Homeomorphisms vs. Diffeomorphisms*

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Introduction

Consider the situation in Figure 1. The radial projection maps the boundary of the square homeomorphically onto the circle. The net effect of this transformation was to smooth out the corners, while leaving the topology unchanged. We can then ask if we can smooth out the corners of any topological manifold, and if so, are there many different ways of doing this. This turns out to be a very subtle problem, and even some of the simplest situations lead to completely unexpected conclusions. I would like to give you a glimpse of this developing line of research. I will begin by rigourously defining what it means to "smooth out the corners", and explain when we should consider two smoothing procedures to be inequivalent. Next I will describe a sample of known facts.

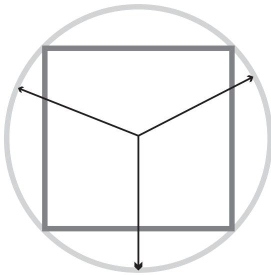

Figure 1: *Smoothing corners*

1 Topological versus smooth manifolds: the fundamental questions

§1.1 Local charts and transition maps. Suppose M is a topological space and m is a point in M. *A local n-dimensional chart near m* is a pair (U, Φ), where U is an open neighborhood of m and $\Phi : U \to \mathbb{R}^n$ is a homeomorphism of U onto an open subset of $\mathbb{R}^n$. For every $u \in U$ we denote by $(x^1(u), \cdots x^n(u))$ the coordinates of the point $\Phi(u) \in \mathbb{R}^n$. The functions

$$U \ni u \mapsto x^i(u) \in \mathbb{R}, \;\; i = 1, \cdots, n$$

are called the *local coordinates* defined by the local chart (U, Φ).

For example, if M is the upper hemisphere

$$M = \Big\{ (x, y, z) \in \mathbb{R}^3; \;\; x^2 + y^2 + z^2 = 1, \;\; z > 0 \Big\},$$

*Notes for the first year graduate student seminar, February 2003, Notre Dame.

$m_0 = (0,0,1) \in M$ is the north pole, and $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ is the orthogonal projection onto the xy-plane then (M, Φ) is a 2-dimensional local chart near m_0. Suppose (U_1, Φ_1) and (U_2, Φ_2)

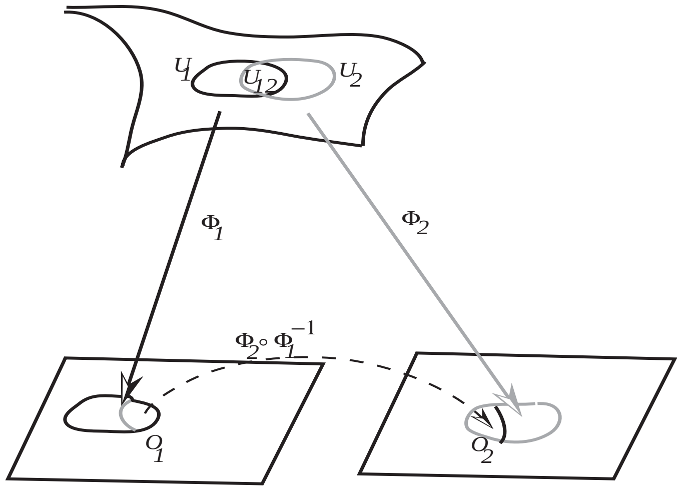

Figure 2: *Transition map.*

are two n-dimensional charts on M near m_1 and respectively m_2 such that

$$U_{12} := U_1 \cap U_2 \neq \emptyset.$$

Φ_1 maps U_{12} homeomorphically to an open set $O_1 \subset \mathbb{R}^n$ while Φ_2 maps U_{12} to another open set $O_2 \subset \mathbb{R}^n$. These two open sets are related by a homeomorphism (see Figure 2)

$$\Phi_{21} := \Phi_2 \circ \Phi_1^{-1}, \quad O_1 \xrightarrow{\Phi_1^{-1}} U_{12} \xrightarrow{\Phi_2} O_2.$$

The homeomorphism Φ_{21} is called the *transition map* between the first coordinate chart to the second coordinate chart. Observe that $\Phi_{12} = \Phi_{21}^{-1}$.

