THE KAC-RICE FORMULA AND THE NUMBER OF CRITICAL POINTS OF RANDOM FUNCTIONS

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Abstract. This survey has two goals. The first is to present of a proof of the multidimensional Kac-Rice formula in the Gaussian case that highlights in as clear a fashion possible the central ideas. The second goal is to present the recent results of Gass-Stecconi and Ancona-Letendre describing sufficient conditions guaranteeing that the nimver of zeros of a random Gaussian map has finite p-momentum.

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Introduction

This is an expository paper and I make no claims of originality.

Given a map $G : \mathbb{R}^d \to \mathbb{R}^d$ and a Borel subset $B \subset \mathbb{R}^d$ we denote by $Z_B(G)$ the number of zeros of G inside the set B. When G is random, $Z_B(G)$ is a random variable. The basic

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Notes for myself and whoever else is curious about this topic.

Kac-Rice formula states that, under certain assumptions, the expectation of $Z_B(G)$ can be described as an integral of B of a certain explicit density called the Kac-Rice density. The first goal of these notes is to describe some simple yet general conditions guaranteeing the validity of the Kac-Rice formula. The second goal is to describe some sufficient conditions guaranteeing that $Z_B(G)$ has finite moments.

There are currently many different proofs of very general versions of the Kac-Rice formula, e.g., [1, 4, 21]. In the first section of the paper I describe in detail the recent shorter proof by Armentao, Azaïs, León [3] with an emphasis on the Gaussian situation.

The second section is devoted to the special case $G = \nabla F$ where F is a random function. In this case $Z_G(B)$ is the number of critical points of F in B and we are interested in describing sufficient conditions guaranteeing that $Z_B(\nabla F)$ has finite moments.

In Subsection 2.3 we show that if F is a.s. C^3 and, for every $u \in \mathcal{U}$ the second jet of F at u is a nondegenerate Gaussian vector, then $Z_B(\nabla F) \in L^2$.

In Subsection 2.4 we investigate the p-th moments of $Z_B(\nabla F)$ and we prove Theorem 2.11 stating that if F is a.s. C^{p+1} and, for any $u \in \mathcal{U}$ the p-th jet of F at u is a nondegenerate Gaussian vector, then $Z_B(\nabla F) \in L^p$. This was proved recently by Gass-Stecconi [10], and Ancoma-Letendre [2]. This subsection is inspired heavily from work of Gass and Stecconi [10].

In the final Section 3 we present another proof of Theorem 2.11 based on a simplified version of the multijet technique introduced by Ancona and Letendre in [2].

Conventions and notations.

- In this paper the set $\mathbb{N}$ of natural numbers is the set of *positive integers*.
- We will we will denote by $\mathbb{E}\big[\, Y \,\|\, X \,\big]$ the conditional expectation of Y given X. It is a *random variable*. The conditional expectation a measurable function of X, $\mathbb{E}\big[\, Y \,\|\, X \,\big] = f(X)$. We will denote the value of f at x by $\mathbb{E}\big[\, Y \,\big|\, X = x\big]$. This is a deterministic quantity.
- We set $\mathbb{I}_n := \{1, \dots, n\}$. For every set U we will think of U^n as functions $\underline{u} : \mathbb{I}_n \to U$. We will often write $\underline{u} = (u_1, \dots, u_n)$, $u_i = \underline{u}(i)$. For any $I \subset \mathbb{I}_n$ we will denote by $\underline{u}_I$ the restriction of $\underline{u}$ to I.
- We will denote by $\mathfrak{S}_n$ the group of permutations of $\mathbb{I}_n$.
- Let $\boldsymbol{U}$ be a finite dimensional Euclidean space. A compact subset $B \subset \boldsymbol{U}$ is called a *box centered at the origin* if there exist Euclidean coordinates (x^i) on $\boldsymbol{U}$ and $r > 0$ such that B is described by
$$\max |x^i| \leq r.$$
A box centered at u_0 is a compact set of the form $u_0 + B_0$, where B_0 is a box centered at the origin.

1. The Euclidean Kac-Rice formula

1.1. **Formulation.** Assume that $\boldsymbol{V}$ and $\boldsymbol{U}$ are finite-dimensional vector spaces of dimensions
$$D = \dim \boldsymbol{V} \geq d = \dim \boldsymbol{U}.$$
Suppose that $\mathcal{V} \subset \boldsymbol{V}$ is an open subset. For any Borel subset $B \subset \mathcal{V}$, any measurable map $F : \mathcal{V} \to \boldsymbol{U}$ and any $u \in \boldsymbol{U}$ we define the level set
$$\mathcal{Z}_B(F, u) = \mathcal{Z}_u(F, B) = F^{-1}(u) \cap B \subset \mathcal{V}.$$
We set
$$\mathcal{Z}_B(F) = F^{-1}(0) \cap B = \mathcal{Z}_B(F, 0).$$

If F is C^1 and u is a regular value of F, then $\mathcal{Z}_u$ is a submanifold of dimension $D-d$ of $\boldsymbol{V}$. We denote by $\mathcal{H}_k$ the k-dimensional Hausdorff measure on $\boldsymbol{V}$.

We equip the space of $\mathcal{X} := C^1(\mathcal{V}, \boldsymbol{U})$ of C^1-maps $\mathcal{V} \to \boldsymbol{U}$ with the topology of uniform convergence on compacts of maps and their first order derivatives. As such, it is a Polish space. We denote by $\operatorname{Prob}(\mathcal{X})$ the space of Borel probability measures on $\mathcal{X}$.

Theorem 1.1 (Kac-Rice formula)**.** *Suppose that $(\Omega, \mathcal{S}, \mathbb{P})$ is a complete probability space and*

$$X : \Omega \times \mathcal{V} \to \boldsymbol{U}, \;\; \Omega \times \mathcal{V} \ni (\omega, v) \mapsto X_\omega(v) \in \boldsymbol{U}$$

*is a $\boldsymbol{U}$-valued random field parametrized by the open subset $\mathcal{V} \subset \boldsymbol{V}$ satisfying the conditions (**A**) and (**B**) in Theorem 1.3 and the additional condition*

($\mathbf{R}_1$) *The random field X is* a.s. *C^1, i.e.*

$$\mathbb{P}\big[\,\big\{\,\omega;\;\; X_\omega : \mathcal{V} \to \boldsymbol{U} \;\; \text{is } C^1\,\big\}\,\big] = 1.$$

It induces a measurable map $X : \Omega \to \mathcal{X}$ and we denote by $\mathbb{P}_X \in \operatorname{Prob}(\mathcal{X})$ the distribution of the process X.

(**T**) *$0 \in \boldsymbol{U}$ is* a.s. *a regular value of X, i.e.,*

$$\mathbb{P}\big[\,\{\,\omega;\; 0 \text{ is a regular value of } X_\omega : \mathcal{V} \to \boldsymbol{U}\,\}\,\big] = 1.$$

($\mathbf{A}_0$) *For each $v \in \mathcal{V}$, the distribution $\mathbb{P}_{X(v)}$ of the random vector $X(v)$ is absolutely continuous with respect the the Lebesgue measure on $\boldsymbol{U}$. We denote by $p_{X(v)}$ its density and we assume that is continuous in u and that for any compact subset $K \subset \mathcal{V}$ we have*

$$\sup_{v \in K} \sup_{u \in \boldsymbol{U}} p_{X(v)}(u) < \infty.$$

(**C**) *For any $u_0 \in \boldsymbol{U}$, $v_0 \in \mathcal{V}$, and any bounded continuous function $\alpha : \mathcal{X} \to \mathbb{R}$ $v_0, v \in \mathcal{V}$ the conditional distribution $\mathbb{P}_{\alpha(X)|X(v_0)=u_0}$ is well defined as a probability measure on $\mathbb{R}$ and depends continuously on u_0 in the topology of weak convergence of measures.*

Then, $\mathcal{H}_{D-d}\big(\mathcal{Z}_B(X)\big)$ is a random variable, i.e., the map

$$\Omega \ni \omega \mapsto \mathcal{H}_{D-d}\big(\,\mathcal{Z}_B(\,X_\omega\,)\,\big) \in [0, \infty]$$

is measurable. Moreover,

$$\mathbb{E}\big[\,\mathcal{H}_{D-d}\big(\,\mathcal{Z}_B(X)\,\big)\,\big] = \int_B \mathbb{E}\big[\,J_X(v)\big|\,X(v) = 0\,\big] p_{X(v)}(u_0) dv, \tag{$\mathbf{KR}$}$$

where $J_X(v)$ denotes the Jacobian of X at $v \in \mathcal{V}$ and $\mathbb{E}\big[\,J_X(v)\big|\,X(v) = u_0\,\big]$ denotes the conditional expectation of $J_X(v)$ given that $X(v) = u_0$. When $D = d$ the Kac-Rice formula takes the form

$$\mathbb{E}\big[\,\#\mathcal{Z}_B(X)\,\big] = \int_B \mathbb{E}\big[\,J_X(v)\,\|\,X(v) = 0\,\big] p_{X(v)}(u_0) dv, \tag{$\mathbf{KR}_0$}$$

Remark 1.2. (a) Condition (**T**) is a transversality condition while ($\mathbf{A}_0$) is closely related to the ampleness conditions in geometry. We can give a more conceptual interpretation of (**C**). We can view X as a measurable map $X : \Omega \to \mathcal{X}$. Fix $v_0 \in \mathcal{V}$ and we obtain another random map

$$\overline{X}_{v_0} : \mathcal{X} \times \boldsymbol{U}, \;\; \omega \mapsto \big(\,X_\omega, X_\omega(v_0)\,\big).$$

The distribution of $\overline{X}_{v_0}$ disintegrates since $\mathcal{X}$ is Polish so there is a kernel $\mathcal{K}_{v_0}$ from $(\boldsymbol{U}, \mathcal{B}_{\boldsymbol{U}})$ to $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$,

$$\mathcal{K}_{v_0} : \boldsymbol{U} \times \mathcal{B}_{\mathcal{X}} \to [0,1], \;\; (u, S) \mapsto \mathcal{K}_{v_0}(u, S)$$

For fixed $u_0 \in \boldsymbol{U}$ the map

$$\mathcal{B}_{\mathcal{X}} \ni B \mapsto \mathcal{K}_{v_0}(u_0, B) \in [0, \infty)$$

is a subprobability Borel measure on $\mathcal{X}$. More precisely

$$\mathcal{K}_{v_0}(u_0, -) = \mathbb{P}_{X \big| X_{v_0} = u_0}.$$

Condition (**C**) states that this measure depends continuously on u_0 and v_0.

(b) The random fields satisfying the conditions ($\mathbf{R}_1$), (**T**), (**C**) are closely related to the z-KROK random fields introduced in [15].

(c) Let us mention that (**KR**) for $D = d$ is satisfied under alternate assumptions, [1, Thm. 11.2.1]. □

Concerning the transversality condition (**T**) we have the following result.

Theorem 1.3 (Transversality)**.** *Suppose that* $(\Omega, \mathcal{S}, \mathbb{P})$ *is a probability space and*

$$X : \Omega \times \mathcal{V} \to \boldsymbol{U},\;\; \Omega \times \mathcal{V} \ni (\omega, v) \mapsto X_\omega(v) \in \boldsymbol{U}$$

is a random field satisfying the following conditions.

($\mathbf{R}_2$) *The random field X is* a.s. C^2, *i.e.*

$$\mathbb{P}\big[\,\{\,\omega;\;\; X_\omega : \mathcal{V} \to \boldsymbol{U} \;\; is\; C^2\,\}\,\big] = 1.$$

($\mathbf{A}_1$) *The random vector*

$$Y : \mathcal{V} \times \boldsymbol{U} \setminus \{0\} \to \boldsymbol{U} \times \boldsymbol{V},\;\; (v, \dot{u}) \mapsto (X(v), X'(v)^*\dot{u})$$

satisfies the condition ($\mathbf{A}_0$)*. Above, $X'(u)^* : \boldsymbol{U} \to \boldsymbol{V}$ is the adjoint of the differential $X'(v) : \boldsymbol{V} \to \boldsymbol{U}$.*

Then the random field satisfies (**T**)*.*

Let us observe that ($\mathbf{A}_1$) $\Rightarrow$ ($\mathbf{A}_0$) and ($\mathbf{R}_2$) $\Rightarrow$ ($\mathbf{R}_1$)

Theorem 1.4. *Suppose that $D = d$, $(\Omega, \mathcal{S}, \mathbb{P})$ is a probability space and*

$$X : \Omega \times \mathcal{V} \to \boldsymbol{U},\;\; \Omega \times \mathcal{V} \ni (\omega, v) \mapsto X_\omega(v) \in \boldsymbol{U}$$

is a random field satisfying ($\mathbf{R}_1$) *and* ($\mathbf{A}_0$)*. Then the random field satisfies* (**T**)*.*

1.2. **Proof of the transversality theorems.** If $F : \mathcal{V} \to \boldsymbol{U}$ is a C^1-map then the fiber $F^{-1}(0)$ is a submanifold of $\mathcal{V}$ of dimension $D - d$ and it seems natural to expect that it is highly unlikely that the equation $F(v) = u_0$ will have no solutions inside a given dimensional subset of $\boldsymbol{V}$ of dimension $d - 1$. This is true a.s. in a much more general context.

Lemma 1.5 (Bulinskaya)**.** *Suppose that $X : \mathcal{V} \to \boldsymbol{U}$ satisfies* ($\mathbf{R}_1$) *and* ($\mathbf{A}_0$)*, $D \geq d$. Let $K \subset \mathcal{V}$ be a compact subset of Hausdorff dimension $< d$. Then*

$$\mathbb{P}\big[\,\mathcal{Z}_K(X, u_0) \neq \emptyset\,\big] = 0.$$

Proof. We follow the argument in the proof of [1, Lemma 11.2.10]. Fix Euclidean coordinates $(v_i)_{1 \leq i \leq D}$ on $\boldsymbol{V}$ and $(u_j)_{1 \leq j \leq d}$ on $\boldsymbol{U}$. We can write X as a collection of random variables $(X^1, \dots, X^d)$. We set

$$C_\omega(v) = \sum_{j=1}^d \left(\,|\,X^j_\omega(v)\,| + \sum_{i=1}^D |\partial_{v_i} X^j_\omega\,|\,\right).$$

For every compact set $S\subset\mathcal{V}$ we set

$$C_\omega(S) := \sup_{v\in S} C_\omega(v).$$

Then $(\mathbf{R}_1)$ implies that

$$\mathbb{P}\big[\,C(S)<\infty\,\big]=1.$$

Hence, for every $\varepsilon>0$ there exists $M_\varepsilon=M_\varepsilon(S)>0$ such that

$$\mathbb{P}\big[\,C(S)<M_\varepsilon\,\big]>1-\varepsilon. \tag{1.1}$$

If we choose S to be a closed ball of radius $r>0$ centered at v_0 and contained in $\mathcal{V}$ we deduce from the mean value theorem that

$$\big|\,X^j_\omega(v)-X^j_\omega(v_0)\,\big|\leq C_\omega(v_0,r)\|v-v_0\|\leq C_\omega(v_0,r)r$$

so that

$$\|\,X_\omega(v)-X_\omega(v_0)\|\leq C_\omega(v_0,r)\sqrt{d}\cdot r.$$

For $r<\operatorname{dist}(K,\partial\mathcal{V})$ we set

$$C_\omega(K,r)=\sup_{v_0\in K}C_\omega(v_0,r)\leq C_\omega(K_r),\;\;K_r:=\{v\in\boldsymbol{V};\;\operatorname{dist}(v,K)\leq r\,\}.$$

For such an r we set

$$\operatorname{osc}_\omega(r):=\sup_{\substack{v_1,v_2\in K\\ \|v_1-v_2\|\leq r}}\|X_\omega(v_1)-X_\omega(v_2)\|.$$

Note that

$$\operatorname{osc}_\omega(r)\leq C_\omega(K,r)\sqrt{d}r.$$

Consider the event

$$E_\varepsilon(r):=\big\{\;\operatorname{osc}_\omega(r)\leq M_\varepsilon(K_r)\sqrt{d}\,r\,\big\}.$$

We set $M_\varepsilon(r):=M_\varepsilon(K_r)$. We deduce from (1.1) that

$$\mathbb{P}\big[\,E_\varepsilon(r)\,\big]>1-\varepsilon.$$

Pick a sequence $\hbar_n\searrow 0$. Since K has Hausdorff dimension $<d$, its d-dimensional Hausdorff measure is zero, and we deduce that there exists a sequence $r_n\searrow 0$ and for any n there exists a finite collection of closed balls $(B_{n,j})_{j\in J_{n,r}}$, of radii $r_{n,j}<r_n$, covering K, such that

$$\sum_{j\in J_{n,\delta}}\big(\,r_{n,j}\,\big)^d\leq\hbar_n.$$

We denote by A the event "*the equation* $X(v)=u_0$ *has a solution* $v\in K$" and we set

$$A_{n,j}=A\cap B_{n,j}.$$

Fix $\varepsilon>0$ and $r>0$ sufficiently small. Then

$$\mathbb{P}\big[\,A\,\big]\leq\sum_j\mathbb{P}\big[\,A_{n,j}\cap E_\varepsilon(r_n)\,\big]+\mathbb{P}\big[\,E_\varepsilon(r_n)^c\,\big]\leq\sum_j\mathbb{P}\big[\,A_{n,j}\cap E_\varepsilon(r_n)\,\big]+\varepsilon. \tag{1.2}$$

Denote by $v_{n,j}$ the center of $B_{n,j}$. Observe that $A_{n,j}\neq\emptyset$ iff there exists v such that $\|v-v_{n_j}\|\leq r_{n,j}$ and $X(v)=u_0$. On $E_\varepsilon(r_n)$ we have

$$\big|\,X(v_{n,j})-u_0\,\big|=\big|\,X(v_{n,j})-X(v)\,\big|\leq M_\varepsilon(r_n)\sqrt{d}\,r_{n,j}.$$

This shows that

$$A_{n,j}\cap E_\varepsilon(r_n)\subset\big\{\,\big|\,X(v_{n,j})-u_0\,\big|<M_\varepsilon(r_n)\sqrt{d}\,r_{n,j}\,\big\}.$$

We denote by $\boldsymbol{\omega}_d$ the volume of the unit d-dimensional ball and we set

$$L := \sup_{v\in K_{r_1}} \sup_{u\in \boldsymbol{U}} p_{X(v)}(u).$$

Assumption ($\mathbf{A}_0$) implies $L<\infty$. We deduce

$$\mathbb{P}\big[\,\big\{\,\big|\,X(v_{n,j})-u_0\,\big|<M_\varepsilon(r_n)\sqrt{d}\,r_{n,j}\,\big\}\,\big]\leq \underbrace{L\boldsymbol{\omega}_d M_\varepsilon(r_n)^d d^{d/2}}_{\Xi_\varepsilon(r_n)} r_{n,j}^d,$$

and

$$\sum_j \mathbb{P}\big[\,A_{n,j}\cap E_\varepsilon(r_n)\,\big]\leq \Xi_\varepsilon(r_n)\sum_j r_{n,j}^d\leq \Xi_\varepsilon(r_n)\hbar_n\leq \Xi_\varepsilon(r_1)\hbar_n.$$

Now choose n such that $\Xi_\varepsilon(r_1)\hbar_n\leq\varepsilon$ to conclude from (1.2) that

$$\mathbb{P}\big[\,A\,\big]\leq 2\varepsilon,\;\;\forall\varepsilon>0.$$

□

Proof of Theorem 1.3 Consider the random field

$$Y:\mathcal{V}\times\boldsymbol{U}\to\boldsymbol{U}\times\boldsymbol{V},\;\;Y(v,\dot{u})\mapsto\big(\,X(v),X'(v)^*\dot{u}\,\big).$$

Fix a box $B\subset\mathcal{V}$ and denote by $S(\boldsymbol{U})$ the unit sphere in $\boldsymbol{U}$. Let us show that a.s., 0 is a regular value of $X|_B$. This means that for any solution $v\in B$ of $X=0$ the transpose of the differential $X'(c)^\top$ is one-to-one, i.e., the equations

$$Y(v,\dot{u})=0\Longleftrightarrow X(v)=0,\;\;X'(v)^*\dot{u}=0,$$

has no solution $(v,\dot{u})\in B\times S(\boldsymbol{U})$. Since $\dim B\times S(\boldsymbol{U})\leq\dim(\boldsymbol{U}\times\boldsymbol{V})$ we deduce from Lemma 1.5 that this happens a.s.. □

Remark 1.6. Sard's transversality theorem requires a bit of regularity. Suppose that

$$F:\mathcal{V}\to\boldsymbol{U}$$

is a C^k map. In [9, Thm. 3.4.3] it is shown that if $k\geq D-d+1$, then the set of critical values of F is negligible in $\boldsymbol{U}$. However, if $k\leq D-d$ there exist C^k-maps $\mathcal{V}\to\boldsymbol{U}$ for which the set of critical values is not negligible in $\boldsymbol{U}$; see [9, Sec. 3.4.4].

In geometry the generic transversality is traditionally obtained as follows. Fix $k\geq D-d+1$. Suppose that N is a positive integer and

$$F:\mathbb{R}^N\times\mathcal{V}\to\boldsymbol{U},\;\;(\lambda,v)\to F_x(v)$$

is a C^k-map. We view it as a family in $C^k(\mathcal{V},\boldsymbol{U})$ parametrized by $\lambda\in\mathbb{R}^N$. We assume that the family is sufficiently large, i.e., satisfies the ampleness condition

$$0 \text{ is a regular value } F. \tag{$*$}$$

Then

$$Z=\big\{\,(\lambda,v)\in\mathbb{R}^N\times\mathcal{V};\;F_\lambda(v)=0\,\big\}$$

is a C^k manifold and the natural projection $\pi:\mathcal{Z}\to\mathbb{R}^N$, $(\lambda,v)\to\lambda$ is a C^k map. Since $\dim\mathcal{Z}-N=D-d$ we deduce from Sard's theorem that most $\lambda\in\mathbb{R}^N$ are regular values of π. One can show that for such λ, 0 is a regular value of F_λ. Thus, a regularity assumption together with an ampleness condition on the family guarantee that 0 is generically a regular value of F_λ.

