

**Topics in topology. Fall 2008.**  
**The signature theorem and some of its applications<sup>1</sup>**

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# Introduction

These are the notes for the *Fall 2008 Topics in topology class* at the University of Notre Dame. I view them as a “walk down memory lane” since I discuss several exciting topological developments that took place during the fifties decade which radically changed the way we think about many issues.

The pretext for this journey is Milnor’s surprising paper [Mi56] in which he constructs exotic smooth structures on the 7-sphere. The main invariant used by Milnor to distinguish between two smooth structures on the same manifold was based on a formula that F. Hirzebruch had recently discovered that related the signature of a smooth manifold to the integral of a certain polynomial in the Pontryagin classes of the tangent bundle.

The goal of this class is easily stated: understand all the details of the very densely packed, beautiful paper [Mi56]. From an academic point of view this has many very useful benefits because it forces the aspiring geometer to learn a large chunk of the basic notions and tools that are part of the everyday arsenal of the modern geometer/topologist: Poincaré duality, Thom isomorphism, Euler, Chern and Pontryagin classes, cobordisms groups, signature formula. Moreover, such a journey has to include some beautiful side-trips into the inner making of several concrete fundamental examples. These can only enhance the appreciation and understanding of the subject.

The goal is a bit ambitious for a one-semester course, and much like the classic [MS] from which I have drawn a lot of inspiration, I had to make some choices. Here is what I decided to leave out:

- The proofs of the various properties of  $\cup$ ,  $\cap$  and  $\times$  products.
- The proofs of the Poincaré duality theorem, the Thom isomorphism theorem, and of the Leray-Hirsch theorem.
- The beautiful work of J.P. Serre on the cohomology of loops spaces, Serre classes of Abelian groups and their applications to homotopy theory.

The first topic ought to be part of a first year graduate course in algebraic topology, and is presented in many easily accessible sources. The right technology for dealing with the last two topics is that of spectral sequences, but given that the spectral sequences have a rather different flavor than the rest of the arguments which are mostly geometrical, I decided it would be too much for anyone to absorb in one semester.



# Singular homology and cohomology

For later use we survey a few basic facts from algebraic topology. For more details we refer to [Bre, Do, Hatch1, Spa].

## 1.1. The basic properties of singular homology and cohomology

A ring  $R$  is called *convenient* if it is isomorphic to one of the rings  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$  or  $\mathbb{F}_p := \mathbb{Z}/p$ ,  $p$  prime. In the sequel we will work exclusively with *convenient rings*.

We denote by  $\Delta_n$  the standard affine  $n$ -simplex

$$\Delta_n := \left\{ (t_0, \dots, t_n) \in \mathbb{R}_{\geq 0}^{n+1}; \sum_{i=0}^n t_i = 1 \right\}.$$

For every topological space  $X$  we denote by  $\mathcal{S}_k(X)$  the set of singular  $k$ -simplices in  $X$ , i.e., the set of continuous maps

$$\sigma : \Delta_k \rightarrow X.$$

We denote by  $C_\bullet(X, R)$  the singular chain complex of  $X$  with coefficients in the ring  $R$ ,

$$C_\bullet(X, R) = \bigoplus_{k \geq 0} C_k(X, R), \quad C_k(X, R) = \bigoplus_{\sigma \in \mathcal{S}_k(X)} R = \bigoplus_{\sigma \in \mathcal{S}_k(X)} R|\sigma.$$

Above, for every singular simplex  $\sigma \in \mathcal{S}_k(X)$  we denoted by  $|\sigma\rangle$  the generator of  $C_k(X, R)$  associated to this singular simplex.

For every  $k \geq 1$ , and every  $i = 0, \dots, k$  we have a natural map

$$\partial_i : \mathcal{S}_k(X) \rightarrow \mathcal{S}_{k-1}(X), \quad \mathcal{S}_k(X) \ni \sigma \mapsto \partial_i \sigma \in \mathcal{S}_{k-1}(X),$$

where for any  $(t_0, \dots, t_{k-1}) \in \Delta_{k-1}$  we have

$$\partial_i \sigma(t_0, \dots, t_{k-1}) = \sigma(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1}).$$

The boundary operator  $\partial : C_k(X, R) \rightarrow C_{k-1}(X, R)$  is then uniquely determined by

$$\partial|\sigma\rangle = \sum_{i=0}^k (-1)^i |\partial_i\sigma\rangle.$$

We denote by  $H_\bullet(X, R)$  the homology of this complex. It is called the *singular homology* of  $X$  with coefficients in  $R$ .

Every subspace  $A \subset X$  defines a subcomplex  $C_\bullet(A, R) \subset C_\bullet(X, R)$  and we denote by  $C_\bullet(X, A; R)$  the quotient complex

$$C_\bullet(X, A; R) := C_\bullet(X, R)/C_\bullet(A, R).$$

We denote by  $H_\bullet(X, A; \mathbb{R})$  the homology of the complex  $C_\bullet(X, A; \mathbb{R})$ . It is called the *singular homology* of the pair  $(X, A)$  with coefficients in  $R$ . When  $R = \mathbb{Z}$  we will drop the ring  $R$  from the notation.

The singular homology enjoys several remarkable properties.

- If  $*$  denotes the topological space consisting of a single point, then

$$H_k(*, R) = \begin{cases} R & k = 0 \\ 0 & k > 0. \end{cases}$$

- A continuous map of pairs  $f : (X, A) \rightarrow (Y, B)$  induces a morphism

$$f_* : H_\bullet(X, A; R) \rightarrow H_\bullet(Y, B, R).$$

Two homotopic maps  $f_0, f_1 : (X, A) \rightarrow (Y, B)$  induce identical morphisms,  $(f_0)_* = (f_1)_*$ . Moreover

$$(\mathbb{1}_{(X,A)})_* = \mathbb{1}_{H_\bullet(X,A;R)}.$$

- If  $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$  are continuous maps then the induced maps in homology satisfy

$$(g \circ f)_* = g_* \circ f_*.$$

- For any topological pair  $(X, A)$  we have a long exact sequence

$$\cdots \xrightarrow{\partial} H_k(A, R) \xrightarrow{i_*} H_k(X, R) \xrightarrow{j_*} H_k(X, A; R) \xrightarrow{\partial} H_{k-1}(A, R) \rightarrow \cdots,$$

where  $i_*$  and  $j_*$  are induced by the natural maps  $(A, \emptyset) \xrightarrow{i} (X, \emptyset) \xrightarrow{j} (X, A)$ . This long exact sequence is natural in the sense that given any continuous maps  $f : (X, A) \rightarrow (Y, B)$  the diagram below is commutative for any  $k \geq 0$ .

$$\begin{array}{ccc} H_{k+1}(X, A; R) & \xrightarrow{\partial} & H_k(A, R) \\ \downarrow f_* & & \downarrow f_* \\ H_{k+1}(Y, B; R) & \xrightarrow{\partial} & H_k(B, R) \end{array} \quad (1.1.1)$$

- If  $X$  is a topological space and  $B \subset A$  are subspaces of  $X$ , then we have a natural long exact sequence

$$\cdots \xrightarrow{\partial} H_k(A, B; R) \xrightarrow{i_*} H_k(X, B; R) \xrightarrow{j_*} H_k(X, A; R) \xrightarrow{\partial} H_{k-1}(A, B; R) \rightarrow \cdots$$



- (*The universal coefficients theorem*) For any topological pair  $(X, A)$  we have a natural isomorphism

$$H_{\bullet}(X, A) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow H_{\bullet}(X, A; \mathbb{R}).$$

Moreover, for every prime number  $p > 1$  we have a non-canonical isomorphism of vector spaces over  $\mathbb{F}_p$

$$H_k(X, A; \mathbb{F}_p) = H_k(X, A) \otimes \mathbb{F}_p \oplus \text{Tor}_p(H_{k-1}(X, A)),$$

where for any Abelian group  $G$ , and any positive integer  $n$  we set

$$\text{Tor}_n(G) := \{g \in G; ng = 0\}.$$

- (*Künneth theorem*) If  $X, Y$  are topological spaces,  $A \subset X, B \subset Y$ , then for every  $k \geq 0$  there exists a natural injection

$$\times : \bigoplus_{i+j=k} H_i(X, A; R) \otimes H_j(Y, B; \mathbb{R}) \rightarrow H_k((X, A) \times (Y, B); R)$$

where

$$(X, A) \times (Y, B) := (X \times Y, A \times Y \cup X \times B).$$

If  $R$  is a field, then the *cross product*  $\times$  defined above is an isomorphism. If  $R \cong \mathbb{Z}$ , then we have a short exact sequence

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=k} H_i(X, A; R) \otimes H_j(Y, B; \mathbb{R}) \xrightarrow{\times} H_k((X, A) \times (Y, B); R) \rightarrow \\ \rightarrow \bigoplus_{i+j=k-1} H_i(X) * H_j(Y) \rightarrow 0, \end{aligned} \quad (1.1.2)$$

where the torsion product  $G * H$  of two Abelian groups  $G, H$  is uniquely determined by the following properties.

$$G * H \cong H * G, \quad G * H = 0, \quad \text{if } G \text{ or } H \text{ is torsion free,}$$

$$\left( \bigoplus_{i \in I} G_i \right) * H = \bigoplus_{i \in I} G_i * H, \quad (\mathbb{Z}/n) * (\mathbb{Z}/m) \cong \text{Tor}_n(\mathbb{Z}/m) \cong \mathbb{Z}/\text{gcd}(m, n), \quad \forall m, n \in \mathbb{Z}_{>1}.$$

The sequence (1.1.2) is *non-canonically split* and thus we have a non-canonical isomorphism

$$H_k(X \times Y) = \left( \bigoplus_{i+j=k} H_i(X) \otimes_{\mathbb{Z}} H_j(Y) \right) \oplus \left( \bigoplus_{i+j=k-1} H_i(X) * H_j(Y) \right).$$

**Remark 1.1.1.** For any two topological spaces  $X, Y$  we have a “reflection”

$$r : X \times Y \rightarrow Y \times X, \quad (x, y) \mapsto (y, x).$$

Let us point out a rather subtle fact. If  $a \in H_i(X, R), b \in H_j(Y, R)$  then

$$r_*(a \times b) = (-1)^{ij} b \times a.$$

In other words, the cross product “ $\times$ ” is not commutative, but rather super-commutative. □

A very important property of singular homology is the *excision property*. This requires a more detailed treatment.

