NOTES ON LINEAR ALGEBRA

LIVIU I. NICOLAESCU

CONTENTS

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1. Multilinear forms and determinants

In this section, we will deal exclusively with *finite dimensional* vector spaces over the field $\mathbb{F}=\mathbb{R},\mathbb{C}$. If $\boldsymbol{U}_1,\boldsymbol{U}_2$ are two $\mathbb{F}$-vector spaces, we will denote by $\operatorname{Hom}(\boldsymbol{U}_1,\boldsymbol{U}_2)$ the space of $\mathbb{F}$-linear maps $\boldsymbol{U}_1\to\boldsymbol{U}_2$.

1.1. Mutilinear maps.

Definition 1.1. Suppose that $\boldsymbol{U}_1,\dots,\boldsymbol{U}_k,\boldsymbol{V}$ are $\mathbb{F}$-vector spaces. A map

$$\Phi:\boldsymbol{U}_1\times\cdots\times\boldsymbol{U}_k\to\boldsymbol{V}$$

is called *k-linear* if for any $1\leq i\leq k$, any vectors $\boldsymbol{u}_i,\boldsymbol{v}_i\in\boldsymbol{U}_i$, vectors $\boldsymbol{u}_j\in\boldsymbol{U}_j$, $j\neq i$, and any scalar $\lambda\in\mathbb{F}$ we have

$$\begin{gathered}\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{i-1},\boldsymbol{u}_i+\boldsymbol{v}_i,\boldsymbol{u}_{i+1},\dots,\boldsymbol{u}_k)\\ =\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{i-1},\boldsymbol{u}_i,\boldsymbol{u}_{i+1},\dots,\boldsymbol{u}_k)+\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{i-1},\boldsymbol{v}_i,\boldsymbol{u}_{i+1},\dots,\boldsymbol{u}_k),\\ \Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{i-1},\lambda\boldsymbol{u}_i,\boldsymbol{u}_{i+1},\dots,\boldsymbol{u}_k)=\lambda\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{i-1},\boldsymbol{u}_i,\boldsymbol{u}_{i+1},\dots,\boldsymbol{u}_k).\end{gathered}$$

In the special case $\boldsymbol{U}_1=\boldsymbol{U}_2=\cdots=\boldsymbol{U}_k=\boldsymbol{U}$ and $\boldsymbol{V}=\mathbb{F}$, the resulting map

$$\Phi:\underbrace{\boldsymbol{U}\times\cdots\times\boldsymbol{U}}_{k}\to\mathbb{F}$$

is called a *k-linear form* on $\boldsymbol{U}$. When $k=2$, we will refer to 2-linear forms as *bilinear forms*. We will denote by $\mathcal{T}^k(\boldsymbol{U}^*)$ the space of k-linear forms on $\boldsymbol{U}$. □

Example 1.2. Suppose that $\boldsymbol{U}$ is an $\mathbb{F}$-vector space and $\boldsymbol{U}^*$ is its dual, $\boldsymbol{U}^*:=\operatorname{Hom}(\boldsymbol{U},\mathbb{F})$. We have a natural bilinear map

$$\langle -,-\rangle:\boldsymbol{U}^*\times\boldsymbol{U}\to\mathbb{F},\;\;\boldsymbol{U}^*\times\boldsymbol{U}\ni(\alpha,\boldsymbol{u})\mapsto\langle\alpha,\boldsymbol{u}\rangle:=\alpha(\boldsymbol{u}).$$

The bilinear map is called the *canonical pairing* between the vector space $\boldsymbol{U}$ and its dual. □

Example 1.3. Suppose that $A=(a_{ij})_{1\leq i,j\leq n}$ is an $n\times n$ matrix with real entries. Define

$$\Phi_A:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R},\;\;\Phi(\boldsymbol{x},\boldsymbol{y})=\sum_{i,j}a_{ij}x_iy_j,$$

$$\boldsymbol{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix},\;\;\boldsymbol{y}=\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}.$$

To show that Φ is indeed a bilinear form we need to prove that for any $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^n$ and any $\lambda\in\mathbb{R}$ we have

$$\Phi_A(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{y})=\Phi_A(\boldsymbol{x},\boldsymbol{y})+\Phi_A(\boldsymbol{z},\boldsymbol{y}),\tag{1.1a}$$

$$\Phi_A(\boldsymbol{x},\boldsymbol{y}+\boldsymbol{z})=\Phi_A(\boldsymbol{x},\boldsymbol{y})+\Phi_A(\boldsymbol{x},\boldsymbol{z}),\tag{1.1b}$$

$$\Phi_A(\lambda\boldsymbol{x},\boldsymbol{y})=\Phi_A(\boldsymbol{x},\lambda\boldsymbol{y})=\lambda\Phi_A(\boldsymbol{x},by).\tag{1.1c}$$

To verify (1.1a) we observe that

$$\begin{gathered}\Phi_A(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{y})=\sum_{i,j}a_{ij}(x_i+z_i),y_j=\sum_{i,j}(a_{ij}x_iy_j+a_{ij}z_iy_j)=\sum_{i,j}a_{ij}x_iy_j+\sum_{ij}a_{ij}z_iy_j\\ =\Phi_A(\boldsymbol{x},\boldsymbol{y})+\Phi_A(\boldsymbol{z},\boldsymbol{y}).\end{gathered}$$

The equalities (1.1b) and (1.1c) are proved in a similar fashion. Observe that if $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the natural basis of $\mathbb{R}^n$, then

$$\Phi_A(\boldsymbol{e}_i, \boldsymbol{e}_j) = a_{ij}.$$

This shows that Φ_A is completely determined by its action on the basic vectors $\boldsymbol{e}_1, \dots \boldsymbol{e}_n$. □

Proposition 1.4. *For any bilinear form $\Phi \in \mathcal{T}^2(\mathbb{R}^n)$ there exists an $n \times n$ real matrix A such that $\Phi = \Phi_A$, where Φ_A is defined as in Example 1.3.* □

The proof is left as an exercise.

1.2. **The symmetric group.** For any finite sets A, B we denote $\mathrm{Bij}(A, B)$ the collection of bijective maps $\varphi : A \to B$. We set $\mathcal{S}(A) := \mathrm{Bij}(A, A)$. We will refer to $\mathcal{S}(A)$ as the *symmetric group on* A and to its elements as *permutations of* A. Note that if $\varphi, \sigma \in \mathcal{S}(A)$ then

$$\varphi \circ \sigma, \;\; \varphi^{-1} \in \mathcal{S}(A).$$

The composition of two permutations is often referred to as the *product* of the permutations. We denote by $\mathbb{1}$, or $\mathbb{1}_A$ the *identity permutation* that does not permute anything, i.e., $\mathbb{1}_A(a) = a$, $\forall a \in A$.

For any finite set S we denote by $|S|$ its cardinality, i.e., the number of elements of S. Observe that

$$\mathrm{Bij}(A, B) \neq \emptyset \Longleftrightarrow |A| = |B|.$$

In the special case when A is the discrete interval $A = \boldsymbol{I}_n = \{1, \dots, n\}$ we set

$$\mathcal{S}_n := \mathcal{S}(\boldsymbol{I}_n).$$

The collection $\mathcal{S}_n$ is called the symmetric group on n objects. We will indicate the elements $\varphi \in \mathcal{S}_n$ by diagrams of the form

$$\left(\begin{array}{cccc} 1 & 2 & \dots & n \\ \varphi_1 & \varphi_2 & \dots & \varphi_n \end{array} \right).$$

For any finite set S we denote by $|S|$ its cardinality, i.e., the number of elements of S.

Proposition 1.5. *(a) If A, B are finite sets and $|A| = |B|$, then*

$$|\mathrm{Bij}(A, B)| = |\mathrm{Bij}(B, A)| = |\mathcal{S}(A)| = |\mathcal{S}(B)|.$$

(b) For any positive integer n we have $|\mathcal{S}_n| = n! := 1 \cdot 2 \cdot 3 \cdots n$.

Proof. (a) Observe that we have a bijective correspondence

$$\mathrm{Bij}(A, B) \ni \varphi \mapsto \varphi^{-1} \in \mathrm{Bij}(B, A)$$

so that

$$|\mathrm{Bij}(A, B)| = |\mathrm{Bij}(B, A)|.$$

Next, fix a bijection $\psi : A \to B$. We get a correspondence

$$F_\psi : \mathrm{Bij}(A, A) \to \mathrm{Bij}(A, B), \;\; \varphi \mapsto F_\psi(\varphi) = \psi \circ \varphi.$$

This correspondence is injective because

$$F_\psi(\varphi_1) = F_\psi(\varphi_2) \Rightarrow \psi \circ \varphi_1 = \psi \circ \varphi_2 \Rightarrow \psi^{-1} \circ (\psi \circ \varphi_1) = \psi^{-1} \circ (\psi \circ \varphi_2) \Rightarrow \varphi_1 = \varphi_2.$$

This correspondence is also surjective. Indeed, if $\phi \in \mathrm{Bij}(A, B)$ then $\psi^{-1} \circ \phi \in \mathrm{Bij}(A, A)$ and

$$F_\psi(\psi^{-1} \circ \phi) = \psi \circ (\psi^{-1} \circ \phi) = \phi.$$

Thus, F_ψ is a bijection so that

$$|\mathcal{S}(A)| = |\mathrm{Bij}(A, B)|.$$

Finally we observe that

$$|\mathcal{S}(B)| = |\operatorname{Bij}(B,A)| = |\operatorname{Bij}(A,B)| = |\mathcal{S}(A)|.$$

This takes care of (a).

To prove (b) we argue by induction. Observe that $|\mathcal{S}_1| = 1$ because there exists a single bijection $\{1\} \to \{1\}$. We assume that $|\mathcal{S}_{n-1}| = (n-1)!$ and we prove that $|\mathcal{S}_n| = n!$. For each $k \in \boldsymbol{I}_n$ we set

$$\mathcal{S}_n^k := \big\{ \varphi \in \mathcal{S}_n;\ \varphi(n) = k \big\}.$$

A permutation $\varphi \in \mathcal{S}_n^k$ is uniquely detetrimined by its restriction to $\boldsymbol{I}_n \setminus \{n\} = \boldsymbol{I}_{n-1}$ and this restriction is a bijection $\boldsymbol{I}_{n-1} \to \boldsymbol{I}_n \setminus \{k\}$. Hence

$$|\mathcal{S}_n^k| = |\operatorname{Bij}(\boldsymbol{I}_{n-1}, \boldsymbol{I}_n \setminus \{k\})| = |\mathcal{S}_{n-1}|,$$

where at the last equality we used part(a). We deduce

$$|\mathcal{S}_n| = |\mathcal{S}_n^1| + \cdots + |\mathcal{S}_n^n| = \underbrace{|\mathcal{S}_{n-1}| + \cdots + |\mathcal{S}_{n-1}|}_{n}$$

$$= n|\mathcal{S}_{n-1}| = n(n-1)!,$$

where at the last step we invoked the inductive assumption. □

Definition 1.6. An *inversion* of a permutation $\sigma \in \mathcal{S}_n$ is a pair $(i,j) \in \boldsymbol{I}_n \times \boldsymbol{I}_n$ with the following properties.

- $i < j$.
- $\sigma(i) > \sigma(j)$.

We denote by $|\sigma|$ the number of inversions of the permutation σ. The *signature* of σ is then the quantity

$$\operatorname{sign}(\sigma) := (-1)^{|\sigma|} \in \{-1, 1\}.$$

A permutation σ is called *even/odd* if $\operatorname{sign}(\sigma) = \pm 1$. We denote by $\mathcal{S}_n^{\pm}$ the collection of even/odd permulations. □

Example 1.7. (a) Consider the permutation $\sigma \in \mathcal{S}_5$ given by

$$\sigma = \left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{array} \right).$$

The inversions of σ are

$$\begin{array}{c} (1,2),\ (1,3),\ (1,4),\ (1,5), \\ (2,3),\ (2,4),\ (2,5), \\ (3,4),\ (3,5),\ \ (4,5), \end{array}$$

so that $|\sigma| = 4+3+2+1 = 10$, $\operatorname{sign}(\sigma) = 1$.

(b) For any $i \neq j$ in $\boldsymbol{I}_n$ we denote by τ_{ij} the permutation defined by the equalities

$$\tau_{ij}(k) = \begin{cases} k, & k \neq i,j \\ j, & k = i \\ i, & k = j. \end{cases}$$

A transposition is defined to be a permutation of the form τ_{ij} for some $i < j$. Observe that

$$|\tau_{ij}| = 2|j-i| - 1,$$

so that

$$\operatorname{sign}(\tau_{ij}) = -1, \quad \forall i \neq j. \tag{1.2}$$

□

Proposition 1.8. *(a) For any $\sigma \in \mathcal{S}_n$ we have*

$$\operatorname{sign}(\sigma) = \prod_{1\leq i<j\leq n} \frac{\sigma(j)-\sigma(i)}{j-i}. \tag{1.3}$$

(b) For any $\varphi, \sigma \in \mathcal{S}_n$ we have

$$\operatorname{sign}(\varphi\circ\sigma) = \operatorname{sign}(\varphi)\cdot\operatorname{sign}(\sigma). \tag{1.4}$$

(c) $\operatorname{sign}(\sigma^{-1}) = \operatorname{sign}(\sigma)$

Proof. (a) Observe that the ratio $\frac{\sigma(j)-\sigma(i)}{j-i}$ negative if and only if (i,j) is inversion. Thus the number of negative ratios $\frac{\sigma(j)-\sigma(i)}{j-i}$, $i<j$, is equal to the number of inversions of σ so that the product

$$\prod_{1\leq i<j\leq n} \frac{\sigma(j)-\sigma(i)}{j-i}$$

has the same sign as the signature of σ. Hence, to prove (1.3) it suffices to show that

$$\left|\prod_{1\leq i<j\leq n} \frac{\sigma(j)-\sigma(i)}{j-i}\right| = |\operatorname{sign}(\sigma)| = 1,$$

i.e.,

$$\prod_{i<j} |\sigma(j)-\sigma(i)| = \prod_{i<j} |j-i|. \tag{1.5}$$

This is now obvious because the factors in the left-hand side are exactly the factors in the right-hand side multiplied in a different order. Indeed, for any $i<j$ we can find a unique pair $i'<j'$ such that

$$\sigma(j')-\sigma(i') = \pm(j-i).$$

(b) Observe that

$$\operatorname{sign}(\varphi) = \prod_{i<j}\frac{\varphi(j)-\varphi(i)}{j-i} = \prod_{i<j}\frac{\varphi(\sigma(j))-\varphi(\sigma(i)}{\sigma(j)-\sigma(i)}$$

and we deduce

$$\operatorname{sign}(\varphi)\cdot\operatorname{sign}(\sigma) = \left(\prod_{i<j}\frac{\varphi(\sigma(j))-\varphi(\sigma(i))}{\sigma(j)-\sigma(i)}\right)\left(\prod_{i<j}\frac{\sigma(j)-\sigma(i)}{j-i}\right)$$

$$\prod_{i<j}\frac{\varphi(\sigma(j))-\varphi(\sigma(i))}{j-i} = \operatorname{sign}(\varphi\circ\sigma).$$

To prove (c) we observe that

$$1 = \operatorname{sign}(\mathbb{1}) = \operatorname{sign}(\sigma^{-1}\circ\sigma) = \operatorname{sign}(\sigma^{-1})\operatorname{sign}(\sigma).$$

□

1.3. **Symmetric and skew-symmetric forms.**

Definition 1.9. Let $\boldsymbol{U}$ be an $\mathbb{F}$-vector space, $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$.
(a) A k-linear form $\Phi\in\mathcal{T}^k(\boldsymbol{U}^*)$ is called *symmetric* if for any $\boldsymbol{u}_1,\dots,\boldsymbol{u}_k\in\boldsymbol{U}$, and any permutation $\sigma\in\mathcal{S}_k$ we have

$$\Phi(\boldsymbol{u}_{\sigma(1)},\dots,\boldsymbol{u}_{\sigma(k)})=\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_k).$$

We denote by $S^k\boldsymbol{U}^*$ the collection of symmetric k-linear forms on $\boldsymbol{U}$.
(b) A k-linear form $\Phi\in\mathcal{T}^k(\boldsymbol{U}^*)$ is called *skew-symmetric* if for any $\boldsymbol{u}_1,\dots,\boldsymbol{u}_k\in\boldsymbol{U}$, and any permutation $\sigma\in\mathcal{S}_k$ we have

$$\Phi(\boldsymbol{u}_{\sigma(1)},\dots,\boldsymbol{u}_{\sigma(k)})=\operatorname{sign}(\sigma)\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_k).$$

We denote by $\Lambda^k\boldsymbol{U}^*$ the space of skew-symmetric k-linear forms on $\boldsymbol{U}$. □

Example 1.10. Suppose that $\Phi\in\Lambda^n\boldsymbol{U}^*$, and $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n\in\boldsymbol{U}$. The skew-linearity implies that for any $i<j$ we have

$$\begin{aligned}&\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{i-1},\boldsymbol{u}_i,\boldsymbol{u}_{i+1},\dots,\boldsymbol{u}_{j-1},\boldsymbol{u}_j,\boldsymbol{u}_{j+1},\dots,\boldsymbol{u}_n)\\ &=-\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{i-1},\boldsymbol{u}_j,\boldsymbol{u}_{i+1},\dots,\boldsymbol{u}_{j-1},\boldsymbol{u}_i,\boldsymbol{u}_{j+1},\dots,\boldsymbol{u}_n).\end{aligned}$$

Indeed, we have

$$\begin{aligned}&\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{i-1},\boldsymbol{u}_j,\boldsymbol{u}_{i+1},\dots,\boldsymbol{u}_{j-1},\boldsymbol{u}_i,\boldsymbol{u}_{j+1},\dots,\boldsymbol{u}_n)\\ &=\Phi(\boldsymbol{u}_{\tau_{ij}(1)},\dots,\boldsymbol{u}_{\tau_{ij}(k)},\dots,\boldsymbol{u}_{\tau_{ij}(n)})\end{aligned}$$

and $\operatorname{sign}(\tau_{ij})=-1$. In particular, this implies that if $i\neq j$, but $\boldsymbol{u}_i=\boldsymbol{u}_j$ then

$$\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)=0.$$ □

Proposition 1.11. *Suppose that $\boldsymbol{U}$ is an n-dimensional $\mathbb{F}$-vector space and $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ is a basis of $\boldsymbol{U}$. Then for any scalar $c\in\mathbb{F}$ there exists a unique skew-symmetric n-linear form $\Phi\in\Lambda^n\boldsymbol{U}^*$ such that*

$$\Phi(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n)=c.$$

Proof. To understand what is happening we consider first the special case $n=2$. Thus $\dim\boldsymbol{U}=2$. If $\Phi\in\Lambda^2\boldsymbol{U}^*$ and $\boldsymbol{u}_1,\boldsymbol{u}_2\in\boldsymbol{U}$ we can write

$$\boldsymbol{u}_1=a_{11}\boldsymbol{e}_1+a_{21}\boldsymbol{e}_2,\;\;\boldsymbol{u}_2=a_{12}\boldsymbol{e}_1+a_{22}\boldsymbol{e}_2,$$

for some scalars $a_{ij}\in\mathbb{F}$, $i,j\in\{1,2\}$. We have

$$\begin{aligned}\Phi(\boldsymbol{u}_1,\boldsymbol{u}_2)&=\Phi(a_{11}\boldsymbol{e}_1+a_{21}\boldsymbol{e}_2,a_{12}\boldsymbol{e}_1+a_{22}\boldsymbol{e}_2)\\ &=a_{11}\Phi(\boldsymbol{e}_1,a_{12}\boldsymbol{e}_1+a_{22}\boldsymbol{e}_2)+a_{21}\Phi(\boldsymbol{e}_2,a_{12}\boldsymbol{e}_1+a_{22}\boldsymbol{e}_2)\\ &=a_{11}a_{12}\Phi(\boldsymbol{e}_1,\boldsymbol{e}_1)+a_{11}a_{22}\Phi(\boldsymbol{e}_1,\boldsymbol{e}_2)+a_{21}a_{12}\Phi(\boldsymbol{e}_2,\boldsymbol{e}_1)+a_{21}a_{22}\Phi(\boldsymbol{e}_2,\boldsymbol{e}_2).\end{aligned}$$

The skew-symmetry of Φ implies that

$$\Phi(\boldsymbol{e}_1,\boldsymbol{e}_1)=\Phi(\boldsymbol{e}_2,\boldsymbol{e}_2)=0,\;\;\Phi(\boldsymbol{e}_2,\boldsymbol{e}_1)=-\Phi(\boldsymbol{e}_1,\boldsymbol{e}_2).$$

Hence

$$\Phi(\boldsymbol{u}_1,\boldsymbol{u}_2)=(a_{11}a_{22}-a_{21}a_{12})\Phi(\boldsymbol{e}_1,\boldsymbol{e}_2).$$

If $\dim\boldsymbol{U}=n$ and $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n\in\boldsymbol{U}$, then we can write

$$\boldsymbol{u}_1=\sum_{i_1=1}^n a_{i_11}\boldsymbol{e}_{i_1},\dots,\boldsymbol{u}_k=\sum_{i_k=1}^n a_{i_kk}\boldsymbol{e}_{i_k}$$

$$\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)=\Phi\left(\sum_{i_1=1}^n a_{i_1 1}\boldsymbol{e}_{i_1},\dots,\sum_{i_n=1}^n a_{i_n n}\boldsymbol{e}_{i_n}\right)$$
$$=\sum_{i_1,\dots,i_n=1}^n a_{i_1 1}\cdots a_{i_n n}\Phi(\boldsymbol{e}_{i_1},\dots,\boldsymbol{e}_{i_n}).$$

Observe that if the indices $i_1,\dots,i_n$ are not pairwise distinct then

$$\Phi(\boldsymbol{e}_{i_1},\dots,\boldsymbol{e}_{i_n})=0.$$

Thus, in the above sum we get contributions only from pairwise distinct choices of indices $i_1,\dots,i_n$, Such a choice corresponds to a permutation $\sigma\in\mathbb{S}_n$, $\sigma(k)=i_k$. We deduce that

$$\Phi(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)=\sum_{\sigma\in\mathbb{S}_n} a_{\sigma(1)1}\cdots a_{\sigma(n)n}\Phi(\boldsymbol{e}_{\sigma(1)},\dots,\boldsymbol{e}_{\sigma(n)}).$$
$$=\left(\sum_{\sigma\in\mathbb{S}_n}\operatorname{sign}(\sigma)a_{\sigma(1)1}\cdots a_{\sigma(n)n}\right)\Phi(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n).$$

Thus, $\Phi\in\Lambda^n\boldsymbol{U}^*$ is uniquely determined by its value on $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n)$.

Conversely, the map

$$(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)\to c\sum_{\sigma\in\mathbb{S}_n}\operatorname{sign}(\sigma)a_{\sigma(1)1}\cdots a_{\sigma(n)n},\ \ \boldsymbol{u}_k=\sum_{i=1}^n a_{ik}\boldsymbol{e}_i,$$

is indeed n-linear, and skew-symmetric. The proof is notationally bushy, but it does not involve any subtle idea so I will skip it. Instead, I'll leave the proof in the case $n=2$ as an exercise.

□

1.4. **The determinant of a square matrix.** Consider the vector space $\mathbb{F}^n$ with canonical basis

$$\boldsymbol{e}_1=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix},\ \ \boldsymbol{e}_2=\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix},\dots,\boldsymbol{e}_n=\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}.$$

According to Proposition 1.11 there exists a unique, n-linear skew-symmetric form Φ on $\mathbb{F}^n$ such that

$$\Phi(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n)=1.$$

We will denote this form by $\det$ and we will refer to it as the *determinant form* on $\mathbb{F}^n$. The proof of Proposition 1.11 shows that if $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n\in\mathbb{F}^n$,

$$\boldsymbol{u}_k=\begin{bmatrix}u_{1k}\\u_{2k}\\\vdots\\u_{nk}\end{bmatrix},\ \ k=1,\dots,n,$$

then

$$\det(\boldsymbol{u}_1,\dots\boldsymbol{u}_n)=\sum_{\sigma\in\mathbb{S}_n}\operatorname{sign}(\sigma)u_{\sigma(1)1}u_{\sigma(2)2}\cdots u_{\sigma(n)n}. \tag{1.6}$$

Note that

$$\det(\boldsymbol{u}_1,\dots\boldsymbol{u}_n)=\sum_{\varphi\in\mathbb{S}_n}\operatorname{sign}(\varphi)u_{1\varphi(1)}u_{2\varphi(2)}\cdots u_{n\varphi(n)}. \tag{1.7}$$

Definition 1.12. Suppose that $A = (a_{ij})_{1\leq i,j\leq n}$ is an $n \times n$-matrix with entries in $\mathbb{F}$ which we regard as a linear operator $A : \mathbb{F}^n \to \mathbb{F}^n$. The determinant of A is the scalar

$$\det A := \det(A\boldsymbol{e}_1, \dots, A\boldsymbol{e}_n)$$

where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the canonical basis of $\mathbb{F}^n$, and $A\boldsymbol{e}_k$ is the k-th column of A,

$$A\boldsymbol{e}_k = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix}, \quad k = 1, \dots, n.$$

□

Thus, according to (1.6) we have

$$\det A = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sign}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} \stackrel{(1.7)}{=} \sum_{\sigma\in\mathcal{S}_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}. \tag{1.8}$$

Remark 1.13. Consider a typical summand in the first sum in (1.8), $a_{\sigma(1)1} \cdots a_{\sigma(n)n}$. Observe that the n entries

$$a_{\sigma(1)1}, a_{\sigma(2)2}, \dots, a_{\sigma(n)n}$$

lie on different columns of A and thus occupy all the n columns of A. Similarly, these entries lie on different rows of A.

A collection of n entries so that no two lie on the same row or the same column is called a *rook placement*.[1] Observe that in order to describe a rook placement, you need to indicate the position of the entry on the first column, by indicating the row $\sigma(1)$ on which it lies, then you need to indicate the position of the entry on the second column etc. Thus, the sum in (1.8) has one term for each rook placement. □

If $A^\dagger$ denotes the transpose of the $n \times n$-matrix A with entries

$$a^\dagger_{ij} = a_{ji}$$

we deduce that

$$\det A^\dagger = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sign}(\sigma) a^\dagger_{\sigma(1)1} \cdots a^\dagger_{\sigma(n)n} = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)} = \det A. \tag{1.9}$$

Example 1.14. Suppose that A is a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

□

Proposition 1.15. *If A is an upper triangular $n \times n$-matrix, then* $\det A$ *is the product of the diagonal entries. A similar result holds if A is lower triangular.*

[1] If you are familiar with chess, a rook controls the row and the column at whose intersection it is situated.

Proof. To keep the ideas as transparent as possible, we carry the proof in the special case $n = 3$. Suppose first that A is upper tringular, Then

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

so that

$$A\boldsymbol{e}_1 = a_{11}\boldsymbol{e}_1, \;\; A\boldsymbol{e}_2 = a_{12}\boldsymbol{e}_1 + a_{22}\boldsymbol{e}_2, \;\; A\boldsymbol{e}_3 = a_{13}\boldsymbol{e}_1 + a_{23}\boldsymbol{e}_2 + a_{33}\boldsymbol{e}_3$$

Then

$$\begin{aligned}
\det A &= \det(A\boldsymbol{e}_1, A\boldsymbol{e}_2, A\boldsymbol{e}_3) \\
&= \det(a_{11}\boldsymbol{e}_1, a_{12}\boldsymbol{e}_1 + a_{22}\boldsymbol{e}_2, a_{13}\boldsymbol{e}_1 + a_{23}\boldsymbol{e}_2 + a_{33}\boldsymbol{e}_3) \\
&= \det(a_{11}\boldsymbol{e}_1, a_{12}\boldsymbol{e}_1, a_{13}\boldsymbol{e}_1 + a_{23}\boldsymbol{e}_2 + a_{33}\boldsymbol{e}_3) + \det(a_{11}\boldsymbol{e}_1, a_{22}\boldsymbol{e}_2, a_{13}\boldsymbol{e}_1 + a_{23}\boldsymbol{e}_2 + a_{33}\boldsymbol{e}_3) \\
&= a_{11}a_{12}\underbrace{\det(\boldsymbol{e}_1, \boldsymbol{e}_1, a_{13}\boldsymbol{e}_1 + a_{23}\boldsymbol{e}_2 + a_{33}\boldsymbol{e}_3)}_{=0} + a_{11}a_{22}\det(\boldsymbol{e}_1, \boldsymbol{e}_2, a_{13}\boldsymbol{e}_1 + a_{23}\boldsymbol{e}_2 + a_{33}\boldsymbol{e}_3) \\
&= a_{11}a_{22}\Big(\underbrace{\det(\boldsymbol{e}_1, \boldsymbol{e}_2, a_{13}\boldsymbol{e}_1)}_{=0} + \underbrace{\det(\boldsymbol{e}_1, \boldsymbol{e}_2, a_{23}\boldsymbol{e}_2)}_{=0} + \det(\boldsymbol{e}_1, \boldsymbol{e}_2, a_{33}\boldsymbol{e}_3)\Big) \\
&= a_{11}a_{22}a_{33}\det(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = a_{11}a_{22}a_{33}.
\end{aligned}$$

This proves the proposition when A is upper triangular. If A is lower triangular, then its transpose $A^\dagger$ is upper triangular and we deduce

$$\det A = \det A^\dagger = a^\dagger_{11}a^\dagger_{22}a^\dagger_{33} = a_{11}a_{22}a_{33}.$$

□

Recall that we have a collection of elementary column (row) operations on a matrix. The next result explains the effect of these operations on the determinant of a matrix.

Proposition 1.16. *Suppose that A is an $n \times n$-matrix. The following hold.*

(a) If the matrix B is obtained from A by multiplying the elements of the i-th column of A by the same nonzero scalar λ, then

$$\det B = \lambda \det A.$$

(b) If the matrix B is obtained from A by switching the order of the columns i and j, $i \neq j$ then

$$\det B = -\det A.$$

(c) If the matrix B is obtained from A by adding to the i-th column, the j-th column, $j \neq i$ then

$$\det B = \det A.$$

(d) Similar results hold if we perform row operations of the same type.

Proof. (a) We have

$$\begin{aligned}
\det B &= \det(B\boldsymbol{e}_1, \dots, B\boldsymbol{e}_n) = \det(A\boldsymbol{e}_1, \dots, \lambda A\boldsymbol{e}_i, A\boldsymbol{e}_n) \\
&= \lambda \det(A\boldsymbol{e}_1, \dots, A\boldsymbol{e}_i, A\boldsymbol{e}_n) = \lambda \det A.
\end{aligned}$$

(b) Observe that for any $\sigma \in \mathcal{S}_n$ we have

$$\det(A\boldsymbol{e}_{\sigma(1)}, \dots, A\boldsymbol{e}_{\sigma(n)}) = \operatorname{sign}(\sigma)\det(A\boldsymbol{e}_1, \dots, A\boldsymbol{e}_{\sigma(n)}) = \operatorname{sign}(\sigma)\det A.$$

Now observe that the columns of B are

$$B\boldsymbol{e}_1 = A\boldsymbol{e}_{\tau_{ij}(1)}, \dots, B\boldsymbol{e}_n = A\boldsymbol{e}_{\tau_{ij}(n)}$$

and $\operatorname{sign}(\tau_{ij}) = 1$.

For (c) we observe that

$$\begin{aligned}\det B &= \det(A\boldsymbol{e}_1,\dots,A\boldsymbol{e}_{i-1},A\boldsymbol{e}_i+A\boldsymbol{e}_j,A\boldsymbol{e}_{i+1},\dots,A\boldsymbol{e}_j,\dots,A\boldsymbol{e}_n)\\ &= \det(A\boldsymbol{e}_1,\dots,A\boldsymbol{e}_{i-1},A\boldsymbol{e}_i,A\boldsymbol{e}_{i+1},\dots,A\boldsymbol{e}_j,\dots,A\boldsymbol{e}_n)\\ &+\underbrace{\det(A\boldsymbol{e}_1,\dots,A\boldsymbol{e}_{i-1},A\boldsymbol{e}_j,A\boldsymbol{e}_{i+1},\dots,A\boldsymbol{e}_j,\dots,A\boldsymbol{e}_n)}_{=0}\\ &= \det A.\end{aligned}$$

Part (d) follows by applying (a), (b), (c) to the transpose of A, observing that the rows of A are the columns of $A^\dagger$ and then using the equality $\det C = \det C^\dagger$.

□

The above results represents one efficient method for computing determinants because we know that by performing elementary row operations on a square matrix we can reduce it to upper triangular form.

Here is a first application of determinants.

Proposition 1.17. *Suppose that A is an $n\times n$-matrix with entries in $\mathbb{F}$. Then the following statements are equivalent.*

(a) *The matrix A is invertible.*
(b) $\det A \neq 0$.

Proof. A matrix A is invertible if and only if by performing elementary row operations we can reduce to an upper triangular matrix B whose diagonal entries are nonzero, i.e., $\det B\neq 0$. By performing elementary row operation the determinant changes by a nonzero factor so that

$$\det A\neq 0 \Longleftrightarrow \det B\neq 0.$$

□

Corollary 1.18. *Suppose that $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n\in\mathbb{F}^n$. The following statements are equivalent.*

(a) *The vectors $\boldsymbol{u}_1,\dots,\boldsymbol{u}_n$ are linearly independent.*
(b) $\det(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)\neq 0$.

Proof. Consider the linear operator $A:\mathbb{F}^n\to\mathbb{F}^n$ given by $A\boldsymbol{e}_i=\boldsymbol{u}_i$, $i=1,\dots,n$. We can tautologically identify it with a matrix and we have

$$\det(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)=\det A.$$

Now observe that $(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)$ are linearly independent if and only if A is invertible and according to the previous propostion, this happens if and only if $\det A\neq 0$.

□

1.5. **Additional properties of determinants.**

Proposition 1.19. *If A,B are two $n\times n$-matrices , then*

$$\det AB=\det A\det B. \tag{1.10}$$

Proof. We have

$$\det AB=\det(AB\boldsymbol{e}_1,\dots,AB\boldsymbol{e}_n)=\det(\sum_{i_1=1}^n b_{i_11}A\boldsymbol{e}_{i_1},\dots\sum_{i_n=1}^n b_{i_nn}A\boldsymbol{e}_{i_n})$$

$$= \sum_{i_1,\dots,i_n=1}^{b} b_{i_1 1}\cdots b_{i_n n}\det(A\boldsymbol{e}_{i_1},\dots A\boldsymbol{e}_{i_n})$$

In the above sum, the only nontrivial terms correspond to choices of pairwise distinct indices $i_1,\dots,i_n$. For such a choice, the sequence $i_1,\dots,i_n$ describes a permutation of $\boldsymbol{I}_n$. We deduce

$$\det AB = \sum_{\sigma\in\mathcal{S}_n} b_{\sigma(1)1}\cdots b_{\sigma(n)n}\underbrace{\det(A\boldsymbol{e}_{\sigma(1)},\dots,A\boldsymbol{e}_{\sigma(n)})}_{=\operatorname{sign}(\sigma)\det A}$$

$$= \det A\sum_{\sigma\in\mathcal{S}_n}\operatorname{sign}(\sigma)b_{\sigma(1)1}\cdots b_{\sigma(n)n} = \det A\det B.$$

□

Corollary 1.20. *If A is an invertible matrix, then*

$$\det A^{-1} = \frac{1}{\det A}.$$

Proof. Indeed, we have

$$A\cdot A^{-1} = \mathbb{1}$$

so that

$$\det A\det A^{-1} = \det\mathbb{1} = 1.$$

□

Proposition 1.21. *Suppose that m,n are positive integers and S is an $(m+n)\times(m+n)$-matrix that has the block form*

$$S = \begin{bmatrix} A & C\\ 0 & B\end{bmatrix},$$

where A is an $m\times m$-matrix, B is an $n\times n$-matrix and C is an $m\times n$-matrix. Then

$$\det S = \det A\cdot\det B.$$

Proof. We denote by s_{ij} the (i,j)-entry of S, $i,j\in\boldsymbol{I}_{m+n}$. From the block description of S we deduce that

$$j\leq m \;\text{ and }\; i>n \Rightarrow s_{ij}=0. \tag{1.11}$$

We have

$$\det S = \sum_{\sigma\in\mathcal{S}_{m+n}}\operatorname{sign}(\sigma)\prod_{i=1}^{m+n}s_{\sigma(i)i}.$$

From (1.11) we deduce that in the above sum the nonzero terms correspond to permutations $\sigma\in\mathcal{S}_{m+n}$ such that

$$\sigma(i)\leq m,\;\;\forall i\leq m. \tag{1.12}$$

If σ is such a permutation, then its restriction to $\boldsymbol{I}_m$ is a permutation α of $\boldsymbol{I}_m$ and its restriction to $\boldsymbol{I}_{m+n}\setminus\boldsymbol{I}_m$ is a permutation of this set, which we regard as a permutation β of $\boldsymbol{I}_n$. Conversely, given $\alpha\in\mathcal{S}_m$ and $\beta\in\mathcal{S}_n$ we obtain a permutation $\sigma=\alpha*\beta\in\mathcal{S}_{m_n}$ satisfying (1.12) given by

$$\alpha*\beta(i) = \begin{cases}\alpha(i), & i\leq m,\\ m+\beta(i-m), & i>m.\end{cases}$$

Observe that

$$\operatorname{sign}(\alpha * \beta) = \operatorname{sign}(\alpha)\operatorname{sign}(\beta),$$

and we deduce

$$\det S = \sum_{\alpha\in\mathcal{S}_m,\ \beta\in\mathcal{S}_n} \operatorname{sign}(\alpha*\beta)\prod_{i=1}^{m+n} s_{\alpha*\beta(i)i}$$

$$= \left(\sum_{\alpha\in\mathcal{S}_m}\operatorname{sign}(\alpha)\prod_{i=1}^{m} s_{\alpha(i)i}\right)\left(\sum_{\beta\in\mathcal{S}_n}\operatorname{sign}(\beta)\prod_{j=1}^{n} s_{m+\beta(j),j+m}\right) = \det A\det B.$$

□

Definition 1.22. If A is an $n\times n$-matrix and $i,j\in\boldsymbol{I}_n$, we denote by $A(i,j)$ the matrix obtained from A by removing the i-th row and the j-th column. □

Corollary 1.23. *Suppose that the j-th column of an $n\times n$-matrix A is sparse, i.e., all the elements on the j-th column, with the possible exception of the element on the i-th row, are equal to zero. Then*

$$\det A = (-1)^{i+j}a_{ij}\det A(i,j).$$

Proof. Observe that if $i=j=1$ then A has the block form

$$A = \left[\begin{array}{cc} a_{11} & * \\ 0 & A(1,1)\end{array}\right]$$

and the result follows from Proposition 1.21.