We approached generic regularity using a different approach. Let N be a (large) positive integer and suppose that, for each $v \in \mathcal{V}$ the collection of C^2-maps

$$\left\{ F_k(v), F'(v)_k^\top \right\}_{1 \le k \le N}$$

spans the vector space $\boldsymbol{U} \times \operatorname{Hom}(\boldsymbol{U}, \boldsymbol{V})$. If we define

$$F_\lambda := \sum_{k=1}^N \lambda_k F_k, \;\; \lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N,$$

then we see that the family $(F_\lambda)_{\lambda \in \mathbb{R}^N}$ satisfies $(*)$. It is however less regular if $D - d > 1$.

Fix independent standard normal random variables $\Lambda_1, \dots, \Lambda_N$ and form the random Gaussian map

$$F_\omega = \sum_k \Lambda_k(\omega) F_k.$$

Equivalently, consider the standard Gaussian measure on $\mathbb{R}^N$ and think of F as a random map. This random map satisfies $(\mathbf{R}_2)$ and $(\mathbf{A}_1)$ and thus 0 is a.s. a regular value of F. This implies 0 a regular value of F_λ for λ in a set of Gaussian probability 1. for generic x. We have thus obtained generic transversality with reduced regularity, C^2 instead of C^{D-d+1}, but the price we had to pay was stronger ampleness assumptions. □

In the remainder of this section we fix Euclidean coordinates $(v_i)_{1 \le i \le d}$ and $(u_j)_{1 \le j \le d}$ on $\boldsymbol{V}$ and respectively on $\boldsymbol{U}$. A *box* in $\boldsymbol{V}$ is a compact set of the form $\prod_{i=1}^d [a_i, b_i]$. The box is called *nondegenerate* if it has nonempty interior.

Proof of Theorem 1.4 We follow the approach in [3, Sec. 4]. Fix a box $B \subset \mathcal{V}$. Consider the quantities

$$T = \liminf_{r \searrow 0} T_r, \;\; T_r(\omega) := \frac{1}{\boldsymbol{\omega}_d r^d} \mathcal{H}_d \big[\{ v \in B; \; \|X(v)\| \le r \} \big],$$

where $\mathcal{H}_d$ denotes the d-dimensional Hausdorff measure on $\boldsymbol{V}$. In this case it coincides with the Lebesgue measure.

Assumption $(\mathbf{A}_0)$ implies that T is a.s. finite. We set

$$\mathcal{Z}^s := \big\{ v \in B; \;\; X(v) = 0, \;\; J_v = 0 \big\}.$$

We will show that $\mathbb{P}\big[\mathcal{Z}^s \neq \emptyset \big] = 0$. Set

$$M := \sup_{v \in B} \|X'(v)\|,$$

$$N(\varepsilon) = \sup_{v \in B, \; 0 < \|\dot{v}\| < \varepsilon} \frac{\|X(v_0 + \dot{v}) - X(v_0) - X'(v_0)\dot{v}\|}{\|\dot{v}\|}.$$

Both random variables M and $N(\varepsilon)$ and $N(\varepsilon) \to 0$, a.s., as $\varepsilon \searrow 0$.

Let $v_0 \in \mathcal{Z}^s$. Lemma 1.5 shows that $v_0 \in \boldsymbol{int}\, B$ a.s.. Set

$$K_0 = \ker X'(v_0) \subset \boldsymbol{V}, \;\; k = \dim K_0^\perp.$$

Since $J_{v_0} = 0$ we deduce that $k \le (n-1)$. Any vector $\dot{v} \in \boldsymbol{V}$ decoposes as

$$\dot{v} = \dot{v}_0 + \dot{v}^\perp, \;\; \dot{v}_0 \in K_0, \;\; \dot{v}^\perp \in K_0^\perp.$$

Then

$$\begin{aligned} \|X(v_0 + \dot{v})\| &\le \|X(v_0 + \dot{v}_0 + \dot{v}^\perp) - X(v_0 + \dot{v}_0)\| + \|X(v_0 + \dot{v}_0)\| \\ &\le M\|\dot{v}^\perp\| + \|\dot{v}_0\| N\big(\|\dot{v}_0\| \big). \end{aligned}$$

Let $\varepsilon > 0$ such that $N(\varepsilon) < 1$ and suppose that

$$\|\dot{v}_0\| \leq \varepsilon, \;\; \|\dot{v}^\perp\| \leq \varepsilon N(\varepsilon). \tag{1.3}$$

We deduce that

$$\|X(v_0 + \dot{v}_0 + \dot{v}^\perp)\| \leq r(\varepsilon) := (M+1)\varepsilon N(\varepsilon).$$

The polydisk

$$P_\varepsilon := \big\{\, v \in B;\;\; v = v_0 + \dot{v},\;\; \dot{v} \text{ satisfies } (1.3)\,\big\}$$

is a.s. contained in B for $\varepsilon > 0$ sufficiently small. Thus

$$\begin{aligned} T_{r(\varepsilon)} &= \frac{1}{\boldsymbol{\omega}_d r(\varepsilon)^d} \mathcal{H}_d\big[\, \{ v \in B;\; \|X(v)\| \leq r(\varepsilon) \}\,\big] \geq \frac{1}{\boldsymbol{\omega}_d r(\varepsilon)^d} \mathcal{H}_d\big[\, P_\varepsilon\,\big] \\ &= \frac{const. \times \varepsilon^d N(\varepsilon)^k}{\boldsymbol{\omega}_d \varepsilon^d N(\varepsilon)^d} = const N(\varepsilon)^{k-d} \to \infty \;\text{ as } \varepsilon \searrow 0. \end{aligned}$$

Hence

$$\mathcal{Z}^s \neq \emptyset \subset \{T = \infty\},$$

so $\mathbb{P}\big[\, \mathcal{Z}^s = \emptyset\,\big] = 1$. □

Remark 1.7. To better understand the idea behind the above proof it helps to have in mind the following suggestive example. Consider the map

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \;\; F(x,y) = (x, y^2).$$

Then

$$T_r := \big\{ \|F\| \leq r \,\big\} = \big\{\, x^2 + y^4 \leq r^2 \,\big\} \supset S_r := \big\{\, |x| \leq 2^{-1/2} r, \;\; |y| \leq 2^{-1/4}\sqrt{r}\,\big\},$$

and

$$\mathcal{H}_2\,(S_r) = 2^{-3/4} r^{3/2}.$$

Hence

$$\frac{\mathcal{H}_2(T_r)}{\pi r^2} \geq 2^{-3/4} r^{-1/2} \nearrow \infty \;\text{ as } r \searrow 0.$$

□

1.3. **Proof of the Kac-Rice formula when $D = d$.** We set

$$Z_B = Z_B(X) = \#\mathcal{Z}_B(X).$$

Lemma 1.8 (Kac's counting formula)**.** *Suppose that $F : \mathcal{V} \to \boldsymbol{U}$ is a C^1-map. Let $B \subset \mathcal{V}$ be a nondegenerate box. Suppose*

(i) *that 0 is a regular value of F and*

(ii)
$$F^{-1}(0) \cap \partial B = \emptyset.$$

Denote by $N(F, B)$ the number of solutions of the equation $F(v) = 0$, $v \in B$. Then, for $r > 0$ sufficiently small we have

$$N_0(F,B) = N_r(F,B) = \frac{1}{\boldsymbol{\omega}_d r^d} \int_B \boldsymbol{I}_{\{|F|<r\}} J_F(v) dv,$$

where J_F denotes the Jacobian of F.

Proof. Since 0 is a regular value and $F^{-1}(0) \cap \partial B = \emptyset$ we deduce from the inverse function theorem function that F has a finite number of zeros in B, $v_1, v_2, \dots, v_n$, $n = N_0(F,B)$, none of them located on ∂B. Choose $\delta > 0$ sufficiently small such that

- The open ball $B_\delta(v_i)$ is contained
- the closure of the balls $B_\delta(v_i)$ are disjoint, and
- the restriction of F to each of the open balls $B_\delta(v_i)$ is a diffeomorphism onto its image.

Set

$$r_0 := \min_{v\in B\setminus\cup_{i=1}^n B_\delta(v_i)} \|F\|.$$

Fix $r\in(0,r_0)$. If $\|u\|<r$ then the equation $F(v)=r$, $u\in B$ has exactly n solutions

$$v_i(u)\in B_\delta(v_i),\;\; i=1,\dots,n.$$

We will use the coarea formula (A.2) where

$$\alpha = \boldsymbol{I}_B,\;\; \beta(u)=\boldsymbol{I}_{\{|u|<r\}}$$

We deduce that

$$\int_{B\cap\{\|F\|<r\}} J_F(v)dv = \int_{\{\|u\|<r\}} \#F^{-1}(u)du = n\boldsymbol{\omega}_d r^d.$$

□

Lemma 1.9 (Continuity of roots)**.** *Fix a box $B\subset\mathcal{V}$. Suppose that $F_\nu:\mathcal{V}\to\boldsymbol{U}$, $\nu\in\mathbb{N}$ is a sequence of C^1 converging in $\mathcal{X}=C^1(\mathcal{V},\boldsymbol{U})$ to a map F that satisfies the conditions (i) and (ii) in Kac's counting formula with respect to the box B. Then*

$$\lim_{\nu\to\infty} N_0(F_\nu,B)=N_0(F,B).$$

Proof. Set

$$\mathcal{Z}=F^{-1}(0)\cap B=\{v_1,\dots,v_n\},\;\; n=\#\mathcal{Z},\;\; \mathcal{Z}_\nu=F_\nu^{-1}(0)\cap B.$$

Choose open balls $B_\delta(v_i)$, $i=1,\dots,n$, as in the proof of Kac's counting formula. Set

$$C:=B\setminus\bigcup_{i=1}^n B_\delta(v_i),$$

$$r_0:=\inf_{v\in C}\|F(v)\|.$$

Since F_ν converges uniformly to F on the compact set C we deduce that there exists $\nu_0>0$ such that

$$\forall\nu\geq\nu_0,\;\; \inf_{v\in C}\|F_\nu(v)\| > r_0/2>0.$$

Thus, for $\nu\geq\nu_0$

$$\mathcal{Z}_\nu\subset\bigcup_{i=1}^n B_\delta(v_i).$$

Set

$$\mathcal{Z}_{\nu,i}:=\mathcal{Z}_\nu\cap B_\delta(v_i).$$

We claim that for each $i=1,\dots,n$, there exists $\nu_i>0$ such that $\#\mathcal{Z}_{\nu,i}=1$, $\forall\nu\geq\nu_i$. We argue by contradiction.

Suppose that there exists a subsequence $\mathcal{Z}_{\nu_k,i}$ such that $\#\mathcal{Z}_{\nu_k,i}\geq 2$. To ease the notation we will write $\mathcal{Z}_{k,i}$ instead of $\mathcal{Z}_{\nu_k,i}$.

Let $v_{0,k}, v_{1,k} \in \mathcal{Z}_{k,i}$, $v_{0,k} \neq v_{1,k}$. Upon extracting subsequences we can assume that $v_{0,k}$ and $v_{1,k}$ converge to $v_{0,\infty}, v_{1,\infty} \in \boldsymbol{cl}\, B_\delta(v_i)$. Clearly $F(v_{0,\infty}) = F(v_{1,\infty}) = 0$ and, since F a single zero, v_i, in $\boldsymbol{cl}\, B_\delta(v_i)$ we deduce

$$v_{0,k},\ v_{1,k} \to v_i \ \text{ as } k \to \infty.$$

Consider the unit vectors

$$w_k := \frac{1}{\|v_{1,k} - v_{0,k}\|}\big(\, v_{1,k} - v_{0,k}\,\big).$$

Upon extracting a subsequence we can assume that w_k converges to the unit vector w. Since the differential $F'(v_i)$ is invertible we deduce that $F'(v_i)w \neq 0$. Choose a linear functional $\xi : \boldsymbol{U} \to \mathbb{R}$ such that

$$\xi\big(\, F'(v_i)w\,\big) = 1. \tag{1.4}$$

Consider now the scalar functions $f_k(v) = \xi\big(\, F_{\nu_k}(v)\,\big)$. From the mean value theorem we deduce that there exists a point p_k on the line segment $[v_{o,k}, v_{1,k}]$ such that

$$0 = f_k(v_{1,k}) - f_k(v_{0,k}) = \|v_{1,k} - v_{0,k}\| df_k(p_k)\big(\, w_k\,\big) = \|v_{1,k} - v_{0,k}\|\xi\big(\, F'_{\nu_k}(p_k)w_k\,\big).$$

In other words

$$\xi\big(\, F'_{\nu_k}(p_k)w_k\,\big) = 0,\ \ \forall k.$$

Note that $p_k \to v_i$. Letting $k \to \infty$ we deduce $\xi\big(\, F'(v_i)w\,\big) = 0$. This contradicts (1.4). □

Corollary 1.10. *Suppose that $(\Omega, \mathcal{S}, \mathbb{P})$ is a complete probability space. Then the map*

$$\Omega \ni \omega \mapsto Z_B(X_\omega) \in [0, \infty]$$

is measurable.

Proof. Lemma 1.5 shows that there exists a $\mathbb{P}_X$-negligible Borel subset $\mathcal{N} \subset \mathcal{X}$ such that any $F \in \mathcal{X}_* = \mathcal{X} \setminus \mathcal{N}$ satisfies the assumptions (i) and (ii) Lemma 1.8. It follows that the map $Z_B : \mathcal{X}_* \to [0, \infty)$ is continuous. Set $Z_B^0 = \boldsymbol{I}_{\mathcal{X}^*} Z_B$. Hence $Z_B^0 : \mathcal{X} \to [0, \infty)$ is measurable and so is $Z_B^0(X)$. Since $Z_B(X) = Z_B^0(X)$ a.s. and $\mathcal{S}$ is $\mathbb{P}$-complete we deduce that $Z_B(X)$ is also measurable. □

Corollary 1.11. *Fix a box $B \subset \mathcal{V}$. Suppose that $X_n : \Omega \times \mathcal{V} \to \boldsymbol{U}$ is a sequence of C^1-random fields such that*

- $X_n \to X$ a.s. *in $C^1(B, \boldsymbol{U})$ and*
- *they satisfy* a.s. *the conditions (i) and (ii) in Lemma 1.8.*

Then

$$N_0(X_n, B) \to N_0(X, B) \ \text{ a.s.}.$$

□

We can finally prove (**KR**) in the case $D = d$. We follow the approach in [3, Sec. 5].

Let $F : [0, \infty) \to [0, 1]$ be the continuous piecewise linear function such that

$$F(x) = \begin{cases} 0, & x \leq 1/2, \\ 1, & x \geq 1. \end{cases}$$

For $n \in \mathbb{N}$ we set $F_n(x) = F(nx)$ and $G_n(x) = 1 - F(x/(2n))$. The functions F_n and G_n are depicted in Figure 1.

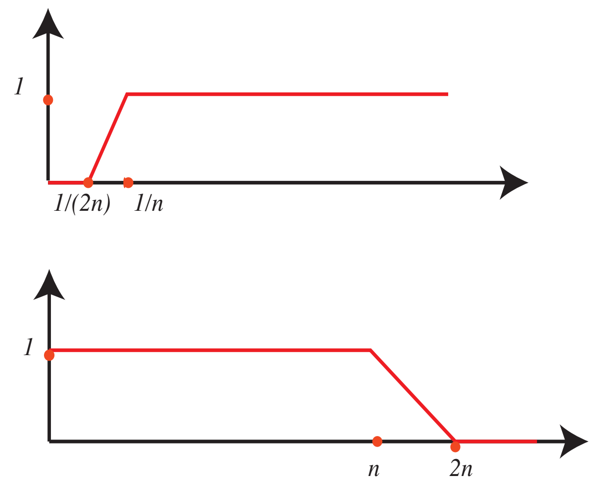

FIGURE 1. *The graphs of F_n (top) and the graph of G_n (bottom).*

For $v \in \mathcal{V}$ we set $J_v := J_X(v)$, i.e., J_v is the Jacobian of X at v. Recall that

$$J_v = \sqrt{\det\big(\, X'(v) X(v)^\top\,\big)} = \det|X'(v)|,$$

where $\det X'(v)$ is computed by identifying $X'(v)$ with a $d \times d$ matrix using the Euclidean coordinates (v_i) and (u_i).

Fix a box $B \subset \mathcal{V}$. For $u \in \boldsymbol{U}$ and $n \in \mathbb{N}$ and $\Phi \in \mathcal{X}$ we set

$$C_u^n(\Phi, B) := \sum_{v \in \Phi^{-1}(u) \cap B} F_n\big(\, J_\Phi(v)\,\big) G_n\big(\, J_\Phi(v)\,\big) F_n\big(\, \operatorname{dist}(v, \partial B)\,\big).$$

Lemma 1.12. *The functions*

$$\mathcal{X} \ni \Phi \mapsto C_u^n(\Phi, B)$$

and $u \mapsto C_u^n(B, \Phi)$ are continuous. □

We proceed assuming the validity of the above lemma. We set

$$C_u^n(B) := C_u^n(B, X) = \sum_{v \in X^{-1}(u) \cap B} F_n\big(\, J_v\,\big) G_n\big(\, J_v\,\big) F_n\big(\, \operatorname{dist}(v, \partial B)\,\big),$$

$$Q_u^n(B) := C_u^n(B) G_n\big(\, C_u^n(B)\,\big).$$

These are measurable as compositions of continuous functions $\mathcal{X} \to \mathbb{R}$ with $X : \Omega \to \mathcal{X}$.

Note that $C_u^n(B)$ is the number of solutions v of the equation $X = u$ in the compact (random) set

$$K_n := \Big\{ v \in B : \;\; J_v, \delta_v \geq \frac{1}{2n}, \;\; J_v \leq 2n \Big\}.$$

Intuitively, $C_u^n(B)$ counts the solutions v of $F(v) = u$ located in B for which that Jacobian J_v is not too small, not too large and they are not too close to the boundary of B. The quantity $Q_u^n(B)$ is a sort of truncation of $C_u^n(B)$. Note that $Q_u^n(B) = 0$ whenever $C_u^n(B) > n$. For simplicity we will write

$$\delta_v := \operatorname{dist}(v, \partial B).$$

Let $g : \boldsymbol{U} \to [0,\infty)$ be a continuous, compactly supported function. The Coarea formula (A.2) implies that

$$\int_{\boldsymbol{U}} g(u) Q^n_u(B) du = \int_B J_v F_n(J_v) G_n(J_v) F_n(\delta_v) G_n\big(\, C^n_{X(v)}(B)\,\big) g\big(\, X(v)\,\big) dv.$$

The above random variables are bounded since the various integrands are bounded. E.g., $Q^n_u(B) \leq 2n$. Taking expectations we deduce

$$\int_{\boldsymbol{U}} g(u) \mathbb{E}\big[\, Q^n_u(B)\,\big] du = \int_B \mathbb{E}\big[\, J_v F_n(J_v) G_n(J_v) F_n(\delta_v) G_n\big(\, C^n_{X(v)}(B)\,\big) g\big(\, X(v)\,\big)\,\big] dv$$

$$= \int_{\boldsymbol{U}} g(u) \left(\int_B \mathbb{E}\big[\, J_v F_n(J_v) G_n(J_v) F_n(\delta_v) G_n\big(\, C^n_{X(v)}(B)\,\big) \,\|\, X(v) = u\,\big] dv \right) p_{X(v)}(u) du$$

Since the above equality holds for any continuous compactly supported function g we deduce

$$\mathbb{E}\big[\, Q^n_u(B)\,\big] = \int_B \mathbb{E}\big[\, J_v F_n(J_v) G_n(J_v) F_n(\delta_v) G_n\big(\, C^n_{X(v)}(B)\,\big) \,\|\, X(v) = u\,\big] p_{X(v)}(u) dv \tag{1.5}$$

for *almost every* $u \in \boldsymbol{U}$. To prove that the above equality holds *for any* u we will show that both sides of (1.5) depend continuously on u.

The random function $u \mapsto C^n_u(B) = C^n_u(B, X)$ is a.s. continuous since

$$u \mapsto C^n_u(B, \Phi)$$

is continuous for any $\Phi \in \mathcal{X}$. Consider

$$\alpha^n_v : \mathcal{X} \to \mathbb{R},\ \ \alpha_v(\Phi) := \underbrace{J_\Phi(v) F_n(J_\Phi(v)) G_n(J_\Phi(v))}_{\leq 2n} F_n(\delta(v)) G_n\big(\, C^n_{\Phi(v)}\,\big).$$

For fixed v it depends continuously with respect to Φ in the topology of $\mathcal{X}$. We can rewrite the right-hand-side of (1.5) as

$$\int_B \mathbb{E}\big[\, \alpha^n_v(X) \big|\, X(v) = u\,\big] p_{X(v)}(u) dv.$$

Conditions ($\mathbf{A}_0$) and ($\mathbf{C}$) show that the integrand depends continuously on u. Clearly it is bounded uniformly in u. The Dominated Convergence Theorem shows that the above integral depends continuously on u. Hence

$$\mathbb{E}\big[\, Q^n_u(B)\,\big] = \int_B \mathbb{E}\big[\, J_v F_n(J_v) G_n(J_v) F_n(\delta_v) G_n\big(\, C^n_{X(v)}(B)\,\big) \big|\, X(v) = u\,\big] p_{X(v)}(u) dv \tag{1.6}$$

for *every* $u \in \boldsymbol{U}$. In particular, for $u = 0$ we deduce

$$\mathbb{E}\big[\, Q^n_0(B)\,\big] = \int_B \mathbb{E}\big[\, J_v F_n(J_v) G_n(J_v) F_n(\delta_v) G_n\big(\, C^n_{X(v)}(B)\,\big) \big|\, X(v) = 0\,\big] p_{X(v)}(0) dv. \tag{1.7}$$

The transversality condition ($\mathbf{T}$) and Lemma 1.5 imply that a.s. 0 is a regular value of X and the equation $X(v) = 0$ has no solutions on ∂B. We deduce that

$$Q^n_0(B) \nearrow Z_B(X) \ \text{ as } n \to \infty.$$

Since $F_n, G_n \nearrow 1$ we can use the Monotone Convergence Theorem in (1.7) as $n \to \infty$ and deduce (**KR**) for $D = d$ assuming the validity of Lemma 1.12.