Suppose  $X$  is a topological spaces and  $A, B$  are subspaces of  $X$ . Note that the complexes  $C_{\bullet}(A)$  and  $C_{\bullet}(B)$  are subcomplexes of  $C_{\bullet}(A \cup B)$  and thus we can form the subcomplex  $C_{\bullet}(A) + C_{\bullet}(B)$

generated by these two subcomplexes. We say that the collection of subsets  $\{A, B\}$  is an *excisive couple* if the natural inclusion

$$C_\bullet(A) + C_\bullet(B) \hookrightarrow C_\bullet(A \cup B)$$

is a *quasi-isomorphism*, i.e., it induces an isomorphism in homology. We have the following *nontrivial* examples of excisive couples.

**Example 1.1.2.** (a) For any subset  $S \subset A \cup B$  we denote by  $\text{int}_{A \cup B} S$  the interior of  $S$  with respect to the subspace topology on  $A \cup B$ . If  $A \cup B = \text{int}_{A \cup B} A \cup \text{int}_{A \cup B} B$ , then  $\{A, B\}$  is an excisive couple.

(b) If there exists a *CW* structure on  $A \cup B$  such that  $A$  and  $B$  are (closed) subcomplexes, then  $\{A, B\}$  is an excisive couple.  $\square$

**Theorem 1.1.3.** *Suppose that  $\{A, B\}$  is an excisive couple. Then the natural morphism*

$$H_\bullet(A, A \cap B; R) \rightarrow H_\bullet(A \cup B, A; R) \quad (1.1.3)$$

*is an isomorphism called excision isomorphism. Moreover, we have a natural long exact sequence*

$$\cdots \rightarrow H_k(A \cap B, R) \xrightarrow{i} H_k(A, R) \oplus H_k(B, R) \xrightarrow{j} H_k(A \cup B, R) \xrightarrow{\partial} H_{k-1}(A \cap B, R) \rightarrow \cdots$$

*called the Mayer-Vietoris sequence.*  $\square$

**Corollary 1.1.4.** *Consider a topological pair  $(X, A)$ , and  $U$  a subset of  $X$  such that  $\text{cl} U \subset \text{int} A$ . Then the natural isomorphism*

$$H_\bullet(X \setminus U, A \setminus U; R) \rightarrow H_\bullet(X, A; R)$$

*is an isomorphism.*

**Proof.** Apply the isomorphism (1.1.3) to the excisive couple  $(A, B) = (A, X \setminus U)$ .  $\square$

More generally, two pairs  $(X_1, A_1), (X_2, A_2)$  of subsets of a topological space  $X$ ,  $A_i \subset X_i \subset X$ ,  $i = 1, 2$  is said to form an excisive couple if each of the couples  $\{X_1, A_1\}, \{X_2, A_2\}$  is excisive. If  $\{(X_1, A_1), (X_2, A_2)\}$  is an excisive couple of pairs, then we have a *relative long-exact Mayer-Vietoris sequence*

$$\begin{aligned} \cdots \rightarrow H_k(X_1 \cap X_2, A_1 \cap A_2; R) &\rightarrow H_k(X_1, A_1; R) \oplus H_k(X_2, A_2; R) \rightarrow \\ &\rightarrow H_k(X_1 \cup X_2, A_1 \cup A_2; R) \rightarrow H_{k-1}(X_1 \cap X_2, A_1 \cap A_2; R) \rightarrow \cdots \end{aligned} \quad (1.1.4)$$

For a topological space  $X$  we define  $C^\bullet(X, R)$  to be the *co-chain* complex of  $R$ -modules dual to  $C_\bullet(X)$ . More precisely, we have

$$C^k(X, R) := \text{Hom}_{\mathbb{Z}}(C_k(X), R).$$

The cochains  $\alpha \in C^k(X, R)$  are said to have degree  $k$  and we write this  $\text{deg } \alpha = k$ . Note that we have an isomorphism of  $R$ -modules

$$C^k(X, R) \cong \text{Hom}_R(C_k(X, R), R).$$

If we denote by

$$\langle -, - \rangle : \text{Hom}_R(C_k(X, R), R) \times C_k(X, R) \rightarrow R,$$

the natural  $R$ -bilinear map defined by

$$\mathrm{Hom}_R(C_k(X, R), R) \times C_k(X, R) \ni (\varphi, c \otimes r) \mapsto r\langle \varphi, c \rangle := r\varphi(c),$$

then the coboundary operator  $\delta : C^k(X, R) \rightarrow C^{k+1}(X, R)$  is uniquely determined by the equality

$$\langle \delta\varphi, c \rangle := \langle \varphi, \partial c \rangle, \quad \forall \varphi \in C^{k+1}(X, R), \quad c \in C_k(X, R).$$

Because of this defining property, we will often find it convenient to denote the coboundary operator as an adjoint of  $\partial$ ,  $\delta = \partial^\dagger$ . The bilinear map  $\langle -, - \rangle$  described above is called the *Kronecker pairing*.

We denote by  $H^\bullet(X, R)$  the cohomology of the complex  $C^\bullet(X, R)$ . It is called the *singular cohomology of  $X$  with coefficients in  $R$* . The Kronecker pairing induces a bilinear map

$$\langle -, - \rangle_\kappa : H^k(X, R) \times H_k(X, R) \rightarrow R,$$

and we will continue to refer to it as the Kronecker pairing. This induces a morphism

$$\kappa : H^k(X, R) \rightarrow \mathrm{Hom}_{\mathbb{Z}}(H_k(X, R), R),$$

called the *Kronecker morphism*

**Theorem 1.1.5** (Universal coefficients theorem for cohomology). *Suppose that the topological space  $X$  is of finite (homological) type, i.e., the homology groups  $H_k(X)$  are finitely generated Abelian groups for any  $k \geq 0$ . We denote by  $T_k(X)$  the torsion subgroup of  $H_k(X)$ . Then the following hold.*

- (a) *If the ring of coefficients  $R$  is a field, then the Kronecker morphism  $\kappa$  is an isomorphism.*  
 (b) *If  $R = \mathbb{Z}$  then we have a short exact sequence*

$$0 \rightarrow \mathrm{Hom}_{\mathbb{Z}}(T_{k-1}(X), \mathbb{Q}/\mathbb{Z}) \rightarrow H^k(X) \xrightarrow{\kappa} \mathrm{Hom}_{\mathbb{Z}}(H_k(X), \mathbb{Z}) \rightarrow 0. \quad (1.1.5)$$

*The sequence splits, but non-canonically, so that we have a non-canonical isomorphism*

$$H^k(X) \cong \mathrm{Hom}_{\mathbb{Z}}(T_{k-1}(X), \mathbb{Q}/\mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(H_k(X), \mathbb{Z}) \quad \square$$

If  $A$  is a subspace of the topological space, then the short exact sequence of chain complexes

$$0 \rightarrow C_\bullet(A) \rightarrow C_\bullet(X) \rightarrow C_\bullet(X, A) \rightarrow 0$$

induces a short exact sequence of co-chain complexes

$$0 \rightarrow \mathrm{Hom}_{\mathbb{Z}}(C_\bullet(X, A), R) \rightarrow C^\bullet(X, R) \rightarrow C^\bullet(A, R) \rightarrow 0.$$

We denote the co-chain complex  $\mathrm{Hom}_{\mathbb{Z}}(C_\bullet(X, A), R)$  by  $C^\bullet(X, A; R)$ . Observe that it can be identified with the *subcomplex* of  $C^\bullet(X)$  consisting of cochains  $\varphi$  such that  $\varphi(c) = 0$  for any chain  $c \in C_\bullet(A)$ .

A continuous map of topological pairs  $f : (X, A) \rightarrow (Y, B)$  induces a morphism

$$f^* : H^\bullet(Y, B; R) \rightarrow H^\bullet(X, A; R)$$

called the *pullback* by  $f$ . Two homotopic maps induce identical pullbacks, and we have the functoriality properties

$$(g \circ f)^* = f^* \circ g^*, \quad \mathbb{1}^* = \mathbb{1}.$$

A topological pair  $(X, A)$  determines a natural long exact sequence

$$\cdots \rightarrow H^k(X, A; R) \rightarrow H^k(X, R) \rightarrow H^k(A, R) \xrightarrow{\delta} H^{k+1}(X, A; R) \rightarrow \cdots$$

where the connecting morphism  $\delta$  satisfies a naturality condition similar to (1.1.1). The excision properties formulated in Theorem 1.1.3 and Corollary 1.1.4 have an obvious cohomological counterpart whose precise formulation can be safely left to the reader. In particular, under appropriate excisive assumptions, we have a relative Mayer-Vietoris sequence

$$\begin{aligned} \cdots \rightarrow H^k(X_1 \cup X_2, A_1 \cup A_2; R) &\rightarrow H^k(X_1, A_1; R) \oplus H^k(X_2, A_2; R) \rightarrow \\ &\rightarrow H^k(X_1 \cap X_2, A_1 \cap A_2; R) \rightarrow H^{k+1}(X_1 \cup X_2, A_1 \cup A_2; R) \rightarrow \cdots \end{aligned} \quad (1.1.6)$$

## 1.2. Products

Although the cohomology groups of a space are completely determined by the homology groups, the cohomology groups possess additional algebraic structures, which have a rather subtle topological origin, and are *not* determined by the homology groups alone.

The first such additional structure is given by the *cup product*. To define it, and formulate its main properties we need to introduce some notation. Fix a topological space  $X$ .

First, let us define the *front* and *back* face operators

$$F_k, B_k : \mathcal{S}_n(X) \rightarrow \mathcal{S}_k(X).$$

Observe that we have two canonical linear inclusions

$$i_k, j_k : \mathbb{R}^{k+1} \hookrightarrow \mathbb{R}^{n+1},$$

$i_k(t_0, \dots, t_k) = (t_0, \dots, t_k, 0, \dots, 0) \in \mathbb{R}^{n+1}$ ,  $j_k(t_0, \dots, t_k) = (0, \dots, 0, t_0, \dots, t_k) \in \mathbb{R}^{n+1}$ , which restrict to linear inclusions  $i_k, j_k : \Delta_k \hookrightarrow \Delta_n$ . For a singular simplex  $\sigma : \Delta_n \rightarrow X$  we set  $F_k\sigma = \sigma \circ i_k$  and  $B_k\sigma = \sigma \circ j_k$ . These face operators induce linear maps

$$F_k, B_k : C_n(X) \rightarrow C_k(X).$$

Given two cochains  $\alpha \in C^k(X, R)$ ,  $\beta \in C^\ell(X, R)$  we define the cup product  $\alpha \cup \beta \in C^{k+\ell}(X, R)$  by the equality<sup>1</sup>

$$\langle \alpha \cup \beta, c \rangle = \langle \alpha, F_k c \rangle \cdot \langle \beta, B_\ell c \rangle, \quad \forall c \in C_{k+\ell}(X).$$

This cup product satisfies the conditions

$$\begin{aligned} \delta(\alpha \cup \beta) &= (\delta\alpha) \cup \beta + (-1)^{\deg \alpha} \alpha \cup (\delta\beta), \\ (\alpha \cup \beta) \cup \gamma &= \alpha \cup (\beta \cup \gamma), \end{aligned}$$

for any cocycles  $\alpha, \beta, \gamma$ . This implies that the cup product induces an associative product

$$\cup : H^\bullet(X, R) \times H^\bullet(X, R) \rightarrow H^\bullet(X, R).$$

This product on *cohomology groups*<sup>2</sup> is super-commutative, i.e.,

$$\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha, \quad \forall \alpha \in H^k(X, R), \quad \beta \in H^\ell(X, R).$$

We have thus obtained a structure of super-commutative ring with 1 on  $H^\bullet(X, R)$ . The ring  $H^\bullet(X)$  is called the *cohomology ring* of  $X$ . This is a much more refined topological invariant of a space, and its computation is much more difficult than the computation of the cohomology *group* structure.