§1.2 Topological and Smooth structures. An *n-dimensional topological manifold* is a Hausdorff space M such that for every $m \in M$ there exists a n-dimensional coordinate chart (U, Φ) near m. An *atlas* on a topological manifold M is a collection of local charts $\big\{(U_\alpha, \Phi_\alpha)\big\}_{\alpha \in A}$ such that

$$\bigcup_{\alpha \in A} U_\alpha = M.$$

An atlas describes how to reconstruct the manifold from the pieces U_α homeomorphic to open subsets in $\mathbb{R}^n$. More precisely U_α is to be glued to U_β along $U_{\alpha\beta}$ using the transition map $\Phi_{\beta\alpha}$.

A *smooth atlas* on the n-dimensional manifold M is an atlas $\mathcal{A} = \big\{(U_\alpha, \Phi_\alpha)\big\}_{\alpha \in A}$ such that *all* the transition maps $\Phi_{\beta\alpha}$ are smooth maps between open subsets in $\mathbb{R}^n$.

For two smooth atlases $\mathfrak{A}$ and $\mathfrak{B}$ we write $\mathfrak{A} \sim \mathfrak{B}$ if $\mathfrak{A} \cup \mathfrak{B}$ is a smooth atlas. "$\sim$" is an equivalence relation on the set of smooth atlases. A *smooth structure* on M is an equivalence class of smooth structures. A *smooth* manifold is a pair (M, τ), where M is a topological manifold and τ is a smooth structure on M. The topological manifolds which admit smooth structures are called *smoothable*.

Roughly speaking a smooth structure is a decomposition of M into tiny open pieces homeomorphic to open sets in $\mathbb{R}^n$ which are glued together via *smooth maps*. We obtain a space "*without corners*".

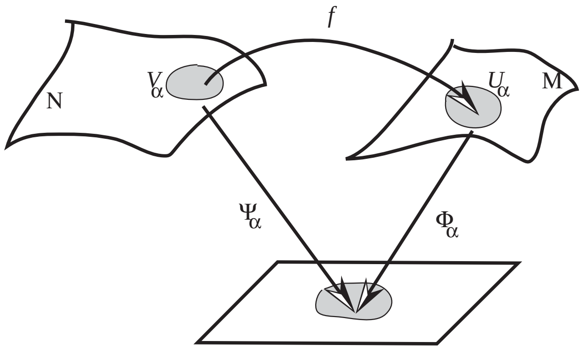

Figure 3: *Pulling pack a smooth structure via a homeomorphism.*

Observe that if $\mathfrak{A} = \Big\{(U_\alpha, \Phi_\alpha)\Big\}$ is a smooth atlas on the topological manifold M and $f : N \to M$ is a homeomorphism, then we get a smooth atlas $f^*\mathfrak{A}$ on N defined by (see Figure 3)

$$f^*\mathfrak{A} := \Big\{\Big(V_\alpha := f^{-1}(U_\alpha), \Psi_\alpha := \Phi_\alpha \circ f\Big)\Big\}_{\alpha\in A}.$$

Note that if $\mathfrak{A} \sim \mathfrak{B}$ then $f^*\mathfrak{A} \sim f^*\mathfrak{B}$. This implies that a homeomorphism of manifolds $N \to M$, and a smooth structure τ on M naturally define a smooth structure $f^*\tau$ on N called the *pullback* of τ via the homeomorphism f.

Two smooth manifolds (M_1, τ_1) and (M_2, τ_2) are called *diffeomorphic* if there exists a homeomorphism $f : M_1 \to M_2$ such that $\tau_1 = f^*\tau_2$.

Example 1.1. $\mathbb{R}^n$ admits a tautological smooth structure defined by the atlas consisting of a single chart $(\mathbb{R}^n, \mathbf{1}_{\mathbb{R}^n})$. □

Example 1.2. The unit sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$ admits a canonical smooth structure defined by the atlas consisting two charts $(U_{north}, \Phi_{north})$ and $(U_{south}, \Phi_{south})$, where $U_{north} = S^n \setminus \{North\ \ Pole\}$, and Φ_{north} is the stereographic projection from the north pole. U_{south} and Φ_{south} are defined similarly. □

§1.3 The fundamental questions. Work of Tibor Rado early in the twentieth century and Edwin Moise in the early fifties showed *every topological manifold of dimension* ≤ 3 *admits a unique smooth structure*. The following questions are thus natural.