Proof of Lemma 1.12. The proof is similar to the proof of Lemma 1.9. Fix $\Phi_0 \in \mathcal{X}$ and $u_0 \in \boldsymbol{U}$. For each $n \in \mathbb{N}$ we consider the compact set

$$K_n := \left\{ v \in B;\ \ \mathrm{dist}(v, \partial B) \geq 1/n,\ \ \frac{1}{2n} \leq J_{\Phi_0}(v) \leq 2n \right\}.$$

Note that $K_n \subset \boldsymbol{int}(K_{n+1})$, $\forall n$. Let

$$Z_n(\Phi_0) = \Phi_0^{-1}(u_0) \cap K_n.$$

The inverse function theorem implies that $Z_n(\Phi_0)$ is finite.

$$Z_n(\Phi_0) := \{v_1, \dots, v_n\}$$

Invoking the inverse function theorem we deduce that there exist $r > 0$ and pairwise disjoint open sets $\mathcal{O}_1, \dots, \mathcal{O}_n$ with the following properties.

- $v_k \in \mathcal{O}_k \subset \boldsymbol{int}\, K_{n+1}$, $\forall k = 1, \dots, n$. We set
$$\mathcal{O} := \bigcup_{k=1}^{n} \mathcal{O}_k.$$
- The restriction of Φ_0 to $\mathcal{O}_k$ is a diffeomorphism onto the open ball $B_r(u_0) \subset \boldsymbol{U}$.

Suppose that $\|\Phi_\nu - \Phi_0\|_{C^1(B)} \to 0$ as $\nu \to \infty$. We claim that

$$\exists N > 0 :\ \ \forall \nu \geq N,\ \ \Phi_\nu^{-1}(u_0) \cap K_n \subset \mathcal{O}.$$

We argue by contradiction. Suppose that there exists a subsequence $\nu_m \nearrow \infty$ and and

$$w_{\nu_m} \in \Phi_{\nu_m}^{-1}(u_0) \cap K_n \setminus \mathcal{O},\ \ \forall m \tag{1.8}$$

Upon extracting a subsequence we can assume that w_{ν_m} converges to $w_* \in K_n$. Letting $m \to \infty$ in the equality $\Phi_{\nu_m}(w_{\nu_m}) = u_0$ we deduce $\Phi_0(w_*) = u_0 \in \mathcal{O}$. This contradicts (1.8).

Arguing as in the proof of Lemma 1.9 we conclude that there exists $N > 0$ such that for any $\nu \geq N$ and any $k = 1, \dots, \nu$ the equation $\Phi_\nu(v) = u_0$ has at most one solution $v \in \mathcal{O}_k$.

Let us now observe that for ν sufficiently large the equation $\Phi_\nu(v) = u_0$ has one solution $v \in \mathcal{O}_k$. Indeed this is an immediate consequence of the theory of degree of a continuous map; see e.g. [19, Chap.1].

This proves that for any continuous function $\varphi : B \to \mathbb{R}$ such that $\mathrm{supp}\,\varphi \subset K_n$ we have

$$\lim_{\nu\to\infty} \sum_{v \in \Phi_\nu^{-1}(u_0)} \varphi(v) = \sum_{v \in \Phi_0^{-1}(u_0)} \varphi(v).$$

This takes care of the first part of Lemma 1.12. The second part of this lemma follows from the first part applied to the maps $\Phi_\nu = \Phi_0 - (u_\nu - u_0)$, where $u_\nu \to u_0$. □

Remark 1.13. For a wide ranging generalization of this result we refer to [21]. □

The case $D > d$ is dealt with in a similar fashion. We will not discuss it here since we will not need it in the applications we have in mind. For details we refer to [4, Chap. 11].

1.4. **Gaussian measures and fields.** The assumption (**C**) is trickiest to verify in practice. In this subsection we will describe a simple yet sufficiently general case when this is satisfied. We need to introduce some Gaussian terminology in a form suitable for geometric applications.

Suppose that $\boldsymbol{X}$ is a finite dimensional vector space. We denote by $\boldsymbol{X}^*$ its dual space and we denote by $\langle -,-\rangle : \boldsymbol{X}^*\times\boldsymbol{X}\to\mathbb{R}$ the natural pairing

$$(\xi,x)\mapsto\langle\xi,x\rangle:=\xi(x).$$

A probability measure $\boldsymbol{\Gamma}$ on $\boldsymbol{X}$ is called *Gaussian* if for any linear functional $\xi\in\boldsymbol{X}^*$ the random variable $\xi:\boldsymbol{X}\to\mathbb{R}$ is Gaussian with mean $m(\xi)$ and variance $v(\xi)$. A random vector $X:(\Omega,\mathcal{S},\mathbb{P})\to\boldsymbol{X}$ is called Gaussian if its distribution is is a Gaussian measure on $\boldsymbol{X}$.

Note that the maps $\boldsymbol{X}^*\ni\xi\to m(\xi)\in\mathbb{R}$ is linear so $m(\xi)\in\boldsymbol{X}^{**}\cong\boldsymbol{X}$. The mean of X is the unique vector $m(X)\in\boldsymbol{X}$ such that

$$\langle\xi,m(X)\rangle=\mathbb{E}\big[\,\langle\xi,X\rangle\,\big),$$

i.e,

$$m(X)=\mathbb{E}\big[\,X\,\big]=\int_\Omega X(\omega)\mathbb{P}\big[\,d\omega\,\big].$$

The covariance form of X is the nonnegative definite symmetric bilinear form

$$C_X=\operatorname{Cov}\big[\,X\,\big]:\boldsymbol{X}^*\times\boldsymbol{X}^*\to\mathbb{R}$$

given by

$$\operatorname{Cov}\big[\,X\,\big](\xi_i,\xi_2)=\operatorname{Cov}\big[\,\langle\xi_1,X\rangle,\langle\xi_2,X\rangle\,\big)\,\big]=\mathbb{E}\big[\,\big(\,\langle\xi_1,X\rangle-m(\xi_1)\,\big)\big(\,\langle\xi_2,X\rangle-m(\xi_2)\,\big)\,\big].$$

Note that the space of bilinear forms on $\boldsymbol{X}^*$ can be identified with $\boldsymbol{X}\otimes\boldsymbol{X}$. Viewed as an element of $\boldsymbol{X}\otimes\boldsymbol{X}$ the covariance form has the simple description

$$C_X=\mathbb{E}\big[\,\big(\,X-m(X)\,\big)\otimes\big(\,X-m(X)\,\big)\,\big].$$

The Gaussian vector X is called *nondegenerate* if $\operatorname{Cov}\big[\,X\,\big]$ is *positive* definite. It is called *centered* if $m(X)=0$

Let us observe that a choice of an inner product on $\boldsymbol{X}$ produces a canonical identification of $\boldsymbol{X}^*$ with $\boldsymbol{X}$ and in this case we can identify $\operatorname{Cov}\big[\,X\,\big]$ either with a symmetric bilinear form on X, or with a symmetric operator $\boldsymbol{X}\to\boldsymbol{X}$ that we denote by $\operatorname{Var}\big[\,X\,\big]$. The Gaussian vector X is nondegenerate iff $\operatorname{Var}\big[\,X\,\big]$ is invertible. In this case the distribution of X is

$$\mathbb{P}_X\big[\,dx\,\big]=\underbrace{\frac{1}{\sqrt{\det\big(\,2\pi\operatorname{Var}\big[\,X\,\big]\,\big)}}e^{-\frac{(\operatorname{Var}[X]^{-1}x,x)}{2}}}_{p_X(x)}\boldsymbol{\lambda}\big[\,dx\,\big]$$

where $(-,-)$ denotes the inner product on $\boldsymbol{X}$ and $\boldsymbol{\lambda}$ denotes the Lebesgue measure on $\boldsymbol{X}$.

If $A:\boldsymbol{X}_0\to\boldsymbol{X}_1$ is a linear operator and $X:(\Omega,\mathcal{S},\mathbb{P})\to\boldsymbol{X}_0$ is a Gaussian vector with mean $m(X)$ and covariance form C_X, then AX is also Gaussian with mean AX and variance operator form

$$\operatorname{Cov}\big[\,AX\,\big](\xi,\eta)=\operatorname{Cov}\big[\,X\,\big](A^\top\xi,A^\top\eta),$$

where $A^\top:\boldsymbol{X}_1^*\to\boldsymbol{X}_0^*$ is the dual map. If we equip $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$ with inner products, then the variance operator of AX is given by

$$\operatorname{Var}\big[\,AX\,\big]=A\operatorname{Var}\big[\,X\,\big]A^*:\boldsymbol{X}_1\to\boldsymbol{X}_1,$$

where $A^*:\boldsymbol{X}_1\to\boldsymbol{X}_1$ is the adjoint of A determined by the chosen metrics.

Suppose that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are finite dimensional vector spaces. Given random vectors

$$X : (\Omega, \mathcal{S}, \mathbb{P}) \to \boldsymbol{X}, \;\; Y : (\Omega, \mathcal{S}, \mathbb{P}) \to \boldsymbol{Y}$$

we define the covariance form of Y and X to be the bilinear form

$$\operatorname{Cov}\big[\, Y, X \,\big] : \boldsymbol{Y}^* \times \boldsymbol{X}^* \to \mathbb{R}$$

given by

$$\operatorname{Cov}\big[\, Y, X \,\big](\eta, \xi) = \operatorname{Cov}\big[\, \langle \eta, Y\rangle, \langle \xi, X\rangle \,\big], \;\; \forall \eta \in \boldsymbol{Y}^*, \; \xi \in \boldsymbol{X}^*.$$

If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are equipped with inner products $(-,-)_{\boldsymbol{X}}$ and respectively $(-,-)_{\boldsymbol{Y}}$, then we can identify $\operatorname{Cov}\big[\, Y, X \,\big]$ as a linear operator $C_{Y,X} : \boldsymbol{X} \to \boldsymbol{Y}$ uniquely determined by the condition

$$\big(\, y, C_{Y,X}x \,\big)_Y = \operatorname{Cov}\big[\, (y, Y)_{\boldsymbol{Y}} (x, X)_{\boldsymbol{X}} \,\big], \;\; \forall x \in \boldsymbol{X}, \;\; y \in \boldsymbol{Y}.$$

Concretely, if $(\boldsymbol{e}_i)_{i\in I}$ and $(\boldsymbol{f}_j)_{j\in J}$ are *orthonormal* bases of $\boldsymbol{X}$ and respectively $\boldsymbol{Y}$, and we set $X_i := (\boldsymbol{e}_i, X)_{\boldsymbol{X}}$, $Y_j := (\boldsymbol{f}_j, Y)_{\boldsymbol{Y}}$, then in these bases the operator $C_{Y,X}$ is describe by matrix $(c_{ji})_{(j,i)\in J\times I}$, where $c_{ji} := \operatorname{Cov}\big[\, Y_j X_i \,\big]$. Hence

$$C_{Y,X}\boldsymbol{e}_i = \sum_j c_{ji}\boldsymbol{f}_j.$$

Let us observe that $C_{X,Y} : \boldsymbol{X} \to \boldsymbol{Y}$ is the adjoint/transpose of $C_{Y,X}$. Note that if $T : \boldsymbol{X} \to \boldsymbol{U}$ is a linear map between Euclidean spaces, then

$$C_{Y,TX} = C_{Y,X} \circ T^* : \boldsymbol{U} \to \boldsymbol{Y}.$$

The random vectors are said to be *jointly Gaussian* if the random vector

$$X \oplus Y : (\Omega, \mathcal{S}, \mathbb{P}) \to \boldsymbol{X} \oplus Y$$

is Gaussian. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are equipped with inner products, then $\boldsymbol{X} \oplus \boldsymbol{Y}$ is equipped th the direct sum of these inner products and in this case $\operatorname{Var}\big[\, X \oplus Y \,\big] : \boldsymbol{X} \oplus \boldsymbol{Y} \to \boldsymbol{X} \oplus \boldsymbol{Y}$ admits the bloc decomposition

$$\operatorname{Var}\big[\, X \oplus Y \,\big] = \left[\begin{array}{cc} \operatorname{Var}\big[\, X \,\big] & C_{X,Y} \\ C_{Y,X} & \operatorname{Var}\big[\, Y \,\big] \end{array}\right].$$

Proposition 1.14 (Gaussian regression formula)**.** *Suppose that X, Y are Gaussian vectors valued in the Euclidean spaces $\boldsymbol{X}$ and respectively $\boldsymbol{Y}$. Assume additionally that*

(i) *the random vectors X, Y are jointly Gaussian and,*
(ii) *X is nondegenerate.*

Define the regression operator

$$R_{Y,X} : \boldsymbol{X} \to \boldsymbol{Y}, \;\; R_{Y,X} := C_{Y,X} \operatorname{Var}[X]^{-1}$$

Then the following hold.

(a) The conditional expectation $\mathbb{E}\big[\, Y \,\|\, X \,\big]$ is the Gaussian vector described by the linear regression formula

$$\mathbb{E}\big[\, Y \,\|\, X \,\big] = m(Y) - R_{Y,X}m(X) + R_{Y,X}X \tag{1.9}$$

(b) For any $x \in X$

$$\mathbb{E}\big[\, Y \,\big|\, X = x \,\big] = m(Y) - R_{Y,X}m(X) + R_{Y,X}x.$$

(c) The random vector vector $Z = Y - \mathbb{E}\big[\, Y \,\|\, X \,\big]$ is Gaussian and independent of X. It has mean 0 and variance operator

$$\Delta_{Y,X} = \operatorname{Var}\big[\, Y \,\big] - D_{Y,X} : \boldsymbol{Y} \to \boldsymbol{Y}, \;\; D_{Y,X} = C_{Y,X} \operatorname{Var}[X]^{-1} C_{X,Y}. \tag{1.10}$$

Moreover, for any bounded measurable function $f : \boldsymbol{Y} \to \mathbb{R}$ and any $x \in \boldsymbol{X}$ we have

$$\mathbb{E}\big[\, f(Y) \big|\, X = x \,\big] = \mathbb{E}\big[\, f\big(\, Z + m(Y) - R_{Y,X} m(X) + R_{Y,X} x \,\big)\,\big]. \tag{1.11}$$

□

For a proof we refer to Appendix B. We will refer to the distribution of the centered Gaussian vector Z as the *regression Gaussian measure*.

A random field $X : \Omega \times \mathcal{V} \to \boldsymbol{U}$ is called Gaussian if for any finite subset $S \subset \mathcal{V}$ the randdom vector

$$X|_S : \Omega \to \boldsymbol{U}^S, \;\; \omega \mapsto \big(\, X_\omega(s) \,\big)_{s \in S}$$

is Gaussian.

1.5. **The Gaussian Kac-Rice formula.** We now return to the setup of Theorem 1.1.

Proposition 1.15 (Azaïs-Wschebor)**.** *Suppose that the random field*

$$X : \Omega \times \mathcal{V} \to \boldsymbol{U}, \;\; (\omega, v) \mapsto X_\omega(v)$$

is Gaussian, it is a.s. C^1 *and,*

$$\textit{for any } v \in \mathcal{V} \textit{ the Gaussian random vector } X(v) : \Omega \to \boldsymbol{U} \textit{ is nondegenerate.} \tag{$\mathbf{N}_0$}$$

Then X satisfies all the conditions of ($\mathbf{R}_1$), ($\mathbf{A}_0$) and ($\mathbf{C}$) in Theorem 1.1.

Proof. The condition ($\mathbf{R}_1$) is tautologically satisfied while ($\mathbf{A}_0$) follows from the nondegeneracy of $X(v)$ for any v.

To prove ($\mathbf{C}$) we will use the regression trick in [4, Eq. (6.11)]. For simplicity we assume that X is centered and $X_\omega \in \mathcal{X}$, $\forall \omega \in \Omega$.

Fix $u_0 \in \boldsymbol{U}$ and $v_0 \in \mathcal{V}$. From the regression formula we deduce that any $v \in V$ we have

$$X(v) = R_{X(v), X(v_0)} X(v_0) + Z(v, v_0), \;\; Z(v, v_0) \perp\!\!\!\perp X(v_0).$$

and we have

$$\partial_v X_\omega(v) = \big(\, \partial_v R_{X(v), X(v_0)} \,\big) X_\omega(v_0) + \partial_v Z_\omega(v, v_0).$$

Hence, for any ω the map

$$v \mapsto Z_\omega(v, v_0)$$

is also C^1. The resulting map $\mathcal{V} \ni v_0 \mapsto Z_\omega(-, v_0) \in \mathcal{X}$ is continuous for any ω.

Fix a continuous and bounded function $\alpha : \mathcal{X} \to \mathbb{R}$. Then the real number

$$\alpha(X_\omega) = \alpha\big(\, R_{X_\omega(-), X_\omega(v_0)} u_0 + Z_\omega(-, v_0) \,\big)$$

depends continuously on (u_0, v_0) for any ω and since α is bounded we deduce from the Dominated Convergence Theorem that

$$\mathbb{E}\big[\, \alpha(X) \big|\, X(v_0) = u_0 \,\big] = \mathbb{E}\big[\, \alpha\big(\, R_{X, X(v_0)} u_0 + Z(-, v_0) \,\big)\,\big]$$

depends continuously on (u_0, v_0). □

We can now formulate the Gaussian Kac-Rice formula in the case $D = d$.

Theorem 1.16. *Suppose that $\boldsymbol{U}$, $\boldsymbol{V}$ are Euclidean spaces of the same dimension d, $\mathcal{V} \subset \boldsymbol{V}$ is an open set and $G : \mathcal{V} \to \boldsymbol{U}$ is a Gaussian random field that is* a.s. C^1 *and satisfies ($\mathbf{N}_0$). For any $v \in \mathcal{V}$ we denote by $p_{G(v)}$ the probability density of the Gaussian vector $G(v)$. For any $S \subset \mathcal{U}$ we set $Z_S(G) = \# G^{-1}(0) \cap S$. Then for any Borel subset $S \subset \mathcal{V}$ we have*

$$\mathbb{E}\big[\, Z_S(G) \,\big] = \int_S \rho_G(v) dv,$$

where

$$\rho_G(u) := \mathbb{E}\big[\, J_G(u) \,\big|\, G(v) = 0 \,\big] p_{F(v)}(0).$$

Moreover, $\mathbb{E}\big[\, Z_S(G) \,\big] < \infty$ *if* S *is compact.*

Proof. We proved the equality when S is a box. In this case the right-hand-side of the above equality is finite. Both sides of the above equality are σ-finite Borel measures on $\mathcal{U}$ that agree on boxes and thus they must agree for any S. □

Remark 1.17. (a) Note that in the special case $\boldsymbol{U} = \boldsymbol{V}$, we have

$$J_G(u) = |\det G'(u)|,$$

where $\det A$ denotes the determinant of a linear map $A : \boldsymbol{U} \to \boldsymbol{U}$. In particular, if G is the gradient of a function F, $G = \nabla F$, then

$$J_G(u) = |\det H_F(u)|,$$

where $H_F(u)$ is the Hessian of the function F at u viewed as a symmetric operator $H_F(u) : \boldsymbol{U} \to \boldsymbol{U}$.

(b) Let G be as in Theorem 1.16. For any continuous compactly supported function $\varphi \in C^0_{\mathrm{cpt}}(\mathcal{V})$ we set

$$Z_\varphi(G) := \sum_{G(u):=0} \varphi(0).$$

The above arguments can be modified to yield the weighted Kac-Rice formula

$$\mathbb{E}\big[\, Z_\varphi(G) \,\big] = \int_{\mathcal{U}} \varphi(u)\rho_G(u)du \tag{1.12}$$

□

1.6. **Zeros of random sections.** Suppose that $\mathcal{U}$ is a smooth d-dimensional manifold and $\pi_E : E \to \mathcal{U}$ is a smooth, rank d, real vector bundle over $\mathcal{U}$. Fix a smooth metric g on $\mathcal{U}$.

There is a high-brow method of defining a random Gaussian section of $\boldsymbol{E}$ (see [17]), but we want avoid technicalities and we take a more pedestrian approach. A Gaussian section is a random map $G : \mathcal{U} \to E$ such that

- $\pi_E \circ F = \mathbb{1}_M$ and,
- for any $p \in \mathbb{N}$, $u_1, \dots, u_p \in \mathcal{U}$ the random vector $\big(G(u_1), \dots, G(u_p) \big)$ is a Gaussian vector valued in $E_{u_1} \times \cdots \times E_{u_p}$.

We can locally view G as a Gaussian random map $G : \boldsymbol{U} \to \mathbb{R}^d$. We will make the following standing assumption

> We assume that G is a.s. C^1 and, $\forall u \in \mathcal{U}$, the random vector $G(0)$ is a centered nondegenerate E_u-valued Gaussian vector.