<sup>1</sup>This definition agrees with the definition in [Hatch1, §3.2], and [Spa, §5.6]. It differs by a sign,  $(-1)^{k\ell}$ , from the definition in [Bre, VI.4] and [MS, App.A].

<sup>2</sup>We want to emphasize that the above equality does not hold at cochain level.

Let us observe that if  $A$  is a subspace of  $X$ ,

$$\alpha \in C^k(X, A; R) \subset C^k(X, R), \quad \beta \in C^\ell(X, A; R) \subset C^\ell(X, R)$$

then the cochain  $\alpha \cup \beta$  also belongs to the subcomplex  $C^\bullet(X, A; R)$ . In particular, we have an associative and super-commutative product

$$\cup : H^\bullet(X, A; R) \times H^\bullet(X, A; R) \rightarrow H^\bullet(X, A; R)$$

We obtain in this fashion a ring structure on  $H^\bullet(X, A; R)$ . The ring  $H^\bullet(X, A)$  is called the *cohomology ring of the pair*  $(X, A)$ .

We can be even more precise. Suppose  $(A, B)$  is an *excisive* couple. If

$$\alpha \in C^k(X, A; R), \quad \beta \in C^\ell(X, B; R),$$

then  $\alpha \cup \beta$  belongs to the subcomplex

$$C^\bullet(X, A; R) \cap C^\bullet(X, B; R) \subset C^\bullet(X, A \cup B; R).$$

Because  $(A, B)$  is excisive, the inclusion  $C^\bullet(X, A; R) \cap C^\bullet(X, B; R) \hookrightarrow C^\bullet(X, A \cup B; R)$  is a quasi-isomorphism. We obtain in this fashion an *extraordinary cup product*

$$\cup : H^k(X, A; R) \times H^\ell(X, B; R) \rightarrow H^{k+\ell}(X, A \cup B; R), \quad (A, B) \text{ excisive couple.} \quad (1.2.1)$$

Finally, we define the *cap product*

$$\cap : C^k(X, R) \times C_m(X, R) \rightarrow C_{m-k}(X, R)$$

uniquely determined by the equality

$$\langle \alpha, \beta \cap c \rangle = \langle \alpha \cup \beta, c \rangle, \quad \forall \alpha \in C^{m-k}(X, R), \quad \beta \in C^k(X, R), \quad c \in C_m(X, R).$$

More precisely, for every singular  $m$ -simplex  $\sigma \in \mathcal{S}_m(X)$  and any  $\beta \in C^k(X, R)$  we have

$$\beta \cap |\sigma\rangle = \langle \beta, B_k \sigma \rangle \cdot |F_{m-k} \sigma\rangle.$$

Then

$$\partial(\beta \cap c) = (-1)^{m-k}(\delta\beta) \cap c + \beta \cap (\partial c), \quad (1.2.2)$$

and thus we obtain a  $R$ -bilinear map

$$\cap : H^k(X, R) \times H_m(X, R) \rightarrow H_{m-k}(X, R).$$

More generally, we have cap products

$$\cap : H^k(X, A; R) \times H_m(X, A; R) \rightarrow H_{m-k}(X, R)$$

and

$$H^k(X, A; R) \times H_m(X, A \cup B; R) \rightarrow H_{m-k}(X, B; R)$$

if the couple  $(A, B)$  is excisive. If we denote by  $\mathbf{1}_X \in H_0(X, R)$  the canonical 0-cocycle (which maps every 0-simplex to 1), then we have the equality

$$\langle \alpha, c \rangle_\kappa = \langle \mathbf{1}_X, \alpha \cap c \rangle_\kappa, \quad \forall \alpha \in H^k(X, R), \quad c \in H_k(X, R), \quad k \geq 0. \quad (1.2.3)$$

which give an alternate description to the Kronecker pairing. We list in the proposition below some basic properties of the cap product.

**Proposition 1.2.1.** *Let  $X$  be a topological space, and  $A_1, A_2, A_3 \subset X$ . Set  $A = A_1 \cup A_2 \cup A_3$ . Then for any  $\alpha_i \in H^\bullet(X, A_i; R)$ ,  $i = 1, 2$  and any  $c \in H_\bullet(X, A; R)$  we have*

$$\alpha_1 \cap (\alpha_2 \cap c) = (\alpha_1 \cup \alpha_2) \cap c,$$

if the appropriate excisive assumptions are made.

If  $f : X \rightarrow Y$  is a continuous map, and  $B_i$  are subsets of  $Y$  such that  $B_i \supset f(A_i)$ ,  $i = 1, 2$ . Then we have projection formula

$$f_*(f^* \beta_1 \cap c) = \beta_1 \cap f_* c, \quad \forall \beta_1 \in H^\bullet(Y, B_1; R), \quad c \in H_\bullet(X, A_1 \cup A_2; R), \quad (1.2.4)$$

if the appropriate excisive assumptions are made.  $\square$

### 1.3. Local homology and cohomology

For any closed subset  $C$  of topological space  $X$ , we set

$$H_\bullet^C(X, R) := H_\bullet(X, X \setminus C; R), \quad H_C^\bullet(X, R) := H^\bullet(X, X \setminus C; R)$$

We say that  $H_\bullet^C(X, R)$  is the *local homology of  $X$  along  $C$*  and that  $H_C^\bullet(X, R)$  is the *local cohomology of  $X$  along  $C$* . Let us justify the usage of the attribute *local*. For any neighborhood  $N$  of  $C$  in  $X$  we have  $X \setminus N \subset X \setminus C$  and thus, excising  $X \setminus N$ , we deduce that the inclusion induced isomorphism

$$H_\bullet(N, N \setminus C; R) \rightarrow H_\bullet(X, X \setminus C; R)$$

is an isomorphism. In other words,

$$H_\bullet^C(N, R) \cong H_\bullet^C(X, R) \quad \text{for any closed neighborhood } N \text{ of } C \text{ in } X.$$

This shows that  $H_\bullet^C(X, R)$  depends only on the behavior of  $X$  in an arbitrarily small neighborhood of  $C$ , whence the attribute *local*.

From the relative long Mayer-Vietoris sequence (1.1.4) we deduce that for any closed sets  $C_1, C_2 \in X$  we have a long exact sequence

$$\cdots \rightarrow H_k^{C_1 \cup C_2}(X, R) \rightarrow H_k^{C_1}(X, R) \oplus H_k^{C_2}(X, R) \rightarrow H_k^{C_1 \cap C_2}(X, R) \rightarrow H_{k-1}^{C_1 \cup C_2}(X, R) \rightarrow \cdots \quad (1.3.1)$$

Observe that if  $C_1 \subset C_2$  then  $X \setminus C_1 \supset X \setminus C_2$ , and thus we have a restriction map

$$\rho_{C_1, C_2} : H_\bullet^{C_2}(X, R) \rightarrow H_\bullet^{C_1}(X).$$

**Example 1.3.1.** Suppose  $X = \mathbb{R}^n$ , and  $C$  is the closed set consisting of a single point  $x_0$ . We set  $H_\bullet^{x_0}(\mathbb{R}^n) := H_\bullet^{\{x_0\}}(\mathbb{R}^n)$ . Then

$$H_k^{x_0}(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases} \quad (1.3.2)$$

We begin with the case  $n = 1$ . We only need to prove that  $H_1^{x_0}(\mathbb{R}) \cong \mathbb{Z}$ . Assume  $x_0 = 0$ . Let  $I = [-1, 1]$ . By excision and homotopy invariance we deduce that

$$H_\bullet^{x_0}(\mathbb{R}) = H_\bullet(I, \partial I).$$

From the long exact sequence of the pair  $(I, \partial I)$  we obtain the exact sequence

$$0 = H_1(I) \rightarrow H_1(I, \partial I) \xrightarrow{\partial} H_0(\partial I) \xrightarrow{i_*} H_0(I) \rightarrow 0 = H_0(I, \partial I).$$

The interval  $I$  is connected and we have a canonical isomorphism

$$\mathbb{Z} \rightarrow H_0(I), \quad \mathbb{Z} \ni n \mapsto n|t),$$

where  $t$  is an arbitrary point in  $I$  regarded in the obvious fashion as a singular 0-simplex. We also have a canonical isomorphism

$$\mathbb{Z}^2 \ni (j, k) \mapsto j|-1\rangle + k|1\rangle \in H_0(\partial I)$$

The morphism  $i_*$  can then be described as

$$j|-1\rangle + k|1\rangle \mapsto (j+k)|t\rangle$$

so that  $\ker i_* \cong \mathbb{Z}$ . This groups has two generators  $\mathbf{g}_+ = |1\rangle - |-1\rangle$  and  $\mathbf{g}_- = |-1\rangle - |1\rangle$ .

The Mayer-Vietoris connecting  $\partial : H_1(I, \partial I) \rightarrow H_0(\partial I)$  induces an isomorphism

$$\partial : H_1(I, \partial I) \rightarrow \ker i_*.$$

Thus, either of the elements  $\partial^{-1}\mathbf{g}_\pm$  is a generator of  $H_1(I, \partial I)$ . Let us observe that  $\partial^{-1}\mathbf{g}_+$  can be represented by the singular chain

$$\mu_1 : \Delta_1 \rightarrow [-1, 1], \quad \Delta_1 \ni (t_0, t_1) \mapsto t_1 - t_0 \in [0, 1], \quad (1.3.3)$$

because  $\partial|\mu_1\rangle = \mathbf{g}_+$ . We say that  $|\mu_+\rangle$  is the *canonical generator* of  $H_1(I, \partial I)$ .