We can reduce the general case to this special case by permuting rows and columns of A. If we switch the j-th column with $(j-1)$-th column we can arrange that the $(j-1)$-th column is the sparse column. Iterating this procedure we deduce after $(j-1)$ such switches that the first column is the sparse column.

By performing $(i-1)$ row-switches we can arrange that the nontrivial element on this sparse column is situated on the first row. Thus, after a total of $i+j-2$ row and column switches we obtain a new matrix A' with the block form

$$A' = \left[\begin{array}{cc} a_{ij} & * \\ 0 & A(i,j)\end{array}\right]$$

We have

$$(-1)^{i+j}\det A = \det A' = a_{ij}\det A(i,j).$$

□

Corollary 1.24 (Row and column expansion)**.** *Fix $j\in\boldsymbol{I}_n$. Then for any $n\times n$-matrix we have*

$$\det A = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det A(i,j) = \sum_{k=1}^{n}(-1)^{i+k}a_{jk}A(j,k).$$

The first equality is referred to as the j-th column expansion *of* $\det A$*, while the second equality is referred to as the j*-th row expansion *of* $\det A$*.*

Proof. We prove only the column expansion. The row expansion is obtained by applying to column expansion to the transpose matrix. For simplicity we assume that $j=1$. We have

$$\det A = \det(A\boldsymbol{e}_1, A\boldsymbol{e}_2,\dots,A\boldsymbol{e}_n) = \det\Big(\sum_{i-1}^{n} a_{i1}\boldsymbol{e}_i, A\boldsymbol{e}_2,\dots,A\boldsymbol{e}_n\Big)$$

$$= \sum_{i-1}^{n} a_{i1} \det\big(\boldsymbol{e}_i, A\boldsymbol{e}_2, \dots, A\boldsymbol{e}_n\big).$$

Denote by A_i the matrix whose first column is the column basic vector $\boldsymbol{e}_i$, and the other columns are the corresponding columns of A, $A\boldsymbol{e}_2, \dots, A\boldsymbol{e}_n$. We can rewrite the last equality as

$$\det A = \sum_{i=1}^{n} a_{i1} \det A_i.$$

The first column of A_i is sparse, and the submatrix $A_i(i,1)$ is equal to the submatrix $A(i,1)$. We deduce from the previous corollary that

$$\det A_i = (-1)^{i+1} \det A_i(i,1) = (-1)^{i+1} \det A(i,1).$$

This completes the proof of the column expansion formula. □

Corollary 1.25. *If $k \neq j$ then*

$$\sum_{i=1}^{n} (-1)^{i+j} a_{ik} \det A(i,j) = 0.$$

Proof. Denote by A' the matrix obtained from A by removing the j-th column and replacing with the k-th column of A. Thus, in the new matrix A' the j-th and the k-th columns are identical so that $\det A' = 0$. On the other hand $A'(i,j) = A(i,j)$ Expanding $\det A'$ along the j-th column we deduce

$$0 = \det A' = \sum_{i=1}^{n} (-1)^{i+j} a'_{ij} \det A(i,j) = \sum_{i=1}^{n} (-1)^{ij} a_{ik} \det A(i,j).$$

□

Definition 1.26. For any $n \times n$ matrix A we define the *adjoint matrix* $\check{A}$ to be the $n \times n$-matrix with entries

$$\check{a}_{ij} = (-1)^{i+j} \det A(j,i), \;\; \forall i,j \in \boldsymbol{I}_n.$$ □

Form Corollary 1.24 we deduce that for any j we have

$$\sum_{i=1}^{n} \check{a}_{ji} a_{ij} = \det A,$$

while Corollary 1.25 implies that for any $j \neq k$ we have

$$\sum_{i=1}^{n} \check{a}_{ji} a_{ik} = 0.$$

The last two identities can be rewritten in the compact form

$$\check{A}A = (\det A)\mathbb{1}. \tag{1.13}$$

If A is invertible, then from the above equality we conclude that

$$A^{-1} = \frac{1}{\det A}\check{A}. \tag{1.14}$$

Example 1.27. Suppose that A is a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then

$$\det A = a_{11}a_{22} - a_{12}a_{21},$$

$$A(1,1) = [a_{22}], \quad A(1,2) = [a_{21}], \quad A(2,1) = [a_{12}], \quad A(2,2) = [a_{11}],$$

$$\breve{a}_{11} = \det A(1,1) = a_{22}, \quad \breve{a}_{12} = -\det A(2,1) = -a_{12},$$

$$\breve{a}_{21} = -\det A(1,2) = -a_{21}, \quad \breve{a}_{22} = \det A(2,2) = a_{11},$$

so that

$$\breve{A} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

and we observe that

$$\breve{A}A = \begin{bmatrix} \det A & 0 \\ 0 & \det A \end{bmatrix}.$$

Hence

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \qquad \square$$

Proposition 1.28 (Cramer's Rule)**.** *Suppose that A is an invertible $n \times n$-matrix and $\boldsymbol{u}, \boldsymbol{x} \in \mathbb{F}^n$ are two column vectors such that*

$$A\boldsymbol{x} = \boldsymbol{u},$$

i.e., $\boldsymbol{x}$ is a solution of the linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & u_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & u_2 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & u_n. \end{cases}$$

Denote by $A_j(\boldsymbol{u})$ the matrix obtained from A by replacing the j-th column with the column vector $\boldsymbol{u}$. Then

$$x_j = \frac{\det A_j(\boldsymbol{u})}{\det A}, \quad \forall j = 1, \dots, n. \tag{1.15}$$

Proof. By expanding along the j-th column of $A_j(\boldsymbol{u})$ we deduce

$$\det A_j(u) = \sum_{k=1}^{n} (-1)^{j+k} u_k \det A(k,j). \tag{1.16}$$

On the other hand,

$$(\det A)\boldsymbol{x} = (\breve{A}A)\boldsymbol{x} = \breve{A}\boldsymbol{u}.$$

Hence

$$(\det A)x_j = \sum_{k=1}^{n} \breve{a}_{jk} u_k = \sum_k (-1)^{k+j} u_k \det A(k,j) \stackrel{(1.16)}{=} \det A_j(\boldsymbol{u}).$$

$\square$

1.6. **Examples.** To any list of complex numbers $(x_1, \dots, x_n)$ we associate the $n \times n$ matrix

$$V(x_1, \dots, x_n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}. \tag{1.17}$$

This matrix is called the *Vandermonde matrix* associated to the list of numbers $(x_1, \dots, x_n)$. We want to compute its determinant. Observe first that

$$\det V(x_1, \dots x_n) = 0.$$

if the numbers $z_1, \dots, z_n$ are not distinct. Observe next that

$$\det V(x_1, x_2) = \det \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} = (x_2 - x_1).$$

Consider now the 3×3 situation. We have

$$\det V(x_1, x_2, x_3) = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix}.$$

Subtract from the 3rd row the second row multiplied by x_1 to deduce

$$\det V(x_1, x_2, x_3) = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ 0 & x_2^2 - x_1 x_2 & x_3^2 - x_3 x_1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ 0 & x_2(x_2 - x_1) & x_3^2 - x_3 x_1 \end{bmatrix}.$$

Subtract from the 2nd row the first row multiplied by x_1 to deduce

$$\det V(x_1, x_2, x_3) = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 \\ 0 & x_2(x_2 - x_1) & x_3(x_3 - x_1) \end{bmatrix} = \det \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ x_2(x_2 - x_1) & x_3(x_3 - x_1) \end{bmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1) \det \begin{bmatrix} 1 & 1 \\ x_2 & x_3 \end{bmatrix} = (x_2 - x_1)(x_3 - x_1) \det V(x_2, x_3).$$

$$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_1).$$

We can write the above equalities in a more compact form

$$\det V(x_1, x_2) = \prod_{1 \le i < j \le 2} (x_j - x_i), \quad \det V(x_1, x_2, x_3) = \prod_{1 \le i < j \le 3} (x_j - x_i). \tag{1.18}$$

A similar row manipulation argument (left to you as an exercise) shows that

$$\det V(x_1, \dots, x_n) = (x_2 - x_1) \cdots (x_n - x_1) \det V(x_2, \dots, x_n). \tag{1.19}$$

We have the following general result.

Proposition 1.29. *For any integer $n \ge 2$ and any complex numbers $x_1, \dots, x_n$ we have*

$$\det V_n(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} (x_j - x_i). \tag{1.20}$$

Proof. We will argue by induction on n. The case $n = 2$ is contained in (1.18). Assume now that (1.20) is true for $n - 1$. This means that

$$\det V(x_2, \dots, x_n) = \prod_{2 \leq i < j \leq n} (x_j - x_i).$$

Using this in (1.19) we deduce

$$\det V_n(x_1, \dots, x_n) = (x_2 - x_1) \cdots (x_n - x_1) \cdot \prod_{2 \leq i < j \leq n} (x_j - x_i) = \prod_{1 \leq i < j \leq n} (x_j - x_i). \qquad \square$$

Here is a simple application of the above computation.

Corollary 1.30. *If $x_1, \dots, x_n$ are* distinct *complex numbers then for any complex numbers $r_1, \dots, r_n$ there exists a polynomial of degree $\leq n - 1$ uniquely determined by the conditions*

$$P(x_1) = r_1, \dots, p(x_n) = r_n. \tag{1.21}$$

Proof. The polynomial P must have the form

$$P(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1},$$

where the coefficients $a_0, \dots, a_{n-1}$ are to be determined. We will do this using (1.21) which can be rewritten as a system of linear equations in which the unknown are the coefficients $a_0, \dots, a_{n-1}$,

$$\begin{cases} a_0 + a_1 x_1 + \cdots + a_{n-1} x_1^{n-1} & = & r_1 \\ a_0 + a_1 x_2 + \cdots + a_{n-1} x_2^{n-1} & = & r_2 \\ \vdots & \vdots & \vdots \\ a_0 + a_1 x_n + \cdots + a_{n-1} x_n^{n-1} & = & r_n. \end{cases}$$

We can rewrite this in matrix form

$$\underbrace{\begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}}_{=V(x_1, \dots, x_n)^\dagger} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}.$$

Because the numbers $x_1, \dots, x_n$ are distinct, we deduce from (1.20) that

$$\det V(x_1, \dots, x_n)^\dagger = V(x_1, \dots, x_n) \neq 0.$$

Hence the above linear system has a unique solution $a_0, \dots, a_{n-1}$. $\square$

1.7. Exercises.

Exercise 1.1. Prove that the map in Example 1.2 is indeed a bilinear map. □

Exercise 1.2. Prove Proposition 1.4. □

Exercise 1.3. Fix $n \in \mathbb{N}$, $n \geq 2$ and set $\boldsymbol{I}_n := \{1, \dots, n\}$

(i) Show that for any $1 \leq i < j \leq n$ we have $\tau_{ij} \circ \tau_{ij} = \mathbb{1}_{\boldsymbol{I}_n}$.

(ii) Prove that for any permutation $\sigma \in \mathcal{S}_n$ there exists a sequence of transpositions $\tau_{i_1 j_1}, \dots, \tau_{i_m j_m}$, $m < n$, such that

$$\sigma \circ \tau_{i_m j_m} \circ \cdots \circ \tau_{i_1 j_1} = \mathbb{1}_{\boldsymbol{I}_n}.$$

Conclude that any permutation is a product of transpositions. **Hint.** Define $k_1 = k_1(\sigma)$ as the unique element of $\boldsymbol{\sigma}_n$ such that $\sigma(k) = 1$. If $k_1(\sigma) \neq 1$ compute $k_1\big(\,\sigma \circ \tau_{1k}\,\big)$.

□

Exercise 1.4. Decompose the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

as a composition of transpositions. □

Exercise 1.5. Suppose that $\Phi \in \mathcal{T}^2(\boldsymbol{U})$ is a *symmetric* bilinear map. Define $Q : \boldsymbol{U} \to \mathbb{F}$ by setting

$$Q(\boldsymbol{u}) = \Phi(\boldsymbol{u}, \boldsymbol{u}), \;\; \forall \boldsymbol{u} \in \boldsymbol{U}.$$

Show that for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$ we have

$$\Phi(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{4}\Big(Q(\boldsymbol{u} + \boldsymbol{v}) - Q(\boldsymbol{u} - \boldsymbol{v}) \Big).$$ □

Exercise 1.6. Prove that the map

$$\Phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \;\; \Phi(\boldsymbol{u}, \boldsymbol{v}) = u_1 v_2 - u_2 v_1,$$

is bilinear, and skew-symmetric. □

Exercise 1.7. (a) Show that a bilinear form $\Phi : \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{F}$ is skew-symmetric if and only if $\Phi(\boldsymbol{u}, \boldsymbol{u}) = 0$, $\forall \boldsymbol{u} \in \boldsymbol{U}$.
Hint: Expand $\Phi(\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v})$ using the bilinearity of Φ.

(b) Prove that an n-linear form $\Phi \in \mathcal{T}^n(\boldsymbol{U})$ is skew-symmetric if and only if for any $i \neq j$ and any vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n \in \boldsymbol{U}$ such that $\boldsymbol{u}_i = \boldsymbol{u}_j$ we have

$$\Phi(\boldsymbol{u}_1, \dots, \boldsymbol{u}_n) = 0.$$

Hint. Use the trick in part (a) and Exercise 1.3. □

Exercise 1.8. Compute the determinant of the following 5×5-matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}.$$

Exercise 1.9. Fix complex numbers x and h. Compute the determinant of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ x & h & -1 & 0 \\ x^2 & hx & h & -1 \\ x^3 & hx^2 & hx & h \end{bmatrix}.$$

Can you generalize thus example? □

Exercise 1.10. Prove the equality (1.19). □

Exercise 1.11. (a) Consider a degree $(n-1)$ polynomial

$$P(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0, \;\; a_{n-1} \neq 0.$$

Compute the determinant of the following matrix.

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ P(x_1) & P(x_2) & \cdots & P(x_n) \end{bmatrix}.$$

(b) Compute the determinants of the following $n \times n$ matrices

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_2x_3\cdots x_n & x_1x_3x_4\cdots x_n & \cdots & x_1x_2\cdots x_{n-1} \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ (x_2+x_3+\cdots+x_n)^{n-1} & (x_1+x_3+x_4+\cdots+x_n)^{n-1} & \cdots & (x_1+x_2+\cdots+x_{n-1})^{n-1} \end{bmatrix}.$$

Hint. To compute $\det B$ it is wise to write $S = x_1 + \cdots + x_n$ so that $x_2 + x_3 + \cdots + x_n = (S - x_1)$, $x_1 + x_3 + \cdots + x_n = S - x_2$ etc. Next observe that $(S-x)^k$ is a polynomial of degree k in x. □

Exercise 1.12. Suppose that A is skew-symmetric $n \times n$ matrix, i.e.,

$$A^\dagger = -A.$$

Show that $\det A = 0$ if n is odd. □

Exercise 1.13. Suppose that $A = (a_{ij})_{1 \leq i,j \leq n}$ is an $n \times n$ matrix with complex entries.

(i) Fix complex numbers $x_1, \dots, x_n$, $y_1, \dots, y_n$ and consider the $n \times n$ matrix B with entries

$$b_{ij} = x_iy_ja_{ij}.$$

Show that

$$\det B = (x_1y_1\cdots x_ny_n)\det A.$$

(ii) Suppose that C is the $n \times n$ matrix with entries
$$c_{ij} = (-1)^{i+j} a_{ij}.$$
Show that $\det C = \det A$.

□

Exercise 1.14. Suppose we are given three sequences of numbers $\underline{a} = (a_k)_{k\geq 1}$, $\underline{b} = (b_k)_{k\geq 1}$ and $\underline{c} = (c_k)_{k\geq 1}$. To these sequences we associate a sequence of *Jacobi matrices*
$$J_n = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_{n-1} & a_n \end{bmatrix}. \tag{$\boldsymbol{J}$}$$

(i) Show that
$$\det J_n = a_n \det J_{n-1} - b_{n-1}c_{n-1} \det J_{n-2}. \tag{1.22}$$
Hint: Expand along the last row.

(ii) Suppose that above we have
$$c_k = 1, \;\; b_k = 2, \;\; a_k = 3, \;\; \forall k \geq 1.$$
Compute $\det J_1, \det J_2$. Using (1.22) determine $\det J_3, \det J_4, \det J_5, \det J_6, \det J_7$. Can you detect a pattern?

□

Exercise 1.15. Suppose we are given a sequence of polynomials with complex coefficients $(P_n(x))_{n\geq 0}$, $\deg P_n = n$, for all $n \geq 0$,
$$P_n(x) = a_n x^n + \cdots, \;\; a_n \neq 0.$$
Denote by $\boldsymbol{V}_n$ the space of polynomials with complex coefficients and degree $\leq n$.

(i) Show that the collection $\{P_0(x), \dots, P_n(x)\}$ is a basis of $\boldsymbol{V}_n$.

(ii) Show that for any $x_1, \dots, x_n \in \mathbb{C}$ we have
$$\det \begin{bmatrix} P_0(x_1) & P_0(x_2) & \cdots & P_0(x_n) \\ P_1(x_1) & P_1(x_2) & \cdots & P_1(x_n) \\ \vdots & \vdots & \vdots & \vdots \\ P_{n-1}(x_1) & P_{n-1}(x_2) & \cdots & P_{n-1}(x_n) \end{bmatrix} = a_0 a_1 \cdots a_{n-1} \prod_{i<j} (x_j - x_i).$$
Hint. Factor out a_0 from the first row, a_1 from the second row etc. Then use row operations to compute the determinant.

□

Exercise 1.16. To any polynomial $P(x) = c_0 + c_1 x + \ldots + c_{n-1}x^{n-1}$ of degree $\leq n-1$ with complex coefficients we associate the $n \times n$ *circulant matrix*
$$C_P = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{bmatrix},$$

Set

$$\rho = e^{\frac{2\pi \boldsymbol{i}}{n}}, \;\; \boldsymbol{i} = \sqrt{-1},$$

so that $\rho^n = 1$. Consider the $n \times n$ Vandermonde matrix $V_\rho = V(1, \rho, \dots, \rho^{n-1})$ defined as in (1.17)

(i) Show that for any $j = 1, \dots, n-1$ we have

$$1 + \rho^j + \rho^{2j} + \cdots + \rho^{(n-1)j} = 0.$$

(ii) Show that

$$C_P \cdot V_\rho = V_\rho \cdot \operatorname{Diag}\big(P(1), P(\rho), \dots, P(\rho^{n-1}) \big),$$

where $\operatorname{Diag}(a_1, \dots, a_n)$ denotes the diagonal $n \times n$-matrix with diagonal entries $a_1, \dots, a_n$.

(iii) Show that

$$\det C_P = P(1)P(\rho) \cdots P(\rho^{n-1}). \qquad \square$$

(iv) Suppose that $P(x) = 1 + 2x + 3x^2 + 4x^3$ so that C_P is a 4×4-matrix with integer entries and thus $\det C_P$ is an integer. Find this *integer*.

(v) Generalize the computation at (iv).

Exercise 1.17. Consider the $n \times n$-matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(i) Find the matrices

$$A^2, A^3, \dots, A^n.$$

(ii) Compute $(I - A)(I + A + \cdots + A^{n-1})$.

(iii) Find the inverse of $(I - A)$.

$\square$

Exercise 1.18. Let

$$P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$$

be a polynomial of degree d with complex coefficients. We denote by $\mathcal{S}$ the collection of sequences of complex numbers, i.e., functions

$$f : \big\{0, 1, 2, \dots\big\} \to \mathbb{C}, \;\; n \mapsto f(n).$$

This is a complex vector space in a standard fashion. We denote by $\mathcal{S}_P$ the subcollection of sequences $f \in \mathcal{S}$ satisfying the *recurrence relation*

$$f(n+d) + a_{d-1}f(n+d-1) + \cdots a_1 f(n+1) + a_0 f(n) = 0, \;\; \forall n \geq 0. \qquad (\boldsymbol{R}_P)$$

(a) Show that $\mathcal{S}_P$ is a vector subspace of $\mathcal{S}$.

(b) Show that the map $\mathcal{I} : \mathcal{S}_P \to \mathbb{C}^d$ which associates to $f \in \mathcal{S}_P$ its initial values $\mathcal{I}f$,

$$\mathcal{I}f = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(d-1) \end{bmatrix} \in \mathbb{C}^d$$

is an isomorphism of vector spaces.

(c) For any $\lambda \in \mathbb{C}$ we consider the sequence f_λ defined by

$$f_\lambda(n) = \lambda^n, \;\; \forall n \geq 0.$$

(Above it is understood that $\lambda^0 = 1$.) Show that $f_\lambda \in \mathcal{S}_P$ if and only if $P(\lambda) = 0$, i.e., λ is a root of P.
(d) Suppose P has d *distinct* roots $\lambda_1, \dots, \lambda_d \in \mathbb{C}$. Show that the collection of sequences $f_{\lambda_1}, \dots, f_{\lambda_d}$ is a basis of $\mathcal{S}_P$.
(e) Consider the *Fibonacci sequence* $\big(f(n) \big)_{n \geq 0}$ defined by

$$f(0) = f(1) = 1, \;\; f(n+2) = f(n+1) + f(n), \;\; \forall n \geq 0.$$

Thus,

$$f(2) = 2, \;\; f(3) = 3, \;\; f(4) = 5, \;\; f(5) = 8, \;\; f(6) = 13, \dots .$$

Use the results (a)–(d) above to find a short formula describing $f(n)$. □

Exercise 1.19. Let b, c be two <u>distinct</u> complex numbers. Consider the $n \times n$ Jacobi matrix

$$J_n = \begin{bmatrix} b+c & b & 0 & 0 & \cdots & 0 & 0 \\ c & b+c & b & 0 & \cdots & 0 & 0 \\ 0 & c & b+c & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & c & b+c & b \\ 0 & 0 & 0 & 0 & \cdots & c & b+c \end{bmatrix}.$$

Find a short formula for $\det J_n$.
Hint: Use the results in Exercises 1.14 and 1.18. □

2. SPECTRAL DECOMPOSITION OF LINEAR OPERATORS

2.1. **Invariants of linear operators.** Suppose that $\boldsymbol{U}$ is an n-dimensional $\mathbb{F}$-vector space. We denote by $L(\boldsymbol{U})$ the space of linear operators (maps) $T : \boldsymbol{U} \to \boldsymbol{U}$. We already know that once we choose a basis $\underline{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ of $\boldsymbol{U}$ we can represent T by a matrix

$$A = \mathcal{M}(\underline{\boldsymbol{e}}, T) = (a_{ij})_{1 \le i,j \le n},$$

where the elements of the k-th column of A describe the coordinates of $T\boldsymbol{e}_k$ in the basis $\underline{\boldsymbol{e}}$, i.e.,

$$T\boldsymbol{e}_k = a_{1k}\boldsymbol{e}_1 + \cdots + a_{nk}\boldsymbol{e}_n = \sum_{j=1}^{n} a_{jk}\boldsymbol{e}_j.$$

A priori, there is no good reason of choosing the basis $\underline{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ over another $\underline{\boldsymbol{f}} = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_n)$. With respect to this new basis the operator T is represented by another matrix

$$B = \mathcal{M}(\underline{\boldsymbol{f}}, T) = (b_{ij})_{1 \le i,j \le n}, \quad T\boldsymbol{f}_k = \sum_{j=1}^{n} b_{jk}\boldsymbol{f}_j.$$

The basis $\underline{\boldsymbol{f}}$ is related to the basis $\underline{\boldsymbol{e}}$ by a *transition matrix*

$$C = (c_{ij})_{1 \le i,j \le n}, \quad \boldsymbol{f}_k = \sum_{j=1}^{n} c_{jk}\boldsymbol{e}_j.$$

Thus the, k-th column of C describes the coordinates of the vector $\boldsymbol{f}_k$ in the basis $\underline{\boldsymbol{e}}$. Then C is invertible and

$$B = C^{-1}AC. \tag{2.1}$$

The space $\boldsymbol{U}$ has lots of bases, so *the same* operator T can be represented by many different matrices. The question we want to address in this section can be loosely stated as follows.

Find bases of $\boldsymbol{U}$ so that, in these bases, the operator T represented by "very simple" matrices.

We will not define what a "very simple" matrix is, but we will agree that the more zeros a matrix has, the simpler it is. The above question is closely related to the concept of *invariant* of a linear operator. An invariant is roughly speaking a quantity naturally associated to the operator that does not change when we change bases.

Definition 2.1. (a) A subspace $\boldsymbol{V} \subset \boldsymbol{U}$ is called an *invariant subspace* of the linear operator $T \in L(\boldsymbol{U})$ if

$$T\boldsymbol{v} \in \boldsymbol{V}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

(b) A *nonzero* vector $\boldsymbol{u}_0 \in \boldsymbol{U}$ is called an *eigenvector* of the linear operator T if and only if the linear subspace spanned by $\boldsymbol{u}_0$ is an invariant subspace of T. $\square$

Example 2.2. (a) Suppose that $T : \boldsymbol{U} \to \boldsymbol{U}$ is a linear operator. Its *null space* or *kernel*

$$\ker T := \{ \boldsymbol{u} \in \boldsymbol{U};\ \ T\boldsymbol{u} = 0 \},$$

is an invariant subspace of T. Its dimension, $\dim \ker T$, is an *invariant* of T because in its definition we have not mentioned any particular basis. We have already encountered this dimension under a different guise.

If we choose a basis $\underline{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ of $\boldsymbol{U}$ and use it to represent T as an $n \times n$ matrix $A = (a_{ij})_{1\le i,j\le n}$, then $\dim\ker T$ is equal to the nullity of A, i.e., the dimension of the vector space of solutions of the linear system

$$A\boldsymbol{x} = 0, \;\; \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{F}^n.$$

The *range*

$$\boldsymbol{R}(T) = \bigl\{ T\boldsymbol{u}; \;\; \boldsymbol{u} \in \boldsymbol{U} \bigr\}$$

is also an invariant subspace of T. Its dimension $\dim \boldsymbol{R}(T)$ can be identified with the rank of the matrix A above. The rank nullity theorem implies that

$$\dim\ker T + \dim \boldsymbol{R}(T) = \dim \boldsymbol{U}. \tag{2.2}$$

(b) Suppose that $\boldsymbol{u}_0 \in \boldsymbol{U}$ is an eigenvector of T. Then $T\boldsymbol{u}_0 \in \operatorname{span}(u_0)$ so that there exists $\lambda \in \mathbb{F}$ such that

$$T\boldsymbol{u}_0 = \lambda \boldsymbol{u}_0. \qquad \square$$

2.2. **The determinant and the characteristic polynomial of an operator.** Assume again that $\boldsymbol{U}$ is an n-dimensional $\mathbb{F}$-vector space. A more subtle invariant of an operator $T \in L(\boldsymbol{U})$ is its determinant. This is a scalar $\det T \in \mathbb{F}$. Its definition requires a choice of a basis of $\boldsymbol{U}$, but the end result *is independent of any choice of basis*. Here are the details.

Fix a basis

$$\underline{\boldsymbol{e}} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$$

of $\boldsymbol{U}$. We use it to represent T as an $n \times n$ real matrix $A = (a_{ij})_{1\le i,j\le n}$. More precisely, this means that

$$T\boldsymbol{e}_j = \sum_{i=1}^n a_{ij}\boldsymbol{e}_i, \;\; \forall j = 1, \dots, n.$$

If we choose another basis of $\boldsymbol{U}$,

$$\underline{\boldsymbol{f}} = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_n),$$

then we can represent T by another $n \times n$ matrix $B = (b_{ij})_{1\le i,j\le n}$, i.e.,

$$T\boldsymbol{f}_j = \sum_{i=1}^n b_{ij}\boldsymbol{f}_i, \;\; j = 1, \dots, n.$$

As we have discussed above the basis $\underline{\boldsymbol{f}}$ is obtained from $\underline{\boldsymbol{e}}$ via a *change-of-basis* matrix $C = (c_{ij})_{1\le i,j\le n}$, i.e.,

$$\boldsymbol{f}_j = \sum_{i=1}^n c_{ij}\boldsymbol{e}_j, \;\; j = 1, \dots, n.$$

Moreover the matrices A, B are related by the *transition rule* (2.1),

$$B = C^{-1}AC.$$

We say that two $n \times n$ matrices A, B are *similar* if there exists an invertible $n \times n$ matrix C such that $B = C^{-1}AC$. Similar matrices represent the same linear operator, but in different bases.

If A, B are similar, then

$$\det B = \det(C^{-1}AC) = \det C^{-1} \det A \det C = \det A.$$

The upshot is that the similar matrices A and B *have the same determinant*. Thus, no mater what basis of $\boldsymbol{U}$ we choose to represent T as an $n \times n$ matrix, the determinant of that matrix *is independent* *of the basis used*. This number, denoted by $\det T$ is an invariant of T called the *determinant* of the *operator* T. Here is a simple application of this concept.

Corollary 2.3.

$$\ker T \neq 0 \Longleftrightarrow \det T = 0. \qquad \square$$

More generally, for any $x \in \mathbb{F}$ consider the operator

$$x\mathbb{1} - T : \boldsymbol{U} \to \boldsymbol{U},$$

defined by

$$(x\mathbb{1} - T)\boldsymbol{u} = x\boldsymbol{u} - T\boldsymbol{u}, \;\; \forall \boldsymbol{u} \in \boldsymbol{U}.$$

Note that if A, B are matrices representing T in different bases, then $B = C^{-1}AC$ for some invertible matrix C. We deduce that

$$x\mathbb{1} - B = C^{-1}x\mathbb{1}C - C^{-1}AC = C^{-1}(x\mathbb{1} - A)C.$$

Thus, the matrices $x\mathbb{1} - A$ and $x\mathbb{1} - B$ are similar for any $x \in \mathbb{F}$ and thus they have the same determinant. We set

$$P_T(x) := \det(x\mathbb{1} - A) = \det(x\mathbb{1} - B)$$

The function $P_T(x)$ is an invariant of T.

Proposition 2.4. *The quantity $P_T(x)$ is a polynomial of degree $n = \dim \boldsymbol{U}$ in the variable x.*

Proof. Choose a basis $\underline{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$. In this basis the operator T is represented by an $n \times n$ matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ and the operator $x\mathbb{1} - T$ is represented by the matrix

$$xI - A = \begin{bmatrix} x - a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & x - a_{22} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & x - a_{nn} \end{bmatrix}.$$

As explained in Remark 1.13, the determinant of this matrix is a sum of products of certain choices of n entries of this matrix, namely the entries that form a rook placement. Since there are exactly n entries in this matrix that contain the variable x, we see that each product associated to a rook placement of entries is a polynomial in x of degree $\leq n$. There exists *exactly one* rook placement so that each of the entries of this placement contain the term x. This placement is easily described, it consists of the terms situated on the diagonal of this matrix, and the product associated to these entries is

$$(x - a_{11}) \cdots (x - a_{nn}).$$

Any other rook placement contains at most $(n-1)$ entries that involve the term x, so the corresponding product of these entries is a polynomial of degree at most $n - 1$. Hence

$$\det(xI - A) = (x - a_{11}) \cdots (x - a_{nn}) + \text{polynomial of degree} \leq n - 1.$$

Hence $P_T(x) = \det(xI - A)$ is a polynomial of degree n in x. $\square$

Definition 2.5. The polynomial $P_T(x)$ is called the *characteristic polynomial* of the operator T. $\square$

Recall that a number $\lambda \in \mathbb{F}$ is called an *eigenvalue* of the operator T if and only if there exists $\boldsymbol{u} \in \boldsymbol{U} \setminus 0$ such that $T\boldsymbol{u} = \lambda \boldsymbol{u}$, i.e.,

$$(\lambda \mathbb{1} - T)\boldsymbol{u} = 0.$$

Thus λ is an eigenvalue of T if and only if $\ker(\lambda I - T) \neq 0$. Invoking Corollary 2.3 we obtain the following important result.

Corollary 2.6. *A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if it is a root of the characteristic polynomial of T, i.e., $P_T(\lambda) = 0$.* □

The collection of eigenvalues of an operator T is called the *spectrum* of T and it is denoted by $\operatorname{spec}(T)$. If $\lambda \in \operatorname{spec}(T)$, then the subspace $\ker(\lambda \mathbb{1} - T) \subset \boldsymbol{U}$ is called the *eigenspace* corresponding to the eigenvalue λ.

From the above corollary and the fundamental theorem of algebra we obtain the following important consequence.

Corollary 2.7. *If $T : \boldsymbol{U} \to \boldsymbol{U}$ is a linear operator on a* <u>complex</u> *vector space $\boldsymbol{U}$, then* $\operatorname{spec}(T) \neq \emptyset$.□

We say that a linear operator $T : \boldsymbol{U} \to \boldsymbol{U}$ is *triangulable* if there exists a basis $\underline{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ of $\boldsymbol{U}$ such that the matrix A representing T in this basis is upper triangular. We will refer to A as a *triangular representation* of T. Such a triangular representation need not be unique. The linear operators over complex vector spaces are triangulable.

Theorem 2.8 (Schur)**.** *Suppose that $\boldsymbol{U}$ is an n-dimensional complex vector space and $T \in L(\boldsymbol{U})$ is a linear operator. Then T is triangulable.*

Proof. We argue by induction on n. The result is trivially true when $n = 1$. So we assume it is true for operators on complex vectors spaces of dimension $n - 1$ and we prove it for a linear operator T on a complex n-dimensional vector space $\boldsymbol{U}$. In this case $\operatorname{spec}(T) \neq \emptyset$. Fix an eigenvalue λ of T and choose a nonzero eigenvector u_1, $Tu_1 = \lambda u_1$. Complete u_1 to a basis $\{u_1, u_2, \dots, u_n\}$ of $\boldsymbol{U}$. The matrix representing T in this basis has the block decomposition

$$A = \begin{bmatrix} \lambda & * & * & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{bmatrix}.$$

Above, the block A_1 is an $(n-1) \times (n-1)$ matrix defining a linear operator on the $(n-1)$-dimensional space

$$\boldsymbol{V} = \operatorname{span}\{u_2, \dots, u_n\}.$$

More concretely, any vector u is described uniquely as a linear combination $u = c_1 u_1 + v$, $c_1 \in \mathbb{C}$, $v \in \boldsymbol{V}$ and then

$$Tv_k = c_k u_1 + A_1 v_k, \quad k = 2, \dots, n.$$

By induction, we can find a basis $v_2, \dots, v_n$ of $\boldsymbol{V}$ such that, in this basis, the operator A_1 is represented by the upper triangular matrix B_1. In the basis $\{u_1, v_2, \dots, v_n\}$ of $\boldsymbol{U}$ the operator T is represented by the upper triangular matrix

$$\begin{bmatrix} \lambda & * & * & * \\ 0 & & & \\ \vdots & & B_1 & \\ 0 & & & \end{bmatrix}.$$

□

Corollary 2.9. *Suppose that $T : \boldsymbol{U} \to \boldsymbol{U}$ is a triangulable operator. Then for any basis $\underline{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ of $\boldsymbol{U}$ such that the matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ representing T in this basis is upper triangular, we have*

$$P_T(x) = (x - a_{11}) \cdots (x - a_{nn}).$$

Thus, the eigenvalues of T are the elements along the diagonal of any triangular representation of T. □

Suppose that T is a complex linear operator. If A and B are two triangular representations of T, then

$$(x - a_{11}) \cdots (x - a_{nn}) = P_T(x) = (x - b_{11}) \cdots (x - b_{nn}) \tag{2.3}$$

We have the following result whose proof is left to you as an exercise.

We say that two lists of numbers $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ and $\vec{\mu} = (\mu_1, \dots, \mu_n)$ *coincide up to a permutation*, and we write this $\vec{\lambda} =_p \vec{\mu}$, if there exists a permutation $\sigma \in \mathcal{S}_n$ such that

$$\mu_k = \lambda_{\sigma(k)}, \;\; \forall k = 1, \dots, n.$$

Example 2.10. (a) Note that $(3, 1, 1, 2) =_p (1, 3, 2, 1) =_p (3, 2, 1, 1)$

(b) Two lists of numbers coincide up to a permutation if one list can be obtained from another by a succession of switches of location of pairs of elements. For example

$$(\boxed{3}, 1, \boxed{1}, 2) =_p (\boxed{1}, 1, \boxed{3}, 2), \;\; (1, \boxed{1}, \boxed{3}, 2) =_p (1, \boxed{3}, \boxed{1}, 2).$$

□

Proposition 2.11. *Suppose $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, $\vec{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$ are such that*

$$(z - \lambda_1) \cdots (z - \lambda_n) = (z - \mu_1) \cdots (z - \mu_n), \;\; \forall z \in \mathbb{C},$$

then $\vec{\lambda} =_p \vec{\mu}$. □

Using (2.3) and Proposition 2.11 we deduce that the list of numbers $b_{11}, \dots, b_{nn}$ is simply a permutation of the list $a_{11}, \dots, a_{nn}$. For any $\lambda \in \mathbb{C}$ we denote by $m_\lambda(T)$ the number of times λ appears in one of these lists. Note that λ is an eigenvalue of T if and only if $m_\lambda(T) > 0$. In this case $m_\lambda(T)$ is called the algebraic multiplicity of the eigenvalue λ.

We deduce from Theorem 2.8 the following important consequence.