Question 1: *Decide whether a given topological manifold M is smoothable.*

Equivalently, this question asks if M is homeomorphic to some smooth manifold. Roughly speaking, this means that we can obtain M by gluing small open subsets in $\mathbb{R}^n$ via *smooth* maps. The manifolds which are not smoothable should have *"weird corners which cannot be smoothed out"*.

Question 2: *Do there exist topological manifolds with more than one smooth structure?*

Equivalently, do there exists smooth manifolds which *are homeomorphic* but which *are not diffeomorphic.* Any homeomorphism between two such smooth manifolds *"must introduce corners".*

Question 3: *Classify all the smooth structures on a topological manifold*

☛ All the above questions are *global* in nature. One cannot answer them by doing experiments and measurements only in the neighborhood of a point, because the neighborhood of any point in any manifold looks like a neighborhood of a point in an Euclidean space. The smoothability issue has to due with how *all* the various local patches are put together and thus we need to understand the manifold as a whole.

2 A smorgarsbord of oddities

§2.1 The higher dimensional world. In 1956 *John Milnor* took the mathematical world by surprise by showing that even "nice" topological manifolds such as the 7-dimensional sphere can have smooth structures other than the canonical one. Such smooth structures have since been dubbed *exotic.*

Observe that the set Θ_n of smooth n-dimensional spheres forms an Abelian group with respect to the connected sum operation. The sphere S^n with its canonical smooth structure is the identity element of this group.

A few years later after Milnor's first example of exotic sphere, *M. Kervaire and J.Milnor* proved another shocker. More precisely, they showed that Θ_7 is a cyclic group of order 28. Moreover, they proved that Θ_{4k-1} contains a cyclic group of order $2^{2k-2}(2^{2k-1}-1)\nu(k)$, where $\nu(k)$ denotes the numerator of $(4B_k/k)$, and B_k denotes the k-th Bernoulli[1] number.

In the mid sixties *E. Brieskorn and F. Hirzebruch* showed that Milnor and Kervaire's exotic 7-spheres can be given an extraordinarily simple description. For every $k>0$ denote by $X_k\subset\mathbb{C}^5$ the singular hypersurface

$$X_k:=\Big\{\vec{z}=(z_0,z_1,z_2,z_3,z_4)\in\mathbb{C}^5;\;\; z_0^{6k-1}+z_1^3+z_2^2+z_3^2+z_4^2=0\Big\}.$$

The origin $\vec{0}\in\mathbb{C}^5$ is a singular point of this hypersurface. If an observer sits at this point and "looks around" in X_k then he will notice a "horizon"

$$H_k:=X_k\cap S^9_\varepsilon,$$

where S^9_ε is the sphere of radius ε in $\mathbb{C}^5$ centered at the origin $\vec{0}$. One can think of this sphere as the"horizon" of an observer in $\mathbb{C}^5$ situated at the origin. A simple application of the implicit function theorem shows H_k is a *smooth* 7-dimensional submanifold of S^9_ε. Thus each of the smooth manifolds H_k defines an element $[H_k]\in\Theta_7$. E. Brieskorn and F. Hirzebruch have shown that $[H_1]$ is a generator of the cyclic group Θ_7, and in fact

$$[H_k]=k\cdot[H_1]\;\text{ in }\;\Theta_7,$$

[1]The Bernoulli numbers are described by the Taylor expansion

$$\frac{t}{(e^t-1)}=1-\frac{t}{2}+\frac{B_1t^2}{2!}-\frac{B_2t^4}{4!}+\cdots,$$

so that $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{43}$, $B_4=\frac{1}{30}$, $B_5=\frac{5}{66}$ etc.

i.e. H_k is diffeomorphic to H_{k+28} and with the connected sum of k-copies of H_1. Moreover, for every $k \not\equiv 0 \bmod 28$, H_k is a manifold *homeomorphic* but *not diffeomorphic* to S^7.

One can then ask whether the Euclidean spaces admit smooth structures not equivalent to the tautological one. In the early sixties it was proved that if $n \neq 4$ the Euclidean space $\mathbb{R}^n$ admits a *single* smooth structure. The case $n = 4$ had to wait twenty more years for a dramatic and unexpected resolution.