Now observe that

$$(\mathbb{R}^n, \mathbb{R}^n \setminus 0) = (\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \setminus 0) \times (\mathbb{R}, \mathbb{R} \setminus 0)$$

and using Künneth theorem we deduce (1.3.2). Moreover, we can inductively describe a canonical generator  $\mu_n$  of  $H_n^{x_0}(\mathbb{R}^n)$  described by the equality

$$\mu_n = \mu_{n-1} \times \mu_1.$$

Note more generally that if  $C$  is a compact convex subset of  $\mathbb{R}^n$ , then for any convenient ring  $R$  we have

$$H_k^C(\mathbb{R}^n, R) \cong \begin{cases} R, & k = n, \\ 0, & k \neq n. \end{cases} \quad (1.3.4)$$

□





# Poincaré duality

In this section, we want to describe, without proofs, several versions of the Poincaré duality theorem, and then discuss a few applications.

## 2.1. Manifolds and orientability

Recall that a *topological manifold* of dimension  $n$  is a Hausdorff paracompact space  $M$  such that every point has an open neighborhood homeomorphic to  $\mathbb{R}^n$ . The integer  $n$  is called the *dimension* of  $M$ .

For every convenient ring  $R$  and any closed subset  $C \subset M$  we set

$$\mathcal{O}_M(C, R) := H_n^C(M, R).$$

As indicated in the previous section, any inclusion  $C_1 \subset C_2$  determines a restriction map

$$\rho_{C_1, C_2} : \mathcal{O}_M(C_2, R) \rightarrow \mathcal{O}_M(C_1, R).$$

Note that for every  $x \in M$  we can choose an open neighborhood  $U_x$  and a homeomorphism

$$\Psi_x : U_x \rightarrow \mathbb{R}^n,$$

and we have

$$\mathcal{O}_M(x, R) = H_n^x(M, R) \cong H_n^x(U_x, R) \stackrel{(1.3.4)}{\cong} R.$$

In particular, the above equality proves that the dimension is a *topological invariant*. More generally, if  $K \subset U_x$  is a compact neighborhood of  $x$  such that  $\Psi_x(K)$  is convex, then

$$\mathcal{O}_M(K, R) \cong R. \tag{2.1.1}$$

We will refer to such neighborhoods of  $x$  as *compact convex neighborhoods*. We have the following result, [Hatch1, Lemma 3.27].

**Lemma 2.1.1.** *For any ring  $R$ , and any compact set  $K \subset M$  we have*

$$H_i^K(M, R) = 0, \quad \forall i > n = \dim M.$$

Moreover if  $\mu \in H_n^K(M, R) = \mathcal{O}_M(K, R)$  then

$$\mu = 0 \iff \rho_{x, K}(\mu) = 0 \in \mathcal{O}_M(x, R), \quad \forall x \in K.$$

□

**Definition 2.1.2.** (a) An  $R$ -orientation of  $M$  at  $x$  is an isomorphism of  $R$ -modules  $\mu_x : R \rightarrow \mathcal{O}_M(x, R)$ . Such an isomorphism is uniquely determined by the element  $\mu_x(1) \in \mathcal{O}_M(x, R)$ . For simplicity, we will identify  $\mu_x$  with  $\mu_x(1)$ .

(b) An orientation of  $M$  along the closed subset  $C \subset M$  is family of orientations

$$\{\mu_c \in \mathcal{O}_M(c, R); c \in C\}$$

depending continuously on  $c$  in the following sense. For every  $c_0 \in C$  there exists a compact convex neighborhood  $K_0$  and  $\mu_{K_0} \in \mathcal{O}_M(K_0, R)$  such that

$$\mu_c = \rho_{c, K_0}(\mu_{K_0}), \quad \forall c \in C \cap K_0.$$

We will denote by  $\mathbf{Or}_R(M, C)$  the collection of  $R$ -orientations of  $M$  along  $C$ . For simplicity, we set  $\mathbf{Or}_R(M) := \mathbf{Or}_R(M, M)$ . We set  $\mathbf{Or}(M) = \mathbf{Or}_{\mathbb{Z}}(M)$ .

(c) The manifold  $M$  is called  $R$ -orientable along  $C$  if it admits  $R$ -orientations along  $C$ . The manifold  $M$  is called  $R$ -orientable if it admits an orientation along itself. When  $R = \mathbb{Z}$  we will drop the prefix  $R$  from the above terminology and we will refer simply as orientation, orientability.  $\square$

**Proposition 2.1.3.** (a) The manifold  $M$  is  $R$ -orientable along a compact connected  $K$  subset if and only if

$$\mathcal{O}_M(K, R) \cong R.$$

Moreover, for any orientation  $\mu \in \mathbf{Or}_M(K, R)$  there exists a unique generator  $\mu_K \in \mathcal{O}_M(K, R)$  such that

$$\mu_x = \rho_{x, K}(\mu_K), \quad \forall x \in K.$$

Conversely, any generator  $\mu_K$  of  $\mathcal{O}_M(K, R)$  defines an orientation  $\mu$  of  $M$  along  $K$  defined by the above equality.

(b) Any manifold is  $\mathbb{F}_2$ -orientable along any closed subset.  $\square$

**Exercise 2.1.4.** Prove Proposition 2.1.3. *Hint:* Use (2.1.1), Lemma 2.1.1 and the Mayer-Vietoris sequence (1.3.1).  $\square$

To any manifold  $M$  we can associate its orientation cover  $\pi : \widetilde{M} \rightarrow M$ . This is a double cover of  $M$  defined as follows.

As a set  $\widetilde{M}$  is the disjoint union

$$\widetilde{M} = \bigsqcup_{x \in M} \mathcal{O}_M(x)^*,$$

where  $\mathcal{O}_M(x)^*$  denotes the set of generators of the infinite cyclic group  $\mathcal{O}_M(x)$ . Hence,  $\mathcal{O}_M(x)^*$  consists of two elements and therefore the natural projection

$$\pi : \bigsqcup_{x \in M} \mathcal{O}_M(x)^* \rightarrow M, \quad \mu \in \mathcal{O}_M(x)^* \mapsto \pi(\mu) = x$$

is two-to-one.

To describe the natural topology on  $\widetilde{M}$  we need to introduce some notations. For every  $\mu_0 \in \widetilde{M}$  and every compact convex neighborhood  $K$  of  $x_0 = \pi(\mu_0)$ , there exists a unique generator  $\mu_K \in \mathcal{O}_M(K)$  such that  $\rho_{x_0, K_0}(\mu_{K_0}) = \mu_0$ . For  $x \in K_0$  we set  $\mu_0(x) = \rho_{x, K_0}(\mu_{K_0})$ , and we define

$$\widetilde{K}_{\mu_0} = \{ \mu_0(x); x \in K_0 \}.$$

We declare a subset  $U \subset \widetilde{M}$  to be open if and only if, for every  $\mu \in U$  there exists a compact convex neighborhood  $K$  of  $\pi(\mu)$  we have  $\widetilde{K}_\mu \subset U$ . With this topology on  $\widetilde{M}$  the map  $\pi : \widetilde{M} \rightarrow M$  is a double cover.

**Proposition 2.1.5.** *A manifold  $M$  is orientable if and only if the orientation double cover  $\widetilde{M} \rightarrow M$  is trivial. Moreover, there exists a bijection between the collection of sections of this cover, and the orientations of  $M$ .*  $\square$

**Exercise 2.1.6.** (a) Prove Proposition 2.1.5.

(b) Prove that any simply connected manifold is orientable.  $\square$

**Proposition 2.1.7.** *Suppose  $M$  is a connected orientable manifold. Consider an orientation  $\mu \in \mathcal{O}r(M)$  described by a family  $\mu_x \in \mathcal{O}_M(x)$ ,  $x \in M$ . For any homeomorphism  $f : M \rightarrow N$  we define  $f_*\mu$  to be the family*

$$f_*\mu := \{ f_*\mu_x \in \mathcal{O}_N(f(x)); x \in M \}$$

*Then the family  $f_*\mu$  is also an orientation on  $N$ .*  $\square$

**Exercise 2.1.8.** Prove Proposition 2.1.7.  $\square$

**Exercise 2.1.9.** Suppose that  $M$  is a connected, orientable manifold. For any orientation  $\mu = \{ \mu_x \in \mathcal{O}_M(x); x \in M \}$  we define

$$-\mu := \{ -\mu_x \in \mathcal{O}_M(x); x \in M \}.$$

A homeomorphism  $f : M \rightarrow M$  is called *orientation preserving* (resp. *reversing*) if there exists an orientation  $\mu$  of  $M$  such that  $f_*\mu = \mu$  (resp.  $f_*\mu = -\mu$ ).

(a) Prove that the following statements are equivalent

(a1) The homeomorphism  $f$  is orientation preserving (resp. reversing).

(a2) For any orientation  $\mu$  on  $M$  we have  $f_*\mu = \mu$  (resp.  $f_*\mu = -\mu$ ).

(a3) There exists an orientation  $\mu$  on  $M$  and  $x \in M$  such that  $f_*\mu_x = \mu_{f(x)}$  (resp.  $f_*\mu_x = -\mu_{f(x)}$ ).

(b) Suppose  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an invertible linear transformation. Prove that  $A$  is orientation preserving if and only if  $\det A = 0$ .  $\square$

**Exercise 2.1.10.** Suppose  $M$  is a connected smooth  $n$ -dimensional manifold. Prove that the following statements are equivalent.

(a) The manifold  $M$  is orientable in the sense of Definition 2.1.2.

- (b) There exists a smooth coordinate atlas  $(U_\alpha)_{\alpha \in A}$  with coordinates  $(x_\alpha^i)$  along  $U_\alpha$  such that on the overlap  $U_{\alpha\beta} = U_\alpha \cap U_\beta$  we have

$$\det \left( \frac{\partial x_\alpha^i}{\partial x_\beta^j} \right)_{1 \leq i, j \leq n} > 0.$$

- (c) There exists a nowhere vanishing degree  $n$ -form  $\omega \in \Omega^n(M)$ .

*Hint:* Consult [N1, Sec. 3.4.2]. □

## 2.2. Various versions of Poincaré duality

Before we state the theorem, we need to recall an algebraic construction. A *directed set* is a partially order set (poset)  $(I, \prec)$  such that

$$\forall i, j \in I, \exists k \in I, k \succ i, j. \quad (2.2.1)$$

An *inductive family* of  $R$ -modules parameterized by a directed set  $I$  is a collection of  $R$ -modules  $(M_i)_{i \in I}$ , and morphisms  $\varphi_{ji} \in \text{Hom}_R(M_i, M_j)$ , one such morphism for each  $i \prec j$ , such that  $\varphi_{ii} = \mathbb{1}_{M_i}$  and  $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$ ,  $\forall i \prec j \prec k$ . The *inductive limit* (or *colimit*) of such a family is the  $R$ -module denoted  $\varinjlim_{i \in I} M_i$  and defined as follows.