Corollary 2.12. *Suppose that T is a linear operator on the finite dimensional complex vector space $\boldsymbol{U}$. Then*

$$\det T = \prod_{\lambda \in \operatorname{spec}(T)} \lambda^{m_\lambda(T)}, \tag{2.4a}$$

$$P_T(x) = \prod_{\lambda \in \operatorname{spec}(T)} (x - \lambda)^{m_\lambda(T)}, \tag{2.4b}$$

$$\sum_{\lambda \in \operatorname{spec}(T)} m_\lambda(T) = \deg P_T = \dim \boldsymbol{U} = n. \tag{2.4c}$$

2.3. **Symmetric polynomials.** A polynomial f in n complex variables $\vec{\lambda}$ is called *symmetric* if

$$\vec{\lambda} =_p \vec{\mu} \;\Rightarrow\; f(\vec{\lambda}) = f(\vec{\mu}).$$

Any symmetric polynomial f in n complex variables defines an invariant of a linear operator T on an n-dimensional complex vector space $\boldsymbol{U}$. More precisely define

$$f(T) = f(a_{11}, \dots, a_{nn}),$$

where $A = (a_{ij})_{1\leq i,j\leq n}$ is any upper triangular matrix representing T in some basis of $\boldsymbol{U}$.

A fundamental collection of symmetric polynomials are the *elementary symmetric polynomials*

$$c_k(\lambda_1, \dots, \lambda_n) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}, \;\; k = 1, \dots, n.$$

For example,

$$c_1(\lambda_1, \dots, \lambda_n) = \lambda_1 + \cdots + \lambda_n = \sum_{i=1}^n \lambda_i,$$

$$c_2(\lambda_1, \dots, \lambda_n) = \sum_{1\leq i<j\leq n} \lambda_i\lambda_j$$

$$c_n(\lambda_1, \dots, \lambda_n) = \lambda_1\cdots\lambda_n.$$

To prove that the polynomials c_k are symmetric observe that

$$(1 + x\lambda_1)\cdots(1 + x\lambda_n) = 1 + c_1(\lambda)x + \cdots + c_k(\lambda)x^k + \cdots + c_n(\lambda)x^n.$$

If we permute the λ's, the left-hand side of the above equality does not change and thus neither does the right-hand side. From the above equality we also deduce.

$$(x + \lambda_1)\cdots(x + \lambda_n) = x^n + c_1(\lambda_1)x^{n-1} + \cdots + c_k(\lambda)x^{n-k} + \cdots + c_n(\lambda).$$

If $\lambda_1, \dots, \lambda_n$ are the eigenvalues (repeated according to their multiplicities) of a linear operator T on a complex n-dimensional space $\boldsymbol{U}$ complex $n \times n$ matrix, then

$$P_T(x) = (x - \lambda_1)\cdots(x + \lambda_n) = x^n - c_1(\lambda_1)x^{n-1} + \cdots + (-1)^k c_k(\lambda)x^{n-k} + \cdots + (-1)^n c_n(\lambda).$$

If the upper triangular complex $n \times n$ matrix A represents the matrix T in some basis, then

$$\lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn},$$

and we deduce

$$\operatorname{tr} T = a_{11} + \cdots + a_{nn} = c_1(\lambda).$$

Similarly

$$c_n(\lambda) = \lambda_1\cdots\lambda_n = a_{11}\cdots a_{nn} = \det T.$$

Another important class of symmetric polynomials is given by

$$p_k(\lambda_1, \dots, \lambda_n) = \sum_{j=1}^n \lambda_j^k, \;\; k = 1, 2, \dots .$$

E.g.,

$$p_1 = c_1, \;\; p_2 = \lambda_1^2 + \cdots + \lambda_n^2, \dots .$$

Theorem 2.13 (Netwon-Giraud)**.** *For $1 \leq m \leq n$ we have*

$$p_m - p_{m-1}c_1 + p_{m-2}c_2 + \cdots + (-1)^{m-1}p_1c_{m-1} + (-1)^m mc_m = 0. \tag{2.5}$$

More explicitly

$$c_1 = p_1, \;\; -2c_2 = p_2 - p_1c_1, \;\; 3c_3 = p_3 - p_2c_1 + p_1c_2, \dots$$

We deduce successively

$$c_1 = p_1, \;\; 2c_2 = p_2 - p_1^2, \;\; 3c_3 = p_3 - p_2p_1 + \frac{1}{2}p_1(p_2 - p_1^2), \dots$$

In other words the polynomials c_k are uniquely determined by the polynomials $p_1, \dots, p_n$.

Proof. The polynomial on the right hand side of (2.5) is a sum of terms of the form

$$T^k_{j,i_1,\dots,i_k} := (-1)^k \lambda_j^{m-k}\lambda_{i_1}\cdots\lambda_{i_k}, \;\; k = 0, \dots, m-1, \;\; 1 \leq i_1 < \cdots < i_k \leq n,$$

and

$$T^m_{j,i_1,\dots,i_{m-1}} := (-1)^m \lambda_j\lambda_{i_1}\cdots\lambda_{i_{m-1}}, \;\; i_1 < \cdots < i_{m-1}, \;\; j \notin \{i_1, \dots, i_{m-1}\}.$$

Note that if $j \in \{i_1, \dots, i_k\}$, say $j = i_\ell$, then

$$T^k_{j,i_1,\dots,i_k} + T^{k-1}_{j,i_1,\dots,i_{\ell-1},i_{\ell+1},\dots,i_k} = 0.$$

If $j \notin \{i_1, \dots, i_k\}$, we arrange the numbers $j, i_1, \dots, i_k$ in increasing order, $j_1 < \cdots < j_k < j_{k+1}$ and then

$$T^k_{j,i_1,\dots,i_k} + T^{k+1}_{j,j_1,\dots,j_{k+1}} = 0.$$

□

2.4. **Generalized eigenspaces.** Suppose that $T : \boldsymbol{U} \to \boldsymbol{U}$ is a linear operator on the n-dimensional $\mathbb{F}$-vector space. Suppose that $\operatorname{spec}(T) \neq \emptyset$. Choose an eigenvalue $\lambda \in \operatorname{spec}(T)$.

Lemma 2.14. *Let k be a positive integer. Then*

$$\ker(\lambda\mathbb{1} - T)^k \subset \ker(\lambda\mathbb{1} - T)^{k+1}.$$

Moreover, if $\ker(\lambda\mathbb{1} - T)^k = \ker(\lambda\mathbb{1} - T)^{k+1}$*, then*

$$\ker(\lambda\mathbb{1} - T)^k = \ker(\lambda\mathbb{1} - T)^{k+1} = \ker(\lambda\mathbb{1} - T)^{k+2} = \ker(\lambda\mathbb{1} - T)^{k+3} = \cdots.$$

Proof. Observe that if $(\lambda\mathbb{1} - T)^k\boldsymbol{u} = 0$, then

$$(\lambda\mathbb{1} - T)^{k+1}\boldsymbol{u} = (\lambda\mathbb{1} - T)(\lambda\mathbb{1} - T)^k\boldsymbol{u} = 0,$$

so that $\ker(\lambda\mathbb{1} - T)^k \subset \ker(\lambda\mathbb{1} - T)^{k+1}$.

Suppose that

$$\ker(\lambda\mathbb{1} - T)^k = \ker(\lambda\mathbb{1} - T)^{k+1}.$$

To prove that $\ker(\lambda\mathbb{1} - T)^{k+1} = \ker(\lambda\mathbb{1} - T)^{k+2}$ it suffices to show that

$$\ker(\lambda\mathbb{1} - T)^{k+1} \supset \ker(\lambda\mathbb{1} - T)^{k+2}.$$

Let $\boldsymbol{v} \in \ker(\lambda\mathbb{1} - T)^{k+2}$. Then

$$(\lambda\mathbb{1} - T)^{k+1}(\mathbb{1} - \lambda T)\boldsymbol{v} = 0,$$

so that $(\mathbb{1} - \lambda T)\boldsymbol{v} \in \ker(\lambda\mathbb{1} - T)^{k+1} = \ker(\lambda\mathbb{1} - T)^k$ so that

$$(\lambda\mathbb{1} - T)^k(\lambda\mathbb{1} - T)\boldsymbol{v} = 0,$$

i.e., $\boldsymbol{v} \in \ker(\lambda\mathbb{1} - T)^{k+1}$. We have thus shown that

$$\ker(\lambda\mathbb{1} - T)^{k+1} = \ker(\lambda\mathbb{1} - T)^{k+2}.$$

The remaining equalities $\ker(\lambda\mathbb{1}-T)^{k+2}=\ker(\lambda\mathbb{1}-T)^{k+3}=\cdots$ are proven in a similar fashion. □

Corollary 2.15. *For any $m\geq n=\dim\boldsymbol{U}$ we have*

$$\ker(\lambda\mathbb{1}-T)^m=\ker(\lambda\mathbb{1}-T)^n, \tag{2.6a}$$

$$\boldsymbol{R}(\lambda\mathbb{1}-T)^m=\boldsymbol{R}(\lambda\mathbb{1}-T)^n. \tag{2.6b}$$

Proof. Consider the sequence of positive integers

$$d_1(\lambda)=\dim_{\mathbb{F}}(\lambda\mathbb{1}-T),\dots,d_k(\lambda)=\dim_{\mathbb{F}}(\lambda\mathbb{1}-T)^k,\dots.$$

Lemma 2.14 shows that

$$d_1(\lambda)\leq d_2(\lambda)\leq\cdots\leq n=\dim\boldsymbol{U}.$$

Thus there must exist k such that $d_k(\lambda)=d_{k+1}(\lambda)$. We set

$$k_0=\min\big\{\,k;\;\; d_k(\lambda)=d_{k+1}(\lambda)\,\big\}.$$

Thus

$$d_1(\lambda)<\cdots<d_{k_0}(\lambda)\leq n,$$

so that $k_0\leq n$. On the other hand, since $d_{k_0}(\lambda)=d_{k_0+1}(\lambda)$ we deduce that

$$\ker(\lambda\mathbb{1}-T)^{k_0}=\ker(\lambda\mathbb{1}-T)^m,\;\;\forall m\geq k_0.$$

Since $n\geq k_0$ we deduce

$$\ker(\lambda\mathbb{1}-T)^n=\ker(\lambda\mathbb{1}-T)^{k_0}=\ker(\lambda\mathbb{1}-T)^m,\;\;\forall m\geq k_0.$$

This proves (2.6a). To prove (2.6b) observe that if $m>n$, then

$$\boldsymbol{R}(\lambda\mathbb{1}-T)^m=(\lambda\mathbb{1}-T)^n\Big(\,(\lambda\mathbb{1}-T)^{m-n}V\,\Big)\subset(\lambda\mathbb{1}-T)^n\big(\,V\,\big)=\boldsymbol{R}(\lambda\mathbb{1}-T)^n.$$

On the other hand, the rank-nullity formula (2.2) implies that

$$\begin{aligned}\dim\boldsymbol{R}(\lambda\mathbb{1}-T)^n&=\dim\boldsymbol{U}-\dim\ker(\lambda\mathbb{1}-T)^n\\&=\dim\boldsymbol{U}-\dim\ker(\lambda\mathbb{1}-T)^m=\dim\boldsymbol{R}(\lambda\mathbb{1}-T)^m.\end{aligned}$$

This proves (2.6b). □

Definition 2.16. Let $T:\boldsymbol{U}\to\boldsymbol{U}$ be a linear operator on the n-dimensional $\mathbb{F}$-vector space $\boldsymbol{U}$.

(i) For any $\lambda\in\operatorname{spec}(T)$ the subspace $\ker(\lambda\mathbb{1}-T)^n$ is called the *generalized eigenspace* of T corresponding to the eigenvalue λ and it is denoted by $E_\lambda(T)$.

(ii) The *depth* of the eigenvalue is the smallest positive integer $k_\lambda=k_\lambda(T)$ such that

$$\dim\ker(\lambda\mathbb{1}-T)^k=\dim\ker(\lambda\mathbb{1}-T)^{k+1}.$$

□

Proposition 2.17. *Let $T\in L(\boldsymbol{U})$, $\dim_{\mathbb{F}}\boldsymbol{U}=n$, and $\lambda\in\operatorname{spec}(T)$. Then the generalized eigenspace $E_\lambda(T)$ is an invariant subspace of T.*

Proof. We need to show that $TE_\lambda(T)\subset E_\lambda(T)$. Let $\boldsymbol{u}\in E_\lambda(T)$, i.e.,

$$(\lambda\mathbb{1}-T)^n\boldsymbol{u}=0.$$

Clearly $\lambda\boldsymbol{u}-T\boldsymbol{u}\in\ker(\lambda\mathbb{1}-T)^n=E_\lambda(T)$. Since $\lambda\boldsymbol{u}\in E_\lambda(T)$ we deduce that

$$T\boldsymbol{u}=\lambda\boldsymbol{u}-(\lambda\boldsymbol{u}-T\boldsymbol{u})\in E_\lambda(T).$$

□

Theorem 2.18. *Suppose that $T : \boldsymbol{U} \to \boldsymbol{U}$ is a triangulable operator on the the n-dimensional $\mathbb{F}$-vector space $\boldsymbol{U}$. Then, for any $\lambda \in \operatorname{spec}(T)$, $\dim E_\lambda(T) = m_\lambda(T)$. We recall that $m_\lambda(T)$ is equal to the number of times λ appears along the diagonal of a triangular representation of T.*

Proof. We will argue by induction on n. For $n = 1$ the result is trivially true. For the inductive step we assume that the result is true for any triangulable operator on an $(n-1)$-dimensional $\mathbb{F}$-vector space V, and we will prove that the same is true for triangulable operators acting on an n-dimensional space $\boldsymbol{U}$.

Let $T \in L(\boldsymbol{U})$ be such an operator. We can then find a basis $\underline{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ of $\boldsymbol{U}$ such that, in this basis, the operator T is represented by the upper triangular matrix

$$A = \begin{bmatrix} \lambda_1 & * & * & * & * \\ 0 & \lambda_2 & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1} & * \\ 0 & \cdots & 0 & 0 & \lambda_n \end{bmatrix}$$

Suppose that $\lambda \in \operatorname{spec}(T)$. For simplicity we assume $\lambda = 0$. Otherwise, we carry the discussion using instead the operator $T' = T - \lambda \mathbb{1}$. Let ν be the number of times 0 appears on the diagonal of A, i.e., $\nu = m_0(T)$. We have to show that

$$\nu = \dim \ker T^n.$$

Denote by $\boldsymbol{V}$ the subspace spanned by the vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1}$. Observe that $\boldsymbol{V}$ is an invariant subspace of T, i.e., $T\boldsymbol{V} \subset \boldsymbol{V}$. If we denote by S the restriction of T to $\boldsymbol{V}$ we can regard S as a linear operator $S : \boldsymbol{V} \to \boldsymbol{V}$.

The operator S is triangulable because in the basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1})$ of $\boldsymbol{V}$ it is represented by the upper triangular matrix

$$B = \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1} \end{bmatrix}.$$

Denote by μ the number of times 0 appears on the diagonal of B. The induction hypothesis implies that

$$\mu = \dim \ker S^{n-1} = \dim \ker S^n.$$

Clearly $\mu \leq \nu$. Note that

$$\ker S^n \subset \ker T^n$$

so that

$$\mu = \dim \ker S^n \leq \dim \ker T^n.$$

We distinguish two cases.

1. $\lambda_n \neq 0$. In this case we have $\mu = \nu$ so it suffices to show that

$$\ker T^n \subset \boldsymbol{V}. \tag{2.7}$$

Indeed, if (2.7) were true, we would conclude that $\ker T^n \subset \ker S^n$, and thus

$$\dim \ker T^n = \dim \ker S^n = \mu = \nu.$$

To prove (2.7) we argue by contradiction. Suppose that there exists $\boldsymbol{u} \in \ker T^n$ such that $\boldsymbol{u} \notin \boldsymbol{V}$. Thus, we can find $\boldsymbol{v} \in \boldsymbol{V}$ and $c \in \mathbb{F} \setminus 0$ such that

$$\boldsymbol{u} = \boldsymbol{v} + c\boldsymbol{e}_n.$$

Note that $T^n \boldsymbol{v} \in \boldsymbol{V}$ and

$$\boldsymbol{e}_n = \lambda_n \boldsymbol{e}_n + \text{vector in } \boldsymbol{V}.$$

Thus

$$T^n c\boldsymbol{e}_n = c\lambda_n^n \boldsymbol{e}_n + \text{vector in } \boldsymbol{V}$$

so that

$$T^n \boldsymbol{u} = c\lambda_n^n \boldsymbol{e}_n + \text{vector in } \boldsymbol{V} \neq 0.$$

This contradiction completes the discussion of Case **1.**

2. $\lambda_n = 0$. In this case we have $\nu = \mu + 1$ so we have to show that

$$\dim \ker T^n = \mu + 1.$$

We need an auxiliary result.

Lemma 2.19. *There exists* $\boldsymbol{u} \in \boldsymbol{U} \setminus \boldsymbol{V}$ *such that* $T^n \boldsymbol{u} = 0$ *so that*

$$\dim(\boldsymbol{V} + \ker T^n) \geq \dim \boldsymbol{V} + 1 = n. \tag{2.8}$$

Proof. Set

$$\boldsymbol{v}_n := T\boldsymbol{e}_n.$$

Observe that $\boldsymbol{v}_n \in \boldsymbol{V}$. From (2.6b) we deduce that $\boldsymbol{R}\, S^{n-1} = \boldsymbol{R}\, S^n$ so that there exists $\boldsymbol{v}_0 \in \boldsymbol{V}$ such that

$$S^{n-1} \boldsymbol{v}_n = S^n \boldsymbol{v}_0.$$

Set $\boldsymbol{u} := \boldsymbol{e}_n - \boldsymbol{v}_0$. Note that $\boldsymbol{u} \in \boldsymbol{U} \setminus \boldsymbol{V}$,

$$T\boldsymbol{u} = \boldsymbol{v}_n - T\boldsymbol{v}_0 = \boldsymbol{v}_n - S\boldsymbol{v}_0,$$

$$T^n \boldsymbol{u} = T^{n-1}(\boldsymbol{v}_n - S\boldsymbol{v}_0) = S^{n-1}\boldsymbol{v}_n - S^n \boldsymbol{v}_0 = 0.$$

$\square$

Now observe that

$$n = \dim \boldsymbol{U} \geq \dim(\boldsymbol{V} + \ker T^n) \overset{(2.8)}{\geq} n,$$

so that

$$\dim(\boldsymbol{V} + \ker T^n) = n.$$

We conclude that

$$n = \dim(\boldsymbol{V} + \ker T^n) = \dim(\ker T^n) + \underbrace{\dim \boldsymbol{V}}_{n-1} - \underbrace{\dim(\boldsymbol{U} \cap \ker T^n)}_{=\mu}$$

$$= \dim(\ker T^n) + n - 1 - \mu,$$

which shows that

$$\dim \ker T^n = \mu + 1 = \nu.$$

$\square$

☞ *In the remainder of this section we will assume that $\mathbb{F}$ is the field of complex numbers, $\mathbb{C}$.*

For any polynomial with complex coefficients

$$p(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{C}[x]$$

and any linear operator T on a complex vector space $\boldsymbol{U}$ we set

$$p(T) = a_0\mathbb{1} + a_1T + \cdots + a_nT^n.$$

Note that if $p(x), q(x) \in \mathbb{C}[x]$, and if we set $r(x) = p(x)q(x)$, then

$$r(T) = p(T)q(T).$$

Theorem 2.20 (Cayley-Hamilton)**.** *Suppose T is a linear operator on the complex vector space $\boldsymbol{U}$. If $P_T(x)$ is the characteristic polynomial of T, then*

$$P_T(T) = 0.$$

Proof. Fix a basis $\underline{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ in which T is represented by the upper triangular matrix

$$A = \begin{bmatrix} \lambda_1 & * & * & * & * \\ 0 & \lambda_2 & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1} & * \\ 0 & \cdots & 0 & 0 & \lambda_n \end{bmatrix}$$

Note that

$$P_T(x) = \det(x\mathbb{1} - T) = \prod_{j=1}^n (x - \lambda_j)$$

so that

$$P_T(T) = \prod_{j=1}^n (T - \lambda_j\mathbb{1}).$$

For $j = 1, \dots, N$ we define

$$\boldsymbol{U}_j := \operatorname{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_j\}.$$

and we set $\boldsymbol{U}_0 = \{0\}$. Since A is upper triangular, we deduce that for any $j = 1, \dots, n$ we have

$$(T - \lambda_j\mathbb{1})\boldsymbol{U}_j \subset \boldsymbol{U}_{j-1}.$$

Thus

$$P_T(T)\boldsymbol{U} = \prod_{j=1}^n (T - \lambda_j)\boldsymbol{U}_n = \prod_{j=1}^{n-1} (T - \lambda_j)\Big((T - \lambda_n\mathbb{1})\boldsymbol{U}_n \Big)$$

$$\subset \prod_{j=1}^{n-1} (T - \lambda_j)\boldsymbol{U}_{n-1} \subset \prod_{j=1}^{n-2} (T - \lambda_j)\boldsymbol{U}_{n-2} \subset \cdots \subset (T - \lambda_1)\boldsymbol{U}_1 \subset \{0\}.$$

In other words,

$$P_T(T)\boldsymbol{u} = 0, \;\; \forall \boldsymbol{u} \in \boldsymbol{U}.$$

□

Example 2.21. Consider the 2×2-matrix

$$A = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

Its characteristic polynomial is

$$P_A(x) = \det(xI - A) = \det \begin{bmatrix} x-3 & -2 \\ 2 & x+1 \end{bmatrix}$$

$$= (x-3)(x+1) + 4 = x^2 - 2x - 3 + 4 = x^2 - 2x + 1.$$

The Cayley-Hamilton theorem shows that

$$A^2 - 2A + 1 = 0.$$

Let us verify this directly. We have

$$A^2 = \begin{bmatrix} 5 & 8 \\ -4 & -3 \end{bmatrix}$$

and

$$A^2 - 2A + I = \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix} - 2 \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.$$

We can rewrite the last equality as

$$A^2 = 2A - I$$

so that

$$A^{n+2} = 2A^{n+1} - A^n,$$

We can rewrite this as

$$A^{n+2} - A^{n+1} = A^{n+1} - A^n = A^n - A^{n-1} = \cdots = A - I.$$

Hence

$$A^n = \underbrace{(A^n - A^{n-1}) + (A^{n-1} - A^{n-2}) + \cdots + (A - I)}_{=n(A-I)} + I = nA - (n-1)I. \qquad \square$$

2.5. **The Jordan normal form of a complex operator.** Let $\boldsymbol{U}$ be a complex n-dimensional vector space and $T : \boldsymbol{U} \to \boldsymbol{U}$. For each eigenvalue $\lambda \in \operatorname{spec}(T)$ we denote by $E_\lambda(T)$ the corresponding generalized eigenspace, i.e.,

$$\boldsymbol{u} \in E_\lambda(T) \Longleftrightarrow \exists k > 0 : \quad (T - \lambda \mathbb{1})^k \boldsymbol{u} = 0.$$

From Proposition 2.17 we know that $E_\lambda(T)$ is an invariant subspace of T of dimension $m_\lambda(T)$. Suppose that the spectrum of T consists of ℓ distinct eigenvalues,

$$\operatorname{spec}(T) = \{ \lambda_1, \dots, \lambda_\ell \}.$$

Proposition 2.22.

$$\boldsymbol{U} = E_{\lambda_1}(T) \oplus \cdots \oplus E_{\lambda_\ell}(T).$$

Proof. It suffices to show that

$$\boldsymbol{U} = E_{\lambda_1}(T) + \cdots + E_{\lambda_\ell}(T), \tag{2.9a}$$

$$\dim \boldsymbol{U} = \dim E_{\lambda_1}(T) + \cdots + \dim E_{\lambda_\ell}(T). \tag{2.9b}$$

The equality (2.9b) follows from (2.4c) since

$$\dim \boldsymbol{U} = m_{\lambda_1}(T) + \cdots + m_{\lambda_\ell}(T) = \dim E_{\lambda_1}(T) + \cdots + \dim E_{\lambda_\ell}(T),$$

so we only need to prove (2.9a). Set

$$\boldsymbol{V} := E_{\lambda_1}(T) + \cdots + E_{\lambda_\ell}(T) \subset \boldsymbol{U}.$$

We have to show that $\boldsymbol{V} = \boldsymbol{U}$.

Note that since each of the generalized eigenspaces $E_\lambda(T)$ are invariant subspaces of T, so is their sum $\boldsymbol{V}$. Denote by S the restriction of T to $\boldsymbol{V}$, which we regard as an operator $S : \boldsymbol{V} \to \boldsymbol{V}$.

If $\lambda \in \operatorname{spec}(T)$ and $\boldsymbol{v} \in E_\lambda(T) \subset \boldsymbol{V}$, then

$$(S - \lambda\mathbb{1})^k \boldsymbol{v} = (T - \lambda\mathbb{1})^k \boldsymbol{v} = 0$$

for some $k \geq 0$. Thus λ is also an eigenvalue of S and $\boldsymbol{v}$ is also a generalized eigenvector of S. This proves that

$$\operatorname{spec}(T) \subset \operatorname{spec}(S),$$

and

$$E_\lambda(T) \subset E_\lambda(S), \;\; \forall \lambda \in \operatorname{spec}(T).$$

In particular, this implies that

$$\dim \boldsymbol{U} = \sum_{\lambda \in \operatorname{spec}(T)} \dim E_\lambda(T) \leq \sum_{\mu \in \operatorname{spec}(S)} \dim E_\mu(S) = \dim \boldsymbol{V} \leq \dim \boldsymbol{U}.$$

This shows that $\dim \boldsymbol{V} = \dim \boldsymbol{U}$ and thus $\boldsymbol{V} = \boldsymbol{U}$. □

For any $\lambda \in \operatorname{spec}(T)$ we denote by S_λ the restriction of T on the generalized eigenspace $E_\lambda(T)$. Since this is an invariant subspace of T we can regard S_λ as a linear operator

$$S_\lambda : E_\lambda(T) \to E_\lambda(T).$$

Arguing as in the proof of the above proposition we deduce that $E_\lambda(T)$ is also a generalized eigenspace of S_λ. Thus, the spectrum of S_λ consists of a single eigenvalue and

$$E_\lambda(T) = E_\lambda(S) = \ker(\lambda\mathbb{1} - S_\lambda)^{\dim E_\lambda(T)} = \ker(\lambda\mathbb{1} - S_\lambda)^{m_\lambda(T)}.$$

Thus, for any $\boldsymbol{u} \in E_\lambda(T)$ we have

$$(\lambda\mathbb{1} - S_\lambda)^{m_\lambda(T)} \boldsymbol{u} = 0,$$

i.e.,

$$(S_\lambda - \lambda\mathbb{1})^{m_\lambda(T)} = (-1)^{m_\lambda(T)} (\lambda\mathbb{1} - S_\lambda)^{m_\lambda(T)} = 0.$$

Definition 2.23. A linear operator $\boldsymbol{N} : \boldsymbol{U} \to \boldsymbol{U}$ is called *nilpotent* if $\boldsymbol{N}^k = 0$ for some $k > 0$. □

If we set $N_\lambda = S_\lambda - \lambda\mathbb{1}$ we deduce that the operator N_λ is nilpotent.

Definition 2.24. Let $\boldsymbol{N} : \boldsymbol{U} \to \boldsymbol{U}$ be a nilpotent operator on a finite dimensional complex vector space $\boldsymbol{U}$. A *tower* of $\boldsymbol{N}$ is an *ordered* collection $\mathcal{T}$ of nonzero vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \in \boldsymbol{U}$$

satisfying the equalities

$$\boldsymbol{N}\boldsymbol{u}_1 = 0, \;\; \boldsymbol{N}\boldsymbol{u}_2 = \boldsymbol{u}_1, \ldots, \boldsymbol{N}\boldsymbol{u}_k = \boldsymbol{u}_{k-1}.$$

The vector $\boldsymbol{u}_1$ is called the *bottom* of the tower, the vector $\boldsymbol{u}_k$ is called the *top* of the tower, while the integer k is called the *height* of the tower. □

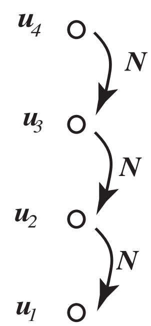

FIGURE 1. *Pancaking a tower of height* 4.

In Figure 1 we depicted a tower of height 4. Observe that the vectors in a tower are generalized eigenvectors of the corresponding nilpotent operator.

Towers interact in a rather pleasant way.

Proposition 2.25. *Suppose that $\boldsymbol{N} : \boldsymbol{U} \to \boldsymbol{U}$ is a nilpotent operator on a complex vector space $\boldsymbol{U}$ and $\mathcal{T}_1, \dots, \mathcal{T}_r$ are towers of $\boldsymbol{N}$ with bottoms $\boldsymbol{b}_1, \dots, \boldsymbol{b}_r$.*

If the bottom vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_r$ are linearly independent, then the following hold.

(i) *The towers $\mathcal{T}_1, \dots, \mathcal{T}_r$ are mutually disjoint, i.e., $\mathcal{T}_i \cap \mathcal{T}_j = \emptyset$ if $i \neq j$.*

(ii) *The union*

$$\mathcal{T} = \mathcal{T}_1 \cup \dots \cup \mathcal{T}_r$$

is a linearly independent family of vectors.

Proof. Denote by k_i the height of the tower $\mathcal{T}_i$ and set

$$k = \max(k_1, \dots, k_r), \quad K := k_1 + \dots + k_r = \#\mathcal{T}.$$

We will argue by induction on k, the sum of the heights of the towers.

For $k = 1$ the result is trivially true. Assume the result is true for all collections of towers with maximum heights $< k$ and linear independent bases, and we will prove that it is true for collection of towers with maximal heights $= k$.

Let $\mathcal{T} = \mathcal{T}_1, \dots, \mathcal{T}_r$ be a collection of towers with maximum height k. Denote by $\mathcal{T}_i'$ the tower obtained by removing the top of the tower $\mathcal{T}_i$ if the height of the tower $\mathcal{T}_i$ is k. Otherwise we set $\mathcal{T}_i' = \mathcal{T}_i$. Define

$$\mathcal{T}' := \mathcal{T}_1' \cup \dots \cup \mathcal{T}_r'.$$

The collection of towers $\mathcal{T}_1', \dots \mathcal{T}_r'$ has maximal height $k - 1$. The induction assumption implies that the collection $\mathcal{T}'$ is linearly independent. Denote by $\boldsymbol{t}_1, \dots, \boldsymbol{t}_\ell$ the tops of $\mathcal{T}$ of height k so that

$$\mathcal{T} = \{\boldsymbol{t}_1, \dots, \boldsymbol{t}_\ell\} \cup \mathcal{T}'.$$

We argue by contradiction. Suppose that $\mathcal{T}$ is linearly dependent. Hence there exist complex numbers

$$c_1, \dots, c_\ell, \ x_{\boldsymbol{u}}, \boldsymbol{u} \in \mathcal{T}',$$

not all equal to zero, such that

$$\sum_{j=1}^{\ell} c_j \boldsymbol{t}_j + \sum_{\boldsymbol{u} \in \mathcal{T}'} x_{\boldsymbol{u}} \boldsymbol{u} = 0.$$

Clearly some of the numbers c_j are nonzero since $\mathcal{T}'$ is linearly independent. Hence

$$\sum_{j=1}^{\ell} c_j \boldsymbol{N}\boldsymbol{t}_j + \sum_{\boldsymbol{u}\in\mathcal{T}'} x_{\boldsymbol{u}} \boldsymbol{N}\boldsymbol{u} = 0.$$

Note that the vectors $\boldsymbol{N}\boldsymbol{t}_j$ are tops of towers in $\mathcal{T}'$, but none of the vectors $\boldsymbol{N}\boldsymbol{u}$, $\boldsymbol{u}\in\mathcal{T}'$ is a top of a tower in $\mathcal{T}'$. Hence

$$\sum_{j=1}^{\ell} c_j \boldsymbol{N}\boldsymbol{t}_j + \sum_{\boldsymbol{u}\in\mathcal{T}'} x_{\boldsymbol{u}} \boldsymbol{N}\boldsymbol{u}.$$

is a *nontrivial* linear combination of vectors in $\mathcal{T}'$ equal to zero. This is impossible. $\square$

Theorem 2.26 (Jordan normal form of a nilpotent operator)**.** *Let $\boldsymbol{N}:\boldsymbol{U}\to\boldsymbol{U}$ be a nilpotent operator on an n-dimensional complex vector space $\boldsymbol{U}$. Then $\boldsymbol{U}$ has a basis consisting of a disjoint union of towers of $\boldsymbol{N}$.*

Proof. We will argue by induction on the dimension n of $\boldsymbol{U}$. For $n=1$ the result is trivially true. We assume that the result is true for any nilpotent operator on a space of dimension $< n$ and we prove it is true for any nilpotent operator $\boldsymbol{N}$ on a space $\boldsymbol{U}$ of dimension n.

Observe that $\boldsymbol{V}=\boldsymbol{R}(\boldsymbol{N})$ is an invariant subspace of $\boldsymbol{N}$. Moreover, since $\ker\boldsymbol{N}\neq 0$, we deduce from the rank-nullity theorem that

$$\dim\boldsymbol{V} = \dim\boldsymbol{U} - \dim\ker\boldsymbol{N} < \dim\boldsymbol{U}.$$

Denote by M the restriction of $\boldsymbol{N}$ to $\boldsymbol{V}$. We view M as a linear operator $M:\boldsymbol{V}\to\boldsymbol{V}$. Clearly M is nilpotent. The induction assumption implies that there exist a basis of $\boldsymbol{V}$ consisting of mutually disjoint towers of M,

$$\mathcal{T}_1,\dots,\mathcal{T}_r.$$

For any $j=1,\dots,r$ we denote by k_j the height of $\mathcal{T}_j$, by $\boldsymbol{b}_j$ the bottom of $\mathcal{T}_j$ and by $\boldsymbol{t}_j$ the top of $\mathcal{T}_j$. By construction

$$\dim\boldsymbol{R}(\boldsymbol{N}) = k_1+\cdots+k_r.$$

Since $\boldsymbol{t}_j\in\boldsymbol{V}=\boldsymbol{R}(\boldsymbol{N})$ there exists $\boldsymbol{u}_j\in\boldsymbol{U}$ such that (see Figure 2)

$$\boldsymbol{t}_j = \boldsymbol{N}\boldsymbol{u}_j.$$

Next observe that the bottoms $\boldsymbol{b}_1,\dots,\boldsymbol{b}_r$ belong to $\ker\boldsymbol{N}$ and are linearly independent, because they are a subfamily of the linearly independent family $\mathcal{T}_1\cup\cdots\cup\mathcal{T}_r$. We can therefore extend the family $\{\boldsymbol{b}_1,\dots,\boldsymbol{b}_r\}$ to a basis of $\ker\boldsymbol{N}$,

$$\boldsymbol{b}_1,\dots,\boldsymbol{b}_r,\boldsymbol{v}_1,\dots,\boldsymbol{v}_s,\;\; r+s=\dim\ker\boldsymbol{N}.$$

We obtain new towers $\hat{\mathcal{T}}_1,\dots,\hat{\mathcal{T}}_r,\hat{\mathcal{T}}_{r+1},\dots,\hat{\mathcal{T}}_{r+s}$ defined by (see Figure 2)

$$\hat{\mathcal{T}}_1 := \mathcal{T}_1\cup\{\boldsymbol{u}_1\},\dots,\hat{\mathcal{T}}_r := \mathcal{T}_r\cup\{\boldsymbol{u}_r\},\;\;\hat{\mathcal{T}}_{r+1} := \{\boldsymbol{v}_1\},\dots,\hat{\mathcal{T}}_{r+s} := \{\boldsymbol{v}_s\}.$$

The sum of the heights of these towers is

$$\begin{aligned}&(k_1+1)+(k_2+1)+\cdots+(k_r+1)+\underbrace{1+\cdots+1}_{s}\\ &=\underbrace{(k_1+\cdots+k_r)}_{=\dim\boldsymbol{R}(\boldsymbol{N})}+\underbrace{(r+s)}_{=\dim\ker\boldsymbol{N}}=\dim\boldsymbol{U}.\end{aligned}$$

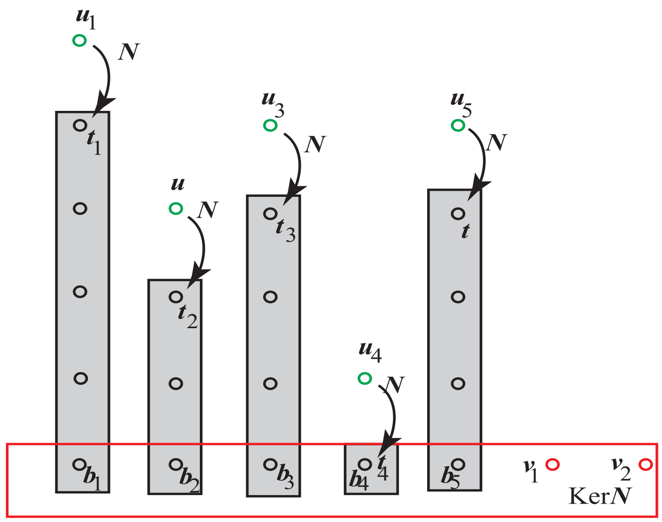

FIGURE 2. *Towers in* $\boldsymbol{R}(\boldsymbol{N})$.