This assumption guarantees that G intersects the zero section transversally a.s.. Thus, the zero set of G is a.s. locally finite set. We denote by $p_{G(u)}$ the probability density of $F(u)$ so that

$$p_{G(u)}(0) = \frac{1}{\sqrt{\det\big(2\pi \operatorname{Var}\big[\, G(u) \,\big] \big)}}.$$

Fix a metric h and connection ∇ on E. Note that for any C^1 -section s of E and any $u \in \mathcal{U}$ we have

$$\nabla s(u) \in T^*_u\mathcal{U} \otimes E_u \cong \operatorname{Hom}\big(T_u\mathcal{U}, E_u \big).$$

We denote by $\operatorname{adj}_s(u)$ the linear map $\nabla s(u) : T_u^*\mathcal{U} \to E_u$. It depends on the choice of the connection ∇ but, if $s(u) = 0$, it is independent of the connection. In this case it is the differential-geometric cousin of the adjunction map in algebraic geometry.

We define the Jacobian of s at u to be the Jacobian of the map linear map $\operatorname{adj}_s(u)$ between two Euclidean spaces

$$J_s(u) = \sqrt{\det\big(\operatorname{adj}_s(u) \circ (\operatorname{adj}_s(u))^*\big)}.$$

For any continuous, compactly supported function $\varphi \in C^0_{\mathrm{cpt}}(\mathcal{U})$ we define the a.s. finite random variable

$$Z_\varphi(G) := \sum_{G(u):=0} \varphi(0).$$

Then, in this case, the Kac-Rice formula reads

$$\mathbb{E}\big[\, Z_\varphi(G)\,\big] = \int_{\mathcal{U}} \varphi(u)\rho_G(u)\, |dV_g(u)|, \tag{1.13}$$

where

$$\rho_G(u) = \mathbb{E}\big[\, J_G(u)\,\big|\, G(u) = 0\,\big] p_{G(u)}(0). \tag{1.14}$$

When E is the trivial vector bundle $\underline{\boldsymbol{V}}_{\mathcal{U}} := \mathcal{U} \times \boldsymbol{V} \to \mathcal{U}$ equipped with the trivial metric and connection we obtain the usual Kac-Rice formula.

Remark 1.18. A priori, the function $\mathbb{E}\big[\, J_G(u)\,\big|\, G(u) = 0\,\big] p_{G(0)}$ depends on the choice of metrics on $\mathcal{U}$ and E and the connection ∇ needed to define $J_G(u)$. However, the dependence of ∇ disappears when we condition $G(u) = 0$.

It is not hard to verify that the Kac-Rice measure (or 1-density in differential-geometric sense, [18, Sec. 3.4.1]) on $\mathcal{U}$

$$KR_G := \rho_G(u)\, |dV_g(u)| \tag{1.15}$$

is independent of the choice of metric on $\mathcal{U}$. Indeed, the Jacobian admits the alternate description via the equality

$$J_G(u)|dV_{T_u\mathcal{U}}| = \operatorname{adj}_s(u)^* |dV_{E_u}|$$

Above, $\operatorname{adj}_u(s)^*$ denotes the pullback between spaces of 1-densities. The 1-density KR_G is also independent of the choice of metric on E since the quantity

$$J_G(u) p_{G(u)}(0) = \frac{J_G(u)}{\sqrt{\det\big(2\pi \operatorname{Var}[G(u)]\big)}}$$

is independent of the metric on E_u.

The Kac-Rice measure is Radon in the sense that it is finite on compact sets when $G(u)$ is nondegenerate. □

2. Critical points of Gaussian random functions

Let $\boldsymbol{U}$ be a d-dimensional Euclidean space and $\mathcal{U} \subset \boldsymbol{U}$. Suppose that $F : \mathcal{U} \to \mathbb{R}$ is a Gaussian random function. This means that, for any $u \in \boldsymbol{U}$ the value $F(u)$ is a Gaussian random variable. For simplicity we assume that its mean is zero

$$\mathbb{E}\big[\, F(u)\,\big] = 0, \;\; \forall u \in \mathcal{U}.$$

Let $\boldsymbol{K} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ be the covariance kernel of F,i.e.,

$$\boldsymbol{K}(u_0, u_1) = \mathbb{E}\big[\, F(u_0)F(u_1)\,\big], \;\; \forall u_0, u_1 \in \boldsymbol{U}.$$

We assume that F is a.s. C^2. Throughout this section we fix Euclidean coordinates $(u^i)_{1\leq i\leq f}$ on $\boldsymbol{U}$. Fix box $B\subset \mathcal{U}$ and denote by Z_B the number of critical points of F in B, i.e.,

$$Z_B := Z_B(\nabla F) = \#\big\{\, u\in B;\ \ \nabla F(u)=0\,\big\}.$$

We want to investigate two basic invariants of this random variable: its mean and its variance.

2.1. **Expectation.** We assume that $\forall u\in\mathcal{U}$

$$\text{the Gaussian vector } G(u)=\nabla F(u) \text{ is nondegenerate.} \tag{$\mathbf{N}_1$}$$

We denote by $p_{F(u)}$ the probability density of the Gaussian vector $\nabla F(u)$ and by $H_F(u)$ the Hessian of F at u. The Gaussian Kac-Rice formula. implies that

$$\mathbb{E}\big[\, Z_B(F)\,\big] = \int_B \rho_F(u)du,$$

where

$$\rho_F(u) = \mathbb{E}\big[\, \big|\,\det H_F(u)\,\big|\,\big|\,\nabla F(u)=0\,\big] p_{\nabla F(0)}(0)$$

$$p_{\nabla F(u)}(0) = \frac{1}{\sqrt{2\pi \det \operatorname{Var}\big[\,\nabla F(u)\,\big]}}.$$

2.2. **An analytic digression: Kergin interpolation.** The one-dimensional case of this technique goes back to Newton. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and $x_1,\dots,x_p$ are distinct points on the real axis, We define inductively the divided differences $f[x_1], f[x_1,x_2],\dots,f[x_1,\dots,x_p]$ by setting

$$f[x]=f(x),\ \ \forall x\in\mathbb{R},$$

$$f[x_1,x_2]=\frac{f[x_1]-f[x_2]}{x_1-x_2}$$

$$f[x_1,x_2,x_3]=\frac{f[x_1,x_2]-f[x_2,x_3]}{x_1-x_3}$$

$$f[x_1,x_2,\dots,x_k,x_{k+1}]=\frac{f[x_1,\dots,x_k]-f[x_2,\dots,x_{k+1}]}{x_1-x_{k+1}}\cdots$$

For simplicity we will write $f[\underline{x}]=f[x_1,\dots,x_p]$. For distinct $x_1,\dots,x_n$ we have the following more explicit description (see [14, Sec. 1.3])

$$f[x_1,\dots,x_n]=\sum_{j=1}^n\frac{f(x_j)}{\prod_{k\neq j}(x_j-x_k)}.$$

If $f\in C^p$, then we have an alternate integral representation of $f[x_0,\dots,x_p]$ called *Hermite-Genocchi formula*

$$f[z_0,x_1,\dots,x_p]=\int_0^1 ds_1\int_0^{s_2}ds_3\cdots\int_0^{s_{p-1}} f^{(p)}(y_p)ds_p, \tag{2.1}$$

where

$$y_p=p_p(s_1,\dots,s_p)=(1-s_1)x_0+(s_1-s_2)x_1+\cdots+(s_{p-1}-s_p)x_p,$$

$$1\geq s_1\geq\cdots\geq s_p\geq 0.$$

We refer to [14, Sec. 16] or [7, Thm. 1.9] for a proof. Note that this formula assumes that f is p-times differentiable. We can rephrase in more revealing terms as follows. Consider the simplex

$$\Delta_p = \Big\{ (t_0, t_1, \dots, t_p) \in [0,1]^{p+1};\ \ \sum_{k=0}^p t_k = 1 \Big\}.$$

It is equipped with an Euclidean volume element $\mu_p\big[\, dt\,\big]$ normalized so that $\mu_p\big[\, \Delta_p\,\big] = \frac{1}{p!}$.

Given $\underline{x} = (x_0, x_1, \dots, x_p) \in \mathbb{R}^{p+1}$ we define

$$\sigma_{\underline{x}} : \Delta_p \to \mathbb{R},\ \ \sigma_{\underline{x}}(t) := \sum_{k=0}^p t_k x_k.$$

Then (2.1) can be rewritten as

$$f[\underline{x}] = \int_{\Delta_p} f^{(p)}\big(\, \sigma_{\underline{x}}(t)\,\big)\mu_p\big[\, dt\,\big]. \tag{2.2}$$

The righ-hand-side of the above equality is symmetric in the variables $x_0, \dots, x_p$, depends continuosly on them and it is well defined even if some of them coincide. This allows us to define $f[x_0, x_1, \dots, x_p]$ even if the numbers $x_0, \dots, x_p$ are not pairwise distinct provided that $f \in C^p$. For example

$$f[x_1, x_1] = \lim_{x_2 \to x_1} f[x_1, x_2] = f'(x_1),$$

$$f[x_1, x_1, x_2] = \frac{f[x_2, x_1] - f'(x_1)}{x_2 - x_1}$$

More generally, if the function $f(x)$ is C^k, then the function $g(x) = f[x, x_2]$ is C^{k-1} and

$$f[x_1, x_2, x_3] = g[x_1, x_3].$$

In general, for distinct $x, x_1, \dots, x_p$, we have the equality (see [14, Sec. 1.1])

$$\begin{split} f(x) = \underbrace{f(x_1) + \sum_{j=1}^{p-1}(x - x_1)\cdots(x - x_j)f[x_1, \cdots, x_{j+1}]}_{=:\boldsymbol{P}_{x_1,\dots,x_p}f(x)} \\ +(x - x_1)\cdots(x - x_p)f[x, x_1, \dots, x_p]. \end{split} \tag{2.3}$$

The term $\boldsymbol{P}_{x_1,\dots,x_p}f(x)$ is a polynomial of degree $\leq (p-1)$ in x and the above formula is called *Newton's interpolation formula*. The above equality shows that

$$\boldsymbol{P}_{x_1,\dots,x_p}f(x_i) = f(x_i),\ \ \forall i = 1, \dots, p.$$

The divided difference $f[x_1, \dots, x_p]$ is well defined even if the numbers $x_1, \dots, x_p$ are not pairwise distinct and thus (2.3) holds for any $x, x_1, \dots, x_p \in \mathbb{R}$, provided that $f \in C^p$. Note that if $x_1 = \cdots = x_m$, then (2.3) implies that

$$\partial_x^k \boldsymbol{P}_{x_1,\dots,x_m}f(x_1) = \frac{1}{k!}\partial_x^k f(x_1)\ \ \forall 0 \leq k < m.$$

If we set

$$[x]_m := \underbrace{x_0, \dots x_0}_{m},$$

then

$$\boldsymbol{P}_{[x_0]_m}(x) = \sum_{j=1}^{m} \frac{1}{(j-1)!} f^{(j-1)}(x-x_0)^{j-1}.$$

is the degree $m-1$ Taylor polynomial of f at x_0.

Let us observe that for f continuous and injective $\underline{x} : \mathbb{I}_p \to \mathbb{R}$ the polynomial $Q = \boldsymbol{P}_{\underline{x}} f$ is the Lagrange interpolation polynomial, i.e., the unique polynomial Q of degree $\leq p-1$ such that

$$Q(x_i) = f(x_i), \;\; \forall i = 1, \dots, p.$$

This proves that $P_{\underline{x}}$ is a projector, i.e.,

$$\boldsymbol{P}_{\underline{x}}^2 f = \boldsymbol{P}_{\underline{x}} f, \;\; \forall f \in C(\mathbb{R}),$$

and that $\boldsymbol{P}_{\underline{x}}$ is invariant under the action of $\mathfrak{S}_p$ on $\mathbb{R}^p$. Moreover, for any $I \subset \mathbb{I}_p$ we have

$$\boldsymbol{P}_{\underline{x}} f(\underline{x}_I) = f(\underline{x}_I).$$

The continuous dependence $\underline{x} \to \boldsymbol{P}_{\underline{x}}$ shows that, for any $\underline{x} \in \mathbb{R}^p$ and any $I \subset \mathbb{I}_p$, is a symmetric projector of $C^{p-1}(\boldsymbol{R})$ i.e., for any permutation $\varphi \in \mathfrak{S}_p$

$$\boldsymbol{P}_{\underline{x}}^2 f = \boldsymbol{P}_{\underline{x}} f = \boldsymbol{P}_{\underline{x} \cdot \varphi} f, \;\; \forall f \in C^{p-1}(\mathbb{R}), \tag{2.4}$$

and

$$\boldsymbol{P}_{\underline{x}} = \boldsymbol{P}_{\underline{x}_I}. \tag{2.5}$$

Formula (2.2) is the basis of the higher dimensional generalization of the above classical facts, [11, 13].

Fix a d-dimensional Euclidean space $\boldsymbol{U}$ and $\mathcal{U} \subset \boldsymbol{U}$ an open convex subset. Given a function $f : C^p(\mathcal{U})$ and $1 \leq k \leq p$, the k-th differential of f at $u \in \mathcal{U}$, denoted by $D^k f(u)$, is a symmetric k-linear form on $\boldsymbol{U}$,

$$D^k f(u) \in \mathbf{Sym}_k(\boldsymbol{U}).$$

Given $\underline{u} = (u_0, u_1, \dots, u_k) \in \mathcal{U}^{k+1}$ we define

$$\sigma_{\underline{u}} = \sigma_{\underline{u}}^k : \Delta_k \to \mathcal{U}, \;\; \sigma_{\underline{u}}(t) := \sum_{i=1}^{k} t_i u_i,$$

and

$$f[\underline{u}] = \int_{\Delta_k} D^k f\big(\, \sigma_{\underline{u}}(t)\,\big) \mu_k\big[\, dt\,\big] \in \mathbf{Sym}_k(\boldsymbol{U}).$$

Given $u_0, u_1, \dots, u_p \in \mathcal{U}$ we define the *Kergin interpolator* of f to be the polynomial of degree $\leq p$ in u,

$$\boldsymbol{P}_{u_0, u_1, \dots, u_p} f(u) = f(u_0) + \sum_{k=1}^{p} f[u_0, \dots, u_k](u-u_0, \dots, u-u_{k-1})$$

Suppose that f is a *ridge function*, i.e., there exists a C^p-function $g : \mathbb{R} \to \mathbb{R}$ and a linear form $\xi \in \boldsymbol{U}^*$ such that $f(u) = g\big(\, \xi(u)\,\big)$. Then

$$f\big[\, u_0, \dots, u_k\,\big] = g\big[\, \xi_0, \dots, \xi_k\,\big], \;\; \xi_k = \xi(u_k), \;\; 0 \leq k \leq p.$$

In particular

$$\boldsymbol{P}_{u_0, u_1, \dots, u_p} f(u) = \boldsymbol{P}_{\xi_0, \dots, \xi_p} g(x), \;\; x = \xi(u)$$

Thus

$$\boldsymbol{P}_{u_0, \dots, u_p} f(u_k) = f(u_k), \;\; \forall 0 \leq k \leq p, \tag{2.6}$$

for any function f that is a linear combination of ridge functions. The linear span of ridge functions contains the space of polynomials (see [7, Lemma 9.11]) which is dense in $C^P(\mathcal{U},\mathbb{R})$ so (2.6) holds for any $f\in C^p(\mathcal{U})$.

A similar argument shows that $\boldsymbol{P}_{u_0,\dots,u_p}f$ is symmetric in the variables $u_0,u_1,\dots,u_p$.

Given $q\leq p$ and $\underline{u}=(u_0,u_1,\dots,u_p)\in\mathcal{U}^{p+1}$, we set $[\underline{u}]_q:=(u_0,\dots,u_q)$. We have

$$\boldsymbol{P}_{[\underline{u}]_q}\boldsymbol{P}_{\underline{u}}=\boldsymbol{P}_{[\underline{u}]_q}. \tag{2.7}$$

Indeed, this is true when $d=1$ and thus it is true for arbitrary d and f a ridge function. The conclusion follows by linearity and density. In particular, when $q=p$ the above equality shows that $\boldsymbol{P}_{\underline{u}}$ is a projector. For this reason we will also refer to $\boldsymbol{P}_{\underline{u}}$ as *Kergin projector*.

Let $p\geq 1$. We denote by $\mathbb{R}_p\big[\,U\,\big]$ the vector space of polynomials of degree $\leq p$ in $u=(u^1,\dots,u^d)$. Define

$$m_i:\mathcal{U}^{p+1}\to\mathbb{N},\;\; m_i(u_0,u_1,\dots,u_p)=\#\big\{\,k;\;\; u_k=u_i\,\big\}.$$

We refer to $m_i(\underline{u})$ the multiplicity of u_i in $\underline{u}=(u_0,\dots,u_p)$, i.e., the number of terms in the sequence of points $u_0,\dots,u_p$ equal to u_i. We have the following result, [11, 13].

Theorem 2.1. *Let $\underline{u}\in\mathcal{U}^p$. The map*

$$\boldsymbol{P}_{\underline{u}}:C^p(\mathcal{U})\to\mathbb{R}_p\big[\,U\,\big]\subset C^p(\mathcal{U}),\;\; f\mapsto\boldsymbol{P}_{\underline{u}}f,$$

is a linear continuous projector, i.e., $\boldsymbol{P}_{\underline{u}}^2=\boldsymbol{P}_{\underline{u}}$. It depends continuously on $\underline{u}$. Moreover, for any $i=0,1,\dots,p$ and any multi-index $\alpha\in\mathbb{N}_0^d$ such that $|\alpha|<m_i(u)$ we have

$$\partial^\alpha\boldsymbol{P}_{\underline{u}}f(u_i)=\partial^\alpha f(u_i). \tag{2.8}$$

□

The Kergin interpolator extends in an obvious way to maps $G:=C^p(\mathcal{O},\boldsymbol{U})$. More precisely, for any $\underline{u}\in\mathcal{U}^{p+1}$ the interpolator $\boldsymbol{P}_{\underline{u}}G$ is the unique polynomial map $\mathcal{U}\to\boldsymbol{U}$ of degree $\leq p$ such that, for any linear functional $\xi\in\boldsymbol{U}^*$ we have

$$\xi\big(\,\boldsymbol{P}_{\underline{u}}G\,\big)=\boldsymbol{P}_{\underline{u}}\xi(G).$$

Even more explicitly, using the Euclidean coordinates $(u^1,\dots,u^d)$ on $\boldsymbol{U}$ we can view G is a d-tuple of functions

$$G=\left[\begin{array}{c}G^1\\ \vdots\\ G^d\end{array}\right],$$

and then

$$\boldsymbol{P}_{\underline{u}}G:=\left[\begin{array}{c}\boldsymbol{P}_{\underline{u}}G^1\\ \vdots\\ \boldsymbol{P}_{\underline{u}}G^d\end{array}\right].$$

In the investigation of the finite of the variance of Z_B we will need the following result of Gass and Stecconi [10, Lemma 2.5].

Lemma 2.2. *Let $\underline{u}^*=(u_0^*,u_1,\dots,u_p^*)\in\mathcal{U}^p$ and $f\in C^{p+1}(\mathcal{U})$. Then for any $k=0,1,\dots,p$ and any $i,j\in\{1,\dots,d\}$ we have*

$$\partial_{u^j}\big(\,\boldsymbol{P}_{\underline{u}^*}\partial_{u^i}f\,\big)=\partial_{u^i}\big(\,\boldsymbol{P}_{\underline{u}^*}\partial_{u^j}f\,\big).$$

In other words the polynomial vector field

$$(V_1,\dots,V_n)=\boldsymbol{P}_{\underline{u}^*}\nabla f=\big(\,\boldsymbol{P}_{\underline{u}^*}\partial_{u_1}f,\dots,\boldsymbol{P}_{\underline{u}^*}\partial_{u_n}f\,\big)$$

is a gradient vector field, i.e., there exists a polynomial $h \in \mathbb{R}_{p+1}\big[\boldsymbol{U}\big]$ *such that* $\nabla h = \boldsymbol{P}_{\underline{u}^*}\nabla f$.

Proof. We first prove that the lemma is true for ridge functions. By choosing the Euclidean coordinates $(u^1,\dots,u^d)$ carefully this means that $f(u)$ has the form $f(u^1,\dots,u^d) = f(u^1)$. In this case the Lemma is obvious since $\boldsymbol{P}_{\underline{u}}f$ it is a polynomial of degree p in u^1. The general case follows from the density in $C^{p+1}(\mathcal{U})$ of the linear span of ridge functions. □

2.3. **Variance.** We now have all the background material needed for proving a sufficient condition for the finiteness of $\operatorname{Var}\big[\,Z_B\,\big]$. We use an ad-hoc method that works only for the 2-momentum of Z_B. For different approaches that work arbitrary moments of Z_B but related approach we refer to [2, 10].

Let $F:\mathcal{U}\to\mathbb{R}$ be a Gaussian random function that is a.s. C^3. Let

$$\mathcal{U}^2_* := \mathcal{U}^2\setminus\Delta,$$

where Δ is the diagonal

$$\Delta := \big\{\,(u_0,u_1)\in\mathcal{U}^2;\;\; u_0=u_1\,\big\}.$$

Define B^2_* in a similar fashion. Consider the random field

$$\widehat{G} = \widehat{G}_F : \mathcal{U}^2_* \to \boldsymbol{U}\oplus\boldsymbol{U},\;\; \hat{G}(u_0,u_1) = \nabla F(u_0)\oplus\nabla F(u_1)$$

For $S\subset\mathcal{U}^2*$ we denote by $Z_S(\widehat{G})$ the number of solution of the equation

$$\widehat{G}(u_0,u_1) = (0,0),\;\; (u_0,u_1)\in S.$$

Note that

$$Z_{B^2_*}(\widehat{G}) = Z_B(F)\big(\,Z_B(F)-1\,\big).$$

We need to impose some conditions in order to apply the Kac-Rice formula. For any $u\in\mathcal{U}$ we denote by $T_2[F,u]\in\mathbb{R}_2\big[\,\boldsymbol{U}\,\big]$ the degree 2 Taylor polynomial of F at u. We will assume that for any $u\in\mathcal{U}$

$$\text{the Gaussian vector } T_2[F,u] \text{ is nondegenerate,} \tag{$\mathbf{N}_2$}$$

We will identify the second differential $D^2F(u)$ of F at u with the Hessian of F at u denoted by $H_F(u)$. Observe that ($\mathbf{N}_2$) implies ($\mathbf{N}_1$).