As a set,  $\varinjlim_{i \in I} M_i$  is the quotient of the disjoint union  $\sqcup_{i \in I} M_i$  modulo the equivalence relation

$$M_i \ni x_i \sim x_j \in M_j \iff \exists k \succ i, j : \varphi_{ki}(x_i) = \varphi_{kj}(x_j).$$

The condition (2.2.1) guarantees that this is an equivalence relation. If we denote by  $[x_i]$  the equivalence class of  $[x_i]$  then we define

$$[x_i] + [x_j] := [\varphi_{ki}(x_i) + \varphi_{kj}(x_j)]$$

where  $k$  is any element  $k \succ i, j$ . Again (2.2.1) implies that this operation is well defined and induces an  $R$ -module structure on  $\varinjlim_{i \in I} M_i$ . The natural map

$$M_i \ni x_i \mapsto [x_i] \in \varinjlim_{i \in I} M_i$$

is a morphism of  $R$ -module. It is denoted by  $\varphi_i$  and it is called the *natural morphism*  $M_i \rightarrow \varinjlim_{i \in I} M_i$ .

The inductive limit construction has the following universality property. If  $\psi_i : M_i \rightarrow M$ ,  $i \in I$ , is a family of morphisms of  $R$ -modules such that for any  $i \prec j$  the diagram below is commutative

$$\begin{array}{ccc} & M & \\ \psi_i \nearrow & & \nwarrow \psi_j \\ M_i & \xrightarrow{\varphi_{ji}} & M_j \end{array}$$

then there exists a unique morphism of  $R$ -modules

$$\psi : \varinjlim_{i \in I} M_i \rightarrow M$$

such that for any  $i \in I$  the diagram below is commutative.

$$\begin{array}{ccc} \varinjlim_{i \in I} M_i & \xrightarrow{\psi} & M \\ & \swarrow \varphi_i \quad \searrow \psi_i & \\ & M_i & \end{array}$$

If  $K$  is a compact subset of a topological manifold  $M$ , then the collection  $\mathcal{N}_K$  of neighborhoods of  $K$  in  $M$  has a natural partial order  $\prec$  satisfying (2.2.1). More precisely, given neighborhoods  $U, V$  of  $K$  we declare  $U \prec V$  if  $U \supset V$ . For every  $i > 0$  we have an inductive family of  $R$ -modules

$$H^i(U, R), \quad U \in \mathcal{N}_K,$$

where for every  $U \supset V \supset K$  we define  $\varphi_{V,U} : H^i(U, R) \rightarrow H^i(V, R)$  to be the pullback induced by the natural inclusion  $V \hookrightarrow U$ . The inductive limit

$$\varinjlim_{U \in \mathcal{N}_K} H^i(U, R)$$

is called the  $i$ -th Čech cohomology of  $K$  with coefficients in  $R$  and it is denoted by  $\check{H}^i(K, R)$ .

If  $K$  is not “too wild” then the Čech cohomology of  $K$  coincides with the singular cohomology. For example, this happens if  $K$  is weakly locally contractible, i.e., for any  $x \in K$  and any neighborhood  $V$  of  $x \in K$  there exists a smaller neighborhood  $U$  of  $x$  in  $K$  such that the inclusion  $U \hookrightarrow V$  is homotopically trivial.

Suppose that  $M$  is a connected  $n$ -dimensional manifold, and  $K$  is a compact subset such that  $M$  is  $R$ -orientable along  $K$ . Fix an  $R$ -orientation  $\mu \in \mathbf{Or}_R(M, K)$  defined by an element  $\mu_K \in H^n(M, M \setminus K; R)$ .

For any neighborhood  $U \in \mathcal{N}_K$  we have an excision isomorphism

$$H_n(U, U \setminus K; R) \rightarrow H_n(M, M \setminus K; R)$$

and thus  $\mu_K$  determines a natural element  $\mu_K^U \in H_n(U, U \setminus K; R)$ . The cap product with  $\mu_K^U$  redetermines a morphism

$$\cap \mu_K^U : H^i(U, R) \rightarrow H_{n-i}(U, U \setminus K; R) \cong H_{n-i}^K(M, R).$$

Moreover, if  $U \supset V$  then the diagram below is commutative

$$\begin{array}{ccc} & H_{n-i}^K(M, R) & \\ \cap \mu_K^U \nearrow & & \nwarrow \cap \mu_K^V \\ H^i(U, R) & \xrightarrow{\rho_{V,U}} & H^i(V, R) \end{array}$$

Passing to inductive limit we obtain a morphism  $\check{H}^i(K, R) \rightarrow H_{n-i}^K(M, R)$  which, for simplicity, we denote by  $\cap \mu_K$ . It coincides with the cap product with  $\mu_K$  when  $H^i = \check{H}^i$ . We have the following fundamental result. For a proof we refer to [Bre, VI.8], [Do, VIII.7] or [Hatch1, Sec. 3.3].

**Theorem 2.2.1** (Poincaré duality). *If the  $n$ -dimensional manifold  $M$  is  $R$ -orientable along the compact set  $K$ , then any orientation  $\mu_K \in \mathbf{Or}_M(K, R)$  determines isomorphisms*

$$\cap \mu_K : \check{H}^i(K, R) \rightarrow H_{n-i}^K(M, R), \quad \forall i = 0, \dots, n. \quad \square$$

**Remark 2.2.2.** (a) Let us observe that if  $K_1 \subset K_2$  are two compact subsets of the manifold  $M$  of dimension  $n$ , then we have a commutative diagram

$$\begin{array}{ccc} \check{H}^i(K_2, R) & \xrightarrow{\cap \mu_{K_2}} & H_{n-i}^{K_2}(M, R) \\ \downarrow & & \downarrow \\ \check{H}^i(K_1, R) & \xrightarrow{\cap \mu_{K_1}} & H_{n-i}^{K_1}(M, R) \end{array} \quad (2.2.2)$$

where the vertical arrows are the restriction maps. Loosely speaking, if a cohomology class  $c \in H^i(K_1)$  “extends” to a cohomology class on  $K_2$ , then the Poincaré dual of  $c$  can also be extended to  $K_2$ .

(b) When the compact  $K$  is a deformation retract of one of its open neighborhoods  $U$  in  $M$ , and  $U$  is  $R$ -orientable then we have another duality

$$H_K^{n-i}(M, R) \xrightarrow{\cap \mu_U} H_i(K).$$

For a proof we refer to [Iv, Thm. IX.4.7].  $\square$

If  $M$  happens to be compact, and we choose  $K = M$  in the above theorem, we deduce the classical version of the Poincaré duality.

**Corollary 2.2.3.** *If  $M$  is a compact,  $R$ -orientable  $n$ -dimensional manifold, and  $\mu_M \in H_n(M, R)$  is an  $R$ -orientation on  $M$  then the map*

$$\cap \mu_M : H^k(M, R) \rightarrow H_{n-k}(M, R)$$

*is an isomorphism for any  $0 \leq k \leq n$ .*  $\square$

Recall that a *manifold with boundary* is a topological pair  $(M, \partial M)$  satisfying the following conditions.

- The set  $\text{int}(M) := M \setminus \partial M$  is a topological manifold. Its dimension is called the dimension of the manifold with boundary.
- The set  $\partial M$  is closed in  $M$  and it is called the *boundary*. It is a manifold of dimension  $\dim \partial M = \dim \text{int}(M) - 1$ .
- There exists an open neighborhood  $N$  of  $\partial M$  in  $M$  and a homeomorphism

$$\Psi : (-1, 0] \times \partial M \rightarrow N$$

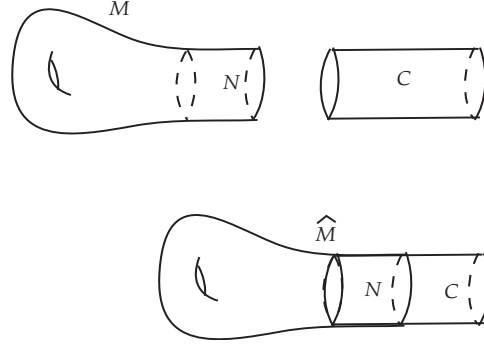
such that  $\Psi(\{0\} \times \partial M) = \partial M$ . Such a neighborhood is called a *neck* of the boundary.

Given a manifold with boundary  $(M, \partial M)$  and a neck  $N$  of  $\partial M$ , we can form the *noncompact manifold*  $\widehat{M}$  by attaching the cylinder  $C = (-1, 1) \times \partial M$  to the neck along the portion  $(-1, 0] \times \partial M$ . We will refer to  $\widehat{M}$  as the *neck extension* of the manifold with boundary.

A manifold with boundary is called  *$R$ -orientable* if it admits an  $R$ -orientable neck extension.

Suppose that  $(M, \partial M)$  is a *compact  $R$ -orientable  $n$ -dimensional manifold with boundary*. An orientation on a neck extension  $\widehat{M}$  induces an orientation on  $M$  described by an element

$$\mu_M \in H_\bullet^M(\widehat{M}, R) \cong H_\bullet(M, \partial M; R).$$



**Figure 2.1.** A neck extension of a manifold with boundary.

If  $\partial : H_n(M, \partial M, R) \rightarrow H_{n-1}(\partial M, R)$  is the connecting morphism in the long exact sequence of the pair  $(M, \partial M)$  then (see [Bre, Lemma VI.9.1]) the element  $\partial\mu_M \in H_{n-1}(\partial M, R)$  defines an orientation on  $\partial M$ . We will refer to it as the *induced orientation* and we will denote it by  $\mu_{\partial M}$ .

**Theorem 2.2.4** (Poincaré-Lefschetz duality). *Suppose that  $(M, \partial M)$  is a compact,  $n$ -dimensional  $R$ -orientable manifold with boundary. If  $\mu_M \in H_n(M, \partial M; R)$  is an  $R$ -orientation class, then for every  $0 \leq k \leq n$  the map*

$$\cap \mu_M : H^k(M, R) \rightarrow H_{n-k}(M, \partial M; R) \tag{2.2.3}$$

is an isomorphism of  $R$ -modules. If  $i : \partial M \rightarrow M$  denotes the natural inclusion, then the diagram below

$$\begin{array}{ccccc} H^k(M, R) & \xrightarrow{i^*} & H^k(\partial M, R) & \xrightarrow{(-1)^{n-k-1}\delta} & H^{k+1}(M, \partial M; R) \\ \cap \mu_M \downarrow & & \cap \mu_{\partial M} \downarrow & & \cap \mu_M \downarrow \\ H_{n-k}(M, \partial M; R) & \xrightarrow{\partial} & H_{(n-1)-k}(\partial M, R) & \xrightarrow{i_*} & H_{n-1-k}(M, R) \end{array}$$

is commutative.