By construction, their bottoms are linearly independent and Proposition 2.25 implies that they are mutually disjoint and their union is a linearly independent collection of vectors. The above computation shows that the number of elements in the union of these towers is equal to the dimension of $\boldsymbol{U}$. Thus, this union is a basis of $\boldsymbol{U}$. □

Definition 2.27. A *Jordan basis* of a nilpotent operator $\boldsymbol{N} : \boldsymbol{U} \to \boldsymbol{U}$ is a basis of $\boldsymbol{U}$ consisting of a disjoint union of towers of $\boldsymbol{N}$ arranged in decreasing order of their heights.. □

Example 2.28. (a) Suppose that the nilpotent operator $\boldsymbol{N} : \boldsymbol{U} \to \boldsymbol{U}$ admits a Jordan basis consisting of a single tower

$$\boldsymbol{e}_1, \dots, \boldsymbol{e}_n.$$

Denote by C_n the matrix representing $\boldsymbol{N}$ in this basis. We use this basis to identify $\boldsymbol{U}$ with $\mathbb{C}^n$ and thus

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots, \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

From the equalities

$$\boldsymbol{N}\boldsymbol{e}_1 = 0, \quad \boldsymbol{N}\boldsymbol{e}_2 = \boldsymbol{e}_1, \quad \boldsymbol{N}\boldsymbol{e}_3 = \boldsymbol{e}_2, \dots$$

we deduce that the first column of C_n is trivial, the second column is $\boldsymbol{e}_1$, the 3-rd column is $\boldsymbol{e}_2$ etc. Thus C_n is the $n \times n$ matrix.

$$C_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The matrix C_n is called a (nilpotent) *Jordan cell of size* n.

(b) Suppose that the nilpotent operator $\boldsymbol{N}:\boldsymbol{U}\to\boldsymbol{U}$ admits a Jordan basis consisting of mutually disjoint towers $\mathcal{T}_1,\dots,\mathcal{T}_r$ of heights $k_1,\dots,k_r$. For $j=1,\dots,r$ we set

$$\boldsymbol{U}_j=\operatorname{span}(\mathcal{T}_j).$$

Observe that $\boldsymbol{U}_j$ is an invariant subspace of $\boldsymbol{N}$, $\mathcal{T}_j$ is a basis of $\boldsymbol{U}_j$ and we have a direct sum decomposition

$$\boldsymbol{U}=\boldsymbol{U}_1\oplus\cdots\oplus\boldsymbol{U}_r.$$

The restriction of $\boldsymbol{N}$ to $\boldsymbol{U}_j$ is represented in the basis $\mathcal{T}_j$ by the Jordan cell C_{k_j} so that in the basis $\mathcal{T}=\mathcal{T}_1\cup\cdots\cup\mathcal{T}_r$ the operator $\boldsymbol{N}$ has the block-matrix description

$$\begin{bmatrix} C_{k_1} & 0 & 0 & \cdots & 0\\ 0 & C_{k_2} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & C_{k_r}\end{bmatrix}. \qquad \square$$

We want to point out that the sizes of the Jordan cells correspond to the heights of the towers in a Jordan basis. While there may be several Jordan bases, the heights of the towers are the same in all of them; see Remark 2.30. In other words, these heights are *invariants* of the nilpotent operator. □

Definition 2.29. The *Jordan invariant* of a nilpotent operator N is the nonincreasing list of the sizes of the Jordan cells that describe the operator in a Jordan basis. □

Remark 2.30 (Algorithmic construction of a Jordan basis)**.** Here is how one constructs a Jordan basis of a nilpotent operator $\boldsymbol{N}:\boldsymbol{U}\to\boldsymbol{U}$ on a complex vector space $\boldsymbol{U}$ of dimension n.

(i) Compute $\boldsymbol{N}^2,\boldsymbol{N}^3,\dots$ and stop at the moment m when $\boldsymbol{N}^m=0$. Set

$$R_0=\boldsymbol{U},\;\; R_1=\boldsymbol{R}(\boldsymbol{N}),\;\; R_2=\boldsymbol{R}(\boldsymbol{N}^2),\dots,R_m=\boldsymbol{R}(\boldsymbol{N}^m)=\{0\}.$$

Observe that $R_1,R_2,\dots,R_m$ are invariant subspaces of $\boldsymbol{N}$, satisfying

$$R_0\supset R_1\supset R_2\supset\cdots,$$

(ii) Denote by N_k the restriction of $\boldsymbol{N}$ to R_k, viewed as an operator $N_k:R_k\to R_k$. Note that $N_{m-1}=0$, $N_0=\boldsymbol{N}$ and

$$R_j=\boldsymbol{R}(N_{j-1}),\;\;\forall j=1,\dots,m.$$

Set $r_j=\dim R_j$, $K_j:=\dim\ker N_j$, $k_j=\dim K_j$ so that $k_j=r_j-r_{j+1}$. Note that

$$R_{m-1}=K_{m-1}\subset K_{m-2}\subset K_{m-2}\subset\cdots.$$

(iii) Construct a basis $\mathcal{B}_{m-1}$ of $R_{m-1}=K_{m-1}$. $\mathcal{B}_{m-1}$ consists of r_{m-1} vectors.

(iv) For each $\boldsymbol{b}\in\mathcal{B}_{m-1}$ find a vector $\boldsymbol{t}(\boldsymbol{b})\in\boldsymbol{U}$ such that

$$\boldsymbol{b}=\boldsymbol{N}^{m-1}\boldsymbol{t}(\boldsymbol{b}).$$

For each $\boldsymbol{b}\in\mathcal{B}_{m-1}$ we thus obtain a tower of height m

$$\mathcal{T}_{m-1}(\boldsymbol{b})=\big\{\,\boldsymbol{b}=\boldsymbol{N}^{m-1}\boldsymbol{t}(\boldsymbol{b}),\boldsymbol{N}^{m-2}\boldsymbol{t}(\boldsymbol{b}),\dots,\boldsymbol{N}\boldsymbol{t}(\boldsymbol{b}),\boldsymbol{t}(\boldsymbol{b})\,\big\},\;\;\boldsymbol{b}\in\mathcal{B}_{m-1}.$$

(v) Extend $\mathcal{B}_{m-1}\subset R_{m-1}\subset R_{m-2}$ to a basis

$$\mathcal{B}_{m-2}=\mathcal{B}_{m-1}\cup\mathscr{C}_{m-2}$$

of K_{m-2}.

(vi) For each $\boldsymbol{b} \in \mathscr{C}_{m-2} \subset R_{m-2}$ find $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{b}) \in \boldsymbol{N}$ such that $\boldsymbol{N}^{m-2}\boldsymbol{t} = \boldsymbol{b}$. For each $\boldsymbol{b} \in \mathscr{C}_{m-2}$ we thus obtain a tower of height $m-1$

$$\mathcal{T}_{m-2}(\boldsymbol{b}) = \left\{ \boldsymbol{b} = \boldsymbol{N}^{m-2}\boldsymbol{t}(\boldsymbol{b}), \dots, \boldsymbol{N}\boldsymbol{t}(\boldsymbol{b}), \boldsymbol{t}(\boldsymbol{b}) \right\}, \quad \boldsymbol{b} \in \mathcal{B}_{m-2}$$

(vii) Extent $\mathcal{B}_{m-2}$ to a basis

$$\mathcal{B}_{m-3} = \mathcal{B}_{m-2} \cup \mathscr{C}_{m-3}$$

of K_{m-3}.

(viii) For each $\boldsymbol{b} \in \mathscr{C}_{m-3} \subset R_{m-3}$, find $\boldsymbol{t}(\boldsymbol{b}) \in \boldsymbol{N}$ such that $\boldsymbol{N}^{m-3}\boldsymbol{t}(\boldsymbol{b}) = \boldsymbol{b}$. For each $\boldsymbol{b} \in \mathscr{C}_{m-3}$ we thus obtain a tower of height $m-2$

$$\mathcal{T}_{m-3}(\boldsymbol{b}) = \left\{ \boldsymbol{b} = \boldsymbol{N}^{m-3}\boldsymbol{t}(\boldsymbol{b}), \dots, \boldsymbol{N}\boldsymbol{t}(\boldsymbol{b}), \boldsymbol{t}(\boldsymbol{b}) \right\}, \quad \boldsymbol{b} \in \mathscr{C}_{m-3}$$

(ix) Iterate the previous two steps

(x) In the end we obtain a basis

$$\mathcal{B}_0 = \mathcal{B}_{m-1} \cup \mathscr{C}_{m-2} \cup \cdots \cup \mathscr{C}_0$$

of $\ker N_0 = \ker \boldsymbol{N}$, vectors $\boldsymbol{t}(\boldsymbol{b})$, $\boldsymbol{b} \in \mathscr{C}_j$, and towers

$$\mathcal{T}_j(\boldsymbol{b}) = \left\{ \boldsymbol{b} = \boldsymbol{N}^j\boldsymbol{t}(\boldsymbol{b}), \dots, \boldsymbol{N}\boldsymbol{t}(\boldsymbol{b}), \boldsymbol{t}(\boldsymbol{b}) \right\}, \quad j = 0, \dots, m-1, \quad \boldsymbol{b} \in \mathscr{C}_j.$$

These towers form a Jordan basis of $\boldsymbol{N}$.

(xi) For uniformity set $\mathscr{C}_{m-1} = \mathcal{B}_{m-1}$, and for any $j = 1, \dots, m$ denote by c_j the cardinality of $\mathscr{C}_{j-1}$. In the above Jordan basis the operator N will be a dirrect sum of c_1 cells of dimension 1, c_2 cells of dimension 2, etc. In terms of towers, there are c_1 towers of height 1, c_2 towers of height 2 etc. We obtain the identities

$$\begin{gathered} r_{m-1} = c_m, \quad r_{m-2} = c_{m-1} + 2c_m, \quad r_{m-3} = c_{m-3} + 2c_{m-2} + 3c_m, \\ r_j = c_{j+1} + 2c_{j+2} + \cdots + (m-j)c_m, \quad \forall j = 0, \dots, m-1. \end{gathered} \tag{2.10}$$

where $r_0 = n$.

Indeed, by construction, $\dim K_j - \dim K_{j+1} = c_{j+1}$ so

$$\dim K_j = c_{j+1} + \cdots + c_m.$$

On the other hand

$$r_j = \dim K_j + r_{j+1}.$$

If we treat the equalities (2.10) as a linear system with unknown $c_1, \dots, c_m$, we see that the matrix of this system is upper triangular with only 1-s along the diagonal. It is thus invertible so that the numbers $c_1, \dots, c_m$ are uniquely determined by the numbers r_j which are *invariants* of the operator N. This shows that the sizes of the Jordan cells are independent of the chosen Jordan basis.

If you are interested only in the sizes of the Jordan cells, all you have to do is find the integers m, $r_1, \dots, r_{m-1}$ and then solve the system (2.10). Exercise 2.17 explains how to explicitly express the c_j-s in terms of the r_j-s. □

Remark 2.31. To find the Jordan invariant of a nilpotent operator N on a complex n-dimensional space proceed as follows.

(i) Find the smallest integer m such that $N^m = 0$.

(ii) Find the ranks r_j of the matrices N^j, $j = 0, \dots, m-1$, where $N^0 := \mathbb{1}$.

(iii) Find the nonnegative integers $c_1, \dots, c_m$ by solving the linear system (2.10).

(iv) The Jordan invariant of N is the list

$$\underbrace{m, \dots, m}_{c_m}, \ \underbrace{(m-1), \dots, (m-1)}_{c_{m-1}}, \dots, \ \underbrace{1, \dots, 1}_{c_1}.$$

□

If $T : \boldsymbol{U} \to \boldsymbol{U}$ is an arbitrary linear operator on a complex n-dimensional space $\boldsymbol{U}$ with spectrum

$$\operatorname{spec}(T) = \{\lambda_1, \dots, \lambda_m\},$$

then we have a direct sum decomposition

$$\boldsymbol{U} = E_{\lambda_1}(T) \oplus \cdots \oplus E_{\lambda_m}(T),$$

where $E_{\lambda_j}(T)$ denotes the generalized eigenspace of T corresponding to the eigenvalue λ_j. The generalized eigenspace $E_{\lambda_j}(T)$ is an invariant subspace of T and we denote by S_{λ_j} the restriction of T to $E_{\lambda_j}(T)$. The operator $N_{\lambda_j} = S_{\lambda_j} - \lambda_j \mathbb{1}$ is nilpotent.

A *Jordan basis* of $\boldsymbol{U}$ is basis obtained as a union of the Jordan bases of the nilpotent operators $N_{\lambda_1}, \dots, N_{\lambda_r}$. The matrix representing T in a Jordan basis is a direct sum of elementary *Jordan λ-cells*

$$C_n(\lambda) = C_n + \lambda I = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

Definition 2.32. The *Jordan invariant* of a complex operator T is a collection of lists, one list for every eigenvalue of of T. The list L_λ corresponding to the eigenvalue λ is the Jordan invariant of the nilpotent operator N_λ, the restriction of $T - \lambda\mathbb{1}$ to $E_\lambda(T)$ arranged in noincreasing order. □

Example 2.33. Consider the 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -4 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

viewed as a linear operator $\mathbb{C}^4 \to \mathbb{C}^4$.

Expanding along the first row and then along the last column we deduce that

$$P_A(x) = \det(xI - A) = (x-1)^2 \det \begin{bmatrix} x+1 & -1 \\ 4 & x-3 \end{bmatrix} = (x-1)^4.$$

Thus A has a single eigenvalue $\lambda = 1$ which has multiplicity 4. Set

$$N := A - I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix N is nilpotent. In fact we have

$$N^2 = 0$$

Upon inspecting N we see that each of its columns is a multiple of the first column. This means that the range of N is spanned by the vector

$$\boldsymbol{u}_1 := N\boldsymbol{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix},$$

where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_4$ denotes the canonical basis of $\mathbb{C}^4$.

The vector $\boldsymbol{u}_1$ is a tower in $\boldsymbol{R}(N)$ which we can extend to a taller tower of N

$$\mathcal{T}_1 = (\boldsymbol{u}_1, \boldsymbol{u}_2), \;\; \boldsymbol{u}_2 = \boldsymbol{e}_1.$$

Next, we need to extend the basis $\{\boldsymbol{u}_1\}$ of $R(N)$ to a basis of $\ker N$. The rank nulity theorem tells us that

$$\dim \boldsymbol{R}(N) + \dim \ker N = 4,$$

so that $\dim \ker N = 3$. Thus, we need to find two more vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ so that the collection $\{\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis of $\ker N$.

To find $\ker N$ we need to solve the linear system

$$N\boldsymbol{x} = 0, \;\; \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

which we do using Gauss elimination, i.e., row operations on N. Observe that

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$\ker N = \big\{ \boldsymbol{x} \in \mathbb{C}^4; \;\; x_1 - 2x_2 + x_3 = 0 \big\},$$

and thus a basis of $\ker N$ is

$$\boldsymbol{f}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \;\; \boldsymbol{f}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \;\; \boldsymbol{f}_3 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Observe that

$$\boldsymbol{u}_1 = 2\boldsymbol{f}_1 + \boldsymbol{f}_2,$$

and thus the collection $\{\boldsymbol{u}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ is also a basis of $\ker N$.

We now have a Jordan basis of N consisting of the towers

$$\mathcal{T}_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}, \;\; \mathcal{T}_2 = \{\boldsymbol{f}_2\}, \;\; \mathcal{T}_3 = \{\boldsymbol{f}_3\}.$$

In this basis the operator is described as a direct sum of three Jordan cells: a cell of dimension 2, and two cells of dimension 1. Thus the Jordan invariant of A consists of single list L_1 corresponding to the single eigenvalue 1. More precisely

$$L_1 = 2, 1, 1. \qquad \square$$

2.6. **Exercises.**

Exercise 2.1. Denote by $\boldsymbol{U}_3$ the space of polynomials of degree ≤ 3 with real coefficients in one variable x. We denote by $\mathcal{B}$ the canonical basis of $\boldsymbol{U}_3$,

$$\mathcal{B} = \{1, x, x^2, x^3\}.$$

(a) Consider the linear operator $D : \boldsymbol{U}_3 \to \boldsymbol{U}_3$ given by

$$\boldsymbol{U}_3 \ni p \mapsto Dp = \frac{dp}{dx} \in \boldsymbol{U}_3.$$

Find the matrix that describes D in the canonical basis $\mathcal{B}$.
(b) Consider the operator $\Delta : \boldsymbol{U}_3 \to \boldsymbol{U}_3$ given by

$$(\Delta p)(x) = p(x+1) - p(x).$$

Find the matrix that describes Δ in the canonical basis $\mathcal{B}$.
(c) Show that for any $p \in \boldsymbol{U}_3$ the function

$$x \mapsto (\mathcal{L}p)(x) = \int_0^\infty e^{-t} \frac{dp(x+t)}{dx} dt$$

is also a polynomial of degree ≤ 3 and then find the matrix that describes $\mathcal{L}$ in the canonical basis $\mathcal{B}$. □

Exercise 2.2. (a) For any $n \times n$ matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ we define the *trace of* A as the sum of the diagonal entries of A,

$$\operatorname{tr} A := a_{11} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ij}.$$

Show that if A, B are two $n \times n$ matrices then

$$\operatorname{tr} AB = \operatorname{tr} BA.$$

(b) Let $\boldsymbol{U}$ be an n-dimensional $\mathbb{F}$-vector space, and $\underline{\boldsymbol{e}}$, $\underline{\boldsymbol{f}}$ be two bases of $\boldsymbol{U}$. Suppose that $T : \boldsymbol{U} \to \boldsymbol{U}$ is a linear operator represented in the basis $\underline{\boldsymbol{e}}$ by the matrix A and the basis $\underline{\boldsymbol{f}}$ by the matrix B. Prove that

$$\operatorname{tr} A = \operatorname{tr} B.$$

(The common value of these traces is called the *trace* of the operator T and it is denoted by $\operatorname{tr} T$.)
Hint: Use part (a) and (2.1).
(c) Consider the operator $A : \mathbb{F}^2 \to \mathbb{F}^2$ described by the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Show that

$$P_T(x) = x^2 - \operatorname{tr} Ax + \det A.$$

(d) Let T be a linear operator on the n-dimensional $\mathbb{F}$-vector space. Prove that the characteristic polynomial of T has the form

$$P_T(x) = \det(x\mathbb{1} - T) = x^n - (\operatorname{tr} T)x^{n-1} + \cdots + (-1)^n \det T. \qquad \square$$

Exercise 2.3. Suppose $T : \boldsymbol{U} \to \boldsymbol{U}$ is a linear operator on the $\mathbb{F}$-vector space $\boldsymbol{U}$, and $\boldsymbol{V}_1, \boldsymbol{V}_2 \subset \boldsymbol{U}$ are invariant subspaces of T. Show that $\boldsymbol{V}_1 \cap \boldsymbol{V}_2$ and $\boldsymbol{V}_1 + \boldsymbol{V}_2$ are also invariant subspaces of T. □

Exercise 2.4. Consider the Jacobi matrix

$$J_n = \begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix}$$

(a) Let $P_n(x)$ denote the characteristic polynomial of J_n,

$$P_n(x) = \det(xI - J_n).$$

Show that

$$P_1(x) = x - 2, \quad P_2(x) = x^2 - 4x + 3 = (x-1)(x-3),$$
$$P_n(x) = (x-2)P_{n-1}(x) - P_{n-2}(x), \quad \forall n \geq 3.$$

(b) Show that all the eigenvalues of J_4 are real and distinct, and then conclude that the matrices $\mathbb{1}, J_4, J_4^2, J_4^3$ are linearly independent. □

Exercise 2.5. Prove Proposition 2.11. **Hint.** Use induction on n. □

Exercise 2.6. Suppose that A and B are complex $n \times n$ matrices. Prove that AB and BA have the same characteristic polynomial.**Hint:** Consider first the case when A is invertible. For the general case show that $\varepsilon \mathbb{1} + A$ is invertible for all sufficiently small ε. □

Exercise 2.7. Suppose that $A = (a_{ij})_{1 \leq i,j \leq n}$ is an $n \times n$-matrix such that

$$a_{1j} + a_{2j} + \cdots + a_{nj} = 1, \quad \forall j = 1, \ldots, n.$$

Let

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

be an eigenvector of A such that $x_1 + \cdots + x_n \neq 0$. Prove that $A\boldsymbol{x} = \boldsymbol{x}$. □

Exercise 2.8. Consider the $n \times n$ matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, where $a_{ij} = 1$, for all i, j, which we regard as a linear operator $\mathbb{C}^n \to \mathbb{C}^n$. Consider the vector

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{C}^n.$$

(a) Compute $A\boldsymbol{c}$.
(b) Compute $\dim \boldsymbol{R}(A)$, $\dim \ker A$ and then determine $\operatorname{spec}(A)$.
(c) Find the characteristic polynomial of A. □

Exercise 2.9. Fix the nonzero complex numbers $z_1, \ldots, z_n$ and denote by $A = (a_{ij})_{1 \leq i,j \leq n}$ be the $n \times n$ matrix with entries $a_{ij} = z_i / z_j$. Find the characteristic polynomial of A. □

Exercise 2.10. Find the eigenvalues and the eigenvectors of the circulant matrix described in Exercise 1.16. □

Exercise 2.11. Let A, B be two complex $n\times n$ matrices such that $\operatorname{tr}(A^m)=\operatorname{tr}(B^m)$, $\forall m=1,\dots,m$.

(i) Let $\vec{\lambda}=(\lambda_1,\dots,\lambda_n)$ and $\vec{\mu}=(\mu_1,\dots,\mu_n)$ be the eigenvalues of A and B respectively, repeated according to their multiplicities. Prove that

$$\operatorname{tr}A^m=\sum_{j=1}^n\lambda_j^m,\ \ \operatorname{tr}B^m=\sum_{j=1}^n\mu_j^m,\ \ \forall m\in\mathbb{N}.$$

(ii) Prove that A and B have the same characteristic polynomial. **Hint.** Use Theorem 2.13.

□

Exercise 2.12. Let $T:\boldsymbol{U}\to\boldsymbol{U}$ be a linear operator on the finite dimensional complex vector space $\boldsymbol{U}$. Suppose that m is a positive integer and $\boldsymbol{u}\in\boldsymbol{U}$ is a vector such that

$$T^{m-1}\boldsymbol{u}\neq 0,\ \text{ but }\ T^m\boldsymbol{u}=0.$$

Show that the vectors

$$\boldsymbol{u},\ T\boldsymbol{u},\dots,T^{m-1}\boldsymbol{u}$$

are linearly independent. □

Exercise 2.13. Let $T:\boldsymbol{U}\to\boldsymbol{U}$ be a linear operator on the finite dimensional complex vector space $\boldsymbol{U}$. Show that if

$$\dim\ker T^{\dim\boldsymbol{U}-1}\neq\dim\ker T^{\dim\boldsymbol{U}},$$

then T is a nilpotent operator and

$$\dim\ker T^j=j,\ \ \forall j=1,\dots,\dim\boldsymbol{U}.$$

Can you construct an example of an operator T satisfying the above properties? □

Exercise 2.14. Let S,T be two linear operators on the finite dimensional complex vector space $\boldsymbol{U}$. Show that ST is nilpotent if and only if TS is nilpotent. □

Exercise 2.15. Find a Jordan basis of the linear operator $\mathbb{C}^4\to\mathbb{C}^4$ described by the $\times 4$ matrix

$$A=\begin{bmatrix}1&0&-1&-1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$ □

Exercise 2.16. Find a Jordan basis of the linear operator $\mathbb{C}^7\to\mathbb{C}^7$ given by the matrix

$$A=\begin{bmatrix}3&2&1&1&0&-3&1\\-1&2&0&0&1&-1&1\\1&1&3&0&-6&5&-1\\0&2&1&4&1&-3&1\\0&0&0&0&3&0&0\\0&0&0&0&-2&5&0\\0&0&0&0&-2&0&5\end{bmatrix}.$$ □

Exercise 2.17. (a) Find the inverse of the $m \times m$ matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & m-1 & m \\ 0 & 1 & 2 & \cdots & m-2 & m-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

(b) Find the Jordan normal form of the above matrix. □

3. EUCLIDEAN SPACES

In the sequel $\mathbb{F}$ will denote either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers. Any complex number has the form $z = a + b\boldsymbol{i}$, $\boldsymbol{i} = \sqrt{-1}$, so that $\boldsymbol{i}^2 = -1$. The real number a is called the *real part* of z and it is denoted by $\mathbf{Re}\, z$. The real number b is called the *imaginary part* of z and its is denoted by $\mathbf{Im}\, z$.

The *conjugate* of a complex number $z = a + b\boldsymbol{i}$ is the complex number

$$\bar{z} = a - b\boldsymbol{i}.$$

In particular, any real number is equal to its conjugate. Note that

$$z + \bar{z} = 2\,\mathbf{Re}\, z, \quad z - \bar{z} = 2\boldsymbol{i}\,\mathbf{Im}\, z.$$

The *norm* or *absolute value* of a complex number $z = a + b\boldsymbol{i}$ is the real number

$$|z| = \sqrt{a^2 + b^2}.$$

Observe that

$$|z|^2 = a^2 + b^2 = (a + b\boldsymbol{i})(a - b\boldsymbol{i}) = z\bar{z}.$$

In particular, if $z \neq 0$ we have

$$\frac{1}{z} = \frac{1}{|z|^2}\bar{z}.$$

3.1. Inner products.

Let $\boldsymbol{U}$ be an $\mathbb{F}$-vector space.

Definition 3.1. An inner product on $\boldsymbol{U}$ is a map

$$\langle -, - \rangle : \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{F}, \quad \boldsymbol{U} \times \boldsymbol{U} \ni (\boldsymbol{u}_1, \boldsymbol{u}_2) \mapsto \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle \in \mathbb{F},$$

satisfying with the following condition.

(i) *Linearity in the first variable*, i.e., $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}_2 \in \boldsymbol{U}$, $x, y \in \mathbb{F}$ we have

$$\langle x\boldsymbol{u} + y\boldsymbol{v}, \boldsymbol{v}_2 \rangle = x\langle \boldsymbol{u}, \boldsymbol{v}_2 \rangle + y\langle \boldsymbol{v}, \boldsymbol{v}_2 \rangle.$$

(ii) *Conjugate linearity in the second variable*, i.e., $\forall \boldsymbol{u}_1, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$, $x, y \in \mathbb{F}$ we have

$$\langle \boldsymbol{u}_1, x\boldsymbol{u} + y\boldsymbol{v} \rangle = \bar{x}\langle \boldsymbol{u}_1, \boldsymbol{u} \rangle + \bar{y}\langle \boldsymbol{u}_1, \boldsymbol{v} \rangle.$$

(iii) *Hermitian property*, i.e., $\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$ we have

$$\langle \boldsymbol{v}, \boldsymbol{u} \rangle = \overline{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}.$$

(iv) *Positive definiteness*, i.e.,

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0, \quad \forall \boldsymbol{u} \in \boldsymbol{U},$$

and

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \Longleftrightarrow \boldsymbol{u} = 0.$$

A vector space $\boldsymbol{u}$ equipped with an inner product is called an *Euclidean* space. □

Example 3.2. (a) ***The standard real n-dimensional Euclidean space.*** The vector space $\mathbb{R}^n$ is equipped with a canonical inner product

$$\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}.$$

More precisely if

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n,$$

then

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + \cdots + u_n v_n = \sum_{k=1}^{n} u_k v_k = \boldsymbol{u}^\dagger \cdot \boldsymbol{v}.$$

You can verify that this is indeed an inner product, i.e., it satisfies the conditions (i)-(iv) in Definition 3.1.

(b) ***The standard complex n-dimensional Euclidean space.*** The vector space $\mathbb{C}^n$ is equipped with a canonical inner product

$$\langle -, - \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}.$$

More precisely if

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{C}^n$$

then

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 \bar{v}_1 + \cdots + u_n \bar{v}_n = \sum_{k=1}^{n} u_k \bar{v}_k.$$

You can verify that this is indeed an inner product.

(c) Denote by $\mathcal{P}_n$ the vector space of polynomials with real coefficients and degree $\leq n$. We can define an inner product on $\mathcal{P}_n$

$$\langle -, - \rangle : \mathcal{P}_n \times \mathcal{P}_n \to \mathbb{R},$$

by setting

$$\langle p, q \rangle = \int_0^1 p(x) q(x) dx, \quad \forall p, q \in \mathcal{P}_n.$$

You can verify that this is indeed an inner product.

(d) Any finite dimensional $\mathbb{F}$-vector space $\boldsymbol{U}$ admits an inner product. Indeed, if we fix a basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ of $\boldsymbol{U}$, then we define

$$\langle -, - \rangle : \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{F},$$

by setting

$$\Big\langle\, u_1 \boldsymbol{e}_1 + \cdots + u_n \boldsymbol{e}_n \,,\, v_1 \boldsymbol{e}_1 + \cdots + v_n \boldsymbol{e}_n \,\Big\rangle = u_1 \bar{v}_1 + \cdots + u_n \bar{v}_n. \qquad \square$$

3.2. **Basic properties of Euclidean spaces.** Suppose that $(\boldsymbol{U}, \langle -, - \rangle)$ is an Euclidean vector space. We define the *norm* or *length* of a vector $\boldsymbol{u} \in \boldsymbol{U}$ to be the nonnegative number

$$\|\boldsymbol{u}\| := \sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle}.$$

Example 3.3. In the standard Euclidean space $\mathbb{R}^n$ of Example 3.2(a) we have

$$\|\boldsymbol{u}\| = \sqrt{\sum_{k=1}^{n} u_k^2}. \qquad \square$$

Theorem 3.4 (Cauchy-Schwarz inequality)**.** *For any $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{U}$ we have*

$$\big|\,\langle\boldsymbol{u},\boldsymbol{v}\rangle\,\big|\leq\|\boldsymbol{u}\|\cdot\|\boldsymbol{v}\|.$$

Moreover, the equality is achieved if and only if the vectors $\boldsymbol{u},\boldsymbol{v}$ are linearly dependent, i.e., collinear.

Proof. If $\langle\boldsymbol{u},\boldsymbol{v}\rangle=0$, then the inequality is trivially satisfied. Hence we can assume that $\langle\boldsymbol{u},\boldsymbol{v}\rangle\neq 0$. In particular, $\boldsymbol{u},\boldsymbol{v}\neq 0$.

From the positive definiteness of the inner product we deduce that for any $x\in\mathbb{F}$ we have

$$\begin{aligned}0\leq\|\boldsymbol{u}-x\boldsymbol{v}\|^2=\langle\boldsymbol{u}-x\boldsymbol{v},\boldsymbol{u}-x\boldsymbol{v}\rangle&=\langle\boldsymbol{u},\boldsymbol{u}-x\boldsymbol{v}\rangle-x\langle\boldsymbol{v},\boldsymbol{u}-x\boldsymbol{v}\rangle\\&=\langle\boldsymbol{u},\boldsymbol{u}\rangle-\bar{x}\langle\boldsymbol{u},\boldsymbol{v}\rangle-x\langle\boldsymbol{v},\boldsymbol{u}\rangle+x\bar{x}\langle\boldsymbol{v},\boldsymbol{v}\rangle\\&=\|\boldsymbol{u}\|^2-\bar{x}\langle\boldsymbol{u},\boldsymbol{v}\rangle-x\langle\boldsymbol{v},\boldsymbol{u}\rangle+|x|^2\|\boldsymbol{v}\|^2.\end{aligned}$$

If we let

$$x_0=\frac{1}{\|\boldsymbol{v}\|^2}\langle\boldsymbol{u},\boldsymbol{v}\rangle,\tag{3.1}$$

then

$$\bar{x_0}\langle\boldsymbol{u},\boldsymbol{v}\rangle+x_0\langle\boldsymbol{v},\boldsymbol{u}\rangle=\frac{1}{\|\boldsymbol{v}\|^2}\overline{\langle\boldsymbol{u},\boldsymbol{v}\rangle}\langle\boldsymbol{u},\boldsymbol{v}\rangle+\frac{1}{\|\boldsymbol{v}\|^2}\langle\boldsymbol{u},\boldsymbol{v}\rangle\overline{\langle\boldsymbol{u},\boldsymbol{v}\rangle}=2\frac{|\langle\boldsymbol{u},\boldsymbol{v}\rangle|^2}{\|\boldsymbol{v}\|^2},$$

$$|x_0|^2\|\boldsymbol{v}\|^2=\frac{|\langle\boldsymbol{u},\boldsymbol{v}\rangle|^2}{\|\boldsymbol{v}\|^2},$$

and thus

$$0\leq\|\boldsymbol{u}-x_0\boldsymbol{v}\|^2=\|\boldsymbol{u}\|^2-2\frac{|\langle\boldsymbol{u},\boldsymbol{v}\rangle|^2}{\|\boldsymbol{v}\|^2}+\frac{|\langle\boldsymbol{u},\boldsymbol{v}\rangle|^2}{\|\boldsymbol{v}\|^2}=\|\boldsymbol{u}\|^2-\frac{|\langle\boldsymbol{u},\boldsymbol{v}\rangle|^2}{\|\boldsymbol{v}\|^2}.\tag{3.2}$$

Thus

$$\frac{|\langle\boldsymbol{u},\boldsymbol{v}\rangle|^2}{\|\boldsymbol{v}\|^2}\leq\|\boldsymbol{u}\|^2$$

so that

$$\big|\,\langle\boldsymbol{u},\boldsymbol{v}\rangle\,\big|^2\leq\|\boldsymbol{u}\|^2\cdot\|\boldsymbol{v}\|^2.$$

Note that if $0=\langle\boldsymbol{u},\boldsymbol{v}\rangle=\|\boldsymbol{u}\|\cdot\|\boldsymbol{v}\|$ then at least one of the vectors $\boldsymbol{u},\boldsymbol{v}$ must be zero.

If $0\neq\langle\boldsymbol{u},\boldsymbol{v}\rangle=\|\boldsymbol{u}\|\cdot\|\boldsymbol{v}\|$, then by choosing x_0 as in (3.1) we deduce as in (3.2) that

$$\|\boldsymbol{u}-x_0\boldsymbol{v}\|=0.$$

Hence $\boldsymbol{u}=x_0\boldsymbol{v}$. □

Remark 3.5. The Cauchy-Schwarz theorem is a rather nontrivial result, which in skilled hands can produce remarkable consequences. Observe that if $\boldsymbol{U}$ is the standard real Euclidean space of Example 3.2(a), then the Cauchy-Schwarz inequality implies that for any real numbers $u_1,v_1,\dots,u_n,v_n$ we have

$$\left|\sum_{k=1}^n u_kv_k\right|\leq\sqrt{\sum_{k=1}^n u_k^2}\cdot\sqrt{\sum_{k=1}^n v_k^2}$$

If we square both sides of the above inequality we deduce

$$\left(\sum_{k=1}^n u_kv_k\right)^2\leq\left(\sum_{k=1}^n u_k^2\right)\cdot\left(\sum_{k=1}^n v_k^2\right).\tag{3.3}$$

□

Observe that if $\boldsymbol{u}, \boldsymbol{v}$ are two nonzero vectors in an Euclidean space $(\boldsymbol{U}, \langle -, - \rangle)$, then the Cauchy-Schwarz inequality implies that

$$\frac{\mathbf{Re}\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\|} \in [-1, 1].$$

Thus there exists a unique $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\mathbf{Re}\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\|}.$$

This angle θ is called the *angle* between the two nonzero vectors $\boldsymbol{u}, \boldsymbol{v}$. We denote it by $\measuredangle(\boldsymbol{u}, \boldsymbol{v})$. In particular, we have, by definition,

$$\cos\measuredangle(\boldsymbol{u}, \boldsymbol{v}) = \frac{\mathbf{Re}\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\|}. \tag{3.4}$$

Note that if the two vectors $\boldsymbol{u}, \boldsymbol{v}$ were perpendicular in the classical sense, i.e., $\measuredangle(\boldsymbol{u}, \boldsymbol{v}) = \frac{\pi}{2}$, then $\mathbf{Re}\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$. This justifies the following notion.

Definition 3.6. Two vectors $\boldsymbol{u}, \boldsymbol{v}$ in an Euclidean vector space $(\boldsymbol{U}, \langle -, - \rangle)$ are said to be *orthogonal* if $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$. We will indicate the orthogonality of two vectors $\boldsymbol{u}, \boldsymbol{v}$ using the notation $\boldsymbol{u} \perp \boldsymbol{v}$. □

☞ *In the remainder of this subsection we fix an Euclidean space* $(\boldsymbol{U}, \langle -, - \rangle)$.

Theorem 3.7 (Pythagora)**.** *If* $\boldsymbol{u} \perp \boldsymbol{v}$*, then*

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2.$$

Proof. We have

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{u} \rangle + \underbrace{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}_{=0} + \underbrace{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}_{=0} + \langle \boldsymbol{v}, \boldsymbol{v} \rangle$$

$$= \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2.$$

□

Theorem 3.8 (Triangle inequality)**.**

$$\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}.$$

Proof. Observe that the inequality is can be rewritten equivalently as

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 \leq \big(\|\boldsymbol{u}\| + \|\boldsymbol{v}\| \big)^2.$$

Observe that

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{u} \rangle + \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{v}, \boldsymbol{u} \rangle + \langle \boldsymbol{v}, \boldsymbol{v} \rangle$$

$$\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2\,\mathbf{Re}\langle \boldsymbol{u}, \boldsymbol{v} \rangle \leq \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2\big|\,\langle \boldsymbol{u}, \boldsymbol{v} \rangle\,\big|$$

(use the Cauchy-Schwarz inequality)

$$\leq \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\| = \big(\|\boldsymbol{u}\| + \|\boldsymbol{v}\| \big)^2.$$

□

3.3. **Orthonormal systems and the Gramm-Schmidt procedure.** In the sequel $\boldsymbol{U}$ will denote an n-dimensional Euclidean $\mathbb{F}$-vector space. We will denote the inner product on $\boldsymbol{U}$ by $\langle -, - \rangle$.