The fact that $T_2F(u)$ is nondegenerate has the following immediate consequence.

Lemma 2.3. *For any compact subset* $K\subset\mathcal{U}$ *the random variable*

$$C(K) := \sum_{|\alpha|\leq 2}\sup_{u\in K}\big|\,\partial^\alpha_u F(u)\,\big]$$

is p*-integrable for any* $p\in[1,\infty)$. □

The differential of $\widehat{G}$ at (u_0,u_1) is the direct sum of the differentials of G' at u_0 and u_1. These coincide with the Hessians of F at u_0 and u_1,

$$\widehat{G}'(u_0,u_1) = G'(u_0)\oplus G'(u_1).$$

The Gaussian random field G is a.s. C^2. To apply the Kac-Rice formula we need to assume that for any $(u_0,u_1)\in\mathcal{U}^2_*$

$$\text{the Gaussian vector } \widehat{G}(u_0,u_1) \text{ is nondegenerate.} \tag{$\mathbf{N}\times\mathbf{N}$}$$

We will see soon that ($\mathbf{N}_2$) implies ($\mathbf{N}\times\mathbf{N}$), at least if u_0 and u_1 are not too far apart.

Set

$$\mathscr{W} = \nabla C^3(\mathcal{U}),\;\; \mathscr{V} := \nabla\big(\,\mathbb{R}_3\big[\,U\,\big]\,\big),\;\; \mathscr{V}_0 = \nabla\big(\,\mathbb{R}_2\big[\,\boldsymbol{U}\,\big]\,\big).$$

Proposition 2.4. *Let $F : \mathcal{U} \to \mathbb{R}$ be a Gaussian random function that is* a.s. C^3 *and satisfies* ($\mathbf{N}_2$)*. Then, for any $u \in \mathcal{U}$ there exists an open neighborhood $\mathcal{O}_u = \mathcal{O}_{u,F}$ of u in $\mathcal{U}$ such that, for any $u_0, u_1 \in \mathcal{O}_u$ the $\mathcal{V}_0$-valued Gaussian vector $\boldsymbol{P}_{u_0,u_1}(G_F)$ is nondegenerate.*

Proof. For $u_0, u_1 \in \mathcal{U}$ consider the $\mathscr{V}_0$-valued Gaussian random vector $\boldsymbol{P}_{u_0,u_1}(G_F)$. Note that $\boldsymbol{P}_{u,u}(f)$ is the degree 1-Taylor polynomial of G_F at u. Its description involves only the derivatives of F of order 1 and 2. The nondegeneracy condition ($\mathbf{N}_2$) implies that the Gaussian vector $\boldsymbol{P}_{u,u}(G_F)$ is nondegenerate. Since the projectors $\boldsymbol{P}_{\underline{u}}$ depend continuously on $\underline{u} \in \mathcal{U}^2$ we deduce that there exists an open neighborhood $\mathcal{O}_u$ of u in $\mathcal{U}$ such that for any $u_0, u_1 \in \mathcal{O}_u$ the Gaussian vector $\boldsymbol{P}_{u_0,u_1}(\nabla f)$ is nondegenerate. □

For any $u \in \mathcal{U}$ and any $u_0, u_1 \in \mathcal{U}$, such that $u_0 \neq u_1$, the map

$$\mathbf{Ev}_{u_0,u_1} : \mathscr{V}_0 \to \boldsymbol{U}^2, \;\; G \mapsto \big(\, G(u_0), G(u_1)\,\big)$$

is onto for any $(u_0, u_1) \in \mathcal{U}^2_*$.

Indeed, given $g_1, g_2 \in \boldsymbol{U}$, there exists $f \in C^3(\mathcal{U})$ such that $\nabla f(u_i) = g_i$, and $\mathbf{Ev}_{u_0,u_1}(\nabla) = (g_1, g_2)$. Next observe that

$$\mathbf{Ev}_{u_0,u_1}(G) = \mathbf{Ev}_{u_0,u_1}\big(\, \boldsymbol{P}_{u_0,u_1} G\,\big)$$

so the restriction of $\mathbf{Ev}_{u_0,u_1}$ to $\mathscr{V}_0$ is onto,

Proposition 2.5. *Fix a box $B \subset \mathcal{U}$ such that $B \subset \mathcal{O}_u$, for some $u \in \mathcal{U}$. Then*

$$\mathbb{E}\big[\, Z_B(G)\big(\, Z_B(G) - 1\,\big)\,\big] < \infty.$$

Proof. Our approach is a slight modification of the arguments in [5, Sec. 4.2]. For any $u_0, u_1 \in \mathcal{O}_u$, $u_0 \neq u_1$, the Gaussian vector

$$\hat{G}(u_0, u_1) = G(u_0) \oplus G(u_1) = \mathbf{Ev}_{u_0,u_1}\big(\, \boldsymbol{P}_{u_0,u_1}(G)\,\big)$$

is nondegenerate since the $\mathscr{V}_0$-valued vector is nondegenerate, and the restriction of $\mathbf{Ev}_{u_0,u_1}$ to $\mathscr{V}_0$ is onto. We denote by $p_{G(u_0),G(u_1)}$ the probability density of $\hat{G}(u_0, u_1)$.

We deduce from Theorem 1.16 that

$$\mathbb{E}\big[\, Z_B(G)\big(\, Z_B(G - 1\,\big)\,\big] = \int_{B^2_*} \rho^{(2)}_G(u_0, u_1) du_0 du_1,$$

where $\rho^{(2)}_F$ is the Kac-Rice density

$$\rho^{(2)}_G(u_0, u_1) := \mathbb{E}\big[\, |\det G'_(u_0) \det G'(u_1)|\; \big|\, G(u_0) = G(u_1) = 0\,\big] p_{G(u_0),G(u_1)}(0). \tag{2.9}$$

Note that

$$p_{G(u_0,u_1)}(0) = \frac{1}{\sqrt{\det\big(\, 2\pi \operatorname{Var}[G(u_0) \oplus G(u_1)]\,\big)}},$$

so $p_{G(u_0,u_1)}(0)$ explodes as (u_0, u_1) approaches the diagonal since $G(u_0) \oplus G(u_0)$ is concentrated on the diagonal of $\boldsymbol{U} \times \boldsymbol{U}$.

We set

$$r(\underline{u}) = \|u_1 - u_0\|, \;\; \Xi(\underline{u}) = \frac{1}{r(\underline{u})}\big(\, G(u_1) - G(u_0)\,\big).$$

Note that

$$\hat{G}(\underline{u}) = 0 \Longleftrightarrow G(u_0) = \Xi(\underline{u}) = 0.$$

Denote by $A(\underline{u})$ the linear map $\boldsymbol{U}^2 \to \boldsymbol{U}^2$ given by

$$A(\underline{u}) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} v_0 \\ v_0 + r(\underline{u}) v_1 \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{\boldsymbol{U}} & 0 \\ \mathbb{1}_{\boldsymbol{U}} & r(\underline{u})\mathbb{1}_{\boldsymbol{U}} \end{bmatrix} \cdot \begin{bmatrix} v_0 \\ v_0 + r(\underline{u}) v_1 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} G(u_0) \\ G(u_1) \end{bmatrix} = A(\underline{u}) \begin{bmatrix} G(u_0) \\ \Xi(\underline{u}) \end{bmatrix}.$$

We deduce

$$p_{G(u_0),G(u_1)} = \frac{1}{\sqrt{\det\big(2\pi \operatorname{Var}[G(u_0), G(u_1)]\big)}} = \frac{1}{|\det A| \sqrt{\det\big(2\pi \operatorname{Var}[G(u_0), \Xi(\underline{u}))]\big)}}$$
$$= t(\underline{u})^{-d} p_{G(u_0),\Xi(\underline{u})}(0).$$

We deduce that for any $\underline{u} \in B_*^2$ we have

$$\rho_F^{(2)}(u_0, u_1) := r(\underline{u})^{-d} \mathbb{E}\big[\, |\det G_F'(u_0) \det G_F'(u_1)| \,\big|\, G_F(u_0) = \Xi(\underline{u}) = 0 \,\big] p_{G(u_0),\Xi(\underline{u})}(0). \tag{2.10}$$

Lemma 2.6. *There exists a constant $C = C(B) > 0$ such that*

$$\Big|\, \mathbb{E}\big[\, |\det G_F'(u_i)|^2 \big|\, G_F(u_0) = \Xi(\underline{u}) = 0 \,\big] \,\Big| \leq C r(\underline{u})^2, \;\; i = 0, 1.$$

Proof. We argue by contradiction. Suppose that there exists a sequence $\underline{u}^n = (u_0^n, u_1)^n$ in B_*^2 converging to $\underline{v} \in B_*^2$ such that

$$\frac{1}{r(\underline{u}^n)^2} \Big|\, \mathbb{E}\big[\, |\det G_F'(u_i^n)|^2 \big|\, G_F(u_0^n) = \Xi(\underline{u}^n) = 0 \,\big] \,\Big| \to \infty. \tag{2.11}$$

Note that $\mathbb{E}\big[\, |\det G_F'(u_i)|^2 \big|\, G_F(u_0) = \Xi(\underline{u}) = 0 \,\big]$ depends continuously on $\underline{u} \in B_*^2$. This shows that the limit point $\underline{v}$ must live on the diagonal $\underline{v} = (v, v)$. Upon extracting a subsequence we can assume that

$$\nu_n := \frac{1}{r(\underline{u}^n)} \big(\, u_n^1 - u_0^n \,\big)$$

converges to a unit vector $\boldsymbol{e}$. Extend $\boldsymbol{e}$ to an orthonormal basis $(\boldsymbol{e}_k)$ of $\boldsymbol{U}$ such that $\boldsymbol{e} = \boldsymbol{e}_1$ Note that

$$\lim_{n\to\infty} \Xi(\underline{u}^n) = \partial_{\boldsymbol{e}} G(v).$$

Moreover, the Gaussian random vector $G(v) \oplus \partial_{\boldsymbol{e}} G(v)$ is also nondegenerate. The regression formula shows that for any continuous and homogeneous function $f = f\big(\, G'(u_0), G'(u_1) \,\big)$ we have

$$\lim_{n\to\infty} \mathbb{E}\big[\, f\big(\, G'(u_0^n), G'(u_1^n) \,\big) \,\big|\, G(u_0^n) = \Xi(\underline{u}^n) = 0 \,\big] = \mathbb{E}\big[\, f(G'(v), G'(v) \,\big) \,\big|\, G(v) = \partial_{\boldsymbol{e}} G'(v) = 0 \,\big].$$

Since $G(u)$ is a.s. C^2 we deduce

$$\big|\, G(u_1^n) - G(u_0^n) - r(\underline{u}^n) \partial_{\nu_n} G(u_0^n) \,\big| \leq K(B) r(\underline{u})^2$$

where, according to Lemma 2.3 the quantity $K(B)$ is a nonnegative random variable that is p-integrable $\forall p \in [1, \infty)$. Thus, $\forall p \in [1, \infty)$ there exists a positive constant $C_p(B)$ such that $\forall n$ we have

$$\mathbb{E}\big[\, \big|\, r(\underline{u}^n) \partial_{\nu_n} G(u_0^n) \,\big|^p \,\big|\, G(u_0^n) = \Xi(\underline{u}^n) = 0 \,\big]$$
$$= \mathbb{E}\big[\, \big|\, G(u_1^n) - G(u_0^n) - r(\underline{u}^n) \partial_{\nu_n} G(u_0^n) \,\big|^p \,\big|\, G(u_0^n) = \Xi(\underline{u}^n) = 0 \,\big] \leq C_p(B) r(\underline{u})^{2p}.$$

Hence

$$\mathbb{E}\big[\, \big|\, \partial_{\nu_n} G(u_0^n) \,\big|^p \,\big|\, G(u_0^n) = \Xi(\underline{u}^n) = 0 \,\big] \leq C_p(B) r(\underline{u})^p. \tag{2.12}$$

Extend (ν_n) to an orthonormal basis $(\boldsymbol{e}_k^n)$ of $\boldsymbol{U}$ such that $\nu_n = \boldsymbol{e}_1^n$ and

$$\lim_{n\to\infty} \boldsymbol{e}_k^n = \boldsymbol{e}_k.$$

Then

$$\bigl|\,\det G'(u_0^n)\,\bigr| = \bigl|\,\det\bigl(\,\partial_{\boldsymbol{e}_1^n} G(u_0^n), \partial_{\boldsymbol{e}_2^n} G(u_0)^n, \dots, \partial_{\boldsymbol{e}_d^n}\,\bigr)\,\bigr| \leq \bigl|\,\partial_{\boldsymbol{e}_1^n} G(u_0^n)\,\bigr| \prod_{k=2}^d \bigl|\,\partial_{\boldsymbol{e}_k^n} G(u_0^k)\,\bigr|.$$

Hence

$$\mathbb{E}\bigl[\,\bigl|\,\det G'(u_0^n)\,\bigr|^2\,\bigl|G(u_0^n) = \Xi(\underline{u}^n) = 0\ \bigr] \leq \prod_{k=1}^d \mathbb{E}\bigl[\,\bigl|\,\partial_{\boldsymbol{e}_k^n} G(u_0^n)\,\bigr|^{2d}\,\bigl|\,\bigl|\,G(u_0^n) = \Xi(\underline{u}^n) = 0\,\bigr]\,\bigr]^{\frac{1}{d}}$$

For $k = 2, \dots, d$ we have

$$\mathbb{E}\bigl[\,\bigl|\,\partial_{\boldsymbol{e}_k^n} G(u_0^n)\,\bigr|^{2d}\,\bigl|\,\bigl|\,G(u_0^n) = \Xi(\underline{u}^n) = 0\,\bigr]^{\frac{1}{d}} = O(1),\ \ \text{as } n \to \infty,$$

since

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\bigl[\,\bigl|\,\partial_{\boldsymbol{e}_k^n} G(u_0^n)\,\bigr|^{2d}\,\bigl|\,\bigl|\,G(u_0^n) = \Xi(\underline{u}^n) = 0\,\bigr] \\ &= \lim_{n\to\infty} \mathbb{E}\bigl[\,\bigl|\,\partial_{\boldsymbol{e}_k^n} G(u_0^n)\,\bigr|^{2d}\,\bigl|\,\bigl|\,G(u_0^n) = \Xi(\underline{u}^n) = 0\,\bigr] \\ &= \mathbb{E}\bigl[\,\bigl|\,\partial_{\boldsymbol{e}_k} G(u)\,\bigr|^{2d}\,\bigl|\,\bigl|\,G(v) = \partial_{\boldsymbol{e}} G(v) = 0\,\bigr] < \infty. \end{aligned}$$

The last equality follows from the regression formula which is valid also in the limit since the limiting condition is also nondegenerate. On the other hand (2.12), shows that

$$\mathbb{E}\bigl[\,\bigl|\,\partial_{\boldsymbol{e}_k^n} G(u_0^n)\,\bigr|^{2d}\,\bigl|\,\bigl|\,G(u_1^n) = G(u_0^n) = 0\,\bigr]^{\frac{1}{d}} = O\bigl(\,r(\underline{u}^n)^2\,\bigr),\ \ \text{as } n \to \infty.$$

$$\mathbb{E}\bigl[\,\bigl|\,\det G'(u_0^n)\,\bigr|^2\,\bigl|\,G(u_1^n) = G(u_0^n) = 0\,\bigr] = O\bigl(\,r(\underline{u}^n)^2\,\bigr),\ \ \text{as } n \to \infty.$$

This contradicts (2.11) and thus completes the proof of Lemma 2.6. □

Lemma 2.6 implies

$$\begin{aligned} &\mathbb{E}\bigl[\,|\det G_F'(u_0)\det G_F'(u_1)|\,\bigl|\,G(u_0) = G(u_1) = 0\,\bigr] \\ &\leq \mathbb{E}\bigl[\,\bigl|\,\det G'(u_0)\,\bigr|^2\,\bigl|\,G(u_0) = G(u_1) = 0\,\bigr]\,\bigr|^{1/2} \\ &\times \mathbb{E}\bigl[\,\bigl|\,\det G'(u_1)\,\bigr|^2\,\bigl|\,G(u_0) = G(u_1) = 0\,\bigr]\,\bigr|^{1/2} \\ &= O\bigl(\,r(\underline{u})^{-2}\,\bigr) \end{aligned}$$

Hence

$$\rho_F^{(2)}(\underline{u}) = O\bigl(\,r(\underline{u})^{2-d}\,\bigr)\ \text{ on } B_*^2.$$

This shows that $\rho_F^{(2)} \in L^1\bigl(\,B_*^2\,\bigr)$ and completes the proof of Proposition 2.5. □

Remark 2.7. If F is a.s. C^4, then one can refine Lemma 2.6 to state that

$$\lim_{u_0,u_1\to u} \frac{1}{r(u_0,u_1)^2}\bigl|\,\mathbb{E}\bigl[\,|\det G_F'(u_i)|^2\bigl|\,G_F(u_0) = \Xi(\underline{u}) = 0\,\bigr]\,\bigr|$$

exists and it is finite. Indeed, in this case use the equality

$$G(u_1^n) = G(u_0^b) + r_n \partial_{\nu_n} G(u_0) + \frac{r_n^2}{2}\partial_{\nu_n}^2 G(u_0^n) + O(r_n^3)$$

Hence

$$r_n\partial_{\nu_n}G(u_0^n) = G(u_1^n) - G(u_0^n) - \frac{r_n^2}{2}\partial^2_{\nu_n}G(u_0^n) + O(r_n^3).$$

Similarly

$$r_n\partial_{\nu_n}G(u_1^n) = G(u_1^n) - G(u_0^n) + \frac{r_n^2}{2}\partial^2_{\nu_n}G(u_1^n) + O(r_n^3).$$

Extend ν_n to an orthonormal basis $(\boldsymbol{e}_1^n, \boldsymbol{e}_2^n, \dots, \boldsymbol{e}_d^n)$ of $\boldsymbol{U}$ that converges to an orthonormal basis of $\boldsymbol{U}$ and $\boldsymbol{e}_1^n = \nu_n$. Note that $\det G'(u_0^n)$ is the determinant of the $d\times d$ matrix with columns

$$\partial^n_{\boldsymbol{e}_k}G(u_0^n),\quad k=1,2,\dots,d,\quad G = \begin{bmatrix} \partial_{\boldsymbol{e}_1^n}F \\ \vdots \\ \partial_{\boldsymbol{e}_d^n}F \end{bmatrix}.$$

Note next that, when conditioned on $G(u_0^n)=0$, the first column satisfies

$$\partial_{\boldsymbol{e}_1^n}G(u_0^n) = -\frac{r_n}{2}\partial^2_{\boldsymbol{e}_1^n}G(u_0^n) + O(r_n).$$

The (random) error $O(r_n)$ can be controlled by the the C^3 norm of $G=\nabla F$.

Let observe that $\Xi(\underline{u})$ is closely related to the divided difference $G[u_0,u_1]$. Let $u_0\neq u_1$ and set

$$\nu := \frac{1}{\|u_1-u_0\|}\big(\, u_1-u_0\,\big).$$

Denote by P the Kergin interpolator

$$P(u) = G(u_0) + G[u_0,u_1](u-u_0).$$

Then

$$G(u_1) - G(u_0) = P(u_1) - P(u_0) = G[u_0,u_1](u_1-u_0)$$

Hence

$$\Xi(\underline{u}) = G[u_0,u_1]\big(\,\nu\,\big).$$

□

Consider now an arbitrary box B and cover it by finitely many open sets of the form $\mathcal{O}_u$. Consider a subdivision of B into finitely many boxes $(B_i)_{i\in I}$ with diameters smaller than the Lebesgue number of the above open cover. Lemma 1.5 shows that a.s.

$$Z_B(F) = \sum_{i\in I} Z_{B_i}(F).$$

Proposition 2.5 shows that each of the random variables Z_{B_i} is L^2 so that Z_B is also L^2. We have thus proved the following result.

Theorem 2.8. *Suppose that $F:\mathcal{U}\to\mathbb{R}$ is a Gaussian function that is* a.s. C^2 *and satisfies the nondegeneracy condition* ($\mathbf{N}_2$)*. Then, for any box $B\subset\mathcal{U}$, the random variable $Z_B(F)$ has finite mean and variance.* □

2.4. **Higher moments.** We want to describe conditions guaranteeing that the p-momentum of $Z_B(G)$ is finite. We follow the approach of Gass and Stecconi [10]. In Section 3 we will present the alternate approach in [2].

Let $p \geq 2$. The *fat diagonal* $\Delta \subset \mathcal{U}^p$ consists of the of points $\underline{u} = (u_1, \dots, u_p) \in \mathcal{U}^p$ such that there $u_i = u_j$ for some $i \neq j$. We set $\mathcal{U}^p_* = \mathcal{U}^p \setminus \Delta$.

For any $q \geq 0$ we introduce the condition N_q

$$F \text{ is a.s. } C^{q+1} \text{ and } \forall u \in \mathcal{U} \text{ the Gaussian vector } T_q\big[\,F, u\,\big] \text{ is nondegenerate,} \tag{$\mathbf{N}_q$}$$

where $T_q[F, u]$ denotes the degree q Taylor polynomial of F at u. Note that $N_{q_1} \Rightarrow N_{q_0}$ if $q_1 \geq q_0$. We set $G := \nabla F$. Assume that F satisfies N_q for some $q \geq p$.