**Proof.** The commutativity is an immediate consequence of the boundary formula (1.2.2) and the definition of the connecting morphism.

Applying Theorem 2.2.1 to the manifold  $\widehat{M}$  and the compact set  $K = M \subset \widehat{M}$  we conclude that the morphism (2.2.3) is an isomorphism.  $\square$

**Corollary 2.2.5.** *If  $(M, \partial M)$  is an compact,  $R$ -orientable,  $n$ -dimensional manifold with boundary and  $\mu_M \in H_n(M, \partial M; R)$  is an orientation class on  $M$  then the morphism*

$$\cap \mu_M : H^k(M, \partial M; R) \rightarrow H_{n-k}(M, R)$$

is an isomorphism.

**Proof.** For simplicity, we drop the ring  $R$  from notation. The isomorphism follows from the five lemma applied to the commutative diagram below relating the homological and cohomological long exact sequences of the pair  $(M, \partial M)$  in which all but the dotted morphisms  $\cap \mu_M : H^j(M, \partial M) \rightarrow$

$H_{n-j}(M)$  are isomorphisms.

$$\begin{array}{ccccccc}
\cdots H^k(M) & \xrightarrow{i^*} & H^k(\partial M) & \xrightarrow{(-1)^{n-k-1}\delta} & H^{k+1}(M, \partial M) & \longrightarrow & H^{k+1}(M) \cdots \\
\downarrow \cap \mu_M & & \downarrow \cap \mu_{\partial M} & & \downarrow \cap \mu_M \text{ (dashed)} & & \downarrow \cap \mu_M \\
\cdots H_{n-k}(M, \partial M) & \xrightarrow{\partial} & H_{(n-1)-k}(\partial M) & \xrightarrow{i_*} & H_{n-1-k}(M) & \longrightarrow & H_{n-k-1}(M, \partial M) \cdots
\end{array}$$

□

### 2.3. Intersection theory

Suppose  $M$  is a compact, connected,  $R$ -orientable  $n$ -dimensional manifold. We fix an orientation class  $\mu_M \in H_n(M, R)$ . We denote by  $PD_M$  the inverse of the Poincaré duality isomorphism

$$\cap \mu_M : H^{n-k}(M, R) \rightarrow H_k(M, R).$$

More precisely  $PD_M$  is a the morphism  $PD_M : H_k(M, R) \rightarrow H^{n-k}(M, R)$  uniquely defined by the equality

$$PD_M(c) \cap \mu_M = c, \quad \forall c \in H_k(M, R).$$

The *Poincaré dual* of a cycle  $c \in H_k(M, R)$  is the cohomology class  $PD_M(c) \in H^{n-k}(M, R)$ . When no confusion is possible we will use the simpler notation  $c^\dagger := PD_M(c)$ . Note that

$$\mu_M^\dagger = \mathbf{1}_M \in H^0(M, R), \quad (2.3.1)$$

where  $\mathbf{1}_M$  is the canonical element in  $H^0(M, R)$ .

**Exercise 2.3.1.** Prove the equality (2.3.1). □

For any  $\alpha \in H^k(M, R)$  we have

$$\langle \alpha, c \rangle_\kappa = \langle \alpha, c^\dagger \cap \mu_M \rangle_\kappa = \langle \alpha \cup c^\dagger, \mu_M \rangle_\kappa.$$

If  $c_i \in H_{k_i}(M, R)$ ,  $i = 0, 1$ , then we define  $c_0 \bullet c_1 \in H_{n-(k_0+k_1)}(M, R)$  by the equality

$$c_0 \bullet c_1 = (c_0^\dagger \cup c_1^\dagger) \cap \mu_M = c_0^\dagger \cap c_1. \quad (2.3.2)$$

The cycle  $c_0 \bullet c_1$  is called the *intersection cycle* of  $c_0$  and  $c_1$ . The resulting map

$$H_{k_0}(M, R) \times H_{k_1}(M, R) \rightarrow H_{n-(k_0+k_1)}(M, R), \quad (c_0, c_1) \mapsto c_0^\dagger \cap c_1.$$

is  $R$ -bilinear and it is called the *intersection pairing* in dimensions  $k_0, k_1$ .

An interesting case arises when  $n$  is even dimensional,  $n = 2k$  and  $k_0 = k_1 = k$ . We obtain an  $R$ -bilinear map

$$\hat{Q}_M : H_k(M, R) \times H_k(M, R) \rightarrow R, \quad \hat{Q}_M(c_0, c_1) = \langle \mathbf{1}_M, c_0^\dagger \cap c_1 \rangle_\kappa = \langle c_0^\dagger \cup c_1^\dagger, \mu_M \rangle_\kappa.$$

Let us point out that if  $z \in H_0(C, R)$  is a 0-cycle, then we can write  $z$  as a finite formal linear combination

$$z = \sum_i r_i |x_i\rangle,$$

where  $r_i \in R$  and  $x_i$  are points in  $M$  regarded in the obvious fashion as 0-simplices. Then

$$\langle \mathbf{1}_M, z \rangle_k = \sum_i r_i.$$



The form  $\hat{Q}_M$  is skew-symmetric if  $k$  is odd and symmetric if  $k$  is even. Observe that if  $c \in H_k(M, R)$  is a torsion element, i.e.,  $r_0 c = 0$  for some  $r_0 \neq 0$  then

$$\hat{Q}_M(c, c') = 0, \quad \forall c' \in H_k(M, R),$$

because the convenient ring  $R$  is an integral domain. If we set

$$\bar{H}_k(M) := H_k(M, R) / \text{Tors}_R H_k(M, R),$$

then we deduce that  $Q_M$  induces an  $R$ -bilinear map

$$Q_M = Q_{M,R} : \bar{H}_k(M, R) \times \bar{H}_k(M, R) \rightarrow R,$$

called the  $R$ -intersection form of  $M$ . When  $R = \mathbb{Z}$  we will refer to  $Q_M$  simply as the *intersection form* of the oriented manifold  $M$ . The intersection form defines an  $R$ -linear map

$$Q_M^\dagger : \bar{H}_k(M, R) \rightarrow \text{Hom}_R(\bar{H}_k(M, R), R),$$

$$\bar{H}_k(M, R) \ni c \mapsto Q_M^\dagger c \in \text{Hom}_R(\bar{H}_k(M, R), R), \quad Q_M^\dagger c(c') := Q_M(c, c'), \quad \forall c' \in \bar{H}_k(M, R).$$

More explicitly,

$$Q_M^\dagger c = \kappa(c^\dagger) = \kappa \circ PD_M(c),$$

where  $\kappa : H^k(M, R) \rightarrow \text{Hom}_R(H_k(M, R), R)$  is the Kronecker morphism.

**Theorem 2.3.2.** *Suppose  $M$  is a compact, connected  $R$ -oriented manifold of dimension  $n = 2k$ . Then the intersection form  $Q_M$  is  $R$ -nondegenerate, i.e., the morphism  $Q_M^\dagger$  is an isomorphism of  $R$ -modules.*

**Proof.** Let us first observe that  $M$  has finite homological type because  $M$  is homotopic to a compact  $CW$ -complex, [Hatch1, Appendix A]. We distinguish two cases.

**1.** *The ring  $R$  is a field  $R \cong \mathbb{Q}, \mathbb{R}, \mathbb{F}_p$ .  $\bar{H}_k(M, R) \cong H_k(M, R)$  is a finite dimensional  $R$ -vector space, and from the universal coefficients theorem we deduce that the Kronecker morphism*

$$\kappa : H^k(M, R) \rightarrow \text{Hom}_R(H_k(M, R), R)$$

is an isomorphism of vector spaces. Hence  $Q_M^\dagger$  is an isomorphism since it is the composition of two isomorphisms.

**2.**  $R \cong \mathbb{Z}$ . We have

$$Q_M^\dagger = \kappa \circ PD_M : H_k(M) \rightarrow H^k(M) \xrightarrow{\kappa} \text{Hom}_{\mathbb{Z}}(H_k(M), \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(\bar{H}_k(M), \mathbb{Z}).$$

The morphism  $PD_M$  induces an isomorphism between  $\bar{H}_k(M) \rightarrow H^k(M)/\text{torsion}$ , and the universal coefficients theorem implies that the Kronecker morphism induces an isomorphism

$$\kappa : H^k(M)/\text{torsion} \rightarrow \text{Hom}_{\mathbb{Z}}(\bar{H}_k(M), \mathbb{Z}).$$

□

Suppose  $M$  is  $\mathbb{Z}$ -orientable,  $\dim M = 4k$ . From the universal coefficients theorem for homology we deduce that  $M$  is also  $\mathbb{R}$ -orientable, and we can regard  $\bar{H}_{n/2}(M)$  as a lattice in  $H_{n/2}(M, \mathbb{R})$ , i.e., a subgroup of the vector space  $H_{n/2}(M, \mathbb{R})$ , such that a  $\mathbb{Z}$ -basis of  $\bar{H}_{n/2}(M)$  is also a  $\mathbb{R}$ -basis of  $H_{n/2}(M, \mathbb{R})$ . For simplicity we set

$$L := \bar{H}_{n/2}(M), \quad L_{\mathbb{R}} = H_{n/2}(M, \mathbb{R}), \quad b_{n/2} := \dim_{\mathbb{R}} H_{n/2}(M, \mathbb{R}).$$

By choosing a  $\mathbb{Z}$ -basis  $\underline{c} := (c_i)$  of  $L$  we can represent  $Q_M$  by a matrix  $Q = (q_{ij})_{1 \leq i, j \leq b_{n/2}}$ . This matrix is symmetric because  $Q_M$  is symmetric. The matrix  $Q$  defines a  $\mathbb{Z}$ -linear map

$$Q^\dagger : \mathbb{Z}^{b_{n/2}} \rightarrow \mathbb{Z}^{b_{n/2}}, \quad \sum_j m_j c_j \mapsto \sum_{i,j} q_{ij} m_j c_i,$$

which can be identified with the linear map  $Q_M^\dagger$ . Thus  $\det Q = \pm 1$ .