Definition 3.9. A family of *nonzero* vectors
$$\big\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\big\} \subset \boldsymbol{U}$$
is called *orthogonal* if
$$\boldsymbol{u}_i \perp \boldsymbol{u}_j, \ \ \forall i \neq j.$$
An orthogonal family
$$\big\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\big\} \subset \boldsymbol{U}$$
is called *orthonormal* if
$$\|\boldsymbol{u}_i\| = 1, \ \ \forall i = 1, \dots, k.$$
A basis of $\boldsymbol{U}$ is called *orthogonal* (respectively *orthonormal*) if it is an orthogonal (respectively orthonormal) family. □

Proposition 3.10. *Any orthogonal family in $\boldsymbol{U}$ is linearly independent.*

Proof. Suppose that
$$\big\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\big\} \subset \boldsymbol{U}$$
is an orthogonal family. If $x_1, \dots, x_k \in \mathbb{F}$ are such that
$$x_1\boldsymbol{u}_1 + \cdots + x_k\boldsymbol{u}_k = 0,$$
then taking the inner product with $\boldsymbol{u}_j$ of both sides in the above equality we deduce
$$0 = x_1\langle \boldsymbol{u}_1, \boldsymbol{u}_j\rangle + \cdots + x_{j-1}\langle \boldsymbol{u}_{j-1}, \boldsymbol{u}_j\rangle + x_j\langle \boldsymbol{u}_j, \boldsymbol{u}_j\rangle + x_{j+1}\langle \boldsymbol{u}_{j+1}, \boldsymbol{u}_j\rangle + \cdots + x_n\langle \boldsymbol{u}_n, \boldsymbol{u}_j\rangle$$
$(\langle \boldsymbol{u}_i, \boldsymbol{u}_j\rangle = 0, \forall i \neq j)$
$$= x_j\|\boldsymbol{u}_j\|^2.$$
Since $\boldsymbol{u}_j \neq 0$ we deduce $x_j = 0$. This happens for any $j = 1, \dots, k$, proving that the family is linearly independent. □

Theorem 3.11 (Gramm-Schmidt). *Suppose that*
$$\big\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\big\} \subset \boldsymbol{U}$$
is a linearly independent *family. Then there exists an* orthonormal *family*
$$\big\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\big\} \subset \boldsymbol{U}$$
such that
$$\operatorname{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_j\big\} = \operatorname{span}\big\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_j\big\}, \ \ \forall j = 1, \dots, k.$$

Proof. We will argue by induction on k. For $k = 1$, if $\{\boldsymbol{u}_1\} \subset \boldsymbol{U}$ is a linearly independent family, then $\boldsymbol{u}_1 \neq 0$ and we set
$$\boldsymbol{e}_1 := \frac{1}{\|\boldsymbol{u}_1\|}\boldsymbol{u}_1.$$
Clearly $\{\boldsymbol{e}_1\}$ is an orthonormal family spanning the same subspace as $\{\boldsymbol{u}_1\}$.

Suppose that the result is true for any linearly independent family of vectors consisting of $(k-1)$ vectors. We need to prove that the result is true for linearly independent families consisting of k vectors. Let
$$\big\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\big\} \subset \boldsymbol{U}$$

be such a family. The induction assumption implies that we can find an orthonormal system

$$\{\, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{k-1} \,\}$$

such that

$$\operatorname{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_j\,\} = \operatorname{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_j\,\}, \;\; \forall j = 1, \dots, k-1.$$

Define

$$\boldsymbol{v}_k := \langle \boldsymbol{u}_k, \boldsymbol{e}_1\rangle \boldsymbol{e}_1 + \dots + \langle \boldsymbol{u}_k, \boldsymbol{e}_{k-1}\rangle \boldsymbol{e}_{k-1},$$
$$\boldsymbol{f}_k := \boldsymbol{u}_k - \boldsymbol{v}_k.$$

Observe that

$$\boldsymbol{v}_k \in \operatorname{span}\{\, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{k-1}\,\} = \operatorname{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_{k-1}\,\}.$$

Since $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ is a linearly independent family we deduce that

$$\boldsymbol{u}_k \notin \{\, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{k-1}\,\}$$

so that $\boldsymbol{f}_k = \boldsymbol{u}_k - \boldsymbol{v}_k \neq 0$. We can now set

$$\boldsymbol{e}_k := \frac{1}{\|\boldsymbol{f}_k\|}\boldsymbol{f}_k.$$

By construction $\|\boldsymbol{e}_k\| = 1$. Also note that if $1 \leq j < k$, then

$$\langle \boldsymbol{f}_k, \boldsymbol{e}_j\rangle = \langle \boldsymbol{u}_k - \boldsymbol{v}_k, \boldsymbol{e}_j\rangle = \langle \boldsymbol{u}_k, \boldsymbol{e}_j\rangle - \langle \boldsymbol{v}_k, \boldsymbol{e}_j\rangle$$
$$= \langle \boldsymbol{u}_k, \boldsymbol{e}_j\rangle - \Big\langle \underbrace{\langle \boldsymbol{u}_k, \boldsymbol{e}_1\rangle \boldsymbol{e}_1 + \dots + \langle \boldsymbol{u}_k, \boldsymbol{e}_{k-1}\rangle \boldsymbol{e}_{k-1}}_{=\boldsymbol{v}_k}, \boldsymbol{e}_j \Big\rangle$$
$$\langle \boldsymbol{u}_k, \boldsymbol{e}_j\rangle - \langle \boldsymbol{u}_k, \boldsymbol{e}_j\rangle \cdot \langle \boldsymbol{e}_j \boldsymbol{e}_j\rangle = 0.$$

This proves that $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{k-1}, \boldsymbol{e}_k\}$ is an orthonormal family.

Finally observe that

$$\boldsymbol{u}_k = \boldsymbol{v}_k + \boldsymbol{f}_k = \boldsymbol{v}_k + \|\boldsymbol{f}_k\|\boldsymbol{e}_k.$$

Since

$$\boldsymbol{v}_k \in \operatorname{span}\{\, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{k-1}\,\}$$

we deduce

$$\boldsymbol{u}_k \in \operatorname{span}\{\, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{k-1}, \boldsymbol{e}_k\,\}$$

and thus

$$\operatorname{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\,\} = \operatorname{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\,\}$$

□

Remark 3.12. The strategy used in the proof of the above theorem is as important as the theorem itself. The procedure we used to produce the orthonormal family $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$. This procedure goes by the name of the *Gram-Schmidt* procedure. To understand how it works we consider a simple case, when $\boldsymbol{U}$ is the space $\mathbb{R}^2$ equipped with the canonical inner product.

Suppose that

$$\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \;\; \boldsymbol{u}_2 := \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Then

$$\|\boldsymbol{u}_1\|^2 = 3^2 + 4^2 = 9 + 16 = 25,$$

so that

$$\boldsymbol{e}_1 := \frac{1}{\|\boldsymbol{u}_1\|}\boldsymbol{u}_1 = \begin{bmatrix} \frac{3}{5} \\ \\ \frac{4}{5} \end{bmatrix}.$$

Next

$$\boldsymbol{v}_2 = \langle \boldsymbol{u}_2, \boldsymbol{e}_1 \rangle \boldsymbol{e}_1 = 3\boldsymbol{e}_1, \quad \boldsymbol{f}_2 = \boldsymbol{u}_2 - \boldsymbol{v}_2 = \begin{bmatrix} 5 \\ \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{9}{5} \\ \\ \frac{12}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{5} \\ \\ -\frac{12}{5} \end{bmatrix} = 4 \begin{bmatrix} \frac{4}{5} \\ \\ -\frac{3}{5} \end{bmatrix}.$$

We see that $\boldsymbol{f}_2 = 4$ and thus

$$\boldsymbol{e}_2 = \begin{bmatrix} \frac{4}{5} \\ \\ -\frac{3}{5} \end{bmatrix}.$$

□

The Gram-Schmidt theorem has many useful consequences. We will discuss a few of them

Corollary 3.13. *Any finite dimensional Euclidean vector space (over $\mathbb{F} = \mathbb{R}, \mathbb{C}$) admits an orthonormal basis.*

Proof. Apply Theorem 3.11 to a basis of the vector space. □

The orthonormal bases of an Euclidean space have certain computational advantages. Suppose that

$$\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$$

is an orthonormal basis of the Euclidean space $\boldsymbol{U}$. Then the coordinates of a vector $\boldsymbol{u} \in \boldsymbol{U}$ in this basis are easily computed. More precisely, if

$$\boldsymbol{u} = x_1 \boldsymbol{e}_1 + \cdots + x_n \boldsymbol{e}_n, \quad x_1, \dots, x_n \in \mathbb{F}, \tag{3.5}$$

then

$$x_j = \langle \boldsymbol{u}, \boldsymbol{e}_j \rangle \ \ \forall j = 1, \dots, n. \tag{3.6}$$

Indeed, the equality (3.5) implies that

$$\langle \boldsymbol{u}, \boldsymbol{e}_j \rangle = \big\langle\, x_1 \boldsymbol{e}_1 + \cdots + x_n \boldsymbol{e}_n, \boldsymbol{e}_j \,\big\rangle = x_j \langle \boldsymbol{e}_j, \boldsymbol{e}_j \rangle = x_j,$$

where at the last step we used the orthonormality condition which translates to

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Applying Pythagora's theorem we deduce

$$\| x_1 \boldsymbol{e}_1 + \cdots + x_n \boldsymbol{e}_n \|^2 = x_1^2 + \cdots + x_n^2 = |\langle \boldsymbol{u}, \boldsymbol{e}_1 \rangle|^2 + \cdots + |\langle \boldsymbol{u}, \boldsymbol{e}_n \rangle\,|^2. \tag{3.7}$$

Example 3.14. Consider the orthonormal basis $\boldsymbol{e}_1, \boldsymbol{e}_2$ of $\mathbb{R}^2$ constructed in Remark 3.12. If

$$\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

then the coordinates x_1, x_2 of $\boldsymbol{u}$ in this basis are given by

$$x_1 = \langle \boldsymbol{u}, \boldsymbol{e}_1 \rangle = \frac{6}{5} + \frac{4}{5} = 2,$$

$$x_2 = \langle \boldsymbol{u}, \boldsymbol{e}_2 \rangle = \frac{8}{5} - \frac{3}{5} = 1,$$

so that

$$\boldsymbol{u} = 2\boldsymbol{e}_1 + \boldsymbol{e}_2.$$

□

Corollary 3.15. *Any orthonormal family*

$$\{\, \boldsymbol{e}_1, \dots, \boldsymbol{e}_k \,\}$$

in a finite dimensional Euclidean vector can be extended to an orthonormal basis *of that space.*

Proof. According to Proposition 3.39, the family

$$\{\, \boldsymbol{e}_1, \dots, \boldsymbol{e}_k \,\}$$

is linearly independent. Therefore, we can extend it to a basis of $\boldsymbol{U}$,

$$\{\, \boldsymbol{e}_1, \dots, \boldsymbol{e}_k, \boldsymbol{u}_{k+1}, \dots, \boldsymbol{u}_n \,\}.$$

If we apply the Gram-Schmidt procedure to the above linearly independent family we obtain an orthonormal basis that extends our original orthonormal family.[2] □

3.4. **Orthogonal projections.** Suppose that $(\boldsymbol{U}, \langle -, - \rangle)$ is a finite dimensional Euclidean $\mathbb{F}$-vector space. If X is a subset of $\boldsymbol{U}$ then we set

$$X^{\perp} = \big\{ \boldsymbol{u} \in \boldsymbol{U};\ \ \langle \boldsymbol{u}, \boldsymbol{x} \rangle = 0,\ \ \forall \boldsymbol{x} \in X \,\big\}.$$

In other words, $X^{\perp}$ consists of the vectors orthogonal to *all* the vectors in X. For this reason we will often write $\boldsymbol{u} \perp X$ to indicate that $\boldsymbol{u} \in X^{\perp}$. The following result is left to the reader.

Proposition 3.16. *(a) The subset $X^{\perp}$ is a vector subspace of $\boldsymbol{U}$.*
(b)

$$X \subset Y \Rightarrow X^{\perp} \supset Y^{\perp}.$$

(c)

$$X^{\perp} = \big(\operatorname{span}(X)\big)^{\perp}.$$ □

Theorem 3.17. *If $\boldsymbol{V}$ is a subspace of $\boldsymbol{U}$, then*

$$\boldsymbol{U} = \boldsymbol{V} \oplus \boldsymbol{V}^{\perp}.$$

Proof. We need to check two things.

$$\boldsymbol{V} \cap \boldsymbol{V}^{\perp} = 0, \tag{3.8a}$$

$$\boldsymbol{U} = \boldsymbol{V} + \boldsymbol{V}^{\perp}. \tag{3.8b}$$

Proof of (3.8a). If $\boldsymbol{x} \in \boldsymbol{V} \cap \boldsymbol{V}^{\perp}$ then

$$0 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$$

which implies that $\boldsymbol{x} = 0$.
Proof of (3.8b). Fix an orthonormal basis of $\boldsymbol{V}$,

$$\mathcal{B} := \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}.$$

Extend it to an orthonormal basis of $\boldsymbol{U}$,

$$\boldsymbol{e}_1, \dots, \boldsymbol{e}_k, \boldsymbol{e}_{k+1}, \dots, \boldsymbol{e}_n.$$

By construction,

$$\boldsymbol{e}_{k+1}, \dots, \boldsymbol{e}_n \in \mathcal{B}^{\perp} = \big(\operatorname{span}(\mathcal{B})\big)^{\perp} = \boldsymbol{V}^{\perp}$$

so that

$$\operatorname{span}\{\boldsymbol{e}_{k+1}, \dots, \boldsymbol{e}_n\} \subset \boldsymbol{V}^{\perp}.$$

[2] Exercise 3.5 asks you to verify this claim.

Clearly any vector $\boldsymbol{u} \in \boldsymbol{U}$ can be written as a sum of two vectors

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \quad \boldsymbol{v} \in \operatorname{span}(\mathcal{B}) = \boldsymbol{V}, \quad \boldsymbol{w} \in \operatorname{span}\{\boldsymbol{e}_{k+1}, \dots, \boldsymbol{e}_n\} \subset \boldsymbol{V}^{\perp}.$$

□

Corollary 3.18. *If $\boldsymbol{V}$ is a subspace of $\boldsymbol{U}$, then*

$$\boldsymbol{V} = (\boldsymbol{V}^{\perp})^{\perp}.$$

Proof. Theorem 3.17 implies that for any subspace $\boldsymbol{W}$ of $\boldsymbol{U}$ we have

$$\dim \boldsymbol{W}^{\perp} = \dim \boldsymbol{U} - \dim \boldsymbol{W}.$$

If we let $\boldsymbol{W} = \boldsymbol{V}^{\perp}$ we deduce that

$$\dim (\boldsymbol{V}^{\perp})^{\perp} = \dim \boldsymbol{U} - \dim \boldsymbol{V}^{\perp}.$$

If we let $\boldsymbol{W} = \boldsymbol{V}$ we deduce

$$\dim \boldsymbol{V}^{\perp} = \dim \boldsymbol{U} - \dim \boldsymbol{V}.$$

Hence

$$\dim \boldsymbol{V} = \dim (\boldsymbol{V}^{\perp})^{\perp}$$

so it suffices to show that

$$\boldsymbol{V} \subset (\boldsymbol{V}^{\perp})^{\perp},$$

i.e., we have to show that any vector $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to any vector $\boldsymbol{w}$ in $\boldsymbol{V}^{\perp}$. Since $\boldsymbol{w} \in \boldsymbol{V}^{\perp}$ we have $\boldsymbol{w} \perp \boldsymbol{v}$ so that $\boldsymbol{v} \perp \boldsymbol{w}$. □

Suppose that $\boldsymbol{V}$ is a subspace of $\boldsymbol{U}$. Then any $\boldsymbol{u} \in \boldsymbol{U}$ admits a unique decomposition

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \quad \boldsymbol{v} \in \boldsymbol{V}, \quad \boldsymbol{w} \in \boldsymbol{V}^{\perp}.$$

We set

$$P_{\boldsymbol{V}} \boldsymbol{u} := \boldsymbol{v}.$$

Observe that if

$$\boldsymbol{u}_0 = \boldsymbol{v}_0 + \boldsymbol{w}_0, \quad \boldsymbol{u}_1 = \boldsymbol{v}_1 + \boldsymbol{w}_1, \quad \boldsymbol{v}_0, \boldsymbol{v}_1 \in \boldsymbol{V}, \quad \boldsymbol{w}_0, \boldsymbol{w}_1 \in \boldsymbol{V}^{\perp},$$

then

$$(\boldsymbol{u}_0 + \boldsymbol{u}_1) = \underbrace{(\boldsymbol{v}_0 + \boldsymbol{v}_1)}_{\in \boldsymbol{V}} + \underbrace{(\boldsymbol{w}_0 + \boldsymbol{w}_1)}_{\in \boldsymbol{V}^{\perp}}$$

and we deduce

$$P_{\boldsymbol{V}}(\boldsymbol{u}_0 + \boldsymbol{u}_1) = \boldsymbol{v}_0 + \boldsymbol{v}_1 = P_{\boldsymbol{V}} \boldsymbol{u}_0 + P_{\boldsymbol{V}} \boldsymbol{u}_1.$$

Similarly, if $\lambda \in \mathbb{F}$ and $\boldsymbol{u} \in \boldsymbol{U}$, then

$$\lambda \boldsymbol{u} = \lambda \boldsymbol{v} + \lambda \boldsymbol{w}$$

and we deduce

$$P_{\boldsymbol{V}}(\lambda \boldsymbol{u}) = \lambda \boldsymbol{v} = \lambda P_{\boldsymbol{V}} \boldsymbol{u}.$$

We have thus shown that the map

$$P_{\boldsymbol{V}} : \boldsymbol{U} \to \boldsymbol{U}, \quad \boldsymbol{u} \mapsto P_{\boldsymbol{V}} \boldsymbol{u}$$

is a linear operator. It is called the *the orthogonal projection* onto the subpace $\boldsymbol{V}$. Observe that

$$\boldsymbol{R}(P_{\boldsymbol{V}}) = \boldsymbol{V}, \quad \ker P_{\boldsymbol{V}} = \boldsymbol{V}^{\perp}. \tag{3.9}$$

Note that $P_{\boldsymbol{V}} \boldsymbol{u}$ is the *unique* vector $\boldsymbol{v}$ in $\boldsymbol{V}$ with the property that $(\boldsymbol{u} - \boldsymbol{v}) \perp \boldsymbol{V}$.

Proposition 3.19. $P_{\boldsymbol{V}^\perp} = \mathbb{1} - P_{\boldsymbol{V}}$.

Proof. Any vector $\boldsymbol{u} \in \boldsymbol{U}$ admits a unique decomposition as a sum

$$\boldsymbol{u} = \boldsymbol{u}^\perp + \boldsymbol{u}^{\perp\perp}, \;\; \boldsymbol{u}^\perp \in \boldsymbol{V}^\perp, \;\; \boldsymbol{u}^{\perp\perp} \in \boldsymbol{V}^{\perp\perp} = \boldsymbol{V}.$$

By definition we have $\boldsymbol{u}^\perp = P_{\boldsymbol{V}^\perp}\boldsymbol{u}$ and $\boldsymbol{u}^{\perp\perp} = P_{\boldsymbol{V}}\boldsymbol{u}$ □

Proposition 3.20. *Suppose $\boldsymbol{V}$ is a subspace of $\boldsymbol{U}$ and $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k$ is an* orthonormal *basis of $\boldsymbol{V}$. Then*

$$P_{\boldsymbol{V}}\boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{e}_1\rangle \boldsymbol{e}_1 + \cdots + \langle \boldsymbol{u}, \boldsymbol{e}_k\rangle \boldsymbol{e}_k, \;\; \forall \boldsymbol{u} \in \boldsymbol{U}.$$

Proof. It suffices to show that the vector

$$\boldsymbol{w} = \boldsymbol{u} - \big(\langle \boldsymbol{u}, \boldsymbol{e}_1\rangle \boldsymbol{e}_1 + \cdots + \langle \boldsymbol{u}, \boldsymbol{e}_k\rangle \boldsymbol{e}_k \big)$$

is orthogonal to all the vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k$ because then it will be orthogonal to any linear combination of these vectors. We have

$$\langle \boldsymbol{w}, \boldsymbol{e}_j\rangle = \langle \boldsymbol{u}, \boldsymbol{e}_j\rangle - \big(\langle \boldsymbol{u}, \boldsymbol{e}_1, \rangle\langle \boldsymbol{e}_1, \boldsymbol{e}_j\rangle + \cdots + \langle \boldsymbol{u}, \boldsymbol{e}_k\rangle\langle \boldsymbol{e}_k, \boldsymbol{e}_j\rangle \big).$$

Since $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k$ is an *orthonormal* basis of $\boldsymbol{V}$ we deduce that

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Hence

$$\big(\langle \boldsymbol{u}, \boldsymbol{e}_1, \rangle\langle \boldsymbol{e}_1, \boldsymbol{e}_j\rangle + \cdots + \langle \boldsymbol{u}, \boldsymbol{e}_k\rangle\langle \boldsymbol{e}_k, \boldsymbol{e}_j\rangle \big) = \langle \boldsymbol{u}, \boldsymbol{e}_j\rangle\langle \boldsymbol{e}_j \boldsymbol{e}_j\rangle = \langle \boldsymbol{u}, \boldsymbol{e}_j\rangle.$$

□

Theorem 3.21. *Let $\boldsymbol{V}$ be a subspace of $\boldsymbol{U}$. Fix $\boldsymbol{u}_0 \in \boldsymbol{U}$. Then*

$$\|\boldsymbol{u}_0 - P_{\boldsymbol{V}}\boldsymbol{u}_0\| \leq \|\boldsymbol{u}_0 - \boldsymbol{v}\|, \;\; \forall \boldsymbol{v} \in \boldsymbol{V},$$

and we have equality if and only if $\boldsymbol{v} = \boldsymbol{v}_0$. In other words, $P_{\boldsymbol{V}}\boldsymbol{u}_0$ is <u>the</u> *vector in $\boldsymbol{V}$ closest to $\boldsymbol{u}_0$.*

Proof. Set $\boldsymbol{v}_0 := P_{\boldsymbol{V}}\boldsymbol{u}_0$, $\boldsymbol{w}_0 := \boldsymbol{u}_0 - P_{\boldsymbol{V}}\boldsymbol{u}_0 \in \boldsymbol{V}^\perp$. Then for any $\boldsymbol{v} \in \boldsymbol{V}$ we have

$$\boldsymbol{u}_0 - \boldsymbol{v} = (\boldsymbol{v}_0 - \boldsymbol{v}) + \boldsymbol{w}_0.$$

Since $\boldsymbol{v}_0 \perp (\boldsymbol{u}_0 - \boldsymbol{v})$ we deduce from Pythagoras' Theorem that

$$\|\boldsymbol{u}_0 - \boldsymbol{v}\|^2 = \|\boldsymbol{v}_0 - \boldsymbol{v}\|^2 + \|\boldsymbol{w}_0\|^2 \geq \|\boldsymbol{w}_0\|^2 = \|\boldsymbol{u}_0 - P_{\boldsymbol{V}}\boldsymbol{u}_0\|^2.$$

Hence

$$\|\boldsymbol{u}_0 - P_{\boldsymbol{V}}\boldsymbol{u}_0\| \leq \|\boldsymbol{u}_0 - \boldsymbol{v}\|,$$

and we have equality if and only if $\boldsymbol{v} = \boldsymbol{v}_0 = P_{\boldsymbol{V}}\boldsymbol{u}_0$. □

Proposition 3.22. *Let $\boldsymbol{V}$ be a subspace of $\boldsymbol{U}$. Then*

$$P_{\boldsymbol{V}}^2 = P_{\boldsymbol{V}},$$

and

$$\|P_{\boldsymbol{V}}\boldsymbol{u}\| \leq \|\boldsymbol{u}\|, \;\; \forall \boldsymbol{u} \in \boldsymbol{U}.$$

Proof. By construction
$$P_{\boldsymbol{V}}\boldsymbol{v} = \boldsymbol{v}, \;\; \forall \boldsymbol{v} \in \boldsymbol{V}.$$
Hence
$$P_{\boldsymbol{V}}\big(P_{\boldsymbol{V}}\boldsymbol{u}\big) = P_{\boldsymbol{V}}\boldsymbol{u}, \;\; \forall \boldsymbol{u} \in \boldsymbol{U}$$
because $P_{\boldsymbol{V}}\boldsymbol{u} \in \boldsymbol{V}$.

Next, observe that for any $\boldsymbol{u} \in \boldsymbol{U}$ we have $P_{\boldsymbol{V}}\boldsymbol{u} \perp (\boldsymbol{u} - P_{\boldsymbol{V}}\boldsymbol{u})$ so that
$$\|\boldsymbol{u}\|^2 = \|P_{\boldsymbol{V}}\boldsymbol{u}\|^2 + \|\boldsymbol{u} - P_{\boldsymbol{V}}\boldsymbol{u}\|^2 \geq \|P_{\boldsymbol{V}}\boldsymbol{u}\|^2.$$
□

3.5. **Linear functionals and adjoints on Euclidean spaces.** Suppose that $\boldsymbol{U}$ is a finite dimensional $\mathbb{F}$-vector space, $\mathbb{F} = \mathbb{R}, \mathbb{C}$. The *dual* of $\boldsymbol{U}$, is the $\mathbb{F}$-vector space of *linear functionals* on $\boldsymbol{U}$, i.e., linear maps
$$\alpha : \boldsymbol{U} \to \mathbb{F}.$$
The dual of $\boldsymbol{U}$ is denoted by $\boldsymbol{U}^*$. The vector space $\boldsymbol{U}^*$ has the same dimension as $\boldsymbol{U}$ and thus they are isomorphic. However, *there is no distinguished isomorphism between these two vector spaces!*

We want to show that if we fix an inner product on $\boldsymbol{U}$, then we can construct in a concrete way an isomorphism between these spaces. Before we proceed with this construction let us first observe that there is a natural bilinear map
$$B : \boldsymbol{U}^* \times \boldsymbol{U} \to \mathbb{F}, \;\; B(\alpha, u) = \alpha(\boldsymbol{u}).$$

Theorem 3.23 (Riesz representation theorem: the real case)**.** *Suppose that $\boldsymbol{U}$ is a finite dimensional real vector space equipped with an inner product*
$$\langle -, - \rangle : \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{R}.$$
To any $\boldsymbol{u} \in \boldsymbol{U}$ we associate the linear functional $\boldsymbol{u}^* \in \boldsymbol{U}^*$ defined by the equality
$$B(\boldsymbol{u}^*, \boldsymbol{x}) = \boldsymbol{u}^*(\boldsymbol{x}) := \langle \boldsymbol{x}, \boldsymbol{u} \rangle, \;\; \forall \boldsymbol{x} \in \boldsymbol{U}.$$
Then the map $\mathcal{D} : \boldsymbol{U} \to \boldsymbol{U}^*$ given by
$$\boldsymbol{U} \ni \boldsymbol{u} \mapsto \boldsymbol{u}^* \in \boldsymbol{U}^*$$
is a linear isomorphism.

Proof. Let us show that the map $\mathcal{D}$ is linear.

For any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x} \in \boldsymbol{U}$ we have
$$\begin{aligned}(\boldsymbol{u} + \boldsymbol{v})^*(\boldsymbol{x}) &= \langle \boldsymbol{x}, \boldsymbol{u} + \boldsymbol{v} \rangle = \langle \boldsymbol{x}, \boldsymbol{u} \rangle + \langle \boldsymbol{x}, \boldsymbol{v} \rangle \\ &= \boldsymbol{u}^*(\boldsymbol{x}) + \boldsymbol{v}^*(\boldsymbol{x}) = (\boldsymbol{u}^* + \boldsymbol{v}^*)(\boldsymbol{x}),\end{aligned}$$
which show that
$$(\boldsymbol{u} + \boldsymbol{v})^* = \boldsymbol{u}^* + \boldsymbol{v}^*.$$
For any $\boldsymbol{u}, \boldsymbol{x} \in \boldsymbol{U}$ and any $t \in \mathbb{R}$ we have
$$(t\boldsymbol{u})^*(\boldsymbol{x}) = \langle \boldsymbol{x}, t\boldsymbol{u} \rangle = t\langle \boldsymbol{x}, \boldsymbol{u} \rangle = t\boldsymbol{u}^*(\boldsymbol{x}),$$
i.e., $(t\boldsymbol{u})^* = t\boldsymbol{u}^*$. This proved the claimed linearity. To prove that it is an isomorphism, we need to show that it is both injective and surjective.

Injectivity. Suppose that $\boldsymbol{u} \in \boldsymbol{U}$ is such that $\boldsymbol{u}^* = 0$. This means that $\boldsymbol{u}^*(\boldsymbol{x}) = 0$, $\forall \boldsymbol{x} \in \boldsymbol{U}$. If we let $\boldsymbol{x} = \boldsymbol{u}$ we deduce
$$0 = \boldsymbol{u}^*(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{u} \rangle = \|\boldsymbol{u}\|^2,$$
so that $\boldsymbol{u} = 0$.

Surjectivity. The dual of $\boldsymbol{U}$ has the same dimension as $\boldsymbol{U}$ and the rank nullity theorem implies that

$$\dim \boldsymbol{R}(\mathcal{D}) = \dim \boldsymbol{U} - \dim \ker \mathcal{D} = \dim \boldsymbol{U} = \dim \boldsymbol{U}^*.$$

Hence $\boldsymbol{R}(\mathcal{D}) = \boldsymbol{U}^*$. □

Theorem 3.24 (Riesz representation theorem: the complex case)**.** *Suppose that $\boldsymbol{U}$ is a finite dimensional complex vector space equipped with an inner product*

$$\langle -,- \rangle : \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{C}.$$

To any $\boldsymbol{u} \in \boldsymbol{U}$ we associate the linear functional $\boldsymbol{u}^* \in \boldsymbol{U}^*$ defined by the equality

$$B(\boldsymbol{u}^*, \boldsymbol{x}) = \boldsymbol{u}^*(\boldsymbol{x}) := \langle \boldsymbol{x}, \boldsymbol{u} \rangle, \ \ \forall \boldsymbol{x} \in \boldsymbol{U}.$$

Then the map $\mathcal{D} : \boldsymbol{U} \to \boldsymbol{U}^*$ given by

$$\boldsymbol{U} \ni \boldsymbol{u} \mapsto \boldsymbol{u}^* \in \boldsymbol{U}^*$$

is bijective and conjugate linear*, i.e., for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$ and any $z \in \mathbb{C}$ we have*

$$(\boldsymbol{u} + \boldsymbol{v})^* = \boldsymbol{u}^* + \boldsymbol{v}^*, \ \ (z\boldsymbol{u})^* = \bar{z}\boldsymbol{u}^*.$$

Proof. Let us first show that the map $\boldsymbol{u} \mapsto \boldsymbol{u}^*$ is conjugate linear.

For any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x} \in \boldsymbol{U}$ we have

$$(\boldsymbol{u} + \boldsymbol{v})^*(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{u} + \boldsymbol{v} \rangle = \langle \boldsymbol{x}, \boldsymbol{u} \rangle + \langle \boldsymbol{x}, \boldsymbol{v} \rangle$$
$$= \boldsymbol{u}^*(\boldsymbol{x}) + \boldsymbol{v}^*(\boldsymbol{x}) = (\boldsymbol{u}^* + \boldsymbol{v}^*)(\boldsymbol{x}),$$

which show that

$$(\boldsymbol{u} + \boldsymbol{v})^* = \boldsymbol{u}^* + \boldsymbol{v}^*.$$

For any $\boldsymbol{u}, \boldsymbol{x} \in \boldsymbol{U}$ and any $z \in \mathbb{C}$ we have

$$(z\boldsymbol{u})^*(\boldsymbol{x}) = \langle \boldsymbol{x}, z\boldsymbol{u} \rangle = \bar{z}\langle \boldsymbol{x}, \boldsymbol{u} \rangle = \bar{z}\boldsymbol{u}^*(\boldsymbol{x}),$$

i.e., $(z\boldsymbol{u})^* = \bar{z}\boldsymbol{u}^*$. This proved the claimed conjugate linearity. We now prove the bijectivity claim.

Injectivity. Suppose that $\boldsymbol{u} \in \boldsymbol{U}$ is such that $\boldsymbol{u}^* = 0$. This means that $\boldsymbol{u}^*(\boldsymbol{x}) = 0$, $\forall \boldsymbol{x} \in \boldsymbol{U}$. If we let $\boldsymbol{x} = \boldsymbol{u}$ we deduce

$$0 = \boldsymbol{u}^*(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{u} \rangle = \|\boldsymbol{u}\|^2,$$

so that $\boldsymbol{u} = 0$.

Surjectivity. We have to show that that for any $\alpha \in \boldsymbol{U}^*$ there exists $\boldsymbol{u} \in \boldsymbol{U}$ such that $\alpha = \boldsymbol{u}^*$.

Let $n = \dim \boldsymbol{U}$. Fix an *orthonormal basis* $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ of $\boldsymbol{U}$. The linear functional α is uniquely determined by its values on $\boldsymbol{e}_i$,

$$\alpha_i = \alpha(\boldsymbol{e}_i).$$

Define

$$\boldsymbol{u} := \sum_{k=1}^{n} \bar{\alpha}_k \boldsymbol{e}_k.$$

Then

$$\boldsymbol{u}^*(\boldsymbol{e}_i) = \langle \boldsymbol{e}_i, \boldsymbol{u} \rangle = \langle \boldsymbol{e}_i, \bar{\alpha}_1\boldsymbol{e}_1 + \cdots + \bar{\alpha}_i\boldsymbol{e}_i + \cdots + \bar{\alpha}_n\boldsymbol{e}_n \rangle$$
$$= \langle \boldsymbol{e}_i, \bar{\alpha}_1\boldsymbol{e}_1 \rangle + \cdots + \langle \boldsymbol{e}_i, \bar{\alpha}_i\boldsymbol{e}_i \rangle + \cdots + \langle \boldsymbol{e}_i, \bar{\alpha}_n\boldsymbol{e}_n \rangle = \langle \boldsymbol{e}_i, \bar{\alpha}_i\boldsymbol{e}_i \rangle = \alpha_i\langle \boldsymbol{e}_i, \boldsymbol{e}_i \rangle = \alpha_i.$$

Thus

$$u^*(\boldsymbol{e}_i) = \alpha(\boldsymbol{e}_i), \ \ \forall i = 1, \dots, n,$$

so that $\alpha = \boldsymbol{u}^*$. □

Suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are two $\mathbb{F}$-vector spaces equipped with inner products $\langle -,-\rangle_{\boldsymbol{U}}$ and respectively $\langle -,-\rangle_{\boldsymbol{V}}$. Next, assume that $T:\boldsymbol{U}\to\boldsymbol{V}$ is a linear map.

Theorem 3.25. *There exists a unique linear map $S:\boldsymbol{V}\to\boldsymbol{U}$ satisfying the equality*

$$\langle T\boldsymbol{u},\boldsymbol{v}\rangle_{\boldsymbol{V}}=\langle\boldsymbol{u},S\boldsymbol{v}\rangle_{\boldsymbol{U}},\;\;\forall\boldsymbol{u}\in\boldsymbol{U},\boldsymbol{v}\in\boldsymbol{V}. \tag{3.10}$$

This map is called the adjoint *of T with respect to the inner products $\langle -,-\rangle_{\boldsymbol{U}}$, $\langle -,-\rangle_{\boldsymbol{V}}$ and it is denoted by T^*. The equality (3.10) can then be rewritten*

$$\langle T\boldsymbol{u},\boldsymbol{v}\rangle_{\boldsymbol{V}}=\langle\boldsymbol{u},T^*\boldsymbol{v}\rangle_{\boldsymbol{U}},\;\;\langle T^*\boldsymbol{v},\boldsymbol{u}\rangle_{\boldsymbol{U}}=\langle\boldsymbol{v},T\boldsymbol{u}\rangle_{\boldsymbol{U}},\;\;\forall\boldsymbol{u}\in\boldsymbol{U},\;\;\boldsymbol{v}\in\boldsymbol{V}. \tag{3.11}$$

Proof. Uniqueness. Suppose there are two linear maps $S_1,S_2:\boldsymbol{V}\to\boldsymbol{U}$ satisfying (3.10). Thus

$$0=\langle\boldsymbol{u},S_1\boldsymbol{v}\rangle_{\boldsymbol{U}}-\langle\boldsymbol{u},S_2\boldsymbol{v}\rangle_{\boldsymbol{U}}=\langle\boldsymbol{u},(S_1-S_2)\boldsymbol{v}\rangle_{\boldsymbol{U}},\;\;\forall\boldsymbol{u}\in\boldsymbol{U},\boldsymbol{v}\in\boldsymbol{V}.$$

For fixed $\boldsymbol{v}\in\boldsymbol{V}$ we let $\boldsymbol{u}=(S_1-S_2)\boldsymbol{v}$ and we deduce from the above equality

$$0=\langle\,(S_1-S_2)\boldsymbol{v},(S_1-S_2)\boldsymbol{v}\,\rangle_{\boldsymbol{U}}=\|(S_1-S_2)\boldsymbol{v}\|^2_{\boldsymbol{U}},$$

so that $(S_1-S_2)\boldsymbol{v}=0$, for any $\boldsymbol{v}$ in $\boldsymbol{V}$. This shows that $S_1=S_2$, thus proving the uniqueness part of the theorem.

