}(C) \).
Using [1, Lemma 1.2] we deduce
\[ J_\beta^2 = \frac{G_g(\beta_s \partial_s, \beta_\varphi \partial_\varphi)}{G_{\hat{g}}(\partial_s, \partial_\varphi)}, \quad J_\alpha^2 = \frac{|\alpha_s \partial_s|^2 \cdot |\partial_\varphi|^2_{\hat{g}}}{G_{\hat{g}}(\partial_s, \partial_\varphi)}. \]
Hence
\[ \frac{J_\beta^2}{J_\alpha^2} = \frac{G_g(\beta_s \partial_s, \beta_\varphi \partial_\varphi)}{|\alpha_s \partial_s|^2 \cdot |\partial_\varphi|^2_{\hat{g}}} = \frac{G_g(\beta_s \partial_s, \beta_\varphi \partial_\varphi)}{|\partial_\varphi|^2_{\hat{g}}}. \]
Using (2.1) we deduce
\[ |\partial_\varphi|^2_{\hat{g}} = 1 + |\partial_\varphi t|^2, \quad G_g(\beta_s \partial_s, \beta_\varphi \partial_\varphi) = \det \left[ \begin{array}{cc} |\beta_s \partial_s|^2 & g(\beta_s \partial_s, \beta_\varphi \partial_\varphi) \\ g(\beta_s \partial_s, \beta_\varphi \partial_\varphi) & |\beta_\varphi \partial_\varphi|^2_{\hat{g}} \end{array} \right] = \det \left[ \begin{array}{cc} |\partial_\varphi t|^2 & \partial_\varphi t \cdot \partial_\varphi t \\ \partial_\varphi t \cdot \partial_\varphi t & 1 + |\partial_\varphi t|^2 \end{array} \right] = |\partial_\varphi t|^2. \]
If \( \mathbf{p} = (x(s_0), y(s_0)) \) then the fiber \( \mathbb{RP}^1(\mathbf{p}) \) admits the parametrization
\[ [0, \pi] \ni \varphi \mapsto (s_0, \varphi, t(s_0, \varphi)) \in \mathcal{J}(C) \]
so that
\[ |d\sigma_\mathbf{p}|^2 = (1 + |\partial_\varphi t|^2) |d\varphi|^2. \]
Hence
\[ J_\beta \frac{d\sigma_p}{J_\alpha} = |\partial_s t| |d\varphi|. \]

Hence
\[ \int_{\mathbb{RP}^1(p)} J_\beta \frac{d\sigma_p}{J_\alpha} = \int_0^\pi |x'(s_0)\cos \varphi + y'(s_0)\sin \varphi| d\varphi = \frac{1}{2} \int_0^{2\pi} |x'(s_0)\cos \varphi + y'(s_0)| d\varphi. \]

At this point we want to invoke the following technical result whose proof we postpone.

**Lemma 2.2.** Suppose that \( v_0 \in \mathbb{R}^2 \) is a nonzero vector. Then
\[ \int_{|r|=1} |v_0 \cdot r| |dr| = 4|v_0|. \]

If we use Lemma 2.2 with \( v_0 = (x'(s_0), y'(s_0)) \) we deduce
\[ \int_0^{2\pi} |x'(s_0)\cos \varphi + y'(s_0)\sin \varphi| d\varphi = 4, \quad \int_{\mathbb{RP}^1(p)} J_\beta \frac{d\sigma_p}{J_\alpha} = 2. \]

Using the last equality in (2.2) we conclude
\[ \int_{\text{Graff}_1(\mathbb{R}^2)} |L \cap C| dV_g = 2 \int_C ds = 2\text{length}(C). \]

**Proof of Lemma 2.2** Choose a new orthonormal basis \((e_1, e_2)\) of \( \mathbb{R}^2 \) such that \( v_0 = |v_0| e_1 \). We denote by \((x_1, x_2)\) the resulting orthonormal coordinates. In these coordinates
\[ v_0 = (|v_0|, 0) \]
and if \( r = (\cos \theta, \sin \theta) \) then
\[ |v_0 \cdot r| = |v_0| \cdot |\cos \theta| \]
and thus
\[ \int_{|r|=1} |v_0 \cdot r| |dr| = |v_0| \int_0^{2\pi} |\cos \theta| d\theta = 4|v_0|. \]

**References**


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