The  $\mathbb{R}$ -intersection form is a symmetric, bilinear nondegenerate form

$$Q_{M,\mathbb{R}} : L_{\mathbb{R}} \times L_{\mathbb{R}} \rightarrow \mathbb{R}.$$

The  $\mathbb{Z}$ -basis  $\underline{c} = (c_i)$  of  $L$  defines a  $\mathbb{R}$ -basis of  $L_{\mathbb{R}}$ , and  $Q_{M,\mathbb{R}}$  is represented by the symmetric matrix  $Q_{\underline{c}}$  as above. The matrix  $Q_{\underline{c}}$  has only real, nonzero eigenvalues. We denote by  $b_{n/2}^\pm(M, \underline{c})$  the number of positive/negative eigenvalues of the matrix  $Q_{\underline{c}}$ . The numbers  $b_{n/2}^\pm(M, \underline{c})$  are independent of the choice of basis  $\underline{c}$  and we will denote them by  $b_{n/2}(M)^\pm$ . Their difference

$$\tau_M := b_{n/2}^+(M) - b_{n/2}^-(M)$$

is called the *signature* of the manifold  $M$ . If  $M$  is not connected, then

$$\tau_M = \sum_{\ell} \tau_{M_\ell},$$

where  $M_\ell$  are the connected components of  $M$ . We want to emphasize that the signature is an invariant of compact, *oriented* manifolds whose dimensions are divisible by 4, and it would be appropriate to denote it by  $\tau_{M, \mu_M}$  to indicate the dependence on the orientation. If we change the orientation to the opposite orientation, then the signature changes sign as well

$$\tau_{M, -\mu_M} = -\tau_{M, \mu_M}.$$

**Remark 2.3.3.** If  $M$  is a smooth, compact, oriented manifold of dimension  $M$ , then the singular cohomology with real coefficients is naturally isomorphic with the DeRham cohomology of  $M$ , i.e., the cohomology of the DeRham complex,

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots$$

Moreover, the cup product is described by the wedge product of forms. The intersection form  $Q_{M, bR} : H^{2k}(M, \mathbb{R}) \times H^{2k}(M, \mathbb{R}) \rightarrow \mathbb{R}$  can be given the alternate description

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta, \quad \forall \alpha, \beta \in \Omega^{2k}(M), \quad d\alpha = d\beta = 0.$$

For a proof we refer to [BT, III. §14]. □

**Proposition 2.3.4** (Thom). *Suppose  $(M, \partial M)$  is a compact, orientable manifold with boundary, and  $\dim M = 4k + 1$ . Fix an orientation class  $\mu_M \in H_{4k+1}(M, \partial M)$  and denote by  $\mu_{\partial M}$  the induced orientation on the boundary. Then*

$$\tau_{\partial M, \mu_{\partial M}} = 0.$$

**Proof.** This equality is a consequence of two facts of rather different natures: an algebraic fact, and a topological fact. First, let us introduce some terminology.

If  $V$  is a finite dimensional real vector space and  $B : V \times V \rightarrow \mathbb{R}$  is a bilinear map, then a subspace  $L \subset V$  is called *lagrangian* with respect to  $B$  if and only if

$$B(v, x) = 0, \quad \forall x \in L \iff v \in L.$$

We have the following algebraic fact. Its proof is left to the reader as an exercise.

**Lemma 2.3.5.** *Suppose  $Q : V \times V \rightarrow \mathbb{R}$  is a bilinear, symmetric, nondegenerate form on the finite dimensional real vector space  $V$ . Then the following statements are equivalent.*

- (a) *The signature of  $Q$  is zero.*
- (b) *There exists a subspace of  $V$  which is lagrangian with respect to  $Q$ .*

□

**Lemma 2.3.6.** *We set  $V = H^{2k}(\partial M, \mathbb{R})$  ( $2k = \frac{1}{2} \dim M$ ), and we denote by  $Q$  the intersection form on  $V$*

$$Q(\alpha, \beta) = \langle \alpha \cup \beta, \mu_{\partial M} \rangle_{\kappa} = \langle \beta \cup \alpha, \mu_{\partial M} \rangle_{\kappa}.$$

*Let  $L \subset V$  be the image of the natural morphism*

$$i^* : H^{2k}(M, \mathbb{R}) \rightarrow H^{2k}(\partial M, \mathbb{R}) = V.$$

*Then  $L$  is lagrangian with respect to  $Q$ .*

**Proof.** We have a commutative diagram

$$\begin{array}{ccccc} H^{2k}(M, \mathbb{R}) & \xrightarrow{i^*} & H^{2k}(\partial M, \mathbb{R}) & \xrightarrow{\delta} & H^{2k+1}(M, \partial M; \mathbb{R}) \\ \cap \mu_M \downarrow & & \cap \mu_{\partial M} \downarrow & & \cap \mu_M \downarrow \\ H_{2k+1}(M, \partial M; \mathbb{R}) & \xrightarrow{\partial} & H_{2k}(\partial M, \mathbb{R}) & \xrightarrow{i_*} & H_{2k}(M, \mathbb{R}) \end{array} \quad (2.3.3)$$

Observe that  $L = \ker \delta = \text{Image}(i^*)$ . We need to prove two things.

**A.** If  $\alpha_0 \in L$ , then  $\langle \alpha_0 \cup \alpha_1, \mu_{\partial M} \rangle_{\kappa} = 0, \forall \alpha_1 \in L$ .

**B.** If  $\alpha_0 \in V$  and  $\langle \alpha_0 \cup \alpha_1, \mu_{\partial M} \rangle_{\kappa} = 0, \forall \alpha_1 \in L$ , then  $\alpha_0 \in L$ .

**A.** Let  $\alpha_j \in L, j = 0, 1$ . Then  $\exists \hat{\alpha}_j \in H^{2k}(M, \mathbb{R})$  such that  $i^* \hat{\alpha}_j = \alpha_j$ . We have

$$\langle i^* \hat{\alpha}_1 \cup i^* \hat{\alpha}_0, \mu_{\partial M} \rangle_{\kappa} = \langle i^* \hat{\alpha}_1, i^* \hat{\alpha}_0 \cap \mu_{\partial M} \rangle_{\kappa}$$

(use the commutativity of the left square in (2.3.3))

$$\langle i^* \hat{\alpha}_1, \partial(\hat{\alpha}_0 \cap \mu_M) \rangle_{\kappa} = \langle \hat{\alpha}_1, i_* \partial(\hat{\alpha}_0 \cap \mu_M) \rangle_{\kappa} = 0,$$

since  $i_* \circ \partial = 0$  due to the exactness of the bottom row in (2.3.3).

**B.** Suppose  $\alpha_0 \in V \setminus L$  and  $\langle \alpha_0 \cup i^* \hat{\alpha}_1, \mu_{\partial M} \rangle_{\kappa} = 0, \forall \hat{\alpha}_1 \in H^{2k}(M, \mathbb{R})$ . Then

$$\langle i^* \hat{\alpha}_1, \alpha_0 \cap \mu_{\partial M} \rangle_{\kappa} = 0, \quad \forall \hat{\alpha}_1 \in H^{2k}(M, \mathbb{R}).$$

so that

$$\langle \hat{\alpha}_1, i_* (\alpha_0 \cap \mu_{\partial M}) \rangle_{\kappa} = 0 \quad \forall \hat{\alpha}_1 \in H^{2k}(M, \mathbb{R}).$$

We deduce that  $i_* (\alpha_0 \cap \mu_{\partial M}) = 0$  because the Kronecker pairing  $H^{2k}(M, \mathbb{R}) \rightarrow H_{2k}(M, \mathbb{R}) \rightarrow \mathbb{R}$  is nondegenerate. From the commutativity of the right square in (2.3.3) we deduce that  $\delta \alpha_0 \cap \mu_M = 0$ .

Using the Poninarcé-Lefschetz duality we deduce  $\delta\alpha_0 = 0$ , and from the exactness of the top line we conclude that  $\alpha_0 \in L$ . □

Proposition 2.3.4 is now a consequence of Lemma 2.3.5 and 2.3.6 □

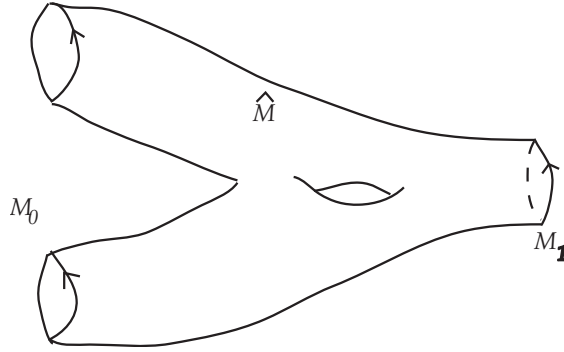
**Exercise 2.3.7.** Prove Lemma 2.3.5. □

**Definition 2.3.8.** (a) Two smooth, compact  $n$ -dimensional manifolds  $M_0, M_1$  are said to be *cobordant* if there exists a smooth compact  $(n + 1)$ -dimensional manifold with boundary  $(\widehat{M}, \partial\widehat{M})$  such that  $\partial\widehat{M}$  is diffeomorphic to  $M_0 \sqcup M_1$ .

(b) Two smooth, compact, oriented  $n$ -dimensional manifolds  $(M_0, \mu_0), (M_1, \mu_1)$  are said to be *orientedly cobordant* if there exists an a compact, oriented manifold with boundary  $(\widehat{M}, \partial\widehat{M}, \hat{\mu})$ , and an orientation preserving diffeomorphism (see Figure 2.2)

$$(\partial\widehat{M}, \partial\hat{\mu}) \cong (M_1, \mu_1) \sqcup (M_0, -\mu_0).$$

We write this as  $M_0 \sim_+ M_1$ . We say that  $\widehat{M}$  is an *oriented cobordism* connecting  $M_0$  to  $M_1$ . The manifold  $M_0$  is sometime referred to as the *incoming* boundary component while  $M_1$  is called the *outgoing* boundary component. □



**Figure 2.2.** An oriented cobordism.

Observe that the cobordism relation is an equivalence relation. We denote by  $\Omega_n$  the cobordism classes of compact  $n$ -dimensional manifolds, and by  $\Omega_n^+$  the collection of oriented cobordism classes of compact, oriented  $n$ -dimensional manifolds. For every compact, oriented  $n$ -dimensional manifold  $(M, \mu_M)$ , we denote by  $[M, \mu_M] \in \Omega_n^+$  its cobordism class.

**Proposition 2.3.9.** We define an operation  $+$  on  $\Omega_n^+$  by setting

$$[M_0, \mu_0] + [M_1, \mu_1] = [M_0 \sqcup M_1, \mu_0 \sqcup \mu_1].$$

Then  $\Omega_n^+$  is an Abelian group with neutral element  $[S^n, \mu_{S^n}]$ , where  $\mu_{S^n}$  denotes the orientation on the sphere  $S^n$  as boundary of the unit ball in  $\mathbb{R}^{n+1}$ . Moreover

$$[M, \mu_M] + [M, -\mu_M] = [S^n, \mu_{S^n}].$$

□

**Exercise 2.3.10.** Prove Proposition 2.3.9. □

**Corollary 2.3.11.** Suppose that  $(M_0, \mu_0)$  and  $(M_1, \mu_1)$  are two compact, oriented manifolds of dimension  $n = 4k$ . If  $(M_0, \mu_0)$  and  $(M_1, \mu_1)$  are orientedly cobordant, then

$$\tau_{M_0, \mu_0} = \tau_{M_1, \mu_1}.$$

In particular, the signature defines a group morphism  $\tau : \Omega_{4k}^+ \rightarrow \mathbb{Z}$ .