Existence. Any $\boldsymbol{v}\in\boldsymbol{V}$ defines a linear functional

$$L_{\boldsymbol{v}}:\boldsymbol{U}\to\mathbb{F},\;\;L_{\boldsymbol{v}}(\boldsymbol{u})=\langle T\boldsymbol{u},\boldsymbol{v}\rangle_{\boldsymbol{V}}.$$

Thus there exists a unique vector $S\boldsymbol{v}\in\boldsymbol{U}$ such that

$$L_{\boldsymbol{v}}=(S\boldsymbol{v})^*\Longleftrightarrow\langle T\boldsymbol{u},\boldsymbol{v}\rangle_{\boldsymbol{V}}=\langle\boldsymbol{u},S\boldsymbol{v}\rangle_{\boldsymbol{U}},\;\;\forall\boldsymbol{u}\in\boldsymbol{U}.$$

One can verify easily that the correspondence $\boldsymbol{V}\ni\boldsymbol{v}\mapsto S\boldsymbol{v}\in\boldsymbol{U}$ described above is a linear map; see Exercise 3.13. □

Example 3.26. Let $T:\boldsymbol{U}\to\boldsymbol{V}$ be as in the statement of Theorem 3.25. Assume $m=\dim_{\mathbb{F}}\boldsymbol{U}$, $n=\dim_{\mathbb{F}}\boldsymbol{V}$. Fix an *orthonormal* basis

$$\underline{\boldsymbol{e}}:=\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_m\}$$

of $\boldsymbol{U}$ and an *orthonormal* basis

$$\underline{\boldsymbol{f}}:=\{\boldsymbol{f}_1,\dots,\boldsymbol{f}_n\}$$

of $\boldsymbol{V}$. With respect to these bases the operator T is represented by an $n\times m$ matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m}\\ a_{21} & a_{22} & \cdots & a_{2m}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nm}\end{bmatrix},$$

while the adjoint operator $T^*:\boldsymbol{V}\to\boldsymbol{U}$ is represented by an $m\times n$ matrix

$$A^*=\begin{bmatrix} a^*_{11} & a^*_{12} & \cdots & a^*_{1n}\\ a^*_{21} & a^*_{22} & \cdots & a^*_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a^*_{m1} & a^*_{m2} & \cdots & a^*_{mn}\end{bmatrix}.$$

The j-th column of A describes the coordinates of the vector $T\boldsymbol{e}_j$ in the basis $\underline{\boldsymbol{f}}$,

$$T\boldsymbol{e}_j=a_{1j}\boldsymbol{f}_1+\cdots a_{nj}\boldsymbol{f}_n.$$

We deduce that

$$\langle T\boldsymbol{e}_j, \boldsymbol{f}_i\rangle_{\boldsymbol{V}} = a_{ij}.$$

Hence

$$\langle \boldsymbol{f}_i, T\boldsymbol{e}_j\rangle_{\boldsymbol{V}} = \overline{\langle T\boldsymbol{e}_j, \boldsymbol{f}_i\rangle_{\boldsymbol{V}}} = \bar{a}_{ij}.$$

On the other hand, the i-th column of A^* describes the coordinates of $T^*\boldsymbol{f}_i$ in the basis $\underline{\boldsymbol{e}}$ so that

$$T^*\boldsymbol{f}_i = a^*_{1i}\boldsymbol{e}_1 + \cdots + a^*_{mi}\boldsymbol{e}_m$$

and we deduce that

$$\langle T^*\boldsymbol{f}_i, \boldsymbol{e}_j\rangle = a^*_{ji}.$$

On the other hand, we have

$$\bar{a}_{ij} = \langle \boldsymbol{f}_i, T\boldsymbol{e}_j\rangle_{\boldsymbol{V}} \stackrel{(3.11)}{=} \langle T^*\boldsymbol{f}_i, \boldsymbol{e}_j\rangle = a^*_{ji}.$$

Thus, A^* is the *conjugate transpose* of A. In other words, *the entries of A^* are the conjugates of the corresponding entries of the transpose of A*,

$$A^* = \overline{A^\dagger}. \qquad \square$$

Definition 3.27. Let $\boldsymbol{U}$ be a finite dimensional Euclidean $\mathbb{F}$-space with inner product $\langle -, -\rangle$. A linear operator $T : \boldsymbol{U} \to \boldsymbol{U}$ is called *selfadjoint* or *symmetric* if $T = T^*$, i.e.,

$$\langle T\boldsymbol{u}_1, \boldsymbol{u}_2\rangle = \langle \boldsymbol{u}_1, T\boldsymbol{u}_2\rangle, \;\; \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{U}. \qquad \square$$

Example 3.28. (a) Consider the standard real Euclidean n-dimensional space $\mathbb{R}^n$. Any real $n \times n$ matrix A can be identified with a linear operator $T_A : \mathbb{R}^n \to \mathbb{R}^n$. The operator T_A is selfadjoint if and only if the matrix A is *symmetric*, i.e.,

$$a_{ij} = a_{ji}, \;\; \forall i, j$$

or, equivalently $A = A^\dagger =$ the transpose of A.

(b) Consider the standard complex Euclidean n-dimensional space $\mathbb{C}^n$. Any complex $n \times n$ matrix A can be identified with a line operator $T_A : \mathbb{C}^n \to \mathbb{C}^n$. The operator T_A is selfadjoint if and only if the matrix A is *Hermitian*, i.e.,

$$a_{ij} = \bar{a}_{ji}, \;\; \forall i, j$$

or, equivalently $A = A^* =$ the conjugate transpose of A.

(c) Suppose that $\boldsymbol{V}$ is a subspace of the finite dimensional Euclidean space $\boldsymbol{U}$. Then the orthogonal projection $P_{\boldsymbol{V}} : \boldsymbol{U} \to \boldsymbol{U}$ is a selfadjoint operator, i.e.,

$$\langle P_{\boldsymbol{V}}\boldsymbol{u}_1, \boldsymbol{u}_2\rangle = \langle \boldsymbol{u}_1, P_{\boldsymbol{V}}\boldsymbol{u}_2\rangle, \;\; \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{U}.$$

Indeed, let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{U}$. They decompose uniquely as

$$\boldsymbol{u}_1 = \boldsymbol{v}_1 + \boldsymbol{w}_1, \;\; \boldsymbol{u}_2 = \boldsymbol{v}_2 + \boldsymbol{w}_2, \;\; \boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{V}, \;\; \boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{V}^\perp.$$

Then $P_{\boldsymbol{V}}\boldsymbol{u}_1 = \boldsymbol{v}_1$ so that

$$\langle P_{\boldsymbol{V}}\boldsymbol{u}_1, \boldsymbol{u}_2\rangle = \langle \boldsymbol{v}_1, \boldsymbol{v}_2 + \boldsymbol{w}_2\rangle = \langle \boldsymbol{v}_1, \boldsymbol{v}_2\rangle + \underbrace{\langle \boldsymbol{v}_1, \boldsymbol{w}_2\rangle}_{\boldsymbol{w}_2 \perp \boldsymbol{v}_1} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2\rangle.$$

Similarly, $P_{\boldsymbol{V}}\boldsymbol{u}_2 = \boldsymbol{v}_2$ and we deduce

$$\langle \boldsymbol{u}_1, P_{\boldsymbol{V}}\boldsymbol{u}_2\rangle = \langle \boldsymbol{v}_1 + \boldsymbol{w}_1, \boldsymbol{v}_2\rangle = \langle \boldsymbol{v}_1, \boldsymbol{v}_2\rangle + \underbrace{\langle \boldsymbol{w}_1, \boldsymbol{v}_2\rangle}_{\boldsymbol{w}_1 \perp \boldsymbol{v}_2} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2\rangle. \qquad \square$$

Proposition 3.29. *Suppose that $\boldsymbol{U}$ is a finite dimensional Euclidean $\mathbb{F}$-space and $T:\boldsymbol{U}\to\boldsymbol{U}$ is a selfadjoint operator. Then*

$$\operatorname{spec} T\subset\mathbb{R}.$$

In other words, the eigenvalues of a selfadjoint operator are real.

Proof. Let λ be an eigenvalue of T and $\boldsymbol{u}\neq 0$ an eigenvector of T corresponding to the eigenvalue λ. We have

$$T\boldsymbol{u}=\lambda\boldsymbol{u}$$

so that

$$\lambda\|\boldsymbol{u}\|^2=\langle\lambda\boldsymbol{u},\boldsymbol{u}\rangle=\langle T\boldsymbol{u},\boldsymbol{u}\rangle\Rightarrow\lambda=\frac{1}{\|\boldsymbol{u}\|^2}\langle T\boldsymbol{u},\boldsymbol{u}\rangle.$$

On the other hand

$$\langle T\boldsymbol{u},\boldsymbol{u}\rangle=\overline{\langle\boldsymbol{u},T\boldsymbol{u}\rangle}.$$

Since T is selfadjoint we deduce

$$\langle\boldsymbol{u},T\boldsymbol{u}\rangle=\langle T\boldsymbol{u},\boldsymbol{u}\rangle$$

so that

$$\langle T\boldsymbol{u},\boldsymbol{u}\rangle=\overline{\langle T\boldsymbol{u},\boldsymbol{u}\rangle}.$$

Hence the inner product $\langle T\boldsymbol{u},\boldsymbol{u}\rangle$ is a real number. From the equality

$$\lambda=\frac{1}{\|\boldsymbol{u}\|^2}\langle T\boldsymbol{u},\boldsymbol{u}\rangle$$

we deduce that λ is a real number as well. □

Corollary 3.30. *If A is an $n\times n$ complex matrix such that $A=A^*$, then all the roots of the characteristic polynomial $P_A(\lambda)=\det(\lambda\mathbb{1}-A)$ are real.*

Proof. The roots of $P_A(\lambda)$ are the eigenvalues of the linear operator $T_A:\mathbb{C}^n\to\mathbb{C}^n$ defined by A. Since $A=A^*$ we deduce that T_A is selfadjoint with respect to the natural inner product on $\mathbb{C}^n$ so that all its eigenvalues are real. □

Theorem 3.31. *Suppose that $\boldsymbol{U},\boldsymbol{V}$ are two finite dimensional Euclidean $\mathbb{F}$-vector spaces and $T:\boldsymbol{U}\to\boldsymbol{V}$ is a linear operator. Then the following hold.*

(a) $(T^*)^*=T$.
(b) $\ker T=\boldsymbol{R}(T^*)^\perp$.
(c) $\ker T^*=\boldsymbol{R}(T)^\perp$.
(d) $\boldsymbol{R}(T)=(\ker T^*)^\perp$.
(e) $\boldsymbol{R}(T^*)=(\ker T)^\perp$.

Proof. (a) The operator $(T^*)^*$ is a linear operator $\boldsymbol{U}\to\boldsymbol{V}$. We need to prove that, for any $\boldsymbol{u}\in\boldsymbol{U}$ we have $\boldsymbol{x}:=(T^*)^*\boldsymbol{u}-T\boldsymbol{u}=0$.

Because $(T^*)^*$ is the adjoint of T^* we deduce from (3.11) that

$$\langle(T^*)^*\boldsymbol{u},\boldsymbol{v}\rangle_{\boldsymbol{V}}=\langle\boldsymbol{u},T^*\boldsymbol{v}\rangle_{\boldsymbol{U}}.$$

Because T^* is the adjoint of T we deduce from (3.11) that

$$\langle\boldsymbol{u},T^*\boldsymbol{v}\rangle_{\boldsymbol{U}}=\langle T\boldsymbol{u},\boldsymbol{v}\rangle_{\boldsymbol{V}}.$$

Hence, *for any* $\boldsymbol{v}\in\boldsymbol{V}$ we have

$$0=\langle(T^*)^*\boldsymbol{u},\boldsymbol{v}\rangle_{\boldsymbol{V}}-\langle T\boldsymbol{u},\boldsymbol{v}\rangle_{\boldsymbol{V}}=\langle\boldsymbol{x},\boldsymbol{v}\rangle_{\boldsymbol{V}}.$$

By choosing $\boldsymbol{v}=\boldsymbol{x}$ we deduce $\boldsymbol{x}=0$.

(b) We need to prove that

$$\boldsymbol{u} \in \ker T \Longleftrightarrow \boldsymbol{u} \perp \boldsymbol{R}(T^*).$$

Let $\boldsymbol{u} \in \ker T$, i.e., $T\boldsymbol{u} = 0$. To prove that $\boldsymbol{u} \perp \boldsymbol{R}(T^*)$ we need to show that $\boldsymbol{u} \perp T^*\boldsymbol{v}$, $\forall \boldsymbol{v} \in \boldsymbol{V}$. For $\boldsymbol{v} \in \boldsymbol{V}$ we have

$$\langle \boldsymbol{u}, T^*\boldsymbol{v} \rangle \stackrel{(3.11)}{=} \langle T\boldsymbol{u}, \boldsymbol{v} \rangle = 0,$$

so that $\boldsymbol{u} \perp T^*\boldsymbol{v}$ for any $\boldsymbol{v} \in \boldsymbol{V}$.

Conversely, let us assume that $u \perp T^*\boldsymbol{v}$ for any $\boldsymbol{v} \in \boldsymbol{V}$. We have to show that $\boldsymbol{x} = Tu = 0$. Observe that $\boldsymbol{x} \in \boldsymbol{V}$ so that $u \perp T^*\boldsymbol{x}$. We deduce

$$0 = \langle \boldsymbol{u}, T^*x \rangle = \langle T\boldsymbol{u}, \boldsymbol{x} \rangle = \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \|\boldsymbol{x}\|^2.$$

Hence $\boldsymbol{x} = 0$.

(c) Set $S := T^*$. From (b) we deduce

$$\ker T^* = \ker S = \boldsymbol{R}(S^*)^{\perp}.$$

From (a) we deduce $S^* = (T^*)^* = T$ and (c) is now obvious.

Part (d) follows from (c) and Corollary 3.18, while (e) follows from (b) and Corollary 3.18. □

Corollary 3.32. *Suppose that $\boldsymbol{U}$ is a finite dimensional Euclidean vector space, and $T : \boldsymbol{U} \to \boldsymbol{U}$ is a selfadjoint operator. Then*

$$\ker T = \boldsymbol{R}(T)^{\perp}, \quad \boldsymbol{R}(T) = (\ker T)^{\perp}, \quad \boldsymbol{U} = \ker T \oplus \boldsymbol{R}(T). \qquad \square$$

Example 3.33 (Least squares approximation)**.** Let $\boldsymbol{U}, \boldsymbol{V}$ be two real Euclidean spaces and $A : \boldsymbol{U} \to \boldsymbol{V}$ an injective linear operator. Given $\boldsymbol{v}_0 \in \boldsymbol{V}$ we seek $\boldsymbol{u}_0 \in \boldsymbol{U}$ such that $A\boldsymbol{u}_0$ is as close to $\boldsymbol{v}_0$ as possible.

Since $A\boldsymbol{u}_0 \in \boldsymbol{R}(A)$ we deduce that $A\boldsymbol{u}_0$ must be the orthogonal projection of $\boldsymbol{v}_0$ on $\boldsymbol{R}(A)$ so that

$$\boldsymbol{v}_0 - A\boldsymbol{u}_0 \in \boldsymbol{R}(A)^{\perp} = \ker A^*.$$

Hence $A^*\boldsymbol{v}_0 - A^*A\boldsymbol{u}_0 = 0$, i.e.,

$$A^*A\boldsymbol{u}_0 = A^*\boldsymbol{v}_0.$$

Now observe that the linear operator $A^*A : \boldsymbol{U} \to \boldsymbol{U}$ is bijective. It suffices to prove that it is injective. Indeed, if $A^*A\boldsymbol{u} = 0$ then

$$0 = (A^*A\boldsymbol{u}, \boldsymbol{u}) = (Au, Au) = \|A\boldsymbol{u}\|^2 = 0 \Rightarrow Au = 0.$$

Since A was assumed injective, we deduce $u = 0$. Hence A^*A is bijective and we deduce

$$\boxed{\boldsymbol{u}_0 = (A^*A)^{-1}A^*\boldsymbol{v}_0}.$$

Suppose for example that we are given three points in the plane

$$P_1 = (x_1, y_1), \quad P_2 = (x_2, y_2), \quad P_3 = (x_3, y_3).$$

We want to find a linear function $f(x) = b + mx$ that is as close as possible to these points, i.e.,

$$\big(y_1 - f(x_1)\big)^2 + \big(y_2 - f(x_1)\big)^2 + \big(y_3 - f(x_3)\big)^2$$

is as small as possible. Consider the operator $A : \mathbb{R}^2 \to \mathbb{R}^3$ given by the matrix

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix}.$$

Set

$$\boldsymbol{v}_0 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3.$$

We seek a vector

$$\boldsymbol{u}_0 = \begin{bmatrix} b \\ m \end{bmatrix} \in \mathbb{R}^2$$

such that

$$A\boldsymbol{u}_0 = \begin{bmatrix} b + mx_1 \\ b + mx_2 \\ b + mx_3 \end{bmatrix}$$

is as close to $\boldsymbol{v}_0$. The answer is

$$\boldsymbol{u}_0 = (A^*A)^{-1}A^*\boldsymbol{v}_0.$$

□

Proposition 3.34. *(a) Suppose that $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are finite dimensional Euclidean $\mathbb{F}$-spaces. If*

$$T : \boldsymbol{U} \to \boldsymbol{V}, \;\; S : \boldsymbol{V} \to \boldsymbol{W}$$

are linear operators, then

$$(ST)^* = T^*S^*.$$

(b) Suppose that $T : \boldsymbol{U} \to \boldsymbol{V}$ is a linear operator between two finite dimensional Euclidean $\mathbb{F}$-spaces. Then T is invertible if and only if the adjoint $T^* : \boldsymbol{V} \to \boldsymbol{U}$ is invertible. Moreover, if T is invertible, then

$$(T^{-1})^* = (T^*)^{-1}.$$

(c) Suppose that $S, T : \boldsymbol{U} \to \boldsymbol{V}$ are linear operators between two finite dimensional Euclidean $\mathbb{F}$-spaces. Then

$$(S+T)^* = S^* + T^*, \;\; (zS)^* = \bar{z}S^*, \;\; \forall z \in \mathbb{F}.$$ □

The proof is left to you as Exercise 3.16.

Proposition 3.35. *Suppose that $\boldsymbol{U}$ is a finite dimensional Euclidean $\mathbb{F}$-space and $S, T : \boldsymbol{U} \to \boldsymbol{U}$ are two selfadjoint operators. Then*

$$S = T \Longleftrightarrow \langle S\boldsymbol{u}, \boldsymbol{u} \rangle = \langle T\boldsymbol{u}, \boldsymbol{u} \rangle, \;\; \forall \boldsymbol{u} \in \boldsymbol{U}.$$

Proof. The implication " $\Rightarrow$ " is obvious so it suffices to prove that

$$\langle S\boldsymbol{u}, \boldsymbol{u} \rangle = \langle T\boldsymbol{u}, \boldsymbol{u} \rangle, \;\; \forall \boldsymbol{u} \in \boldsymbol{U} \Rightarrow S = T.$$

We set $A := S - T$. Then A is a selfadjoint operator and it suffices to show that

$$\langle A\boldsymbol{u}, \boldsymbol{u} \rangle = 0, \;\; \forall \boldsymbol{u} \in \boldsymbol{U} \Rightarrow A = 0.$$

We distinguish two cases.
(a) $\mathbb{F} = \mathbb{R}$. For any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$ we have

$$\begin{aligned} 0 &= \langle A(\boldsymbol{u}+\boldsymbol{v}), \boldsymbol{u}+\boldsymbol{v} \rangle = \langle A\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v} \rangle + \langle A\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v} \rangle \\ &= \underbrace{\langle A\boldsymbol{u}, \boldsymbol{u} \rangle}_{=0} + \langle A\boldsymbol{u}, \boldsymbol{v} \rangle + \langle A\boldsymbol{v}, \boldsymbol{u} \rangle + \underbrace{\langle A\boldsymbol{v}, \boldsymbol{v} \rangle}_{=0} \\ &= \langle A\boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{v}, A\boldsymbol{u} \rangle = 2\langle A\boldsymbol{u}, \boldsymbol{v} \rangle. \end{aligned}$$

Hence
$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = 0, \ \ \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}.$$
If in the above equality we let $\boldsymbol{v} = A\boldsymbol{u}$ we deduce
$$\|A\boldsymbol{u}\|^2 = 0, \ \ \forall \boldsymbol{u} \in \boldsymbol{U},$$
i.e., $A = 0$.
(b) $\mathbb{F} = \mathbb{C}$. For any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$ we have
$$\begin{aligned} 0 = \langle A(\boldsymbol{u}+\boldsymbol{v}), \boldsymbol{u}+\boldsymbol{v}\rangle &= \langle A\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v}\rangle + \langle A\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}\rangle \\ &= \underbrace{\langle A\boldsymbol{u}, \boldsymbol{u}\rangle}_{=0} + \langle A\boldsymbol{u}, \boldsymbol{v}\rangle + \langle A\boldsymbol{v}, \boldsymbol{u}\rangle + \underbrace{\langle A\boldsymbol{v}, \boldsymbol{v}\rangle}_{=0} \\ &= \langle A\boldsymbol{u}, \boldsymbol{v}\rangle + \langle \boldsymbol{v}, A\boldsymbol{u}\rangle = \langle A\boldsymbol{u}, \boldsymbol{v}\rangle + \overline{\langle A\boldsymbol{u}, \boldsymbol{v}\rangle} = 2\,\mathbf{Re}\langle A\boldsymbol{u}, \boldsymbol{v}\rangle. \end{aligned}$$
Hence
$$\mathbf{Re}\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = 0, \ \ \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}. \tag{3.12}$$
Similarly for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$ we have
$$\begin{aligned} 0 = \langle A(\boldsymbol{u}+\boldsymbol{i}\boldsymbol{v}), \boldsymbol{u}+\boldsymbol{i}\boldsymbol{v}\rangle &= \langle A\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{i}\boldsymbol{v}\rangle + \boldsymbol{i}\langle A\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{i}\boldsymbol{v}\rangle \\ &= \underbrace{\langle A\boldsymbol{u}, \boldsymbol{u}\rangle}_{=0} - \boldsymbol{i}\langle A\boldsymbol{u}, \boldsymbol{v}\rangle + \boldsymbol{i}\langle A\boldsymbol{v}, \boldsymbol{u}\rangle - \boldsymbol{i}^2\underbrace{\langle A\boldsymbol{v}, \boldsymbol{v}\rangle}_{=0} \\ &= -\boldsymbol{i}\langle A\boldsymbol{u}, \boldsymbol{v}\rangle + \boldsymbol{i}\langle \boldsymbol{v}, A\boldsymbol{u}\rangle = -\boldsymbol{i}\Big(\langle A\boldsymbol{u}, \boldsymbol{v}\rangle - \overline{\langle A\boldsymbol{u}, \boldsymbol{v}\rangle}\Big) = 2\,\mathbf{Im}\langle A\boldsymbol{u}, \boldsymbol{v}\rangle. \end{aligned}$$
Hence
$$\mathbf{Im}\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = 0, \ \ \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}. \tag{3.13}$$
Putting together (3.12) and (3.13) we deduce that
$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = 0, \ \ \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}.$$
If we now let $\boldsymbol{v} = A\boldsymbol{u}$ in the above equality we deduce as in the real case that $A\boldsymbol{u} = 0$, $\forall \boldsymbol{u} \in \boldsymbol{U}$. □

Definition 3.36. Let $\boldsymbol{U}, \boldsymbol{V}$ be two finite dimensional Euclidean $\mathbb{F}$-vector spaces. A linear operator $T : \boldsymbol{U} \to \boldsymbol{V}$ is called an *isometry* if for any $\boldsymbol{u} \in \boldsymbol{U}$ we have
$$\|T\boldsymbol{u}\|_{\boldsymbol{V}} = \|\boldsymbol{u}\|_{\boldsymbol{U}}.$$ □

Proposition 3.37. *A linear operator $T : \boldsymbol{U} \to \boldsymbol{V}$ between two finite dimensional Euclidean vector spaces is an isometry if and only if*
$$T^*T = \mathbb{1}_{\boldsymbol{U}}.$$ □

The proof is left to you as Exercise 3.17.

Definition 3.38. Let $\boldsymbol{U}$ be a finite dimensional Euclidean $\mathbb{F}$-space. A linear operator $T : \boldsymbol{U} \to \boldsymbol{U}$ is called an *orthogonal* operator if T is an isometry. We denote by $O(\boldsymbol{U})$ the space of orthogonal operators on $\boldsymbol{U}$. □

Proposition 3.39. *Let $\boldsymbol{U}$ be a finite dimensional Euclidean $\mathbb{F}$-space. Then*
$$T \in O(\boldsymbol{U}) \Longleftrightarrow T^*T = TT^* = \mathbb{1}_{\boldsymbol{U}}.$$

Proof. The implication " $\Leftarrow$ " follows from Proposition 3.37. To prove the opposite implication, assume that T is an orthogonal operator. Hence

$$\|T\boldsymbol{u}\| = \|\boldsymbol{u}\|, \;\; \forall \boldsymbol{u} \in \boldsymbol{U}.$$

This implies in particular that $\ker T = 0$, so that T is invertible. If we let $\boldsymbol{u} = T^{-1}\boldsymbol{v}$ in the above equality we deduce

$$\|T^{-1}\boldsymbol{v}\| = \|\boldsymbol{v}\|, \;\; \forall \boldsymbol{v} \in \boldsymbol{U}.$$

Hence T^{-1} is also an isometry so that

$$(T^{-1})^* T^{-1} = \mathbb{1}_{\boldsymbol{U}}.$$

Using Proposition 3.34 we deduce $(T^{-1})^* = (T^*)^{-1}$. Hence

$$(T^*)^{-1} T^{-1} = \mathbb{1}_{\boldsymbol{U}}.$$

Taking the inverses of both sides of the above equality we deduce

$$\mathbb{1}_{\boldsymbol{U}} = \big((T^*)^{-1} T^{-1} \big)^{-1} = (\, T^{-1} \,)^{-1} (\, (T^*)^{-1} \,)^{-1} = TT^*.$$

$\square$

3.6. **Exercises.**

Exercise 3.1. Prove the claims made in Example 3.2 (a), (b), (c). □

Exercise 3.2. Let $\big(\boldsymbol{U}, \langle -,-\rangle\big)$ be an Euclidean space.
(a) Show that for any $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{U}$ we have

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2+\|\boldsymbol{u}-\boldsymbol{v}\|^2 = 2\big(\,\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2\,\big).$$

(b) Let $\boldsymbol{u}_0,\dots,\boldsymbol{u}_n\in\boldsymbol{U}$. Prove that

$$\|\boldsymbol{u}_0-\boldsymbol{u}_n\|\leq\|\boldsymbol{u}_0-\boldsymbol{u}_1\|+\|\boldsymbol{u}_1-\boldsymbol{u}_2\|+\cdots+\|\boldsymbol{u}_{n-1}-\boldsymbol{u}_n\|.$$ □

Exercise 3.3. Show that for any complex numbers $z_1,\dots,z_n$ we have

$$\big(\,|z_1|+\cdots+|z_n|\,\big)^2\leq n\big(|z_1|^2+\cdots+|z_n|^2\,\big).$$

□

Exercise 3.4. Consider the space $\mathcal{P}_3$ of polynomials with real coefficients and degrees ≤ 3 equipped with the inner product

$$\langle P,Q\rangle=\int_0^1 P(x)Q(x)dx,\;\;\forall P,Q\in\mathcal{P}_3.$$

Construct an orthonormal basis of $\mathcal{P}_3$ by using the Gramm-Schmidt procedure applied to the basis of $\mathcal{P}_3$ given by

$$E_0(x)=1,\;\;E_1(x)=x,\;\;E_2(x)=x^2,\;\;E_3(x)=x^3.$$ □

Exercise 3.5. Fill in the missing details in the proof of Corollary 3.15. □

Exercise 3.6. Suppose that $T:\boldsymbol{U}\to\boldsymbol{U}$ is a linear operator on a finite dimensional complex Euclidean vector space. Prove that there exists an *orthonormal* basis of U, such that, in this basis T is represented by an upper triangular matrix. □

Exercise 3.7. Prove Proposition 3.16. □

Exercise 3.8. Consider the standard Euclidean space $\mathbb{R}^3$. Denote by $\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3$ the canonical orthonormal basis of $\mathbb{R}^3$ and by $\boldsymbol{V}$ the subspace generated by the vectors

$$\boldsymbol{v}_1=12\boldsymbol{e}_1+5\boldsymbol{e}_3,\;\;\boldsymbol{v}_2=\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3.$$

Find the matrix representing the orthogonal projection $P_{\boldsymbol{V}}:\mathbb{R}^3\to\mathbb{R}^3$ in the canonical basis $\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3$. □

Exercise 3.9. Consider the space $\mathcal{P}_3$ of polynomials with real coefficients and degrees ≤ 3. Find $P\in\mathcal{P}_3$ such that $p(0)=p'(0)=0$ and

$$\int_0^1\big|\,2+3x-p(x)\,\big|^2dx$$

is as small as possible.
Hint Observe that the set

$$\boldsymbol{V}=\big\{p\in\mathcal{P}_3;\;\;p(0)=p'(0)=0\,\big\}$$

is a subspace of $\mathcal{P}_3$. Then compute $P_{\boldsymbol{V}}$, the orthogonal projection onto $\boldsymbol{V}$ with respect to the inner product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx, \;\; p, q \in \mathcal{P}_3.$$

The answer will be $P_{\boldsymbol{V}}q$, $q = 2 + 3x \in \mathcal{P}_3$. □

Exercise 3.10. Suppose that $(\boldsymbol{U}, \langle -, -\rangle)$ is a finite dimensional real Euclidean space and $P : \boldsymbol{U} \to \boldsymbol{U}$ is a symmetric linear operator such that

$$P^2 = P, \;\; \|P\boldsymbol{u}\| \leq \|\boldsymbol{u}\|, \;\; \forall \boldsymbol{u} \in \boldsymbol{U}. \tag{P}$$

Show that there exists a subspace $\boldsymbol{V} \subset \boldsymbol{U}$ such that $P = P_{\boldsymbol{V}} =$ the orthogonal projection onto $\boldsymbol{V}$.

Hint: Let $\boldsymbol{V} = \boldsymbol{R}(P) = P(\boldsymbol{U})$, $\boldsymbol{W} := \ker P$. Using (P) argue by contradiction that $\boldsymbol{V} \subset \boldsymbol{W}^{\perp}$ and then conclude that $P = P_{\boldsymbol{V}}$. □

Exercise 3.11. Let $\mathcal{P}_2$ denote the space of polynomials with real coefficients and degree ≤ 2. Describe the polynomial $p_0 \in \mathcal{P}_2$ uniquely determined by the equalities

$$\int_{-\pi}^{\pi} \cos x q(x)dx = \int_{-\pi}^{\pi} p_0(x)q(x)dx, \;\; \forall q \in \mathcal{P}_2. \qquad \square$$

Exercise 3.12. Let k be a positive integer, and denote by $\mathcal{P}_k$ denote the space of polynomials with real coefficients and degree $\leq k$. For $k = 2, 3, 4$, describe the polynomial $p_k \in \mathcal{P}_k$ uniquely determined by the equalities

$$q(0) = \int_{-1}^{1} p_k(x)q(x)dx, \;\; \forall q \in \mathcal{P}_k. \qquad \square$$

Exercise 3.13. Finish the proof of Theorem 3.25. (Pay special attention to the case when $\mathbb{F} = \mathbb{C}$.) □

Exercise 3.14. Let $\mathcal{P}_2$ denote the space of polynomials with real coefficients and degree ≤ 2. We equip it with the inner product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

Consider the linear operator $T : \mathcal{P}_2 \to \mathcal{P}_2$ defined by

$$Tp = \frac{dp}{dx}, \;\; \forall p \in \mathcal{P}_2.$$

Describe the adjoint of T. □

Exercise 3.15. Let $\boldsymbol{U}$ be a finite dimensional Euclidean space and $T : \boldsymbol{U} \to \boldsymbol{U}$ a linear operator. Prove that the following are equivalent.

(i) T is surjective.
(ii) T^* is injective.
(iii) TT^* is injective.

□

Exercise 3.16. Prove Proposition 3.34. □

Exercise 3.17. Suppose that $\boldsymbol{U}$ is a finite dimensional Euclidean $\mathbb{F}$-space and $T : \boldsymbol{U} \to \boldsymbol{V}$ is an orthogonal operator. Prove that $\lambda \in \operatorname{spec}(T) \Rightarrow |\lambda| = 1$. □

Exercise 3.18. Let $\boldsymbol{U}$ be a finite dimensional Euclidean $\mathbb{F}$-vector space and $T : \boldsymbol{U} \to \boldsymbol{U}$ is a linear operator. Prove that the following statements are equivalent.

(i) The operator $T : \boldsymbol{U} \to \boldsymbol{U}$ is orthogonal.
(ii) $\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$, $\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$.
(iii) For any orthonormal basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ of $\boldsymbol{U}$, the collection $T\boldsymbol{e}_1, \dots, T\boldsymbol{e}_n$ is also an orthonormal basis of $\boldsymbol{U}$.

□

Exercise 3.19. Denote by $(-,-)$ the natural inner product on $\mathbb{R}^n$,

$$(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} x_i, y_i, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be the canonical basis of $\mathbb{R}^n$. Let $\boldsymbol{U}$ be a real euclidean space with inner product $\langle -,- \rangle$ and associated norm $\| - \|_U$. Fix vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ and consider the linear operator

$$T : \mathbb{R}^n \to \boldsymbol{U}, \quad T\boldsymbol{x} = x_1\boldsymbol{u}_1 + \cdots + x_n\boldsymbol{u}_n..$$

(i) Describe explicitly the adjoint operator $T^* : \boldsymbol{U} \to \boldsymbol{U}$.
(ii) Describe explicitly the operator $G = T^*T : \mathbb{R}^n \to \mathbb{R}^n$.
(iii) Show that
$$\forall \boldsymbol{x} \in \mathbb{R}^n : \langle Gx, x \rangle = \big\| x_1\boldsymbol{u}_1 + \cdots + x_n\boldsymbol{u}_n \big\|_U^2.$$

□

Exercise 3.20. Let H be a finite dimensional real Euclidean space with inner product $\langle -,- \rangle$. Given $n \in \mathbb{N}$ and $u_1, \dots, u_n \in H$ we define the *Gram determinant* of $u_1, \dots, u_n$ to be the determinant of the *Gramian* matrix

$$G(u_1, \dots, u_n) = \big[\, \langle u_i, u_j \rangle \,\big]_{1 \le i,j \le n}$$

(i) Fix any *orthonormal* basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ of $\operatorname{span}\{u_1, \dots, u_n\}$. Denote by A the $m \times n$ matrix with entries $a_{ij} = \langle \boldsymbol{e}_i, \boldsymbol{u}_j \rangle$, $1 \le i \le m$, $1 \le j \le n$. Show that
$$G(u_1, \dots, x_n) = A^\top A,$$
where $A^\top$ is the transpose of A. **Hint.** Define $T : \mathbb{R}^n \to H$, $T\boldsymbol{x} = \sum_{i=1}^n x_i u_i$ and use Exercise 3.19.
(ii) Prove that $\det G(u_1, \dots, u_n) \ge 0$ with equality if and only if the vectors $u_1, \dots, u_n$ are linearly dependent. **Hint** Use (i) to prove that all the e-values of G are nonnegative real numbers. Prove that $\ker A = \ker G$.
(iii) Suppose that $u_1, \dots, u_n$ are linearly independent and set $U := \operatorname{span}\{u_1, \dots, u_n\}$. Let $y \in H$ and denote by y_0 the orthogonal projection of y on U. Prove that
$$\|y - y_0\|^2 = \frac{\det G(y, u_1, \dots, u_n)}{\det G(u_1, \dots, u_n)}.$$
Hint. Observe that $\langle y - y_0, u_i \rangle = 0$, $\forall i$. Use Exercise 3.19. Compute $G(y - y_0, u_1, \dots, u_n)$.

□

4. Spectral theory of normal operators

4.1. Normal operators. Let $\boldsymbol{U}$ be a finite dimensional complex Euclidean space. A linear operator $T : \boldsymbol{U} \to \boldsymbol{U}$ is called *normal* if

$$T^*T = TT^*.$$

Example 4.1. (a) A selfadjoint operator $T : \boldsymbol{U} \to \boldsymbol{U}$ is a normal operator. Indeed, we have $T = T^*$ so that

$$T^*T = T^2 = TT^*.$$

(b) An orthogonal operator $T : \boldsymbol{U} \to \boldsymbol{U}$ is a normal operator. Indeed Proposition 3.39 implies that

$$T^*T = TT^* = \mathbb{1}_{\boldsymbol{U}}. \qquad \square$$

(c) If $T : \boldsymbol{U} \to \boldsymbol{U}$ is a normal operator and $\lambda \in \mathbb{C}$ then $\lambda\mathbb{1}_{\boldsymbol{U}} - T$ is also a normal operator. Indeed

$$(\lambda\mathbb{1}_{\boldsymbol{U}} - T)^* = (\lambda\mathbb{1}_{\boldsymbol{U}})^* - (T^*) = \bar{\lambda}\mathbb{1}_{\boldsymbol{U}} - T^*$$

and we have

$$(\lambda\mathbb{1}_{\boldsymbol{U}} - T)^*(\lambda\mathbb{1}_{\boldsymbol{U}} - T) = (\bar{\lambda}\mathbb{1}_{\boldsymbol{U}} - T^*) \cdot (\lambda\mathbb{1}_{\boldsymbol{U}} - T). \qquad \square$$

Proposition 4.2. *If $T : \boldsymbol{U} \to \boldsymbol{U}$ is a normal operator then so are any of its powers T^k, $k > 0$.*

Proof. To see this we invoke Proposition 3.34 and we deduce

$$(T^k)^* = (T^*)^k$$

Then

$$(T^k)^*T^k = \underbrace{(T^* \cdots T^*)}_{k} \cdot \underbrace{(T \cdots T)}_{k} = \underbrace{(T^* \cdots T^*)}_{k-1} \cdot \underbrace{(T \cdots T)}_{k} \cdot T^*$$

$$= \underbrace{(T^* \cdots T^*)}_{k-2} \cdot \underbrace{(T \cdots T)}_{k} \cdot (T^*)^2 = \cdots = \underbrace{(T \cdots T)}_{k}(T^*)^k = T^k(T^*)^k = T^k(T^k)^*.$$

$\square$

Definition 4.3. Let $\boldsymbol{U}$ be a finite dimensional Euclidean $\mathbb{F}$-space. A linear operator $T : \boldsymbol{U} \to \boldsymbol{U}$ is called *orthogonally diagonalizable* if there exists an orthonormal basis of $\boldsymbol{U}$ such that, in this basis, the operator T is represented by a diagonal matrix. $\square$

We can unravel a bit the above definition and observe that a linear operator T on an n-dimensional Euclidean $\mathbb{F}$-space is orthogonally diagonalizable if and only if there exists an orthonormal basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ and numbers $a_1, \dots, a_n \in \mathbb{F}$ such that

$$T\boldsymbol{e}_k = a_k\boldsymbol{e}_k, \;\; \forall k.$$

Thus the above basis is rather special: it is orthonormal, and it consists of eigenvectors of T. The numbers a_k are eigenvalues of T.