Due to the efforts of several mathematicians, by late sixties it was understood that the answer to **Question 1** for a topological manifold of dimension ≥ 5 can be decided by homotopic theoretic methods. However, homotopy theory has had limited success in dimension 4. This has nothing to do with the lack of human ingenuity. It is now understood that the smoothability issue for 4-manifolds goes beyond homotopy theory. The lack of 4-dimensional counterparts of higher dimensional theorems has now a simple explanation: most of the obvious 4-dimensional counterparts are not true. The elegant machinery of homotopy theory breaks in this dimension.

§2.2 The wild world of four manifolds. For simplicity I will concentrate only on simply connected topological 4-manifolds. Suppose M is such a *compact* 4-manifold. Then $H_2(M, \mathbb{Z})$ is a *free* Abelian group of rank b_2-the second Betti number of M. Poincaré duality shows that there is a *nonsingular, symmetric*, $\mathbb{Z}$-bilinear map

$$I_M : H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \to \mathbb{Z}, \quad (u, v) \mapsto u \cdot v.$$

I_M is called the *intersection form* of M. By choosing a basis of $H_2(M, \mathbb{Z})$ we can represent I_M as a $b_2 \times b_2$-matrix A with integral entries. Then the nonsingularity condition is equivalent with the condition $\det A = \pm 1$. (We say that A is *unimodular*.) The symmetry translates into the symmetry of A, $A = A^t$. By changing $\mathbb{Z}$-bases, A changes to a new matrix SAS^t for some $S \in GL(n, \mathbb{Z})$. We say that A and SAS^t are equivalent over $\mathbb{Z}$. Contrary to the case of symmetric matrices with real entries, the integral ones cannot be diagonalized by an appropriate choice of basis. For example, there exist *infinitely many unimodular, symmetric, positive definite and pairwise inequivalent matrices.*

J.W.C Whitehead had shown in the forties that two compact, simply connected topological 4-manifolds are homotopy equivalent iff they have equivalent intersection forms.

The following question then suggests itself.

☞ *Given a symmetric, unimodular matrix A, does there exist a compact simply connected topological 4-manifold whose intersection form is described by A?*

In the late seventies, after a remarkable tour de force, *Michael Freedman* proved the following result.

Theorem 2.1. *For any symmetric, unimodular matrix A there exists at least one simply connected 4-manifold with intersection form represented by A. Moreover, there can exists at most two such manifolds.*

□

Around the same time (very early eighties) a young mathematician by the name of *Simon Donaldson* proved a completely unexpected negative result.

Theorem 2.2. *If A is a unimodular, symmetric positive (or negative) definite matrix which* **cannot be diagonalized** *over $\mathbb{Z}$, then none of Freedman's manifolds with intersection form A admits smooth structures. In particular, there exist infinitely many such 4-manifolds.* $\square$

Remark 2.3. A majority of these nonsmoothable manifolds can be produced by analyzing the singularities[2] of polynomials of three complex variables. The polynomials $P_{2,3,6k-1} = x^{6k-1} + y^3 + z^2$ which also appear in the Brieskorn-Hirzebruch construction of the exotic 7-spheres play a special part. Here is how one can use the polynomial $P_{2,3,5}$ to construct an example of topological manifold which admits no smooth structure.

Consider the *Milnor fiber* of the polynomial $P_{2,3,5}$

$$M_{2,3,5} := \Big\{ \vec{z} \in \mathbb{C}^3;\ \ |\vec{z}| \le r,\ \ P_{2,3,5}(\vec{z}) = \varepsilon \Big\},$$

where r, ε are two small numbers. This is a 4-manifold with boundary

$$\Sigma(2,3,5) := \Big\{ \vec{z} \in \mathbb{C}^3;\ \ |\vec{z}| = r,\ \ P_{2,3,5}(\vec{z}) = \varepsilon \Big\}.$$

The 3-manifold has an illustrious history. It is known as the *Poincaré homology sphere.* Its integral homology is isomorphic to the integral homology of the 3-sphere S^3 and Poincaré thought it is in fact homeomorphic to S^3. He succeeded in computing its fundamental group which turned out to be a finite group of order 120 so this cannot be homeomorphic to the sphere. He then amended his guess and stated what now is one of the most famous unsolved problems, namely the *Poincaré conjecture*: every compact simply connected 3-manifold is homeomorphic to the 3-sphere. A century later we still don't have an answer to this question.