For $q, m \geq 1$ we set

$$\mathscr{W} := \nabla C^{q+1}(\mathcal{U}),\;\; \mathscr{V}_m =: \nabla \mathbb{R}_{m+1}\big[\,\boldsymbol{U}\,\big].$$

For each $\underline{u} \in \mathcal{U}^p_*$ we have surjections

$$\mathbf{Ev}_{\underline{u}} : \mathscr{W} \to \boldsymbol{U}^p,\;\; \mathbf{Ev}_{\underline{u}}\, G = \big(\, G(u_1), \dots, G(u_p)\,\big),$$

and

$$E = E_{\underline{u}} = \mathscr{V}_{2p} \to \boldsymbol{U}^p,\;\; E_{\underline{u}}(G) = \big(\,(G(u_1), \dots, G(u_p)\,\big).$$

The operator $E_{\underline{u}}$ is onto is onto since $\mathbf{Ev}_{\underline{u}}$ is onto, $\boldsymbol{P}_{\underline{u}} : \mathscr{W} \to \mathscr{V}_{p-1} \subset \mathscr{V}_{2p}$ is onto and

$$E_{\underline{u}}\big(\,\boldsymbol{P}_{\underline{u}} G\,\big) = \mathbf{Ev}_{\underline{u}}(G),\;\; \forall G \in \mathscr{W}.$$

Fix once and for all an inner product on $\mathscr{V}_{2p}$. This induces inner products in all the subspaces $\mathscr{V}_q$, $q \leq 2p$.

Using ($\mathbf{N}_q$) with $q \geq p$ we deduce that for any $u \in \mathcal{U}$ that there exists an open neighborhood $\mathcal{O}_u \subset \mathcal{U}$ of u such that, $\forall u_1 \dots, u_q \in \mathcal{O}_u$, the $\mathscr{V}_{q-1}$-valued Gaussian vector $\boldsymbol{P}_{u_1,\dots u_q} G$ is nondegenerate. In particular, the vector

$$\boldsymbol{P}_{u_1,\dots,u_p}(G) = \boldsymbol{P}_{u_1,\dots,u_p}\big(\,\boldsymbol{P}_{u_1,\dots u_q} G\,\big)$$

is nondegenerate.

Fix a *small box* B, i.e., a box B contained in a neighborhood $\mathcal{O}_u$ for some $u \in \mathcal{U}$. For any $\underline{u} \in B^p_*$ the Gaussian vector $E_{\underline{u}}\big(\,P_{\underline{u}} G\,\big)$ is nondegenerate and we denote by $p_{E_{\underline{u}}(G)}$ its probability density. Note that $E_{\underline{u}}\big(\,\boldsymbol{P}_{\underline{u}} G\,\big))$ $\mathbf{Ev}_{\underline{u}}(G)$ and have the same distribution.

Set

$$B = B_{\underline{u}} := E_{\underline{u}} E^*_{\underline{u}} : \boldsymbol{U}^p \to \boldsymbol{U}^p.$$

The operator $B_{\underline{u}}$ is symmetric and positive definite. We set

$$L_{\underline{u}} = B^{-1/2} E_{\underline{u}}.$$

Note that $\ker L_{\underline{u}} = \ker E_{\underline{u}}$ and since $B^{1/2}$ commutes with EE^* we have

$$L_{\underline{u}} L^*_{\underline{u}} = \mathbb{1}_{\boldsymbol{U}^p}$$

This proves that the map $L^*_{\underline{u}} : \boldsymbol{U}^p \to \mathscr{V}_{2p}$ is an isometry.

The Gaussian vector $L_{\underline{u}}(G_{\underline{u}})$ is also nondegenerate and since $B^{1/2}$ commutes with EE^* we deduce that variance operator is

$$\operatorname{Var}\big[\,L_{\underline{u}}(G_{\underline{u}})\,\big] = B^{-1/2}_{\underline{u}} E_{\underline{u}} E^*_{\underline{u}} B^{-1/2}_{\underline{u}} = L_{\underline{u}} L^*_{\underline{u}} = \mathbb{1}.$$

We denote by $p_{L_{\underline{u}}(G)}$ its probability density. We have

$$\operatorname{Var}\big[\,E_{\underline{u}}(G)\,\big] = B^{1/2} \operatorname{Var}\big[\,L_{\underline{u}}(G)\,\big] B^{1/2},$$

so that

$$p_{E_{\underline{u}}(G)}(0) = \frac{1}{\det B_{\underline{u}}^{1/2}} p_{L_{\underline{u}}(\vec{G})(0)}$$

We have

$$\mathbb{E}\big[\, \big[\, Z_B(F)\,\big]_p \,\big] = \int_{B^{p+1}*} \rho_F^{(p)}(\underline{u}) |du_1 \cdots du_p|$$

where, $[x]_p$ denotes the falling factorial $[x]_p := x(x-1)\cdots(x-p+1)$ and

$$\rho_F^{(p)}(\underline{u}) = \mathbb{E}\Big[\, \prod_{i=1}^p \big|\, \det G'(u_i)\,\big| \,\Big|\, \mathbf{Ev}_{\underline{u}}(G) = 0 \,\Big] p_{\mathbf{Ev}_{\underline{u}}(G)}(0)$$

Define

$$\Phi_{\underline{u}} : \mathscr{W} \to \mathbb{R}, \;\; \Phi_{\underline{u}}(G) = \prod_{i=1}^p |\Phi_{u_i}(G)\,\big|, \;\; \Phi_{u_i}(G) = \det G'(u_i),$$

$$\phi_{\underline{u}} := \Phi_{\underline{u}}\, |_{\mathscr{V}_{2p}} \,.$$

The function ϕ_{u_i} is a degree-d homogeneous polynomial function on $\mathscr{V}_{2p}$

Lemma 2.9. *For any $\underline{u} \in \mathfrak{U}_*^p$ and any $i = 1, \dots, p$ the restriction of Φ_{u_i} to $\mathscr{V}_p \cap \ker E_{\underline{u}}$ is nonzero.*

Proof. To prove that the restriction of $\phi_{\underline{u}}$ to $\ker E_{\underline{u}}$ is nontrivial observe that given any symmetric operators $A_1, \dots, A_p : \boldsymbol{U} \to \boldsymbol{U}$ there exists a function $F \in C^p(\mathfrak{U})$ such that

$$\nabla F(u_i) = 0, \;\; H_F(u_i) = A_i.$$

Then $G_{\underline{u}} := \boldsymbol{P}_{\underline{u},\underline{u}} \nabla F \in \mathscr{V}_{2p} \cap \ker \mathbf{Ev}_{\underline{u}}$ and $G'_{\underline{u}}(u_i) = A_i$. □

We set

$$\lambda_i(\underline{u}) := \sup\big\{\, \big|\, \Phi_{u_i}(G)\,\big|, \;\; G \in \mathscr{V}_{2p}, \; \|G\| = 1, \;\; E_{\underline{u}}(G) = 0 \,\big\},$$

$$\lambda(\underline{u}) := \prod_{i=1}^p \lambda_i(\underline{u})$$

Above $\| - \|$ denotes the norm on $\mathscr{V}_{2p}$ induced by our chosen inner product. Lemma 2.9 implies that $\lambda(\underline{u}) > 0$. We set

$$\bar{\Phi}_{u_i} := \frac{1}{\lambda(u_i)} \Phi_{u_i}, \;\; \bar{\Phi}_{\underline{u}} = \prod_{i=1}^p \bar{\Phi}_{u_i}.$$

Observe that for any $\underline{u} \in B_*^p$ and any $1 = 1, \dots, p$ we have

$$\begin{gathered} \sup\big\{\, \big|\, \bar{\Phi}_{u_i}(G)\,\big|, \;\; G \in \mathscr{V}_p, \; \|G\| = 1, \;\; E_{\underline{u}}(G) = 0 \,\big\} \\ \leq \sup\big\{\, \big|\, \bar{\Phi}_{u_i}(G)\,\big|, \;\; G \in \mathscr{V}_{2p}, \; \|G\| = 1, \;\; E_{\underline{u}}(G) = 0 \,\big\} = 1. \end{gathered} \tag{2.13}$$

We deduce that

$$\begin{aligned} \rho_F^{(p)}(\underline{u}) &= \mathbb{E}\big[\, |\Phi_{\underline{u}}\big(\, G\,\big)| \;\big|\; \mathbf{Ev}_{\underline{u}}(G) = 0 \,\big] p_{\mathbf{Ev}_{\underline{u}}(G)}(0) \\ &= \frac{1}{\det B_{\underline{u}}^{1/2}} \cdot \mathbb{E}\big[\, |\Phi_{\underline{u}}\big(\, G\,\big)| \;\big|\; \mathbf{Ev}_{\underline{u}}(G) = 0 \,\big] p_{L_{\underline{u}}(G)}(0) \\ &= \underbrace{\frac{\lambda(\underline{u})}{\det B_{\underline{u}}^{1/2}}}_{=:w_F(\underline{u})} \cdot \underbrace{\mathbb{E}\big[\, |\bar{\Phi}_{\underline{u}}\big(\, G\,\big)| \;\big|\; \mathbf{Ev}_{\underline{u}}(G) = 0 \,\big] p_{L_{\underline{u}}(G)}(0)}_{=:\sigma_F(\underline{u})}. \end{aligned}$$

Note that w_F is independent of F!

Lemma 2.10 (Gass-Stecconi)**.** *Suppose that F satisfies ($\mathbf{N}_q$) for some $q \geq p$. Fix a small box B, i.e., a box such that $\boldsymbol{P}_{u_1,\dots,u_q}(\nabla F)$ is a nondegenerate Gaussian vector for any $u_1,\dots,u_q \in B$.*

(i) *There exists $C_F > 0$ that depends only on the distribution of F and B such that*

$$\sigma_F(\underline{u}) < C_F, \;\; \forall \underline{u} \in B_*^p.$$

(ii) *If $q \geq 2p$, then there exists $c_F > 0$ such that that depends only on the distribution of F and B*

$$\sigma_F(\underline{u}) > c_F, \;\; \forall \underline{u} \in B_*^p.$$

Proof. (i) For $\underline{u} \in B^p$ define

$$A_{\underline{u}} : \mathscr{W} \to \widehat{\mathscr{V}}_p := \mathscr{V}_p^p, \;\; G \mapsto \big(\, \boldsymbol{P}_{\underline{u},u_i} G\,\big)_{1 \leq i \leq p}.$$

Its image is contained in the subspace

$$\widehat{\mathscr{V}}_p(\underline{u}) = \big\{\, (G_1,\dots,G_p) \in \widehat{\mathscr{V}}_p;\;\; \boldsymbol{P}_{\underline{u}} G_1 = \cdots = \boldsymbol{P}_{\underline{u}} G_p \,\big\}.$$

Observe that since $\boldsymbol{P}_{\underline{u}} : \mathscr{V}_p \to \mathscr{V}_{p-1}$ is onto, the subspace $\widehat{\mathscr{V}}_p(\underline{u})$ has codimension equal $(p-1)\dim \mathscr{V}_{p-1}$, i.e., the codimension of the diagonal in $\mathscr{V}_{p-1}^p$ in $\mathscr{V}_{p-1}^p$. Since $\boldsymbol{P}_{\underline{u}}$ depends continuously on $\underline{u}$, the subspace $\widehat{\mathscr{V}}_p(\underline{u})$ varies continuously with $\underline{u}$ in the Grassmannian of subspaces of $\mathscr{V}_p^p$ of appropriate codimension.

Note that

$$A_{\underline{u}}\big(\, G\big) = A_{\underline{u}}\big(\, \boldsymbol{P}_{\underline{u},\underline{u}} G\,\big), \;\; \boldsymbol{P}_{\underline{u},\underline{u}}\big(\, \mathscr{W}\,\big) = \mathscr{V}_{2p}$$

so that

$$A_{\underline{u}}\big(\, \mathscr{W}\,\big) = A_{\underline{u}}\big(\, \mathscr{V}_{2p}\,\big).$$

We have a map

$$\widehat{\mathbf{Ev}}_{\underline{u}} : \widehat{\mathscr{V}}_p(\underline{u}) \to \boldsymbol{U}^p, \;\; \widehat{\mathbf{Ev}}_{\underline{u}}(G_1,\dots,G_p) = \big(\, G_1(u_1),\dots,G_p(u_p)\,\big).$$

Note that

$$\widehat{\mathbf{Ev}}_{\underline{u}} \circ A_{\underline{u}} = \mathbf{Ev}_{\underline{u}}\,.$$

We have a natural surjection $\boldsymbol{P}_{\underline{u}} : \widehat{\mathscr{V}}_p \to \mathscr{V}_{p-1}$ and we set

$$\widehat{L}_{\underline{u}} : \widehat{\mathscr{V}}_p \to \boldsymbol{U}^p = L_{\underline{u}} \circ \boldsymbol{P}_{\underline{u}} = B_{\underline{u}} \widehat{\mathbf{Ev}}_{\underline{u}}.$$

Moreover,

$$\ker \widehat{\mathbf{Ev}}_{\underline{u}} = \ker \hat{L}_{\underline{u}}.$$

Define

$$\widehat{\Phi}_{u_i} : \widehat{\mathscr{V}}_p \to \mathbb{R}, \;\; \widehat{\Phi}_{u_i}(G_1,\dots,G_p) = \bar{\phi}_{u_i}(G_i), \;\; \Phi_{\underline{u}} = \prod_i \widehat{\Phi}_{u_i}.$$

We set

$$\hat{G}_{\underline{u}} := A_{\underline{u}}(G).$$

and we observe that

$$\bar{\Phi}_{\underline{u}}\big(\, G\big) = \widehat{\Phi}_{\underline{u}}\big(\, \hat{G}_{\underline{u}}\,\big).$$

The function $\widehat{\Phi}_{\underline{u}}$ is supported on $A_{\underline{u}}\big(\, \mathscr{V}_{2p}\,\big)$ and

$$A_{\underline{u}}(\mathscr{V}_{2p}) \cap \ker \widehat{\mathbf{Ev}}_{\underline{u}} = A_{\underline{u}}(\mathscr{V}_{2p} \cap \ker \mathbf{Ev}_{\underline{u}}\,).$$

We deduce that from (2.13) the restriction of the homogeneous function $|\widehat{\Phi}_{\underline{u}}|$ to the unit ball of

$$A_{\underline{u}}(\mathscr{V}_{2p}) \cap \ker \widehat{\mathbf{Ev}}_{\underline{u}} = A_{\underline{u}}(\mathscr{V}_{2p}) \cap \ker \hat{L}_{\underline{u}}$$

is bounded above by a constant M independent of $\underline{u}\in B^p_*$.

The Gaussian measure on $\mathscr{W}$ induces via $A_{\underline{u}}$ a Gaussian measure on $\widehat{\mathscr{V}}_p$. Thus we can regard $\widehat{\mathscr{V}}_p$ as a probability space and regard $\widehat{\Phi}_{\underline{u}}$ as a random variable defined on this probability space. The Gaussian vectors $L_{\underline{u}}(G)$ and $\hat{L}_{\underline{u}}(\hat{G}_{\underline{u}})$ have the same distribution. We deduce

$$\sigma_F(\underline{u})=\mathbb{E}\big[\,\big|\,\widehat{\Phi}_{\underline{u}}\big(\,\hat{G}_{\underline{u}}\,\big)\,\big|\,\big|\,\hat{L}_{\underline{u}}\big(\,\hat{G}_{\underline{u}}\,\big)=0\,\big]p_{\hat{L}_{\underline{u}}(\hat{G}_{\underline{u}})}(0).$$

The Gaussian regression formula (B.1) implies

$$\sigma_F(\underline{u})=\Big(\int_{\ker\hat{L}_{\underline{u}}}\big|\,\widehat{\Phi}_{\underline{u}}\big|\,\big|d\Gamma_{\underline{u}}\Big)p_{\hat{L}_{\underline{u}}(\hat{G}_{\underline{u}})}(0),$$

where $\Gamma_{\underline{u}}$ is a Gaussian measure on $\ker\hat{L}_{\underline{u}}$ that depends continuously on surjection

$$L_{\underline{u}}:\mathscr{V}_{2p}(\underline{u})\to\boldsymbol{U}^p.$$

Note that this surjection has the property that its adjoint $L^*_{\underline{u}}:\boldsymbol{U}^p\to\mathscr{V}_{2p}$ is an isometry since

$$L^*_{\underline{u}}L_{\underline{u}}=\mathbb{1}.$$

We argue by contradiction. Suppose that there exists a sequence $(\underline{u}^n)$ in B^p_* such that

$$\lim_{n\to\infty}\sigma_F\big(\,\underline{u}^n\,\big)=\infty.$$

Upon extracting a subsequence we can assume that as $n\to\infty$ the following hold.

- $\underline{u}^n\to\underline{u}^\infty\in B^p$.
- The isometries $L^*_{\underline{u}^n}$ converges to an isometry $L^*_{\underline{u}^\infty}$.
- $\widehat{\Phi}_{\underline{u}^n}\,|_{\ker\hat{L}_{\underline{u}^n}}$ converges to a homogeneous polynomial $\widehat{\Phi}_\infty$ function on $\ker\hat{L}_{\underline{u}^\infty}$.

From Proposition B.1 we deduce that the Gaussian measures $\Gamma_{\underline{u}^n}$ on $\ker\hat{L}_{\underline{u}^n}$ converge to the regression Gaussian measure on $\ker\hat{L}_{\underline{u}^\infty}$ determined by the distribution of $G_{\underline{u}}$ on $\mathscr{V}$ and the operator L_∞. We deduce that

$$\lim_{n\to\infty}\sigma_F(\underline{u}^n)=\int_{\ker\hat{L}_\infty}\widehat{\Phi}_\infty\big(\,\hat{G}_{\underline{u}^\infty}\,\big)d\Gamma_\infty\neq\infty.$$

(ii) We assume that F satisfies ($\mathbf{N}_q$) with $q=2p$. For any $\underline{u}\in\mathfrak{U}_p$ define

$$A_{\underline{u}}:\mathscr{W}\to\mathscr{V}_{2p},\;\;A_{\underline{u}}(G)=\boldsymbol{P}_{\underline{u},\underline{u}}(G).$$

We set

$$\widehat{\mathbf{Ev}}_{\underline{u}}=\mathbf{Ev}_{\underline{u}}\,|_{\mathscr{V}_{2p}}=E_{\underline{u}},\;\;\widehat{\Phi}_{\underline{u}}=\bar{\Phi}_{\underline{u}}\,|_{\mathscr{V}_{2p}}=\bar{\phi}_{\underline{u}}.$$

We deduce from (N_{2p}) that if B is a sufficiently small box, then for any $\underline{u}\in B^p$, the projection $A_{\underline{u}}$ induces a *nondegenerate* Gaussian measure on $\mathscr{V}_{2p}$ and thus the $\mathscr{V}_{2p}$-valued vector the Gaussian vector $\hat{G}_{\underline{u}}=A_{\underline{u}}(G)$ is nondegenerate.

We have

$$\sigma_F(\underline{u})=\Big(\int_{\ker\hat{L}_{\underline{u}}}|\widehat{\Phi}_{\underline{u}}|d\Gamma_{\underline{u}}\Big)p_{\hat{L}_{\underline{u}}(G_{\underline{u}})}(0),$$

where $\Gamma_{\underline{u}}$ is a Gaussian measure on $\ker\hat{L}_{\underline{u}}$ that depends continuously on surjection

$$L_{\underline{u}}:\mathscr{V}_p(\underline{u})\to\boldsymbol{U}^p.$$

By construction, the supremum of the restriction of $|\widehat{\Phi}_{u_i}|$ to the unit ball of

$$\ker\hat{L}_{\underline{u}}=\mathscr{V}_{2p}\cap\ker\mathbf{Ev}_{\underline{u}}$$

is equal to 1.

We argue again by contradiction. Suppose that there exists a sequence $(\underline{u}^n)$ in B_*^p such that

$$\lim_{n\to\infty} \sigma_F\big(\,\underline{u}^n\big) = 0.$$

Upon extracting a subsequence we can assume that as $n\to\infty$ the following hold.

- $\underline{u}^n \to \underline{u}^\infty \in B^p$.
- The isometries $L^*_{\underline{u}^n}$ converges to and isometry $L^*_{\underline{u}^\infty}$.
- $\widehat{\Phi}_{u_i}\,|_{\ker \hat{L}_{\underline{u}^n}}$ converges to a nonzero continuous homogeneous polynomial $\widehat{\Phi}_{i,\infty}$ function on $\ker \hat{L}_{\underline{u}^\infty}$.

We deduce that the polynomials $\widehat{\Phi}_{\underline{u}^\nu}$ converge to the polynomial

$$\widehat{\Phi}_\infty = \prod_i \widehat{\Phi}_{i,\infty}.$$

The polynomial $\widehat{\Phi}_\infty$ is also nonzero since the rings of polynonmials in any number of variables are integral domains.

From Proposition B.1 we deduce that the Gaussian measures $\Gamma_{\underline{u}^n}$ on $\ker \hat{L}_{\underline{u}^n}$ converge to the *nondegenerate* regression Gaussian measure on $\ker \hat{L}_{\underline{u}^\infty}$ determined by the *nondegenerate* distribution of $\hat{G}_{\underline{u}^\infty}$ on $\mathscr{V}_{2p}$ and the operator $\hat{L}_\infty$. We deduce that

$$\lim_{n\to\infty}\sigma_F(\underline{u}^n) = \int_{\ker \hat{L}_\infty} \widehat{\Phi}_\infty\big(\,\hat{G}_{\underline{u}^\infty}\,\big) d\Gamma_\infty \neq \infty.$$

□

Theorem 2.11. *Suppose that the Gaussian function $F:\mathcal{U}\to\mathbb{R}$ satisfies ($\mathbf{N}_q$) with $q\geq p$. For any box $B\subset\mathcal{U}$ we denote by $Z_B(\nabla F)$ the number of critical points of F in B. Then $Z_B(\nabla F)\in L^p$.*

Proof. Fix a innerg product on $\mathscr{V}_{2p}$. This determines a Gaussian measure $\mathbf{\Gamma}$ on $\mathscr{V}_{2p}$ with variance operator $\mathbb{1}$. We have a linear bijection

$$\Omega_: = \mathbb{R}_{2p}\big[\,\boldsymbol{U}\,\big] \to \mathbb{R}\times\mathscr{V},\;\; \mathbb{R}_{2p}\big[\,\boldsymbol{U}\,\big]\ni\omega\mapsto\big(\,\omega(0),\nabla\omega\,\big).$$

We have nondegenerate Gaussian measure $\hat{\mathbf{\Gamma}} = \boldsymbol{\gamma}_1\times\mathbf{\Gamma}$, where $\boldsymbol{\gamma}_1$ is the standard Gaussian measure on $\mathbb{R}$, mean 0, variance 1. This induces a nondegenerate Gaussian measure on Ω. We obtain a Gaussian random function

$$\mathcal{E}:\Omega\times\mathcal{U}\to\mathbb{R},\;\; F_\omega(u)=\omega(u).$$

This satisfies the nondegeneracy condition ($\mathbf{N}_q$) with $q=2p$.