Note that the converse is also true. If $\boldsymbol{U}$ admits a n orthonormal basis consisting of eigenvectors of T, then T is orthogonally diagonalizable.

Proposition 4.4. *Suppose that $\boldsymbol{U}$ is a complex Euclidean space of dimension n and $T : \boldsymbol{U} \to \boldsymbol{U}$ is orthogonally diagonalizable. Then T is a normal operator.*

Proof. Fix an orthonormal basis

$$\underline{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$$

of $\boldsymbol{U}$ such that, in this basis, the operator T is represented by the diagonal matrix

$$D = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}, \quad a_1, \dots, a_n \in \mathbb{C}.$$

The computations in Example 3.26 show that the operator T^* is represented in the basis $\underline{\boldsymbol{e}}$ by the matrix D^*. Clearly

$$DD^* = D^*D = \begin{bmatrix} |a_1|^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & |a_2|^2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & |a_n|^2 \end{bmatrix}.$$

□

4.2. **The spectral decomposition of a normal operator.** We want to show that the converse of Proposition 4.4 is also true. This is a nontrivial and fundamental result of linear algebra.

Theorem 4.5 (Spectral Theorem for Normal Operators)**.** *Let $\boldsymbol{U}$ be an n-dimensional complex Euclidean space and $T : \boldsymbol{U} \to \boldsymbol{U}$ a* normal *operator. Then T is orthogonally diagonalizable, i.e., there exists an orthonormal basis of $\boldsymbol{U}$ consisting of eigenvectors of T.*

Proof. The key fact behind the Spectral Theorem is contained in the following auxiliary result.

Lemma 4.6. *Let $\lambda \in \operatorname{spec}(T)$. Then*

$$\ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T)^2 = \ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T).$$

We first complete the proof of the Spectral Theorem assuming the validity of the above result. Invoking Lemma 2.14 we deduce that

$$\ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T) = \ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T)^2 = \ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T)^3 = \cdots$$

so that the generalized eigenspace of T corresponding to an eigenvalue λ coincides with the eigenspace $\ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T)$,

$$E_\lambda(T) = \ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T).$$

Suppose that

$$\operatorname{spec}(T) = \{ \lambda_1, \dots, \lambda_\ell \}.$$

From Proposition 2.22 we deduce that

$$\boldsymbol{U} = \ker(\lambda_1 \mathbb{1}_{\boldsymbol{U}} - T) \oplus \cdots \oplus \ker(\lambda_\ell \mathbb{1}_{\boldsymbol{U}} - T). \tag{4.1}$$

The next crucial observation is contained in the following elementary result.

Lemma 4.7. *Suppose that λ, μ are two* distinct *eigenvalues of T, and $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$ are eigenvectors*

$$T\boldsymbol{u} = \lambda \boldsymbol{u}, \;\; T\boldsymbol{v} = \mu \boldsymbol{v}.$$

Then

$$T^*\boldsymbol{u} = \bar{\lambda}\boldsymbol{u}, \;\; T^*\boldsymbol{v} = \bar{\mu}\boldsymbol{v},$$

and

$$\boldsymbol{u} \perp \boldsymbol{v}.$$

Proof. Let $S_\lambda = T - \lambda \mathbb{1}_{\boldsymbol{U}}$ so that $S_\lambda \boldsymbol{u} = 0$. Note that $S_\lambda^* = T^* - \bar{\lambda}\mathbb{1}_{\boldsymbol{U}}$ so that we have to show that $S_\lambda^* \boldsymbol{u} = 0$. As explained in Example 4.1(c), the operator S_λ is normal. We deduce that

$$0 = S_\lambda^* S_\lambda \boldsymbol{u} = S_\lambda S_\lambda^* \boldsymbol{u}.$$

Hence

$$0 = \langle S_\lambda S_\lambda^* \boldsymbol{u}, \boldsymbol{u} \rangle = \langle S_\lambda^* \boldsymbol{u}, S_\lambda^* \boldsymbol{u} \rangle = \| S_\lambda^* \boldsymbol{u} \|^2.$$

This proves that $T^* \boldsymbol{u} = \bar{\lambda} \boldsymbol{u}$. A similar argument shows that $T^* \boldsymbol{v} = \bar{\mu} \boldsymbol{v}$.

From the equality $T\boldsymbol{u} = \lambda \boldsymbol{u}$ we deduce

$$\lambda \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle T\boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{u}, T^* \boldsymbol{v} \rangle = \langle \boldsymbol{u}, \bar{\mu} \boldsymbol{v} \rangle = \mu \langle \boldsymbol{u}, \boldsymbol{v} \rangle.$$

Hence

$$(\lambda - \mu) \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0.$$

Since $\lambda \neq \mu$ we deduce $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$. □

From the above result we conclude that the direct summands in (4.1) are mutually orthogonal. Set $d_k = \dim \ker(\lambda_k \mathbb{1}_U - T)$. We fix an orthonormal basis

$$\underline{\boldsymbol{e}}(k) = \boldsymbol{e}_1(k), \dots, \boldsymbol{e}_{d_k}(k)$$

of $\ker(\lambda_k \mathbb{1}_U - T)$. By construction, the vectors in this basis are eigenvectors of T. Since the spaces $\ker(\lambda_k \mathbb{1}_{\boldsymbol{U}} - T)$ are mutually orthogonal we deduce from (4.1) that the union of the orthonormal bases $\underline{\boldsymbol{e}}_k$ is an orthonormal basis of $\boldsymbol{U}$ consisting of eigenvectors of T. This completes the proof of the Spectral Theorem, modulo Lemma 4.6. □

Proof of Lemma 4.6. The operator $S = \lambda \mathbb{1}_{\boldsymbol{U}} - T$ is normal so that the conclusion of the lemma follows if we prove that for any normal operator S we have

$$\ker S^2 = \ker S.$$

Note that $\ker S \subset \ker S^2$ so that it suffices to show that $\ker S^2 \subset \ker S$.

Let $\boldsymbol{u} \in \boldsymbol{U}$ such that $S^2 \boldsymbol{u} = 0$. We have to show that $S\boldsymbol{u} = 0$. Note that

$$0 = (S^*)^2 S^2 \boldsymbol{u} = S^* S^* S S \boldsymbol{u} = S^* S S^* S \boldsymbol{u}.$$

Set $A := S^* S$. Note that A is selfadjoint, $A = A^*$ and we can rewrite the above equality as $0 = A^2 \boldsymbol{u}$. Hence

$$0 = \langle A^2 \boldsymbol{u}, \boldsymbol{u} \rangle = \langle A\boldsymbol{u}, A\boldsymbol{u} \rangle = \| A\boldsymbol{u} \|^2.$$

The equality $A\boldsymbol{u} = 0$ now implies

$$0 = \langle A\boldsymbol{u}, \boldsymbol{u} \rangle = \langle S^* S\boldsymbol{u}, \boldsymbol{u} \rangle = \langle S\boldsymbol{u}, S\boldsymbol{u} \rangle = \| S\boldsymbol{u} \|^2.$$

This completes the proof of Lemma 4.6. □

4.3. **The spectral decomposition of a real symmetric operator.** We begin with the real counterpart of Proposition 4.4

Proposition 4.8. *Suppose that $\boldsymbol{U}$ is a real Euclidean space of dimension n and $T:\boldsymbol{U}\to\boldsymbol{U}$ is orthogonally diagonalizable. Then T is a symmetric operator.*

Proof. Fix an orthonormal basis

$$\underline{\boldsymbol{e}} = (\boldsymbol{e}_1,\dots,\boldsymbol{e}_n)$$

of $\boldsymbol{U}$ such that, in this basis, the operator T is represented by the diagonal matrix

$$D = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0\\ 0 & a_2 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix},\quad a_1,\dots,a_n\in\mathbb{R}.$$

Clearly $D=D^*$ and the computations in Example 3.26 show that T is a selfadjoint operator. □

We can now state and prove the real counterpart of Theorem 4.5

Theorem 4.9 (Spectral Theorem for Real Symmetric Operators)**.** *Let $\boldsymbol{U}$ be an n-dimensional real Euclidean space and $T:\boldsymbol{U}\to\boldsymbol{U}$ a* symmetric *operator. Then T is orthogonally diagonalizable, i.e., there exists an orthonormal basis of $\boldsymbol{U}$ consisting of eigenvectors of T.*

Proof. We argue by induction on $n=\dim\boldsymbol{U}$. For $n=1$ the result is trivially true. We assume that the result is true for real symmetric operators action on Euclidean spaces of dimension $<n$ and we prove that it holds for a symmetric operator T on a real n-dimenasional Euclidean space $\boldsymbol{U}$.

To begin with let us observe that T has at least one real eigenvalue. Indeed, if we fix an orthonormal basis $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ of $\boldsymbol{U}$, then in this basis the operator T is represented by a symmetric $n\times n$ real matrix A. As explained in Corollary 3.30, all the roots of the characteristic polynomial $\det(\lambda\mathbb{1}-A)$ are real, and they coincide with the eigenvalues of T.

Fix one such eigenvalue $\lambda\in\operatorname{spec}(T)\subset\mathbb{R}$ and denote by E_λ the corresponding eigenspace

$$E_\lambda := \ker(\lambda\mathbb{1}-T)\subset\boldsymbol{U}.$$

Lemma 4.10. *The orthogonal complement $E_\lambda^\perp$ is an invariant subspace of T, i.e.,*

$$\boldsymbol{u}\perp E_\lambda \Rightarrow T\boldsymbol{u}\perp E_\lambda.$$

Proof. Let $\boldsymbol{u}\in E_\lambda^\perp$. We have to show that $T\boldsymbol{u}\perp E_\lambda$, i.e., $T\boldsymbol{u}\perp\boldsymbol{v}$, $\forall\boldsymbol{v}\in E_\lambda$. Given such a $\boldsymbol{v}$, we have

$$T\boldsymbol{v}=\lambda\boldsymbol{v}.$$

Next observe that $\boldsymbol{u}\perp\boldsymbol{v}$ since $\boldsymbol{u}\in E_\lambda^\perp$. Hence

$$\langle T\boldsymbol{u},\boldsymbol{v}\rangle = \langle\boldsymbol{u},T\boldsymbol{v}\rangle = \lambda\langle\boldsymbol{u},\boldsymbol{v}\rangle = 0.$$

□

The restriction S_λ of T to $E_\lambda^\perp$ is a symmetric operator $E_\lambda^\perp\to E_\lambda^\perp$ and the induction hypothesis implies that we can find and orthonormal basis of $E_\lambda^\perp$ such that, in this basis, the operator S_λ is represented by a diagonal matrix D_λ. Fix an arbitrary basis $\underline{\boldsymbol{e}}_\lambda$ of $E_\lambda^\perp$. The union of these $\underline{\boldsymbol{e}}_\lambda$'s is an orthonormal basis $\underline{\boldsymbol{f}}$ of $\boldsymbol{U}$. In this basis T is represented by the block matrix

$$\begin{bmatrix} \mathbb{1}_{E_\lambda} & 0\\ 0 & D_\lambda\end{bmatrix}.$$

The above matrix is clearly diagonal. □

4.4. **Nonnegative operators.** Suppose that $\boldsymbol{U}$ is a finite dimensional Euclidean $\mathbb{F}$-vector space.

Definition 4.11. A linear operator $T : \boldsymbol{U} \to \boldsymbol{U}$ is called *nonnegative* if the following hold.

(i) T is selfadjoint, $T^* = T$.
(ii) $\langle T\boldsymbol{u}, \boldsymbol{u}\rangle \geq 0$, for all $\boldsymbol{u} \in \boldsymbol{U}$.

The operator is called *positive* if it is nonnegative and $\langle T\boldsymbol{u}, \boldsymbol{u}\rangle = 0 \Longleftrightarrow \boldsymbol{u} = 0$. □

Example 4.12. Suppose that $T : \boldsymbol{U} \to \boldsymbol{U}$ is a linear operator. Then the operator $S := T^*T$ is nonnegative. Indeed, it is selfadjoint and

$$\langle S\boldsymbol{u}, \boldsymbol{u}\rangle = \langle T^*T\boldsymbol{u}, \boldsymbol{u}\rangle = \langle T\boldsymbol{u}, T\boldsymbol{u}\rangle = \|T\boldsymbol{u}\|^2 \geq 0.$$

Note that S is positive if and only $\ker S = 0$ so that S is injective. □

Definition 4.13. Suppose that $T : \boldsymbol{U} \to \boldsymbol{U}$ is a linear operator on the finite dimensional $\mathbb{F}$-space $\boldsymbol{U}$. A *square root* of T is a linear operator $S : \boldsymbol{U} \to \boldsymbol{U}$ such that $S^2 = T$. □

Theorem 4.14. *Let $T : \boldsymbol{U} \to \boldsymbol{U}$ be a linear operator on the n-dimensional Euclidean $\mathbb{F}$-space. Then the following statements are equivalent.*

(i) *The operator T is nonnegative.*
(ii) *The operator T is selfadjoint and all its eigenvalues are nonnegative.*
(iii) *The operator T admits a nonnegative square root.*
(iv) *The operator T admits a selfadjoint root.*
(v) *there exists an operator $S : \boldsymbol{U} \to \boldsymbol{U}$ such that $T = S^*S$.*

Proof. (i) ⇒ (ii) The operator T being nonnegative is also selfadjoint. Hence all its eigenvalues are real. If λ is an eigenvalue of T and $\boldsymbol{u} \in \ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T) \setminus 0$, then

$$\lambda\|\boldsymbol{u}\|^2 = \langle T\boldsymbol{u}, \boldsymbol{u}\rangle \geq 0.$$

This implies $\lambda \geq 0$.

(ii) ⇒ (iii) Since T is selfadjoint, there exists an orthonormal basis $\underline{\boldsymbol{e}} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ of $\boldsymbol{U}$ such that, in this basis, the operator T is represented by the diagonal matrix

$$A = \operatorname{Diag}(\lambda_1, \dots, .\lambda_n),$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of T. They are all nonnegative so we can form a new diagonal matrix

$$B = \operatorname{Diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}).$$

The matrix B defines a selfadjoint linear operator S on $\boldsymbol{U}$ which is represented by the matrix B in the basis $\underline{\boldsymbol{e}}$. More precisely

$$S\boldsymbol{e}_i = \sqrt{\lambda_i}\boldsymbol{e}_i, \ \ \forall i = 1, \dots, n.$$

If $\boldsymbol{u} = \sum_{i=1}^n u_i\boldsymbol{e}_i$, then

$$S\boldsymbol{u} = \sum_{i=1}^n \sqrt{\lambda_i} u_i \boldsymbol{e}_i, \ \ \langle S\boldsymbol{u}, \boldsymbol{u}\rangle = \sum_{i=1}^n \sqrt{\lambda_i}|u_i|^2 \geq 0.$$

Hence S is nonnegative. From the obvious equality $A = B^2$ we deduce $T = S^2$ so that S is nonnegative square root of T.

The implication (iii) $\Rightarrow$ (iv) is obvious because any nonnegative square root of T is automatically a selfadjoint square root. To prove the implication (iv) $\Rightarrow$ (v) observe that if S is a selfadjoint square root of T then

$$T = S^2 = S^*S.$$

The implication (v) $\Rightarrow$ (i) was proved in Example 4.12. $\square$

Proposition 4.15. *Let $\boldsymbol{U}$ be a finite dimension Euclidean $\mathbb{F}$-space. Then any nonnegative operator $T : \boldsymbol{U} \to \boldsymbol{U}$ admits a* unique *nonnegative square root.*

Proof. We have an orthogonal decomposition

$$\boldsymbol{U} = \bigoplus_{\lambda \in \operatorname{spec}(T)} \ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T)$$

so that any vector $\boldsymbol{u} \in \boldsymbol{U}$ can be written uniquely as

$$\boldsymbol{u} = \sum_{\lambda \in \operatorname{spec}(T)} \boldsymbol{u}_\lambda, \quad \boldsymbol{u}_\lambda \in \ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T). \tag{4.2}$$

Moreover

$$T\boldsymbol{u} = \sum_{\lambda \in \operatorname{spec}(T)} \lambda \boldsymbol{u}_\lambda.$$

Suppose that S is a nonnegative square root of T. If $\mu \in \operatorname{spec}(S)$ and $\boldsymbol{u} \in \ker(\mu \mathbb{1}_{\boldsymbol{U}} - S)$, then

$$T\boldsymbol{u} = S^2\boldsymbol{u} = S(S\boldsymbol{u}) = S(\mu\boldsymbol{u}) = \mu^2\boldsymbol{u}.$$

Hence

$$\mu^2 \in \operatorname{spec}(T)$$

and

$$\ker(\mu \mathbb{1}_{\boldsymbol{U}} - S) \subset \ker(\mu^2 \mathbb{1}_{\boldsymbol{U}} - T).$$

We have a similar orthogonal decomposition

$$\boldsymbol{U} = \bigoplus_{\mu \in \operatorname{spec}(S)} \ker(\mu \mathbb{1}_{\boldsymbol{U}} - S) \subset \bigoplus_{\mu \in \operatorname{spec}(S)} \ker(\mu^2 \mathbb{1}_{\boldsymbol{U}} - T) \subset \bigoplus_{\lambda \in \operatorname{spec}(T)} \ker(\lambda \mathbb{1}_{\boldsymbol{U}} - T) = \boldsymbol{U}.$$

This implies that

$$\operatorname{spec}(T) = \big\{ \mu^2;\ \mu \in \operatorname{spec}(S) \big\}, \quad \ker(\mu \mathbb{1}_{\boldsymbol{U}} - S) = \ker(\mu^2 \mathbb{1}_{\boldsymbol{U}} - T), \quad \forall \mu \in \operatorname{spec}(S).$$

Since all the eigenvalues of S are nonnegative we deduce that for any $\lambda \in \operatorname{spec}(T)$ we have $\sqrt{\lambda} \in \operatorname{spec}(S)$. Thus if $\boldsymbol{u}$ is decomposed as in (4.2),

$$\boldsymbol{u} = \sum_{\lambda \in \operatorname{spec}(T)} \boldsymbol{u}_\lambda,$$

then

$$S\boldsymbol{u} = \sum_{\lambda \in \operatorname{spec}(T)} \sqrt{\lambda}\boldsymbol{u}_\lambda.$$

The last equality determines S uniquely. $\square$

Definition 4.16. If T is a nonnegative operator, then its unique nonnegative square root is denoted by $\sqrt{T}$. $\square$

4.5. **Exercises.**

Exercise 4.1. Let $\boldsymbol{U}$ be a finite dimensional complex Euclidean vector space and $T : \boldsymbol{U} \to \boldsymbol{U}$ a normal operator. Prove that for any complex numbers $a_0, \dots, a_k$ the operator

$$a_0 \mathbb{1}_{\boldsymbol{U}} + a_1 T + \cdots + a_k T^k$$

is a normal operator. □

Exercise 4.2. Let $\boldsymbol{U}$ be a finite dimensional complex vector space and $\boldsymbol{T} : \boldsymbol{U} \to \boldsymbol{U}$ a *normal* operator. Show that the following statements are equivalent.

(i) The operator T is orthogonal.
(ii) If λ is an eigenvalue of T, then $|\lambda| = 1$. □

Exercise 4.3. (a) Prove that the product of two orthogonal operators on a finite dimensional Euclidean space is an orthogonal operator.
(b) Is it true that the product of two selfadjoint operators on a finite dimensional Euclidean space is also a selfadjoint operator? □

Exercise 4.4. Suppose that $\boldsymbol{U}$ is a finite dimensional Euclidean space and $P : \boldsymbol{U} \to \boldsymbol{U}$ is a linear operator such that $P^2 = P$. Show that the following statements are equivalent.

(i) P is the orthogonal projection onto a subspace $\boldsymbol{V} \subset \boldsymbol{U}$.
(ii) $P^* = P$.

□

Exercise 4.5. Suppose that $\boldsymbol{U}$ is a *real* Euclidean space of dimension $2k-1$, $k \in \mathbb{N}$, and $T : \boldsymbol{U} \to \boldsymbol{U}$ is an orthogonal operator. Prove that there exists a one-dimensional subspace $L \subset \boldsymbol{U}$ such that $TL \subset L$. □

Exercise 4.6. Suppose that $\boldsymbol{U}$ is a finite dimensional complex Euclidean space and $T : \boldsymbol{U} \to \boldsymbol{U}$ is a normal operator. Show that

$$\boldsymbol{R}(T) = \boldsymbol{R}(T^*).$$ □

Exercise 4.7. Does there exists a symmetric operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \;\; T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}?$$ □

Exercise 4.8. Show that a normal operator on a complex Euclidean space is selfadjoint if and only if all its eigenvalues are real. □

Exercise 4.9. Suppose that T is a normal operator on a complex Euclidean space $\boldsymbol{U}$ such that $T^7 = T^5$. Prove that T is selfadjoint and $T^3 = T$. □

Exercise 4.10. Let $\boldsymbol{U}$ be a finite dimensional complex Euclidean space and $T : \boldsymbol{U} \to \boldsymbol{U}$ be a selfadjoint operator. Suppose that there exists a vector $\boldsymbol{u}$, a complex number μ, and a number $\varepsilon > 0$ such that

$$\|T\boldsymbol{u} - \mu\boldsymbol{u}\| < \varepsilon\|\boldsymbol{u}\|.$$

Prove that there exists an eigenvalue λ of T such that $|\lambda - \mu| < \varepsilon$. **Hint.** Use an orthonormal basis that diagonalizes T. □

Exercise 4.11. Let $\boldsymbol{U}$ be a finite dimensional complex Euclidean space and $S, T : \boldsymbol{U} \to \boldsymbol{U}$ be two selfadjoint operators such that $ST = TS$. Prove that there exists an orthonormal basis of $\boldsymbol{U}$ so that in this basis both operators are represented by diagonal matrices. Give an example of an Euclidean space $\boldsymbol{U}$ and selfadjoint operators $S, T : \boldsymbol{U} \to \boldsymbol{U}$ such that there exists no orthonormal basis of $\boldsymbol{U}$ in which both operators are represented by diagonal matrices. □

Exercise 4.12. Let $\boldsymbol{U}$ be a finite dimensional complex Euclidean space and $T : \boldsymbol{U} \to \boldsymbol{U}$ a selfadjoint operator. Prove that for any $u \in \boldsymbol{U}$ the number $\langle Tu, u\rangle$ is real. Set

$$\lambda^* := \sup_{\|u\|=1} \langle Tu, u\rangle, \quad \lambda_* := \inf_{\|u\|=1} \langle Tu, u\rangle.$$

Prove that λ_* and λ^* are eigenvalues of T and $\operatorname{spec}(T) \subset [\lambda_*, \lambda^*]$. **Hint.** Use Theorem 4.9. □

Exercise 4.13. Suppose that $\boldsymbol{U}$ is a *real* Euclidean space and $J : \boldsymbol{U} \to \boldsymbol{U}$ is a linear operator such that

$$J = -J^* \text{ and } J^2 = -\mathbb{1}.$$

(i) Show that $\boldsymbol{U}$ is even dimensional $\dim_{\mathbb{R}} \boldsymbol{U} = 2m$, $m \in \mathbb{R}$. **Hint.** Show that J does not have real eigenvalues.
(ii) Prove that $\boldsymbol{u} \perp J\boldsymbol{u}$, for any $\boldsymbol{u} \in \boldsymbol{U}$.
(iii) Suppose that $\boldsymbol{u}_1 \in \boldsymbol{U}$ is a unit vector, i.e., $\|\boldsymbol{u}_1\| = 1$. Set $\boldsymbol{v}_1 = J\boldsymbol{u}_1$ and

$$\boldsymbol{U}_1 = \operatorname{span}_{\mathbb{R}}\{\boldsymbol{u}_1, \boldsymbol{v}_1\} \;\; \boldsymbol{V}_1 = \boldsymbol{U}_1^{\perp}.$$

Prove that $\boldsymbol{U}_1$ and $\boldsymbol{V}_1$ are invariant subspaces of J.
(iv) Prove that there exists an orthonormal basis $\boldsymbol{e}_1, \boldsymbol{f}_1, \dots, \boldsymbol{e}_m, \boldsymbol{f}_m$ of $\boldsymbol{U}$ such that in this basis J has the block form

$$\begin{bmatrix} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} \end{bmatrix}.$$

□

Exercise 4.14. Suppose that $\boldsymbol{U}$ is a real Euclidean space and $A : \boldsymbol{U} \to \boldsymbol{U}$ is a skew-symmetric operator, i.e., $A^* = -A$.

(i) Prove that $\mathbb{1} + A$ is invertible.
(ii) Show that the operator $T = (\mathbb{1} - A)(\mathbb{1} + A)^{-1}$ is orthogonal.
(iii) Prove that $\mathbb{1} + T$ is invertible and $A = (\mathbb{1} - T)(\mathbb{1} + T)^{-1}$.

□

Exercise 4.15. Suppose that $\boldsymbol{U}$ is a finite dimensional real vector space and $T : \boldsymbol{U} \to \boldsymbol{U}$ is linear operator. Prove that the following statements are equivalent.

(i) There exists an inner product $\langle -, -\rangle$ on $\boldsymbol{U}$ such that

$$\langle T\boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, T\boldsymbol{v}\rangle, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}.$$

(ii) All the eigenvalues of T are real, $\operatorname{spec}(T) \subset \mathbb{R}$, and the operator T is diagonalizable.

$\square$

5. Applications

5.1. **Symmetric bilinear forms.** Suppose that $\boldsymbol{U}$ is a finite dimensional *real* vector space. Recall that a symmetric bilinear form on $\boldsymbol{U}$ is a bilinear map

$$Q : \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{R}$$

such that

$$Q(\boldsymbol{u}_1, \boldsymbol{u}_2) = Q(\boldsymbol{u}_1, \boldsymbol{u}_2).$$

We denote by $\mathrm{Sym}(\boldsymbol{U})$ the space of symmetric bilinear forms on $\boldsymbol{U}$.

Suppose that $\underline{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ is a basis of the real vector space $\boldsymbol{U}$. This basis associated to any symmetric bilinear form $Q \in \mathrm{Sym}(\boldsymbol{U})$ a symmetric matrix

$$A = (a_{ij})_{1 \leq i,j \leq n}, \;\; a_{ij} = Q(\boldsymbol{e}_i, \boldsymbol{e}_j) = Q(\boldsymbol{e}_j, \boldsymbol{e}_i) = a_{ji}.$$

Note that the form Q is completely determined by the matrix A. Indeed if

$$\boldsymbol{u} = \sum_{i=1}^{n} u_i \boldsymbol{e}_i, \;\; \boldsymbol{v} = \sum_{j=1}^{n} v_j \boldsymbol{e}_j,$$

then

$$Q(\boldsymbol{u}, \boldsymbol{v}) = Q\left(\sum_{i=1}^{n} u_i \boldsymbol{e}_i, \sum_{j=1}^{n} v_j \boldsymbol{e}_j \right) = \sum_{i,j=1}^{n} u_i v_j Q(\boldsymbol{e}_i, \boldsymbol{e}_j) = \sum_{i,j=1}^{m} a_{ij} u_i v_j.$$

The matrix A is called *the symmetric matrix associated to the symmetric form Q in the basis $\underline{\boldsymbol{e}}$.*

Conversely any symmetric $n \times n$ matrix A defines a symmetric bilinear form $Q_A \in \mathrm{Sym}(\mathbb{R}^n)$ defined by

$$Q(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i,j=1}^{m} a_{ij} u_i v_j, \;\; \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n.$$

If $\langle -, - \rangle$ denotes the canonical inner product on $\mathbb{R}^n$, then we can rewrite the above equality in the more compact form

$$Q_A(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, A\boldsymbol{v} \rangle.$$

Definition 5.1. Let $Q \in \mathrm{Sym}(\boldsymbol{U})$ be a symmetric bilinear form on the finite dimensional real space $\boldsymbol{U}$. The *quadratic* form associated to Q is the function

$$\Phi_Q : \boldsymbol{U} \to \mathbb{R}, \;\; \Phi_Q(\boldsymbol{u}) = Q(\boldsymbol{u}, \boldsymbol{u}), \;\; \forall \boldsymbol{u} \in \boldsymbol{U}. \qquad \square$$

Observe that if $Q \in \mathrm{Sym}(\boldsymbol{U})$, then we have the *polarization identity*

$$Q(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{4}\Big(\Phi_Q(\boldsymbol{u} + \boldsymbol{v}) - \Phi_Q(\boldsymbol{u} - \boldsymbol{v}) \Big).$$

This shows that a symmetric bilinear form is completely determined by its associated quadratic form.

Proposition 5.2. *Suppose that $Q \in \mathrm{Sym}(\boldsymbol{U})$ and*

$$\underline{\boldsymbol{e}} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}, \;\; \underline{\boldsymbol{f}} = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\}$$

are two bases of $\boldsymbol{U}$. Denote by S the matrix describing the transition from the basis $\underline{\boldsymbol{e}}$ to the basis $\underline{\boldsymbol{f}}$. In other words, the j-th column of S describes the coordinates of $\boldsymbol{f}_j$ in the basis $\underline{\boldsymbol{e}}$, i.e.,

$$\boldsymbol{f}_j = \sum_{i=1}^{n} s_{ij} \boldsymbol{e}_i.$$

Denote by A (respectively B) the matrix associated to Q by the basis $\underline{\boldsymbol{e}}$ (respectively $\underline{\boldsymbol{f}}$). Then

$$B = S^{\dagger}AS, \tag{5.1}$$

where $S^{\dagger}$ denotes the transpose of S.

Proof. We have

$$b_{ij} = Q(\boldsymbol{f}_i, \boldsymbol{f}_j) = Q\left(\sum_{k=1}^{n} s_{ki}\boldsymbol{e}_k, \sum_{\ell=1}^{n} s_{\ell j}\boldsymbol{e}_\ell\right)$$
$$= \sum_{k,\ell=1}^{n} s_{ki}s_{\ell j}Q(\boldsymbol{e}_k, \boldsymbol{e}_\ell) = \sum_{k,\ell=1}^{n} s_{ki}a_{k\ell}s_{\ell j}$$

If we denote by $s^{\dagger}_{ij}$ the entries of the transpose matrix $S^{\dagger}$, $s^{\dagger}_{ij} = s_{ji}$, we deduce

$$b_{ij} = \sum_{k,\ell=1}^{n} s^{\dagger}_{ik}a_{k\ell}s_{\ell j}.$$

The last equality shows that b_{ij} is the (i,j)-entry of the matrix $S^{\dagger}AS$. □

☞ *We strongly recommend the reader to compare the change of base formula (5.1) with the change of base formula (2.1).*

Theorem 5.3. *Suppose that Q is a symmetric bilinear form on a finite dimensional real vector space $\boldsymbol{U}$. Then there exist at least one basis of $\boldsymbol{U}$ such that the matrix associated to Q by this basis is a diagonal matrix.*

Proof. We will employ the spectral theory of real symmetric operators. For this reason with fix an Euclidean inner product $\langle -,-\rangle$ on $\boldsymbol{U}$ and the choose a basis $\underline{\boldsymbol{e}}$ of $\boldsymbol{U}$ which is *orthonormal* with respect to the above inner product. We denote by A the symmetric matrix associated to Q by this basis, i.e.,

$$a_{ij} = Q(\boldsymbol{e}_i, \boldsymbol{e}_j).$$

This matrix defines a symmetric operator $T_A : \boldsymbol{U} \to \boldsymbol{U}$ by the formula

$$T_A\boldsymbol{e}_j = \sum_{i=1}^{n} a_{ij}\boldsymbol{e}_i.$$

Let us observe that

$$Q(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, T_A\boldsymbol{v}\rangle, \;\; \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}. \tag{5.2}$$

To verify the above equality we first notice that both sides of the above equalities are bilinear on $\boldsymbol{u}, \boldsymbol{v}$ so that it suffices to check the equality in the special case when the vectors $\boldsymbol{u}, \boldsymbol{v}$ belong to the basis $\underline{\boldsymbol{e}}$. We have

$$\langle \boldsymbol{e}_i, T_A\boldsymbol{e}_j\rangle = \left\langle \boldsymbol{e}_i, \sum_k a_{kj}\boldsymbol{e}_k \right\rangle = a_{ij} = Q(\boldsymbol{e}_i, \boldsymbol{e}_j).$$

The spectral theorem for real symmetric operators implies that there exists an orthonormal basis $\underline{\boldsymbol{f}}$ of $\boldsymbol{U}$ such that the matrix B representing T_A in this basis is diagonal. If S denotes the matrix describing the transition to the basis $\underline{\boldsymbol{e}}$ to the basis $\underline{\boldsymbol{f}}$ then the equality (2.1) implies that

$$B = S^{-1}AS.$$

Since the biases $\underline{\boldsymbol{e}}$ and $\underline{\boldsymbol{f}}$ are orthonormal we deduce from Exercise 3.18 that the matrix S is orthogonal, i.e., $S^\dagger S = SS^\dagger = \mathbb{1}$. Hence $S^{-1} = S^\dagger$ and we deduce that

$$B = S^* A S.$$

The above equality and Proposition 5.2 imply that the diagonal matrix B is also the matrix associated to Q by the basis $\underline{\boldsymbol{f}}$. □

☞ **Warning.** Suppose that Q is a symmetric bilinear form on $\boldsymbol{U}$. As shown above if we choose an inner product $(-,-)_1$ on $\boldsymbol{U}$, then we can identify Q with a symmetric operator A_1. If we choose another inner product $(-,-)_2$ on $\boldsymbol{U}$, then we can identify Q with a symmetric operator A_2. Typically *these two operators are different!* They may not even have the same spectra!

Theorem 5.4 (The law of inertia)**.** *Let $\boldsymbol{U}$ be a real vector space of dimension n and Q a symmetric bilinear form on $\boldsymbol{U}$. Suppose that $\underline{\boldsymbol{e}}$, $\underline{\boldsymbol{f}}$ are bases of $\boldsymbol{U}$ such that the matrices associated to Q by these bases are diagonal.*[3] *Then these matrices have the same number of positive (respectively negative, respectively zero) entries on their diagonal.*

Proof. We will show that these two matrices have the same number of positive elements and the same number of negative entries on their diagonals. Automatically then they must have the same number of trivial entries on their diagonals.

We take care of the positive entries first. Denote by A the matrix associated to Q by $\underline{\boldsymbol{e}}$ and by B the matrix associated by $\underline{\boldsymbol{f}}$. We denote by p the number of positive entries on the diagonal of A and by q the number of positive entries on the diagonal of B. We have to show that $p = q$. We argue by contradiction and we assume that $p \neq q$, say $p > q$.

We can label the elements in the basis $\underline{\boldsymbol{e}}$ so that

$$a_{ii} = Q(\boldsymbol{e}_i, \boldsymbol{e}_i) > 0, \;\; \forall i \leq p, \;\; a_{jj} = Q(\boldsymbol{e}_j, \boldsymbol{e}_j) \leq 0, \;\; \forall j > p. \tag{5.3}$$

Observe that since A is diagonal we have

$$Q(\boldsymbol{e}_i, \boldsymbol{e}_j) = 0, \;\; \forall i \neq j. \tag{5.4}$$

Similarly we can label the elements in the basis $\underline{\boldsymbol{f}}$ so that

$$b_{kk} = Q(\boldsymbol{f}_k, \boldsymbol{f}_k) > 0, \;\; \forall k \leq q, \;\; b_{\ell\ell} = Q(\boldsymbol{f}_\ell, \boldsymbol{f}_\ell) \leq 0, \;\; \forall \ell > q. \tag{5.5}$$

Since B is diagonal we have

$$Q(\boldsymbol{f}_i, \boldsymbol{f}_j) = 0, \;\; \forall i \neq j. \tag{5.6}$$

Denote $\boldsymbol{V}$ the subspace spanned by the vectors $\boldsymbol{e}_i$, $i = 1, \dots, p$, and by $\boldsymbol{W}$ the subspace spanned by the vectors $\boldsymbol{f}_{q+1}, \dots, \boldsymbol{f}_n$. From the equalities (5.3), (5.4), (5.5), (5.6) we deduce that

$$Q(\boldsymbol{v}, \boldsymbol{v}) > 0, \;\; \forall \boldsymbol{v} \in \boldsymbol{V} \setminus 0, \tag{5.7a}$$

$$Q(\boldsymbol{w}, \boldsymbol{w}) \leq 0, \;\; \forall \boldsymbol{w} \in \boldsymbol{W} \setminus 0. \tag{5.7b}$$

On the other hand, we observe that $\dim \boldsymbol{V} = p$, $\dim \boldsymbol{W} = n - q$. Hence

$$\dim \boldsymbol{V} + \dim \boldsymbol{W} = n + p - q > n > \dim \boldsymbol{U}$$

so that there exists a vector

$$\boldsymbol{u} \in (\boldsymbol{V} \cap \boldsymbol{W}) \setminus 0.$$

The vector $\boldsymbol{u}$ cannot simultaneously satisfy both inequalities (5.7a) and (5.7b). This contradiction implies that $p = q$.

[3] Such a bases are called *diagonalizing* bases of Q.

Using the above argument for the form $-Q$ we deduce that A and B have the same number of negative elements on their diagonals. □

The above theorem shows that no matter what diagonalizing basis of Q we choose, the diagonal matrix representing Q in that basis will have the same number of positive negative and zero elements on its diagonal. We will denote these common numbers by $\mu_+(Q)$, $\mu_-(Q)$ and respectively $\mu_0(Q)$. These numbers are called the *indices of inertia* of the symmetric form Q. The integer $\mu_-(Q)$ is called *Morse index* of the symmetric form Q and the difference

$$\sigma(Q) = \mu_+(Q) - \mu_-(Q).$$

is called the *signature* of the form Q.