M. Freedman observed a strange phenomenon: $\Sigma(2,3,5)$ bounds a contractible topological 4-manifold[3] Z !!! Now glue Z to $M(2,3,5)$ along $\Sigma(2,3,5)$ to produce a simply connected 4-manifold X as depicted in Figure 4.

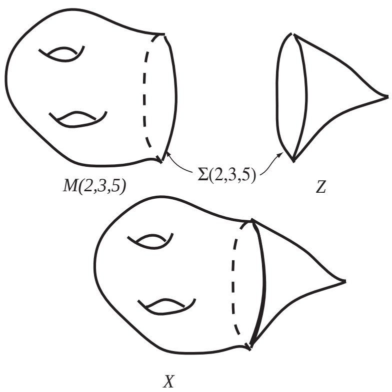

Figure 4: *Constructing a nonsmoothable 4-manifold.*

[2] In fact all of these examples are intimately related to singularities of 2-dimensional complex analytic varieties.

[3] Z cannot be the 4-ball since $\Sigma(2,3,5)$ is not homeomorphic to S^3. In particular this result of Freedman shows that there exist contractible 4-manifolds *not homeomorphic* to the 4-ball!!!

A computation based on ideas going back to Poincaré shows that the intersection form of X is negative definite and cannot be diagonalized over $\mathbb{Z}$. Thus X is a *non-smoothable* topological 4-manifold.

$\square$

To put Donaldson's result in perspective let us mention a result of *Frank Quinn* which states that every topological 4-manifold is smoothable away from an arbitrary point. Loosely speaking this means that we can smooth out all "corners" with the possible exception of one. Donaldson theorem thus produced examples of manifolds for which this one last corner cannot be removed.

Soon after Donaldson's result *Michael Freedman* noticed that by combining his techniques with Donaldson's conclusion he can prove the following.

Theorem 2.4. *There exists a smooth* 4*-manifold* X *with the following properties.*

(i) X *is homeomorphic to* $\mathbb{R}^4$.
(ii) There exists a compact set $C \subset X$ *with the property that it cannot be surrounded by a smoothly embedded* 3*-sphere.*

$\square$

In particular, this result shows that $\mathbb{R}^4$ admits exotic smooth structures[4] because in the tautological $\mathbb{R}^4$ every compact set is surrounded by a sufficiently large round sphere so it cannot be diffeomorphic to X.

Perhaps, as surprising as the result itself is the technique Donaldson used to prove it. It relies on some nonlinear differential equations arising in the theoretic physics. The equations are called the *Yang-Mills equations*, and Donaldson's technique opened up a new and very eclectic branch of mathematics called *gauge theory.* Since its inception it has helped dispel many long held beliefs. In the mid nineties, the Donaldson theory was substantially simplified by the introduction of a new set of equations arising in string theory. These equations are called the *Seiberg-Witten equations*, and according to general physical principles, these equations should contain the same information as the original Yang-Mills equations. The new equations are much more user friendly, and have lead to new striking discoveries.

What next?

The 4-dimensional world turned out to be quite unruly, and although a few patterns have been observed, it is still a jungle out there, with intriguing and still unexplained connections with most branches of mathematics.

If you want to learn more about this addictive subject stop by my office for a chat, or have a look at any of the references below.

[4] It is now known that $\mathbb{R}^4$ admits uncountably many exotic smooth structures.

References

[1] S.K. Donaldson, P.B. Kronheimer: *The Geometry of Four-Manifolds*, Clarendon Press, Oxford, 1990.

[2] D.S. Freed, K. Uhlenbeck: *Instantons and 4-Manifolds*, Springer-Verlag, 1985.

[3] F. Hirzebruch: *Singularities and exotic spheres*, Séminare Bourbaki, 1966/67, no. 314, 1-19.

[4] J.W. Milnor: *On manifolds homeomorphic to the 7-sphere*, Ann. of Math., **64**(1956), 399-405.

[5] L.I. Nicolaescu: *Notes on Seiberg-Witten Theory*, Graduate Studies in Mathematics, vol.28, Amer. Math. Soc., 2000.

`http://www.nd.edu/~lnicolae/new1.pdf`