For every $u\in\mathcal{U}$ there exists an open neighborhood $\mathcal{O}_u$ of u such that, forall $u_1,\dots,u_2p\in\mathcal{O}_u$, the Gaussian vectors

$$\boldsymbol{P}_{u_1,\dots,u_p}(\nabla F)\;\text{ and }\;\boldsymbol{P}_{u_1,\dots u_{2p}}(\nabla\mathcal{E})$$

are nongenenerate. Suppose that $B\subset\mathcal{U}$ is a small box, i.e., contained in an open set $\mathcal{O}_u$ for some $u\in\mathcal{U}$.

Bézout's theorem [6, Lemma 11.5.1] implies that $Z_B(\nabla\mathcal{E})\in L^\infty$. Thus $\rho^{(p)}_{\mathcal{E}}$ is integrable on B^p_*. We have

$$\rho^{(p)}_{\mathcal{E}} = w\cdot\sigma_{\mathcal{E}},$$

and Lemma 2.10 implies

$$\int_{B_*^p} \rho_F^{(p)}(\underline{u})\,|du_1\cdots du_p| = \int_{B_*^p} w(\underline{u})\sigma_F(\underline{u})\,|du_1\cdots du_p|$$

$(\frac{\sigma_F}{C_F} \leq 1 \leq \frac{\sigma_{\mathcal{E}}}{c_{\mathcal{E}}})$

$$\leq \frac{C_F}{c_{\mathcal{E}}}\int_{B_*^p} w(\underline{u})\sigma_{\mathcal{E}}(\underline{u})\,|du_1\cdots du_p| = \frac{C_F}{c_{\mathcal{E}}}\int_{B_*^p} \rho_{\mathcal{E}}^{(p)}(\underline{u})\,|du_1\cdots du_p| < \infty.$$

We deduce that $\mathbb{E}\big[\, Z_P^p(\nabla F)\,\big] < \infty$ for any small box.

If $B \subset \mathcal{U}$ is an arbitrary box, then it can be decomposed as a finite union of boxes

$$B = \bigcup_{i=1}^N B_i$$

where the boxes B_i are small and have disjoint interiors. Then

$$Z_B(F) = \sum_{i=1}^N Z_{B_i}(\nabla F) \in L^p.$$

□

3. Multijets

In this section we want the present the ideas of Ancona and Letendre [2]. We will stick to the simplest context and will skip some technical details.

3.1. The setup. Suppose that $\boldsymbol{U}, \boldsymbol{V}$ are d-dimensional Euclidean spaces, $\mathcal{U} \subset \boldsymbol{U}$ is an open set and $G : \mathcal{U} \to \boldsymbol{V}$ is a Gaussian random map. We denote by $Z_B(G)$ the number of zeros of G inside the Borel set $B \subset \mathcal{U}$.

Fix $p \geq 2$. The "fat" diagonal in $\boldsymbol{U}^p$, denoted by Δ_p, consists of noninjective maps

$$\mathbb{I}_p \to \boldsymbol{U}.$$

We set $\mathcal{U}^* := \mathcal{U} \setminus \Delta_p$. We have a map

$$\hat{G} : \mathcal{U}^p \to \boldsymbol{V}^p,\;\; \mathcal{U}_+^p \ni \underline{u} \mapsto \big(\, G(u_1), \dots, G(u_p)\,\big).$$

Note that

$$Z_{B_*^p}\big(\,\hat{G}\,\big) = \big[\, Z_B(G)\,\big]_p,\;\; [x]_p := \prod_{j=0}^{p-1}(x-j).$$

We set $W_k = C^k(\mathcal{U}, \boldsymbol{V})$ and we define

$$\mathbf{Ev} : W_0 \times \mathcal{U}^p \to \boldsymbol{V}^p,\;\; (F, \underline{u}) \mapsto \mathbf{Ev}_{\underline{u}}(F) = \big(\, F(u_1), \cdots, F(u_p)\,\big).$$

Assume that for any $\underline{u} \in \mathcal{U}_*^p$ the Gaussian vector $\mathbf{Ev}_{\underline{u}}(G)$ is nondegenerate. Then,

$$\mathbb{E}\big[\, Z_{B_*^p}\,\big] = \int_{B_*^p} \rho_G^{(p)}(\underline{u})\,|d\underline{u}|,$$

where

$$\rho_G^{(p)}(\underline{u}) = \mathbb{E}\big[\, J_G(\underline{u}) \,\big|\, \mathbf{Ev}_{\underline{u}}(G) = 0\,\big]\frac{1}{\big(\,\det 2\pi\,\mathrm{Var}[\,\mathbf{Ev}_{\underline{u}}(G)\,]\,\big)^{1/2}},$$

and

$$J_G(\underline{u}) := \prod_{K=1}^p |\det G'(u_k)G'(u_k)^*|^{1/2}.$$

3.2. **Renormalizing the diagonal singularities.** The integrand $\rho_G^{(p)}(\underline{u})$ might not be integrable because $\det \operatorname{Var}\big[\,\mathbf{Ev}_{\underline{u}}(G)\,\big] \to 0$ as $\underline{u} \to \Delta_p$. In Subsections 2.3 and 2.4 we use related approaches to get a handle of the degeneration of $\rho_{\hat{G}}$ near the diagonal. In both cases, the first step was a appropriate renormalization.

This renormalization is a gauge transformation $T : \mathcal{U}_*^p \to \operatorname{GL}\big(\boldsymbol{V}^p\big)$. The renormalized random field $\bar{G}(\underline{u}) := T_{\underline{u}}\big(\hat{G}(\underline{u})\big)$ has the same zero set as $\hat{G}(\underline{u})$, so counting the zeros of $\hat{G}$ is equivalent to counting the zeros of $\bar{G}$. The new field $\bar{G}$ has a different Kac-Rice density $\rho_{\bar{G}}$ which could be more manageable if the renormalization $T_{\underline{u}}$ is chosen judiciously.

For example, in Subsection 2.3 we discussed the case $p = 2$ and we used the renormalization

$$T_{u_1,u_2} : \mathcal{U}_*^2 \to \operatorname{GL}(\boldsymbol{V}^2),\;\; T_{u_1,u_2}\left[\begin{array}{c} v_1 \\ v_2 \end{array}\right] = \left[\begin{array}{c} v_1 \\ \frac{1}{\|u_2-u_1\|}(v_2 - v_1) \end{array}\right].$$

Related renormalization are used in [5, Lemma 4.8] to investigate the 3-moments.

In Subsection 2.4 we used different renormalizations. Denote by $\operatorname{Poly}_k(\boldsymbol{U},\boldsymbol{V})$ the space of maps $\boldsymbol{U} \to \boldsymbol{V}$ that are polynomial of degree $\leq k$ in the variable $u \in \boldsymbol{U}$. For any $\underline{u} \in \mathcal{U}_*^p$ we have an evaluation map

$$E_{\underline{u}} : \operatorname{Poly}_{2p-1}(\boldsymbol{U},\boldsymbol{V}) \to \boldsymbol{V}^p,\;\; E_{\underline{u}}(P) = \big(\,P(u_1), \dots, P(u_p)\,\big).$$

Then, as gauge renormalization we used map $T_{\underline{u}} = \big(\,E_{\underline{u}}E_{\underline{u}}^*\,\big)^{-1/2}$, where the adjoint $E_{\underline{u}}^*$ is defined in terms of suitable inner product on on $\operatorname{Poly}_{2p-1}(\boldsymbol{U}, V)$.

Ancona and Letendre [2] propose a different way of dealing with the diagonal singularites. It involves clever renormalizations hidden in a geometric cloak. Let us describe a baby example to give a taste of this principle.

Think of a function $f : \boldsymbol{U} \to \mathbb{R}$ not as a function, but as a section of the trivial vector bundle $\underline{\mathbb{R}}_{\boldsymbol{U}} = \mathbb{R} \times \boldsymbol{U} \to \boldsymbol{U}$. We can trivialize this bundle over the punctured space $\boldsymbol{U}^*$ using the frame

$$\boldsymbol{e} : \boldsymbol{U}^* \to \underline{\mathbb{R}}_{\boldsymbol{U}^*},\;\; u \mapsto \boldsymbol{e}(u) = \|u\|$$

Over the unit ball $B(\boldsymbol{U})$ of $\boldsymbol{U}$ we use the canonical frame $\boldsymbol{e}_0(u) = 1$. A function $f : \boldsymbol{U} \to \mathbb{R}$ can be viewed as a section of $\underline{\mathbb{R}}_{\boldsymbol{U}}$. Using the above trivializations we can view it as a pair of functions

$$s : \boldsymbol{U}^* \to \mathbb{R},\;\; s_0 : B(\boldsymbol{U}) \to \mathbb{R}$$

satisfying the compatibility equation

$$s_0 = \|u\|s(u),\;\; \forall 0 < \|u\| < 1.$$

The perfectly nice constant function $f = 1$ is then represented by the pair of functiosn

$$s_0 = 1,\;\; s(u) = \frac{1}{\|u\|}.$$

Note that $s(u)$ a singularity at the origin although is hides a nicely behaved object. It can me resolved by an approriate change of gauge, or renormalization.

The Kac-Rice density $\rho_G^{(p)}(\underline{u})|dV_{\boldsymbol{U}^p}|$ is also a section of real line bundle. This section depends only on G; see Remark 1.18. Ancona and Letendre showed in [2] that its singularities along the diagonal singularities are only "apparent" and are due to a similar, but much more complicated gauge renormalization phenomenon. In the next subsection we describe a simplified version of their approach. We will omit some technical details that can be found in [2].

3.3. **Multijet desingularization.** We need to introduce some notations. For a finite set I we have a space $\boldsymbol{U}^I$ of functions $\underline{u}: I \to \boldsymbol{U}$ and a configuration space [1]

$$\mathscr{C}_I := \mathscr{C}_I(\boldsymbol{U}) \subset \boldsymbol{U}^I$$

consisting of injective maps $I \to \boldsymbol{U}$. For $I = \mathbb{I}_p := \{1, \dots, p\}$ we write

$$\mathscr{C}_p = \mathscr{C}_{\mathbb{I}_p} = \boldsymbol{U}^p_*.$$

We denote by $\mathscr{F}_p$ the space of polynomial maps $f : \boldsymbol{U} \to \mathbb{R}$ of degree $p-1$. Each $\underline{u} \in \mathscr{C}_p$ defines a surjective map

$$\mathbf{Ev}_{\underline{u}} : \mathscr{F}_p \to \mathbb{R}^p,\;\; f \mapsto \big(\, f(u_1), \dots, f(u_p)\,\big).$$

We denote its kernel by $K_{\underline{u}}$. It is a codimension-p subspace of $\mathscr{F}_p$. We denote by $\mathbf{Gr}^p$ the Grassmannian of codimension p subspaces of $\mathscr{F}_p$. We have thus have a smooth map

$$K : \mathscr{C}_p \to \mathbf{Gr}^p\,.$$

We denote by Σ the graph of K, $\Sigma \subset \mathscr{C}_p \times \mathbf{Gr}^p$. We have a natural projection

$$\pi : \Sigma \to \mathscr{C}_p \subset \boldsymbol{U}^p,\;\; \Pi : \Sigma \to \mathbf{Gr}^p\,.$$

We denote by $\bar{\bar{\Sigma}}$ the closure of Σ in $\boldsymbol{U}^p \times \mathbf{Gr}^p$. We have a natural projections

$$\pi : \bar{\bar{\Sigma}} \to \boldsymbol{U}^p$$

which is proper, and surjective. We can be more precise [2, Sec. 5.1].

Proposition 3.1. *The following hold.*

(i) Σ *is a smooth real algebraic manifold and the projection* $\pi : \Sigma \to \mathscr{C}_p$ *is a diffeomorphism.*
(ii) $\bar{\bar{\Sigma}}$ *is a real algebraic variety and the map* $\pi : \bar{\bar{\Sigma}} \to \boldsymbol{U}^p$ *is proper and surjecfive.*
(iii) *The singular locus of* $\bar{\bar{\Sigma}}$ *is contained in* $\bar{\Delta} : \pi^{-1}(\Delta) = \bar{\bar{\Sigma}} \setminus \Sigma$.

□

Invoking Hironaka's (embedded) resolution of singularities theorem one can prove the following result, [2, Sec. 5.1].

Theorem 3.2. *There exists a smooth manifold* $\widehat{\widehat{\Sigma}}$ *and a proper smooth map*

$$\mathcal{R} : \widehat{\widehat{\Sigma}} \to \boldsymbol{U}^p \times \mathbf{Gr}^p$$

with the following properties.

(i) $\dim \widehat{\widehat{\Sigma}} = \dim \mathscr{C}_p = p \dim \boldsymbol{U}$.
(ii) $\mathcal{R}\big(\widehat{\widehat{\Sigma}}\big) = \bar{\Sigma}$.
(iii) *The set* $\mathcal{R}^{-1}\big(\Sigma\big)$ *is open and dense in* $\hat{\Sigma}$ *and the restriction of* $\mathcal{R}$ *to* $\mathcal{R}^{-1}\big(\Sigma\big) \to \Sigma$ *is a diffeomorphism onto* Σ.
(iv) *The map* $\hat{\pi} := \pi \circ \mathcal{R} : \hat{\Sigma} \to \boldsymbol{U}^p$ *is smooth and proper. Set* $\hat{\Delta} := \hat{\pi}^{-1}(\Delta)$

[1] Configuration of distinct points in $\boldsymbol{U}$ labelled by I.

(v) *The map*

$$K\circ\hat{\pi}:\hat{\Sigma}\setminus\hat{\Delta}\to \boldsymbol{U}^p\times \mathbf{Gr}^p\stackrel{\Pi}{\longrightarrow}\mathbf{Gr}^p$$

admits a smooth extensions to maps $\hat{K}:\hat{\Sigma}\to\mathbf{Gr}^p$. *The extension* $\hat{K}$ *is the composition* $\widehat{\Sigma}\stackrel{\mathcal{R}}{\longrightarrow}\mathscr{G}_p\stackrel{\Pi}{\longrightarrow}\mathbf{Gr}^p$.

□

The pair $(\hat{\Sigma},\mathcal{R})$ with the above properties is not unique. Fix a choice that we denote by $(\widehat{\mathscr{C}_p},\mathcal{R})$. We set $\hat{\pi}=\pi\circ\mathcal{R}$,

$$\widehat{\mathscr{C}_p^*}=\widehat{\mathscr{C}_p}\setminus\hat{\Delta}$$

and we can identify $\widehat{\mathscr{C}_p^*}$ with $\mathscr{C}_p$ using the diffeomeorphism $\hat{\pi}:\widehat{\mathscr{C}_p^*}\to\mathscr{C}_p$. For any $\underline{u}\in\boldsymbol{U}^p$ we will denote by $\hat{\underline{u}}$ a point in $\hat{\pi}^{-1}(\underline{u})\in\widehat{\mathscr{C}_p}$. If $\underline{u}\in\mathscr{C}_p$, there is only one such $\hat{\underline{u}}$.

The map $\hat{K}$ determines by pullback a smooth subbundle $\mathscr{K}^p$ of codimension p of the trivial bundlle with fiber $\mathscr{F}_p$ over $\widehat{\mathscr{C}_p}$

$$\underline{\mathscr{F}_p}_{\widehat{\mathscr{C}_p}}=\mathscr{F}_p\times\widehat{\mathscr{C}_p}\to\widehat{\mathscr{C}_p}.$$

We denote by $\mathscr{M}_p$ the quotient bundle

$$\mathscr{M}_p:=\underline{\mathscr{F}_p}_{\widehat{\mathscr{C}_p}}/\mathscr{K}^p\to\widehat{\mathscr{C}_p}.$$

The vector bundle $\mathscr{M}_p$ is the *bundle of p-multijets*.

To a function a function $f\in C^p(\boldsymbol{U})$ we can associate a C^1-section of $\underline{\mathscr{F}_p}_{\widehat{\mathscr{C}_p}}$ namely

$$\widehat{\mathscr{C}_p}\ni\hat{\underline{u}}\mapsto\boldsymbol{P}_{\hat{\pi}(\hat{\underline{u}})}f\in\mathscr{F}_p.$$

This projects to a C^1 section $\mu_p[f]$ of the bundle of multijets $\mathscr{M}_p$. Note that for any $\underline{u}\in\mathscr{C}_p$ we have

$$\mathbf{Ev}_{\underline{u}}(f)=0\Longleftrightarrow\mu_p[f]\big(\,\hat{\underline{u}}\,\big)=0$$

The multijet $\mu_p[f]$ achieves the renormalization alluded to in the previous subsection, and it comes with many other "gifts".

More generally given a finite dimensional Euclidean space $\boldsymbol{V}$ we can define a bundle of multijets

$$\mathscr{M}_p(\boldsymbol{V})=\mathscr{M}_p\otimes\boldsymbol{V}\to\widehat{\mathscr{C}_p},$$

and a C^p-map $F:\boldsymbol{U}\to\boldsymbol{V}$ a multijet $\mu_p[F]$; this is a section of the multijetbundle. ote that

$$\dim\mathscr{C}_p=p\times\dim\boldsymbol{U},\;\;\operatorname{rank}\mathscr{M}_p(\boldsymbol{V})=p\dim\boldsymbol{V}.$$

The map F defines a map

$$F^{\times p}:\mathscr{C}_p\to\boldsymbol{V}^p,\;\;F^{\times p}(u_1,\dots,u_p)=\big(\,F(u_1,\dots,F(u_p)\,\big),$$

and $\hat{\pi}$ defines a bijection

$$\{\mu_p[F]=0\}\cap\widehat{\mathscr{C}_p^*}\to\big\{\,F^{\times p}=0\,\big\}.$$

In particular, if $\dim\boldsymbol{V}=\dim\boldsymbol{U}$ and $B\subset\boldsymbol{U}$ is a box, then

$$\#Z_{B^p_*}\big(\,F^{\times p}\,\big)\leq\#Z_{\hat{\pi}^{-1}(B^p)}\big(\,\mu_p(F)\,\big).$$

3.4. **Higher moments again.** Fix an open subset $\mathcal{U}\subset\boldsymbol{U}$ and an Euclidean space $\boldsymbol{V}$ of the same dimension as $\boldsymbol{U}$.

Suppose that $G:\mathcal{U}\to\boldsymbol{V}$ is a Gaussian random map such that

- G is a.s. C^p and,
- for every $u\in\mathcal{U}$ the Gaussian vector $T_{p-1}\big(G,u\big)$ is a nondegenerate Gaussian vector. Here $T_q\big(G,u\big)$ denotes the degree q Taylor polynomial of G at u.

Equivalently, if we set

$$[u]_p=(\underbrace{u,\dots,u}_{p})\in\mathcal{U}^p,$$

then the Gaussian vector described by the Kergin projector $\boldsymbol{P}_{[u]_p}(G)$ is nondegenerate. In particular, this means that there exists an open neighborhood $\mathcal{O}_u$ of u in $\mathcal{U}$ such that, for any $\underline{u}\in\mathcal{O}_u^p$, the Gaussian vector $\boldsymbol{P}_{\underline{u}}(F)$ is nondegenerate.

The *thin diagonal* of $\mathcal{U}^p$, denoted by Δ_0 is the subset

$$\Delta_0:=\big\{\,\underline{u}\in\mathcal{U}^p;\;\;u_1=\cdots=u_p\,\big\}.$$

Equivalently, Δ_0 is the image of $\mathcal{U}$ in $\mathcal{U}^p$ via the diagonal map $u\mapsto[u]_p$ .Set

$$\mathcal{O}:=\bigcup_{u\in\mathcal{U}}\mathcal{O}_u^p$$

The set $\mathcal{O}$ is an open neighborhood of the thin diagonal and, for any $\underline{u}\in\mathcal{O}$, the Gaussian vector $\boldsymbol{P}_{\underline{u}}(G)$ is nondegenerate.

The multijet random section $\mu_p[G]$ is a.s. C^1. The above discussion shows that for any $\underline{\tilde{u}}\in\hat{\mathcal{O}}:=\hat{\pi}^{-1}(\mathcal{O})$ the Gaussian vector $\mu_p[G](\underline{\tilde{u}})$ is nondegenerate as the image of the nondegenerate vector $\boldsymbol{P}_{\underline{u}}(G)$, $\underline{u}=\hat{\pi}(\underline{u})$, via the linear projection $\mathscr{F}_p\otimes\boldsymbol{V}\to\big(\mathscr{F}_p/\mathscr{K}_{\underline{u}}^p\big)\otimes\boldsymbol{V}$.