Definition 5.5. A symmetric bilinear form $Q :\in \mathrm{Sym}(U)$ is called *positive definite* is

$$Q(\boldsymbol{u}, \boldsymbol{u}) > 0, \;\; \forall \boldsymbol{u} \in \boldsymbol{U} \setminus 0.$$

It is called *positive semidefinite* if

$$Q(\boldsymbol{u}, \boldsymbol{u}) \geq 0, \;\; \forall \boldsymbol{u} \in \boldsymbol{U}.$$

It is called *negative (semi)definite* if $-Q$ is positive (semi)definite. □

We observe that a symmetric bilinear form on an n-dimensional real space $\boldsymbol{U}$ is positive definite if and only if $\mu_+(Q) = n = \dim \boldsymbol{U}$.

Definition 5.6. A real symmetric $n \times n$ matrix A is called *positive definite* if and only if the associated symmetric bilinear form $Q_A \in \mathrm{Sym}(\mathbb{R}^n)$ is positive definite. □

5.2. **Stochastic matrices.** First some terminology. Let $A \in M_{m\times n}(\mathbb{C})$,

$$A = (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}.$$

We. introduce the following notation and conventions.

- We denote by $|A|$ the matrix

$$|A| = (|a_{ij}|)_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} \in M_{m\times n}(\mathbb{R}).$$

- We denote by $R_i(A)$ the i-th row of A

$$R_i(A) = \big[\, a_{i1}, a_{i2}, \dots, a_{in} \,\big].$$

- We denote by $C_j(A)$ the j-th column of A

$$c_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

- We write $A \geq 0$ to indicate that $a_{ij} \geq 0$, $\forall i, j$. Wer write $A > 0$ to indicate that that $a_{ij} > 0$, $\forall i, j$.

If $A, B \in M_{m\times n}(\mathbb{R})$, then

$$A \geq B \stackrel{def}{\Longleftrightarrow} A - B \geq 0$$

and

$$A > B \stackrel{def}{\Longleftrightarrow} A - B > 0.$$

We will refer to $1 \times n$ matrices as *rows* or *weights* of dimension n and to $n \times 1$ matrices as *columns* or *observables* of dimension n. Observe that if R and C are respectively a row and a column of the same dimension then $R \cdot C$ is a scalar,

Definition 5.7. A weight

$$w = \begin{bmatrix} w_1, \dots, w_n \end{bmatrix}$$

is said to be a *distribution* if it is nonnegative $w \geq 0$ and

$$w_1 + \cdots + w_n = 1,$$

The distribution w is called positive if $w > 0$. □

We ought to explain the above terminology. Suppose that we have a box with a large number of balls that come in n colors. We denote by p_i the proportion of balls of color i. Then the row

$$\pi := \begin{bmatrix} p_1, \dots, p_n \end{bmatrix}$$

is a positive distribution: the distribution of colors in the box. The number p_i is the probably that a randomly drawn ball has color 1.

One can think of an observable as a numerical attribute associate to a color, e.g., the value or the mass of a ball of a certain color. If

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

is an observable: v_i is the "value" of a ball of color i. The scalar

$$\pi \cdot \boldsymbol{v} = \sum_{k=1}^{n} p_k v_k$$

is the expected value of a randomly drawn ball.

Set

$$\vec{\mathbf{1}} := \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Observe that a weight $w = \begin{bmatrix} w_1, \dots, w_n \end{bmatrix}$ is a distribution iff $w \geq 0$ and

$$w \cdot \vec{\mathbf{1}} = w_1 + \cdots + w_n = 1.$$

Definition 5.8. A matrix $A \in M_N(\mathbb{R})$ is called *stochastic* if each of its rows is a distribution, i.e., $A \geq 0$ and

$$a_{i1} + \cdots + a_{iN} = 1, \ \ \forall 1 \leq i \leq N.$$

□

Observe that a nonnegative matrix A is stochastic iff

$$A\vec{\mathbf{1}} = \vec{\mathbf{1}}.$$

Indeed

$$A\vec{\mathbf{1}} = \begin{bmatrix} R_1(A) \cdot \vec{\mathbf{1}} \\ \vdots \\ R_1(A) \cdot \vec{\mathbf{1}} \end{bmatrix}.$$

We have thus proved the following result.

Proposition 5.9. *If $A \in M_N(\mathbb{R})$ is a stochastic matrix then $1 \in \operatorname{spec}(A)$.* □

Lemma 5.10. *Let $A, B \in M_N(\mathbb{R})$ be stochastic matrices.*

(i) *If $w = \left[\, w_1, \dots, w_N \,\right]$ is a distribution then $w \cdot A$ is also a distribution.*

(ii) *$A \cdot B$ is a stochastic matrix.*

□

Proof. (i) Set $\pi := w \cdot A = \left[\, \pi_1, \dots, \pi_N \,\right|$ where

$$\pi_i = w \cdot C_i(A) = \sum_j w_j a_{ji}.$$

Then obviously $\pi \geq 0$ and

$$\sum_i \pi_i = \sum_i \sum_i w_j a_{ji} = \sum_j \sum_i w_j a_{ji} = \sum_i w_j \underbrace{\left(\sum_i a_{ji} \right)}_{=1}$$

$$= \sum_j w_j = 1.$$

Hence π is a distribution.

(ii) We have

$$A \cdot B = \begin{bmatrix} R_1(A \cdot B) \\ \vdots \\ R_N(A \cdot B) \end{bmatrix} = \begin{bmatrix} R_1(A) \cdot B \\ \vdots \\ R_N(A) \cdot B \end{bmatrix}.$$

We deduce from (i) that $R_i(A) \cdot B$ is a distribution for any i since the row $R_i(A)$ is a distribution and B is stochastic. □

To proceed further we need to introduce a but more terminology. For any N dimensional real observable

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathbb{R}^N$$

we set

$$\max \boldsymbol{u} = \max_{1 \leq k \leq N} u_k, \quad \min \boldsymbol{u} = \min_{1 \leq k \leq N} u_k,$$

Lemma 5.11. *If w is a N dimensional distribution and $\mathbf{u}$ is an N-dimensional observable real observable, then*

$$\min \boldsymbol{u} \leq w \cdot \boldsymbol{u} \leq \max \boldsymbol{u}. \tag{5.8}$$

If w is positive and $\min \boldsymbol{u} \neq \max \boldsymbol{u}$, then the above inequalities are strict.

Proof. Let

$$w = \left[\, w_1, \dots, w_N \,\right]$$

Then

$$w \cdot \boldsymbol{u} = w_1 u_1 + w_2 u_2 + \cdots + w_N u_N$$

$$\leq w_1(\max \boldsymbol{u}) + \cdots + w_N(\max \boldsymbol{u}) = (w_1 + \cdots + w_N) \max \boldsymbol{u} = \max \boldsymbol{u}.$$

If $w > 0$ and $\min \boldsymbol{u} < \max \boldsymbol{u}$, then there exists i such that $w_i u_i < w_i \max \boldsymbol{u}$ so

$$w \cdot \boldsymbol{u} < \max \boldsymbol{u}.$$

The lower inequality is dealt with in a similar fashion. □

Lemma 5.12. *Suppose that w is a distribution and $\boldsymbol{u}$ is a complex observable of the same dimension. Then*

$$\big|\, w \cdot \boldsymbol{u} \,\big| \leq w \cdot \big|\, \boldsymbol{u} \,\big| \leq \max \big|\, \boldsymbol{u} \,\big|. \tag{5.9}$$

Moreover, if $w > 0$, then we have equality $\big|\, w \cdot \boldsymbol{u} \,\big| = \max \big|\, \boldsymbol{u} \,\big|$ iff $\boldsymbol{u}$ is a multiple of $\vec{\mathbf{1}}$.

Proof. We have

$$\big|\, w \cdot \boldsymbol{u} \,\big| = \big|\, w_1 u_1 + \cdots + w_N u_N \,\big| \leq w_1 |u_1| + \cdots + w_N |u_N| = w \cdot \big|\, \boldsymbol{u} \,\big| \stackrel{(5.8)}{\leq} \max \big|\, \boldsymbol{u} \,\big|.$$

If $w > 0$ and we have equality, then we deduce from Lemma 5.11 that

$$|u_1| = \cdots = |u_N| = \max \big[\, \boldsymbol{u} \,\big|.$$

Thus, the n complex numbers $u_1, \ldots, u_N$ are situated on a circle of radius $r = \max \big|\, \boldsymbol{u} \,\big|$. If two of them are different, say $u_1 \neq u_2$, then the point

$$z = \frac{w_1}{w_1 + w_2} u_1 + \frac{w_2}{w_1 + w_2} u_2,$$

is in the interior of the the line segment connecting these two points so z lies strictly in the interior of the circle of radius r", $|z| < r$ so that

$$\big|\, w_1 u_1 + w_2 u_2 \,\big| = (w_1 + w_2)|z| < (w_1 + w_2) r.$$

Hence

$$\begin{aligned} \big|\, w \cdot \boldsymbol{u} \,\big| &\leq \big|\, w_1 u_1 + w_2 u_2 \,\big| + w_3 |u_3| + \cdots w_N |u_N| \\ &< (w_1 + w_2) r + w_3 |u_3| + \cdots w_N |u_N| = r. \end{aligned}$$

Thus if $\big|\, w \cdot \boldsymbol{u} \,\big| = \max \big|\, \boldsymbol{u} \,\big|$, then $u_1 = u_2 = \cdots = u_N$, i.e., $\boldsymbol{u}$ is a multiple of $\vec{\mathbf{1}}$. □

Corollary 5.13. *Let $A \in M_N(\mathbb{R})$ be a stochastic matrix. If $\lambda \in \operatorname{spec}(A)$, then $|\lambda| \leq 1$.*

Proof. Suppose

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathbb{C}^N \setminus \{0\}.$$

be an eigenvector corresponding to the eigenvalue λ. Then $\max \big|\, \boldsymbol{u} \,\big| > 0$. Note that

$$\lambda \boldsymbol{u} = A \boldsymbol{u} = \begin{bmatrix} R_1(A) \cdot \boldsymbol{u} \\ \vdots \\ R_N(A) \cdot \boldsymbol{u} \end{bmatrix}.$$

There exists k such that $|u_k| = \max \big|\, \boldsymbol{u} \,\big]$. Then

$$|\lambda| \cdot \max \big|\, \boldsymbol{u} \,\big| = |\lambda| \cdot |u_k| = \big|\, R_k(A) \cdot \boldsymbol{u} \,\big| \stackrel{(5.9)}{\leq} \max \big|\, \boldsymbol{u} \,\big|.$$

Hence

$$|\lambda| \leq 1.$$

□

Theorem 5.14. *If $A \in M_N(\mathbb{R})$ is a* positive *stochastic matrix then the following hold.*

(i) 1 *is a simple eigenvalue of A and* $\ker\big(\mathbb{1} - A\big) = \mathrm{span}(\vec{\mathbf{1}})$.

(ii) *If $\lambda \in \mathrm{spec}(A) \setminus \{1\}$, then $|\lambda| < 1$.*

Proof. We have to prove that 1 has algebraic multiplicity 1. Let us first show that $\ker(\mathbb{1} - A)$ is spanned by $\vec{\mathbf{1}}$ and thus it is 1-dimensional. Let

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \ker(\mathbb{1} - A).$$

There exists k such that $u_k = \max \nu$. Form the equality $\boldsymbol{u} = A\boldsymbol{u}$ we deduce

$$\max \boldsymbol{u} = u_k = R_k(A) \cdot \boldsymbol{u} \stackrel{5.8)}{\leq} \max \boldsymbol{u}.$$

Hence $R_k(A) \cdot \boldsymbol{u} = \max \boldsymbol{u}$ we deduce from Lemma 5.11 that $u_1 = \cdots = u_N$ so $\boldsymbol{u}$ is a multiple of $\vec{\mathbf{1}}$.

We will show that $\ker(\mathbb{1} - A)^2 = \ker(\mathbb{1} - A)$. Let V denote the subspace of $\mathbb{R}^N$ consisting of observable $\boldsymbol{v}$ such that

$$v_1 + \cdots + v_N = 0.$$

Any $\boldsymbol{u} \in \mathbb{R}^n$ decomposes uniquely as a sum $c\vec{\mathbf{1}} + \boldsymbol{v}$, $\boldsymbol{v} \in V$. If $\boldsymbol{u} \in \ker(\mathbb{1} - A)^2$ then $(\mathbb{1} - A)\boldsymbol{u} \in \ker(\mathbb{1} - A)$. We have

$$(\mathbb{1} - A)\boldsymbol{u} = (\mathbb{1} - A)\big(c\vec{\mathbf{1}} + \boldsymbol{v}\big) = \boldsymbol{v} - A\boldsymbol{v}.$$

Hence $\boldsymbol{v} - A\boldsymbol{v} \in \ker(\mathbb{1} - A)$ so $\boldsymbol{v} - A\boldsymbol{v}$ is a multiple of $\vec{\mathbf{1}}$, say $\boldsymbol{v} - A\boldsymbol{v} = c\vec{\mathbf{1}}$, $c \geq 0$. We will argue by contradiction that $\boldsymbol{v} = 0$. Suppose $\boldsymbol{v} \neq 0$. Since

$$v_1 + \cdots + v_N = 0$$

we deduce $\min \boldsymbol{v} < 0$. Fix k such that $v_k = \min \boldsymbol{v}$. From the equality $\boldsymbol{v} - A\boldsymbol{v} = c\vec{\mathbf{1}}$ we deduce

$$c = v_k - R_k(A)\boldsymbol{v}.$$

Since $A > 0$ we deduce from Lemma 5.11 that

$$0 \leq c = v_k - R_k(A)\boldsymbol{u} < v_k - \min \boldsymbol{v} = 0.$$

Let $\lambda \in \mathrm{spec}(A) \setminus \{1\}$. Then $|\lambda| \leq 1$. We will prove by contradiction that $|\lambda| < 1$.

Suppose that $|\lambda| = 1$ and $\boldsymbol{u} \in \ker(\lambda\mathbb{1} - A)$ is a non-zero eigenvector. Fix k such that

$$|u_k| = \max \big| \boldsymbol{u} \big] > 0.$$

Since $A > 0$ and $\boldsymbol{u}$ is not a multiple of $\vec{\mathbf{1}}$ we deduce from Lemma 5.12 that

$$0 < \max \big| \boldsymbol{u} \big| = |\lambda| \cdot |u_k| = \big| R_k(A)\boldsymbol{u} \big| \big| < \max \big[\boldsymbol{u} \big|.$$

This contradiction show that $|\lambda| < 1$. □

Definition 5.15. Let $A \in M_N(\mathbb{R})$ be stochastic matrix .

(i) The matrix A is called *irreducible* if for any $1 \leq i, j \leq N$ there exists $k = k(i,j) \in \mathbb{N}$ such that A^k_{ij}, where A^k_{ij} denotes the entry of A^k at location (i,j).

(ii) The matrix A is called *primitive* if there exists $k \in \mathbb{N}$ such that $A^k_{ij} > 0$, $\forall i, j$.

□

Clearly a primitive stochastic matrix is irreducible.

Corollary 5.16 (Perron)**.** *If $A \in M_N(\mathbb{R})$ is a primitive stochastic matrix, then 1 is a simple eigenvalue of A. Moreover, if $\lambda \in \operatorname{spec}(A) \setminus \{1\}$, then $|\lambda| < 1$.*

Proof. Let $k \in \mathbb{N}$ such that $A^k > 0$. Then $A^m > 0$, $\forall m \geq k$. Note that A^k is stochastic

$$\operatorname{spec}(A^k) = \big\{ \lambda^k;\ \ \lambda \in \operatorname{spec}(A) \big\}.$$

We deduce from Theorem 5.14 that 1 is a simple eigenvalue and the corresponding eigenspace is spanned by $\vec{\mathbf{1}}$. Moreover if $\mu \in \operatorname{spec}(A^k) \setminus \{1\}$, then $|\mu| < 1$. Thus, if $\lambda \in \operatorname{spec}(A) \setminus \{1\}$, then either $\lambda^k = 1$ or $|\lambda| < 1$. If $\lambda^k = 1$ and $\lambda \neq 1$, then there exists a nonzero vector $\boldsymbol{u}$ such that $\lambda \boldsymbol{u} = A\boldsymbol{u}$. Clearly $\boldsymbol{u}$ is not a multiple of $\vec{\mathbf{1}}$ since $\lambda \neq 1$. We deduce that $\boldsymbol{u} = A^k \boldsymbol{u}$ contradicting the fact that $\dim \ker(\mathbb{1} - A) = 1$. The characteristic polynomial of A^k is

$$P_{A^k}(t) = \prod_{\lambda \in \operatorname{spec}(A)} \big(t - \lambda^k \big)^{m_A(\lambda)},$$

where $m_A(\lambda)$ is the algebraic multiplicity of $\lambda \in \operatorname{spec}(A)$. Since $\lambda^k = 1$ iff $\lambda = 1$ we deduce

$$m_A(\lambda) = m_{A^k}(\lambda) = 1.$$

□

Corollary 5.17. *If $A \in M_N(\mathbb{R})$ is an irreducible stochastic matrix, then 1 is a simple eigenvalue of A.*

Proof. Set $B = \frac{1}{2}\big(\mathbb{1} + A \big)$. Then B is a stochastic matrix. Moreover, $\forall m \in \mathbb{N}$

$$B^m = \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} A^j.$$

Since A is irreducible we deduce that there exists $m \in \mathbb{N}$ such that $B^m > 0$, i.e., B is a primitive stochastic matrix. Thus 1 is a simple eigenvalue of B. Since

$$\operatorname{spec}(B) = \left\{ \frac{1}{2}(1 + \lambda);\ \ \lambda \in \operatorname{spec}(A) \right\}$$

we deduce that 1 is also a simple eigenvalue of A. □

Theorem 5.18. *Let $A \in M_N(\mathbb{R})$ be an irreducible stochastic matrix. The there exists a unique distribution w^∞ such that $w^\infty \cdot A = w^\infty$. This distribution is called the* invariant *distribution of A.*

Proof. Note that

$$w \cdot A = w \Longleftrightarrow A^\top w^\top = w^\top \Longleftrightarrow w^\top \in \ker\big(\mathbb{1} - A^\top \big)$$

Since A and $A^\top$ have the same characteristic polynomial we deduce that 1 is a simple eigenvalue of $A^\top$. Hence the vector space

$$\big\{ w;\ \ w = w \cdot A \big\}.$$

is one dimensional. It particular, this space contains at most one distribution.

Let w be a weight such that $w \cdot A = w$. We argue by contradiction that $\big| w \big| \cdot A = \big| w \big|$. We have

$$w_j = \sum_i w_i A_{ij}, \ \ \forall j$$

so that

$$|w_j| \leq \sum_j |w_i| A_{ij}, \ \ \forall j$$

If $\big|\,w\,\big|\cdot A\neq\big|\,w\,\big|$, then there exists j_0 such that

$$|w_{j_0}|<\sum_i|w_i|A_{ij_0}.$$

so

$$\sum_j|w_j|>\sum_j\sum_i|w_i|A_{ij}=\sum_i|w_i|\underbrace{\left(\sum_j A_{ij}\right)}_{=1}=\sum_i|w_i|.$$

This contradiction shows that $\big|\,w\,\big|=\big|\,w\,\big|\cdot A$.

Set $B=\frac{1}{2}\big(\mathbb{1}+A\big)$. Note that $\big|\;\big|\cdot A=\big|\,w\,\big|$ iff $\big|\,w\,\big|\cdot B=\big|\,w\,\big|$. In particular we deduce $\big|\,w\,\big|\cdot B^m=\big|\,w\,\big|$, $\forall m\in\mathbb{N}$. Choose m such that $B^m>0$ and set $M=B^m$. Since $|\big|\,w\,\big|\cdot M=\big|\,w\,\big|$ and $M>0$ we deduce $\big|\,w\,\big|>0$ and $\big|\,w\,\big|=\big|\,w\,\big|\cdot A$. Hence

$$w^\infty:=\frac{1}{|w_1|+\cdots+|w_N|}\big[\,|w_1|,\dots,|w_N|\,\big]$$

is a positive distribution satisfying $w^\infty\cdot A=w^\infty$. Since there is at most one distribution with this property, we deduce that w^∞ is *the unique* distribution with this property. □

Theorem 5.19. *Let $A\in M_N(\mathbb{R})$ be a primitive stochastic matrix. Denote by w^∞ its invariant distribution. Then the sequence of stochastic matrices $(A^n)_{n\in\mathbb{N}}$ converges as $n\to\infty$. Its limit, denoted by A^∞ satisfies*

$$R_i(A^\infty)=w^\infty,\ \ \forall i.$$

In other words

$$\lim_{n\to\infty}R_i(A^n)=w^\infty,\ \ \forall i=1,\dots,N.$$

1st Proof. Assume first that the sequence A^n converges and denote by A_∞ its limit. Hence

$$\lim_{n\to\infty}R_i(A^n)=R_i(A_\infty),\ \ \forall i.$$

Since the rows $R_i(A^n)$ are distributions so are their limits. Letting $n\to\infty$ in the equality $A^{n+1}=A^n\cdot A$ we deduce

$$A^\infty=A^\infty\cdot A.$$

We have

$$R_i(A_\infty)=R_i(A^\infty)\cdot A.$$

so the distribution $R_i(A_\infty)$ is a solution of the equation $w\cdot A=w$. The only distribution satisfying this equation is w^∞ so that

$$R_i(A)=w^\infty,\ \ \forall i.$$

To conclude the proof of the theorem it suffices to show that A^n converges as $n\to\infty$. We will achieve this using the Jordan decomposition of A.

The space $\mathbb{C}^N$ decomposes as a direct sum

$$\mathbb{C}^n=\bigoplus_{\lambda\in\operatorname{spec}(A)}V_\lambda$$

where V_λ is the generalized eigenspace corresponding to the eigenvalue λ. These are invariant spaces of A. The space V_1 is one-dimensional and spanned by $\vec{\mathbf{1}}$. The restriction of A to V_λ has the form $\lambda\mathbb{1}+N_\lambda$, where N_λ is a nilpotent operator.

Any $\boldsymbol{u} \in \mathbb{C}^n$ decomposes uniquely as a sum

$$\boldsymbol{u} = \boldsymbol{u}_1 + \sum_{\lambda \in \operatorname{spec}(A) \setminus \{1\}} \boldsymbol{u}_\lambda, \;\; \boldsymbol{u}_\lambda \in V^\lambda.$$

Then

$$A^n \boldsymbol{u} = \boldsymbol{u}_1 + \sum_{\lambda \in \operatorname{spec}(A) \setminus \{1\}} \big(\lambda \mathbb{1} + N_\lambda \big)^n \boldsymbol{u}_\lambda$$

Since A is primitive, any $\lambda \in \operatorname{spec}(A) \setminus \{1\}$ satisfies $\lambda| < 1$ and we deduce that

$$\lim_{n \to \infty} \big(\lambda \mathbb{1} + N_\lambda \big)^n \boldsymbol{u}_\lambda = 0, \;\; \forall \lambda \in \operatorname{spec}(A) \setminus \{1\}.$$

Hence, for any $\boldsymbol{u} \in \mathbb{C}^n$ the sequence $A^n \boldsymbol{u}$ is convergent and its limit is $\boldsymbol{u}_1$.

Let $\big(\boldsymbol{e}_j \big)_{1 \leq j \leq N}$ be the canonical basis of $\mathbb{C}^n$. We deduce that for any jb the sequence $\big(A^n \boldsymbol{e}_j \big)_{n \in \mathbb{N}}$ is convergent. Observing that $A^n \boldsymbol{e}_j$ is the j-th column of A^n we deduce that the sequence $(A^n)_{n \in \mathbb{N}}$ is convergent. □

2nd Proof.[4] For any matrix $M \in M_N(\mathbb{C})$ we set

$$\|M\| := \sup_{i,j} \big| M_{ij} \big|.$$

Denote by A^∞ the stochastic matrix with all rows equal to w^∞. Note that $A^\infty > 0$ and

$$A^\infty \cdot A = A \cdot A^\infty = A^\infty.$$

Fix k such that $A^k > 0$ and set

$$c = \min_{i,j} \frac{A^k_{ij}}{A^\infty_{ij}} = \min_{i,j} \frac{A^k_{ij}}{w^\infty_j} > 0.$$

From the equality $w^\infty = w^\infty A^k$ we deduce

$$w^\infty_j = w^\infty \cdot C_j(A^k) \geq \min C_j(A^k) = \min_i A^k_{ij}$$

Hence

$$\min_i \frac{A^k_{ij}}{w^\infty_j} \leq 1$$

so that $c \in (0, 1]$. Note that if $c = 1$ then we deduce from Lemma 5.11 that

$$A^k_{ij} = w_j, \;\; \forall i$$

and thus in this case $A^k = A^\infty$ so that

$$A^n = A^\infty, \;\; \forall n \geq k.$$

In this case the theorem is trivially true. Suppose that $c \in (0, 1)$. Note that $A^k - cA^\infty \geq 0$ and the matrix

$$B = \frac{1}{1-c} \big(A^k - cA^\infty \big)$$

is stochastic. Moreover

$$B^m A^\infty = A^\infty B^m = A^\infty = A^\infty \cdot A^\infty, \;\; \forall m \in \mathbb{N}$$

[4] This argument is due to Wolfgang Doeblin (1915-1940). During his life cut short by WW2 he had major mathematical contributions, many made public only in the year 2000. https://link.springer.com/content/pdf/10.1007/s780-002-8399-0.pdf.

Set $D := B - A^\infty$. We claim that

$$D^m = \big(B - A^\infty \big)^m = B^m - A^\infty, \;\; \forall m \in \mathbb{N}. \tag{5.10}$$

We argue by induction. The result is obviously true for $m = 1$. As for the inductive step we have

$$\begin{aligned} D^{m+1} &= D^m \cdot D = (B^m - A^\infty)(B - A^\infty) \\ &= B^{m+1} - B^m A^\infty - A^\infty - A^\infty \cdot A^\infty = B^{m+1} - A^\infty. \end{aligned}$$

Arguing in a similar fashion we deduce

$$\big(A^k - A^\infty \big)^m = A^{km} - A^\infty. \tag{5.11}$$

Note that

$$D^m := B^m - A^\infty = B^m - B^m A^\infty = B^m (\mathbb{1} - A^\infty).$$

We deduce from Lemma 5.12 that

$$|D^m_{ij}| = \big| \, R_i(B^m) C_j(\mathbb{1} - A^\infty) \, \big| \leq 1, \;\; \forall i, j,$$

i.e.,

$$\| D^m \| \leq 1.$$

On the other hand,

$$(A^k - A^\infty) = (1-c)(B - A^\infty) = (1-c)D.$$

Hence for any $m \in \mathbb{N}$

$$A^{km} - A^\infty = \big(A^k - A^\infty \big)^m = (1-c)^m D^m.$$

Hence

$$\| A^{km} - A^\infty \| \leq (1-c)^m \| D^m \| \leq (1-c)^m. \tag{5.12}$$

On the other hand from the equality $A^{\ell+1} - A^\infty = A\big(A^\ell - A^\infty \big)$ we deduce that

$$\big| \, \big(A^{\ell+1} - A^\infty \big)_{ij} \, \big| = \big| \, R_i(A) \cdot C_j \big(A^\ell - A^\infty \big) \, \big| \stackrel{(5.9)}{\leq} \| A^\ell - A^\infty \|, \;\; \forall i, j.$$

In other words,

$$\| A^{\ell+1} - A^\infty \| \leq \| A^\ell - A^\infty \|,$$

so the function $\ell \mapsto r(\ell) := \| A^\ell - A^\infty \|$ is nonincreasing. The inequality (5.12) implies that

$$\lim_{m \to \infty} r(km) = 0$$

and thus

$$\lim_{\ell \to \infty} r(\ell) = 0.$$

This completes the proof of the theorem. □

How hard it is to find the invariant distribution of an irreducible stochastic matrix $A \in M_N(\mathbb{R})$? If the size N of the matrix, then this is a nearly impossible task. There are however situations frequently occurring in concrete applications when this is possible, and quite easily.

Proposition 5.20. *We say that $P \in M_N(\mathbb{R})$ is a stochastic matrix and $w_1, \dots, w_N$ are positive numbers such that*

$$w_i p_{ij} = w_j p_{ji}, \;\; \forall i, j \tag{5.13}$$

If we set $w := \big[\, w_1, \dots, w_N \, \big]$, then $wP = w$. In particular, if P is irreducible, its stationary distribution is

$$w_i^\infty = \frac{1}{W} w_i, \;\; W = \sum_{i=1}^N w_i.$$

Proof. Note that

$$w \cdot P = \big[\, w \cdot C_1(P), \dots, w \cdot C_N(P) \,\big]$$

Hence

$$(w \cdot P)_j = w \cdot C_j(P) = \sum_i w_i p_{ij} \stackrel{(5.13)}{=} \sum_i w_j p_{ji} = w_j \sum_i p_j i = w_j.$$

Note that

$$\overline{w} = \frac{1}{W} w$$

is a distribution satisfying

$$\overline{w} P = \overline{w}.$$

If P is irreducible, there is only one such distribution, the stationary distribution w^∞. □

Definition 5.21. A stochastic matrix $P \in M_N(\mathbb{R})$ is called *reversible* if there exist positive numbers $w_1, \dots, w_N$ satisfying the condition (5.13) in Proposition 5.20. □

5.3. **Exercises.**

Exercise 5.1. Let Q be a symmetric bilinear form on an n-dimensional real vector space. Prove that there exists a basis of $\boldsymbol{U}$ such that the matrix associated to Q by this basis is diagonal and all the entries belong to $\{-1, 0, 1\}$. □

Exercise 5.2. Let Q be a symmetric bilinear form on an n-dimensional real vector space $\boldsymbol{U}$ with indices of inertia μ_+, μ_-, μ_0. We say Q is positive definite on a subspace $\boldsymbol{V} \subset \boldsymbol{U}$ if

$$Q(\boldsymbol{v}, \boldsymbol{v}) > 0, \ \ \forall \boldsymbol{v} \in \boldsymbol{V} \setminus 0.$$

(a) Prove that if Q is positive definite on a subspace $\boldsymbol{V}$, then $\dim \boldsymbol{V} \leq \mu_+$.
(b) Show that there exists a subspace of dimension μ_+ on which Q is positive definite.

Hint: (a) Choose a diagonalizing basis $\underline{\boldsymbol{e}} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ of Q. Assume that $Q(\boldsymbol{e}_i, \boldsymbol{e}_i) > 0$, for $i = 1, \dots, \mu_+$ and $Q(\boldsymbol{e}_j, \boldsymbol{e}_j) \leq 0$ for $j > \mu_+$. Argue by contradiction that

$$\dim \boldsymbol{V} + \dim \operatorname{span}\{\boldsymbol{e}_j; \ \ j > \mu_+\} \leq \dim \boldsymbol{U} = n.$$ □

Exercise 5.3. Let Q be a symmetric bilinear form on an n-dimensional real vector space $\boldsymbol{U}$. with indices of inertia μ_+, μ_-, μ_0. Define

$$\operatorname{Null}(Q) := \left\{ \boldsymbol{u} \in \boldsymbol{U}; \ \ Q(\boldsymbol{u}, \boldsymbol{v}) = 0, \ \ \forall \boldsymbol{v} \in \boldsymbol{U}. \right\}.$$

Show that $\operatorname{Null}(Q)$ is a vector subspace of $\boldsymbol{U}$ of dimension μ_0. □

Exercise 5.4 (Jacobi)**.** For any $n \times n$ matrix M we denote by M_i the $i \times i$ matrix determined by the first i rows and columns of M.

Suppose that Q is a symmetric bilinear form on the real vector space $\boldsymbol{U}$ of dimension n. Fix a basis $\underline{\boldsymbol{e}} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ of $\boldsymbol{U}$. Denote by A the matrix associated to Q by the basis $\underline{\boldsymbol{e}}$ and assume that

$$\Delta_i := \det A_i \neq 0, \ \ \forall i = 1, \dots, n.$$

(a) Prove that there exists a basis $\underline{\boldsymbol{f}} = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_n)$ of $\boldsymbol{U}$ with the following properties.

(i) $\operatorname{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_i\} = \operatorname{span}\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_i\}$, $\forall i = 1, \dots, n$.
(ii) $Q(\boldsymbol{f}_k, \boldsymbol{e}_i) = 0$, $\forall 1 \leq i < k \leq n$, $Q(\boldsymbol{f}_k, \boldsymbol{e}_k) = 1$, $\forall k = 1, \dots, n$.

Hint: For fixed k, express the vector $\boldsymbol{f}_k$ in terms of the vectors $\boldsymbol{e}_i$,

$$\boldsymbol{f}_k = \sum_{i=1}^{n} s_{ik} \boldsymbol{e}_i$$

and then show that the conditions (i) and (ii) above uniquely determine the coefficients s_{ik}, again with k fixed.
(b) If $\underline{\boldsymbol{f}}$ is the basis found above, show that

$$Q(\boldsymbol{f}_k, \boldsymbol{f}_i) = 0, \ \ \forall i \neq k,$$
$$Q(\boldsymbol{f}_k, \boldsymbol{f}_k) = \frac{\Delta_{k-1}}{\Delta_k}, \ \forall k = 1, \dots, n, \ \ \Delta_0 := 1.$$

(c) Show that the Morse index $\mu_-(Q)$ is the number of sign changes in the sequence

$$1, \Delta_1, \Delta_2, \dots, \Delta_n.$$ □

Exercise 5.5 (Sylvester)**.** For any $n \times n$ matrix M we denote by M_i the $i \times i$ matrix determined by the first i rows and columns of M.

Suppose that Q is a symmetric bilinear form on the real vector space $\boldsymbol{U}$ of dimension n. Fix a basis $\underline{\boldsymbol{e}} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ of $\boldsymbol{U}$ and denote by A the matrix associated to Q by the basis $\underline{\boldsymbol{e}}$. Prove that the following statements are equivalent.

(i) Q is positive definite.
(ii) $\det A_i > 0, \forall i = 1, \dots, n$. □

Exercise 5.6. (a) Let $f : [0,1] \to [0,\infty)$ by a continuous function which is not identically zero. For any $k = 01, 2, \dots$ we define the *k-th momentum* of f to be the real number

$$\mu_k := \mu_k(f) = \int_0^1 x^k f(x)dx.$$

Prove that the symmetric $(n+1) \times (n+1)$-matrix symmetric matrix

$$A = (a_{ij})_{0 \leq i,j \leq n}, \quad a_{ij} = \mu_{i+j}.$$

is positive definite.
Hint: Associate to any vector

$$\boldsymbol{u} = \left[\begin{array}{c} u_0 \\ u_1 \\ \vdots \\ u_n \end{array}\right]$$

the polynomial $P_{\boldsymbol{u}}(x) = u_0 + u_1 x + \cdots + u_n x^n$ and then express the integral

$$\int_0^1 P_{\boldsymbol{u}}(x) P_{\boldsymbol{v}}(x) f(x)$$

in terms of A.
(b) Prove that the symmetric $n \times n$ symmetric matrix

$$B = (b_{ij})_{1 \leq i,j \leq n}, \quad b_{ij} = \frac{1}{i+j}$$

is positive definite.
(c) Prove that the symmetric $(n+1) \times (n+1)$ symmetric matrix

$$C = (c_{ij})_{0 \leq i,j \leq n}, \quad c_{ij} = (i+j)!,$$

where $0! := 1$, $n! = 1 \cdot 2 \cdots n$.
Hint: Show that for any $k = 0, 1, 2, \dots$ we have

$$\int_0^\infty x^k e^{-x} dx = k!,$$

and then due the trick in (a). □

Exercise 5.7. (a) Show that the 2×2-symmetric matrix

$$A = \left[\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array}\right]$$

is positive definite.
(b) Denote by Q_A the symmetric bilinear form on $\mathbb{R}^2$ defined by then above matrix A. Since Q_A is positive definite, it defines an inner product $\langle -, - \rangle_A$ on $\mathbb{R}^2$,

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_A = Q_A(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, A\boldsymbol{v} \rangle$$

where $\langle -, - \rangle$ denotes the canonical inner product on $\mathbb{R}^2$. Denote by $T^\#$ the adjoint of T with respect to the inner product $\langle -, - \rangle_A$, i.e.,

$$\langle T\boldsymbol{u}, \boldsymbol{v} \rangle_A = \langle \boldsymbol{u}, T^\# \boldsymbol{v} \rangle_A, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2. \tag{\#}$$

Show that

$$T^{\#} = A^{-1}T^{*}A,$$

where T^* is the adjoint of T with respect to the canonical inner product $\langle -,-\rangle$. What does this formula tell you in the special case when T is described by the symmetric matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}?$$

Hint: In (#) use the equalities $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_A = \langle \boldsymbol{x}, A\boldsymbol{y}\rangle$, $\langle T\boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{x}, T^*\boldsymbol{y}\rangle$, $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. □

Exercise 5.8. Suppose that $\boldsymbol{U}$ is a finite dimensional Euclidean $\mathbb{F}$-space and $T : \boldsymbol{U} \to \boldsymbol{U}$ is an invertible operator. Prove that $\sqrt{T^*T}$ is invertible and the operator $S = T(\sqrt{T^*T})^{-1}$ is orthogonal. □

Exercise 5.9. (a) Suppose that $\boldsymbol{U}$ is a finite dimensional real Euclidean space and $Q \in \mathrm{Sym}(\boldsymbol{U})$ is a positive definite symmetric bilinear form. Prove that there exists a unique positive operator

$$T : \boldsymbol{U} \to \boldsymbol{U}$$

such that

$$Q(\boldsymbol{u}, \boldsymbol{v}) = \langle T\boldsymbol{u}, T\boldsymbol{v}\rangle, \;\; \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}.$$ □

Exercise 5.10. Let $\boldsymbol{U}$ be a complex Euclidean space and $T : \boldsymbol{U} \to \boldsymbol{U}$ a selfadjoint operator.

(i) Prove that T^2 is nonnegative definite.
(ii) Set $|T| = \sqrt{T^2}$. Prove that $|T|$ and T commute, i.e., $T \cdot |T| = |T| \cdot T$.
(iii) Prove that if T is invertible then so is $|T|$ and then operator $|T|^{-1}T$ is orthogonal.

□

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556-4618.
Email address: nicolaescu.1@nd.edu
URL: http://www.nd.edu/~lnicolae/