Using the Kac-Rice formula for the number of zeros of random sections (Subsection 1.6) we deduce that for any compact set $K\subset\hat{\mathcal{O}}:=\hat{\pi}^{-1}(\mathcal{O})$, the number of zeros of $\mu_p\big[\,G\,\big]$ in K has finite mean, i.e.,

$$\mathbb{E}\big[\,Z_K\big(\,\mu_p[G]\,\big)\,\big]<\infty.$$

Suppose that B is a small box, i.e., a box contained in some $\mathcal{O}_u$. Then $B^p\subset\mathcal{O}$ and the set

$$\widehat{B^p}:=\hat{\pi}^{-1}\big(\,B^p\,\big)\subset\hat{\mathcal{O}}$$

is compact. We deduce

$$\mathbb{E}\big[\,[Z_B(G)]_p\,\big]=\mathbb{E}\big[\,Z_{B_*^p}(F^{\times p})\,\big]=\mathbb{E}\big[\,Z_{\hat{\pi}^{-1}(B_*^p)}(\mu_p[G])\,\big]\le\mathbb{E}\big[\,Z_{\widehat{B^p}}(\mu_p[G])\,\big]<\infty.$$

As argued at the end of Subsection 2.3 , for any box $B\subset\mathcal{U}$ we can find a finite collection of small boxes $(B_i)_{i\in I}$ such that

$$Z_B(G)=\sum_{i\in I}Z_{B_i}(G)$$

and we conclude that $Z_B\in L^p$ for any box $B\subset\mathcal{U}$.

Remark 3.3. (a) The multijet bundle described in this section is a simplified version of the construction of Ancona and Letendre but it is based on the same technique they introduced in [2].

The random multijet $\mu_p[G]$ we described above is nondegenerate only on an open neighborhood $\hat{\mathcal{O}}$ of $\hat{\pi}^{-1}(\Delta_0)$. It is very likely that this neighborhood does not contain the "exceptional divisor" $\hat{\pi}^{-1}(\Delta_p)$.

The more sophisticated multijet constructed in [2] is nondegenerate over an open neighborhood of this exceptional divisor. This allowed the authors to prove the more refined result, namely, that the expectation of p-th combinatorial momentum of the random measure

$$\nu_G = \sum_{G(u)=0} \delta_u$$

(see [2, Sec. 6.3]) is a Radon measure measure over $\mathcal{U}^p$.

The small box localization trick has allowed us to bypass that more sophisticated multijet construction but we proved an apparently weaker result, namely, for any compactly supported continuous function φ on $\mathcal{U}$ the random variable

$$Z_\varphi(G) = \int_{\mathcal{U}} \varphi(u) \nu_G\big[\, du\,\big]$$

is p-integrable. However, as shown in [2, Prop. 6.25], these properties are equivalent.

(b) I want to comment on the renormalization implicit in the multijet approach versus the renormalizations used in Subsections 2.3 and 2.4.

As indicate earlier, the source of headaches is the degeneration of the Gaussian vectors $\mathbf{Ev}_{\underline{u}}(G)$ as $\underline{u} \to \Delta_p$. The renormalizations used in Subsections 2.3 and 2.4 take care only of singularity of $\operatorname{Var}\big[\, \mathbf{Ev}_{\underline{u}}(G)\,\big]$ as $\underline{u} \to \Delta_p$. Moreover these renormalizations are dependent on the way $\underline{u}$ approaches Δ_p. In terms of the resolution constructed above, these renormalizations depend on the limit points of $\hat{\underline{u}}_\nu \in \hat{\Delta}_p$ as $\underline{u}_\nu \to \Delta_p$. The Gass-Stecconi technique described in Subsection 2.4 uses the Grassmannian of codimension p-subspaces of $\mathscr{F}_p$ in disguised as the subspaces $\ker\big(\, \mathbf{Ev}_{\underline{u}} : \mathscr{V} \to \boldsymbol{U}^p\,\big)$.

The multijet method shows that on small boxes B the Kac-Rice 1-density $KR_{G^{\times p}}$ on B^p_* (see (1.15)) is the restriction of a Radon 1-density/measure over a smooth manifold $\widehat{\mathscr{C}_p}$ where B^p_* embeds and the image of this embedding is contained in a compact subset of this manifold. Its singularities are due to a "wrong" choice of trivialization over B^p_* of the line bundle of 1-densities over $\widehat{\mathscr{C}_p}$.

A good analogy to keep in mind is the description in polar coordinates of the Euclidean area density $|dA| = r|drd\theta|$. This makes no sense at $r = 0$. This is because the trivialization of $T\mathbb{R}^2$ given by $\partial_r, \frac{1}{r}\partial_\theta$ does not extend over the origin. The deficiency is addressed by a renormalization: pass from polar to Euclidean coordinates.

$$\partial_r = (\cos\theta)\partial_x + (\sin\theta)\partial - Y,\;\; \partial_\theta = -(r\sin\theta)\partial_x + (r\cos\theta)\partial_y.$$

□

Appendix A. Jacobians and the Coarea formula

.

Suppose that U, Y are smooth manifolds such that $\dim U \geq \dim Y$ and $\Phi : U \to Y$ is a C^1-map. For $u \in U$ we denote by $\Phi'(u)$ the differential of F at u. This is a linear map

$$\Phi'(u) : T_u U \to T_{F(u)} Y.$$

If we fix Riemann metrics g^U, g^Y on U and respectively Y we can associate a *Jacobian* to the map Φ. This is a function

$$J_\Phi : U \to [0, \infty),\;\; J_\Phi(u) = \det\big(\, \Phi'(u)\Phi'(u)^*\,\big),$$

where $\Phi'(u) : T_{F(u)}Y \to T_uU$ is the adjoint of $\Phi'(u)$ determined by the inner products g^U_u on T_uU and $g^Y_{F(u)}$ on $T_{F(u)}Y$. Note that $\Phi'(u)$ is surjective if and only if $J_\Phi(u) \neq 0$.

A Riemann metric g_U on U determines a family of Borel measures on U, $\big(\mathcal{H}_s\big)_{0\leq s\leq \dim U}$. The measure $\mathcal{H}_s = \mathcal{H}^U_s$ is usually referred to as the s-dimensional Hausdorff measure on U. When $s = n = \dim U$ this coincides with the Borel measure dV_{g^U}determined by the metric g_U. In local coordinates, if

$$g^U = \sum_{i,j} g_{ij} du^i du^j,$$

then

$$dV_{g^U}\big[\,du\,\big] = \sqrt{\det(g_{ij})}\,|du^1\cdots du^n|.$$

If $X \subset U$ is a d-dimensional submanifold, and g_X is the metric on X induced by g_U, then the restriction to X of $\mathcal{H}^U_d$ coincides with the (volume measure) measure $\mathcal{H}^X_d$ induced g_X. We refer for more details to [9, Sec. 3.2.46].

We have to following useful result. For a proof we refer to [16, Sec. 3]

Theorem A.1 (Coarea formula)**.** *Suppose that (U, g^U) and (Y, g^Y) are smooth Riemann manifolds, $\dim U = n \geq \dim Y = m$. Let $\Phi : U \to Y$ be a C^1-map. Then for any nonnegative Borel measurable functions $\alpha : U \to \mathbb{R}$ and $\beta : Y \to \mathbb{R}$ such that α has compact support we have*

$$\int_U J_\Phi(u)\alpha(u)\Phi^*\beta(u)\mathcal{H}_n\big[\,du\,\big] = \int_Y \left(\int_{\Phi^{-1}(y)} \alpha(u)\mathcal{H}_{n-m}\big[\,du\,\big]\right)\beta(y)d\mathcal{H}_m\big[\,dy\,\big]. \tag{A.1}$$

The two sides of the above equality are simultaneously finite or infinite. If $\dim U = \dim Y = n$, then the above equality reads

$$\int_U J_\Phi(u)\alpha(u)\Phi^*\beta(u)\mathcal{H}_n\big[\,du\,\big] = \int_Y \left(\sum_{\Phi(u)=y} \alpha(u)\right)\beta(y)\mathcal{H}_n\big[\,dy\,\big] \tag{A.2}$$

Remark A.2. If set denote by U_Φ the set of regular points of Φ, $U_\Phi = U \setminus \Sigma_\Phi$, then equality [16, Lemma 4.2] shows that

$$\int_U J_\Phi(u)\alpha(u)\Phi^*\beta(u)\mathcal{H}_n\big[\,du\,\big] = \int_{U_\Phi} J_\Phi(u)\alpha(u)\Phi^*\beta(u)\mathcal{H}_n\big[\,du\,\big]$$

and

$$\int_Y \left(\sum_{\Phi(u)=y} \alpha(u)\right)\beta(y)\mathcal{H}_n\big[\,dy\,\big] = \int_{Y\setminus\Delta_\Phi} \left(\sum_{\Phi(u)=y} \alpha(u)\right)\beta(y)\mathcal{H}_n\big[\,dy\,\big]$$

Appendix B. Gaussian regression

Let us first present the proof of the regression formula.

Proof of Proposition 1.14. Assume first that both X and Y are centered. Set

$$Z = Y - R_{Y,X}X,\ \ R_{Y,X} = C_{Y,X}\operatorname{Var}\big[\,X\,\big]^{-1}.$$

Assumption (i) implies that Z is also a centered Gaussian vector.

Let $(\boldsymbol{e}_i)_{i\in I}$ and $(\boldsymbol{f}_\alpha)_{\alpha\in A}$ are orthonormal bases of $\boldsymbol{X}$ and respectively $\boldsymbol{Y}$, and we set $X_i := (\boldsymbol{e}_i, X)_{\boldsymbol{X}}$, $Y_\alpha := (\boldsymbol{f}_\alpha, Y)_{\boldsymbol{Y}}$, $Z_\alpha := (\boldsymbol{f}_\alpha, Z)_{\boldsymbol{Y}}$. We set

$$V(X)_{ij} := \mathbb{E}\big[\,X_iX_j\,\big],\ \ C_{\alpha i} := \mathbb{E}\big[\,Y_\alpha X_i\,\big] = C_{i\alpha},\ \ V(Y)_{\alpha\beta} := \mathbb{E}\big[\,Y_\alpha Y_\beta\,\big].$$

The matrix $\big(V(X)_{ij}\big)_{i,j\in I}$ describes the variance operator of X, the matrix $\big(V(Y)_{\alpha\beta}\big)_{\alpha,\beta\in A}$ describes the variance operator of Y and the matrix $\big(C_{\alpha i}\big)_{\alpha\in A, i\in I}$ defines the covariance operator $\mathrm{Cov}_{Y,X}$. We denote by $V(X)^{-1}_{ij}$ the entries of $\mathrm{Var}\big[\,X\,\big]^{-1}$ and by $D_{\alpha\beta}$ the entries of $D_{Y,X} = C_{Y,X}\,\mathrm{Var}[X]^{-1}C_{X,Y}$. We have

$$R_{X,Y}X = \sum_\alpha \Big(\sum_i R_{\alpha i}X_i\Big)X_i\Big)\boldsymbol{f}_\alpha,$$

where

$$R_{\alpha i} = \sum_j C_{\alpha j}V(X)^{-1}_{ji}.$$

Hence

$$Z_\alpha = Y_\alpha - \sum_i R_{\alpha i}X_i,\ \ Z_\beta = Y_\beta - \sum_j R_{\beta j}X_j,$$

$$\mathbb{E}\big[\,Z_\alpha Z_\beta\,\big] = V(Y)_{\alpha\beta} - \sum_j R_{\beta j}C_{\alpha j} - \sum_i R_{\alpha i}C_{i\beta} + \sum_{i,j} R_{\alpha i}V_{ij}R_{\beta j}.$$

We have

$$\sum_i\sum_j R_{\alpha i}V_{ij}R_{\beta j} = \sum_i\sum_j\Big(\sum_k C_{\alpha k}V(X)^{-1}_{ki}V_{ij}\Big)R_{\beta_j}$$
$$= \sum_j\Big(\sum_k C_{\alpha k}\delta_{kj}\Big)R_{\beta j} = \sum_k C_{\alpha k}R_{\beta_k} = \sum_k R_{\beta_k}C_{k\alpha} = D_{\beta\alpha} = D_{\alpha\beta}.$$

A similar but simpler computation shows that

$$\sum_j R_{\beta j}C_{\alpha j} = D_{\beta\alpha} = D_{\alpha\beta} = \sum_i R_{\alpha i}C_{i\beta}.$$

Thus $\Delta_{Y,X} = \mathrm{Var}\big[\,Y\,\big] - D_{Y,X}$ is the covariance operator of Z.

An elementary computation shows that.

$$\mathbb{E}\big[\,Z_\alpha X_i\,\big] = 0,\ \ \forall \alpha, i$$

and assumption (i) implies that X and Z are independent centered Gaussian vectors. Clearly Z is an X-measurable random vector. If S is and X-measurable event, then

$$\mathbb{E}\big[\,Z\boldsymbol{I}_F\,\big] = \mathbb{E}\big[\,Z\,\big]\mathbb{P}\big[\,F\,\big] = 0.$$

Hence

$$\mathbb{E}\big[\,Y\boldsymbol{I}_F\,\big] - \mathbb{E}\big[\,R_{Y,XS}\boldsymbol{I}_F\,\big] = \mathbb{E}\big[\,Z\boldsymbol{I}_F\,\big] = 0$$

so that

$$R_{Y,X}X = \mathbb{E}\big[\,Y\,\|\,X\,\big]$$

and

$$\mathbb{E}\big[\,Y\big|\,X = x\,\big] = R_{Y,X}x.$$

Now let $f : \boldsymbol{Y} \to \mathbb{R}$ be a bounded measurable function. Then $Y = \mathbb{E}\big[\,Y\,\|\,X\,\big] + Z$, with $\mathbb{E}\big[\,Y\,\|\,X\,\big], Z$ independent. Then

$$\mathbb{E}\big[\,f(Y)\,\|\,X = x\,\big] = \mathbb{E}\big[\,f\big(\,Z + \mathbb{E}[Y\,\|\,X]\,\big)\,\|\,X = x\,\big]$$
$$= \mathbb{E}\big[\,f(Z + \mathbb{E}[Y\,\|\,X = x])\,\big] = \mathbb{E}\big[\,f(Z + R_{Y,X}x)\,\big].$$

This proves the Proposition 1.14 when both X and Y are centered.

We now reduce the general case to the centered case. Consider the centered vectors

$$\bar{X} := X - m(X),\ \ \bar{Y} = Y - m(Y).$$

Then

$$R_{Y,X} = R_{\bar{Y},\bar{X}},$$

$$\mathbb{E}\big[\, Y \,\|\, X \,\big] = m(Y) + \mathbb{E}\big[\, \bar{Y} \,\|\, X \,\big] = m(Y) + \mathbb{E}\big[\, \bar{Y} \,\|\, \bar{X} \,\big]$$
$$= m(Y) + R_{Y,X}\bar{X} = m(Y) - R_{Y,X}m(X) + R_{Y,X}X.$$

If we set

$$\bar{Z} = \bar{Y} - R_{Y,X}\bar{X} = Y - m(Y) + R_{Y,X}m(X) - R_{Y,X}X = Y - \mathbb{E}\big[\, Y \,\|\, X \,\big],$$

then $\bar{Z}$ is independent of $\bar{X}$ and thus also of X. □

Proposition B.1. *Suppose that $\mathscr{V}, \boldsymbol{U}$ are finite dimensional Euclidean spaces, V is a centered, $\mathscr{V}$-valued Gaussian vector, and $E : \mathscr{V} \to \boldsymbol{U}$ a linear surjection. Assume that the $\boldsymbol{U}$-valued Gaussian vector $E(V)$ is nondegenerate. Define $\boldsymbol{Y} = \ker E$, $\boldsymbol{X} = \boldsymbol{Y}^{\perp}$. Set*

$$L = (EE^*)^{-1/2}E.$$

Denote by X and respectively Y the components of V along $\boldsymbol{X}$ and respectively $\boldsymbol{Y}$. Then the following hold

(i) *The Gaussian vectors LV and X are nondegenerate.*
(ii) *The Gaussian vectors $Y - \mathbb{E}\big[\, Y \,\|\, X \,\big]$ and $Y - \mathbb{E}\big[\, Y \,\|\, LX \,\big]$ have the same distribution and their common variance operator is $\Delta_{Y,X} : \boldsymbol{Y} \to \boldsymbol{Y}$ described in (1.10) . They are nondegenerate if and only if V is nondegenerate. Denote by $\Gamma_{\Delta_{Y,X}}$ the regression Gaussian measure, i.e., the centered Gaussian measure on $\boldsymbol{Y}$ with variance operator $\Delta_{Y,X}$.*
(iii) *If $f : \mathscr{V} \to \mathbb{R}$ is integrable with respect to the distribution of V, then*

$$\mathbb{E}\big[\, f(V) \,\big|\, EV = 0 \,\big] = \int_{\boldsymbol{Y}} f(y)\Gamma_{\Delta_{Y,X}}\big[\, dy \,\big] = \int_{\ker E} f(y)\Gamma_{\Delta_{Y,X}}\big[\, dy \,\big]. \tag{B.1}$$

In particular, if the Gaussian vector V is nondegenerate. and $f : \mathcal{V} \to (0, \infty)$ is a nonnegative, continuous homogeneous function whose restriction to $\ker E = \boldsymbol{Y}$ is nonzero, then

$$\mathbb{E}\big[\, f(V) \big|\, L(V) = \int_{\ker E} f(y)\Gamma_{\Delta_{Y,X}}\big[\, dy \,\big] = 0 \,\big] > 0. \tag{B.2}$$

Proof. The vectors $L(V) = (EE^*)^{-1/2}E(V)$, $(EE^*)^{-1/2}$ is surjective, $E\,|_{\boldsymbol{X}}: \boldsymbol{X} \to \boldsymbol{U}$ is an isomorphism and $E\,|_{\boldsymbol{X}}\,(X) = E(V)$.

Denote by P the orthogonal projection onto $\boldsymbol{X}$.Then $X = P(V)$, $Y = V - X$ and

$$E(V) = E(PV) = E(X).$$

Note that $E^*\boldsymbol{s}(U) = \boldsymbol{X}$. Set $B := EE^* : \boldsymbol{U} \to \boldsymbol{U}$. The operator B is symmetric and positive definite. Observe that $L := B^{-1/2}E$.

Lemma B.2. *The operator of L^* induces an isometry $\boldsymbol{U} \hookrightarrow \mathscr{V}$. Note that $L^*(\boldsymbol{U}) = (\ker L)^{\perp} = (\ker E)^{\perp} = \boldsymbol{X}$. Moreover $LL^* = \mathbb{1}_{\boldsymbol{U}}$.*

Proof. Let $u_1, u_2 \in \boldsymbol{U}$. We have

$$(L^*u_1, L^*u_2) = (E^*B^{-1/2}u_1, E^*B^{-1/2}u_2)$$
$$= (EE^*B^{-1/2}u_1, B^{-1/2}u_2) = (B^{1/2}u_1, B^{-1/2}u_2) = (u_1, u_2).$$

Note that $LL^* = B^{-1/2}LL^*B^{-1/2} = \mathbb{1}$. □

If A denotes the variance operator of X then the variance operator of $L(V) = L(X)$ is LAL^*,

$$\mathrm{Var}\left[\,L(X)\,\right] = L\,\mathrm{Var}\left[\,X\,\right]L^*.$$

Moreover, $C_{Y,L(X)} = C_{Y,X}L^*$.

Denote by Q the variance operator of V. With respect to the decomposition $\mathscr{V} = \boldsymbol{X} \oplus \boldsymbol{Y}$ Q has the block form

$$Q = \begin{bmatrix} A & C^\top \\ C & B \end{bmatrix},$$

where $C = C_{Y,X}$, and B is the variance operator of Y. Since X is nondegenerate, the operator A is invertible. Form the operator

$$\Delta_{Y,X} := \mathrm{Var}\left[\,Y\,\right] - C_{Y,X}\,\mathrm{Var}\left[\,X\,\right]^{-1} C_{Y,X} = B - CA^{-1}C^*$$

Then *Schur's complement formula* [20, Prop. 3.9] shows that $\det Q = \det A \cdot \det \Delta_{Y,X}$, so that $\det \Delta_{Y,X} \neq 0$ if and only if $\det Q \neq 0$, i.e., V is nondegenerate. Similarly

$$\begin{aligned} \Delta_{Y,LX} &= \mathrm{Var}\left[\,Y\,\right] - C_{Y,LX}\,\mathrm{Var}\left[\,LX\,\right]^{-1} C_{LX,Y} \\ &= B - CL^*(LAL^*\,)^{-1}LC^* = B - CA^{-1}C = \Delta_{Y,X}. \end{aligned}$$

since $LL^* = \mathbb{1}_{\boldsymbol{U}}$. This proves (ii).

From the equality

$$\mathbb{E}\big[\,V \,\|\, X\,\big] = \mathbb{E}\big[\,X + Y \,\|\, X\,\big] = \mathbb{E}\big[\,Y \,\|\, X\,\big] + X,$$

we deduce

$$Z = V - \mathbb{E}\big[\,V \,\|\, X\,\big] = Y - \mathbb{E}\big[\,Y \,\|\, X\,\big]$$

so Z is $\boldsymbol{Y}$-valued and its distribution is the centered Gaussian measure on $\boldsymbol{Y}$ with variance operator $\Delta_{Y,X}$. The equality (B.1) now follows from the regression formula (1.11).

To prove (B.2) observe that, since $\Gamma_{\Delta_{X,Y}}$ is nondegenerate, we have $\Gamma_{\Delta_{X,Y}}\left[\,\mathcal{O}\,\right] > 0$, for any open subset $\mathcal{O}$ of $\ker L$. Choose $c > 0$ such that the open set $\big\{ f\,|_{\ker L} > c \big\}$ is nonempty. Then

$$\int_{\ker L} f(y)\Gamma_{\Delta_{Y,X}}\left[\,dy\,\right] > c\Gamma_{\Delta_{X,Y}}\left[\,\{f > c\} \cap \ker L\,\right] > 0.$$

□

Remark B.3. The nondegeneracy of $\Gamma_{\Delta_{Y,X}}$ is important. If $\Gamma_{\Delta_{Y,X}}$ were concentrated on a proper subspace $S \subset \ker L$ it would still be possible that f is nontrivial yet $f\,|_S = 0$. □

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