**Proof.** Suppose  $(\widehat{M}, \partial\widehat{M}, \hat{\mu})$  is an oriented cobordism between  $(M_0, \mu_0)$  and  $(M_1, \mu_1)$  so that

$$(\partial\widehat{M}, \partial\hat{\mu}) \cong (M_1, \mu_1) \sqcup (M_0, -\mu_0).$$

From Proposition 2.3.4 we deduce

$$0 = \tau_{\partial\widehat{M}, \partial\hat{\mu}} = \tau_{M_1, \mu_1} + \tau_{M_0, -\mu_0} = \tau_{M_1, \mu_1} - \tau_{M_0, \mu_0}.$$

□

For uniformity we define the signature morphism  $\tau : \Omega_n^+ \rightarrow \mathbb{Z}$  to be zero if  $n$  is not divisible by 4. We can now organize the direct sum

$$\Omega_{\bullet}^+ := \bigoplus_{n \geq 0} \Omega_n^+$$

as a ring, in which the product operation is given by the cartesian product of oriented manifolds.

**Exercise 2.3.12.** Suppose  $(M_0, \mu_0), (M_1, \mu_1)$  are compact oriented manifolds. Then

$$\tau_{(M_0 \times M_1, \mu_0 \times \mu_1)} = \tau_{(M_0, \mu_0)} \tau_{(M_1, \mu_1)}.$$

□

The above exercise implies that the signature defines a morphism of rings

$$\tau : \Omega_{\bullet}^+ \rightarrow \mathbb{Z}.$$

In the remainder of this class we will try to elucidate the nature of this morphism. At this point we do not have enough technology to compute the signature of even the most basic of manifolds such as  $\mathbb{C}\mathbb{P}^{2k}$ .

Here is a brief preview of things to come. We will discuss the Thom isomorphism and Thom class, notions associated to vector bundles. We define the Euler class of a vector bundle and describe its role in the Gysin sequence. As an application, we will compute the cohomology rings of projective spaces.

This will allow us to introduce the Chern classes of a complex vector bundle using the elegant approach due to Grothendieck. The Pontryagin classes are then easily defined in terms of Chern classes. We then use the Pontryagin classes to construct Pontryagin numbers, which are oriented cobordism invariants of oriented manifolds, or more formally, morphisms of groups

$$\Omega_n^+ \rightarrow \mathbb{Q}$$

which are expressible as integrals of certain canonical forms on manifolds. A certain (infinite) linear combination of such Pontryagin numbers produces a ring morphism

$$L : \Omega_{\bullet}^+ \rightarrow \mathbb{Q}$$

such that

$$L(\mathbb{C}\mathbb{P}^{2k}) = \tau_{\mathbb{C}\mathbb{P}^{2k}}$$

A clever argument using Thom's cobordism work will allow us to conclude that  $L(M) = \tau_M$ , for any compact oriented manifold  $M$ .

# Vector bundles and classifying spaces

In this chapter we want to describe a few fundamental facts concerning vector bundles. *Throughout this chapter, the topological spaces will be tacitly assumed to be Hausdorff and paracompact.*

## 3.1. Definition and examples of vector bundles

Loosely speaking, a topological vector bundle over a topological space  $B$  is a “continuous” family  $(E_b)_{b \in B}$  of vector spaces parameterized by the topological space  $B$ . The formal definition is a mouthful, and requires a bit of extra terminology.

Given two continuous maps  $p_i : X_i \rightarrow Y$ ,  $i = 0, 1$  we can form *topological fiber product* is the topological space

$$X_0 \times_Y X_1 := \{(x_0, x_1) \in X_0 \times X_1; p_0(x_0) = p_1(x_1)\},$$

where the topology, is the topology as a subspace of  $X_0 \times X_1$ . Observe that the fiber product is equipped with natural maps

$$q_i : X_0 \times_Y X_1 \rightarrow X_i, \quad i = 0, 1,$$

and the resulting diagram below is commutative.

$$\begin{array}{ccc} X_0 \times_Y X_1 & \xrightarrow{q_1} & X_1 \\ q_0 \downarrow & & \downarrow p_1 \\ X_0 & \xrightarrow{p_0} & Y \end{array}$$

We denote by  $p_0 \times_Y p_1$  the natural map  $p_0 q_0 = p_1 q_1 : X_0 \times_Y X_1 \rightarrow Y$ . We will refer to the above diagram as the *Cartesian diagram associated to the fibered product*.

We let  $\mathbb{K}$  denote one of the fields  $\mathbb{R}$  or  $\mathbb{C}$ . For every finite dimensional  $\mathbb{K}$  vector space  $V$ , and any topological space  $B$  we set

$$\underline{V}_B := V \times B,$$

and we denote by  $\pi = \pi_{V,B}$  the natural projection  $\underline{V}_B \rightarrow B$ .

**Definition 3.1.1.** Let  $r$  be a positive integer. A  $\mathbb{K}$ -vector bundle of rank  $r$  over a topological space  $B$  is a triplet  $(E, p, B)$ , where  $E$  and  $B$  are topological spaces, and  $p : E \rightarrow B$  is a continuous map satisfying the following conditions.

- (a) For every  $b \in B$ , the fiber  $E_p = p^{-1}(b)$  is equipped with a structure of  $\mathbb{K}$ -vector space of dimension  $r$  depending continuously on  $b$ . More precisely, this means that there exist continuous maps

$$+ : E \times_B E \rightarrow E, \quad \cdot : \mathbb{K}_B \times_B E \rightarrow E$$

such that, for any  $b \in B$ , the map  $+$  sends the fiber  $E_b \times E_b = (p \times_B p)^{-1}(b)$  to the fiber  $E_b = p^{-1}(b)$ , the map  $\cdot$  sends the fiber  $\mathbb{K} \times E_b = (\pi \times_B p)^{-1}(b) \subset \mathbb{K}_B \times_B E$  to  $E_p \subset E$ , and the resulting structure  $(E_b, +, \cdot)$  is a  $\mathbb{K}$ -vector space of dimension  $r$ .

- (b) For any point  $b \in B$ , there exists an open neighborhood  $U$  of  $b$  and a continuous map  $\Psi_U : p^{-1}(U) \rightarrow \mathbb{K}_U^r$  such that for any  $b$  in  $B$  the map  $\Psi$  sends the fiber  $E_b$  to the fiber  $\mathbb{K}^r \times b$  and the resulting map is a linear isomorphism  $E_p \rightarrow \mathbb{K}^r$ . The map  $\Psi$  is called a *local trivialization*.

The space  $E$  is called the *total space* of the bundle, the space  $B$  is called the *base* of the bundle, while  $p$  is called the *canonical projection*. Condition (b) is usually referred to as *local triviality*. A *line bundle* is a rank 1 vector bundle.  $\square$

The notion of vector bundle is best understood by looking at a few examples.

**Example 3.1.2** (Trivial vector bundles). For any topological space  $B$  the trivial  $\mathbb{K}$ -vector bundle of rank  $r$  over  $r$  is  $\mathbb{K}_B^r \xrightarrow{\pi} B$ . This can be visualized as the trivial family of vector spaces parameterized by  $B$ : the same space  $\mathbb{K}^r$  for every  $b \in B$ . In general, for any finite dimensional vector space  $V$  we denote by  $\underline{V}_B$  the trivial vector bundle  $V \times B \rightarrow B$ .  $\square$

**Example 3.1.3.** Suppose  $p : E \rightarrow B$  is a vector bundle. For any subset  $S \subset B$ , we denote by  $E|_S \rightarrow S$  the *restriction* of  $E$  to  $S$ . This is the vector bundle with total space  $E|_S = p^{-1}(S)$ , and canonical projection given by the restriction of  $P$  to  $E|_S$ .  $\square$

**Example 3.1.4.** Suppose  $U, V$  are  $\mathbb{K}$ -vector spaces of dimensions  $\dim V = n, \dim U = n+r, r > 0$ . Suppose that  $A : S \rightarrow \text{Hom}_{\mathbb{K}}(U, V), s \mapsto A_s$ , is a continuous map such that for any  $s \in S$  the linear map  $A_s$  is onto. Then the family of vector spaces  $(\ker A_s)_{s \in S}$  can be organized as a vector bundle of rank  $r$  over  $S$ . Define

$$E = \{ (u, s) \in U \times S; A_s u = 0 \}.$$

Then  $E$  is a closed subset of  $U \times B$  and it is equipped with a canonical projection  $p : E \rightarrow B$ . The fiber of  $p$  over  $B$  is naturally identified with the vector space  $\ker A_s$ . We want to prove that  $E \xrightarrow{p} S$  is a vector bundle. The condition (a) in Definition 3.1.1 is clearly satisfied so we only need to check the local triviality condition (b).

Let  $s_0 \in S$ . Fix a basis  $(v_1, \dots, v_n)$  of  $V$  and a basis  $(e_1, \dots, e_n; f_1, \dots, f_r)$  of  $U$  such that

$$A_{s_0} e_i = v_i, \quad A_{s_0} f_j = 0, \quad \forall 1 \leq i \leq n, \quad 1 \leq j \leq r.$$

These choices of bases produce isomorphisms  $U \cong \mathbb{K}^n \oplus \mathbb{K}^r, V \cong \mathbb{K}^n$ . A linear map  $A : U \rightarrow V$  can be identified via these isomorphisms with a pair of linear maps

$$\hat{A} = A|_{\mathbb{K}^n} : \mathbb{K}^n \rightarrow \mathbb{K}^n, \quad A^0 = A|_{\mathbb{K}^r} : \mathbb{K}^r \rightarrow \mathbb{K}^n.$$



We have  $\hat{A}_{s_0} = \mathbb{1}_{\mathbb{K}^n}$  and  $A^0 = 0$ . For  $A : U \rightarrow V$  we have

$$\ker A = \{(\hat{x}, x^0) \in \mathbb{K}^n \oplus \mathbb{K}^r; \hat{A}\hat{x} + A^0x^0 = 0\}.$$

We can find a small open neighborhood  $\mathcal{O}$  of  $s_0$  in  $S$  such that  $\hat{A}_s$  is invertible for any  $s \in \mathcal{O}$ . If  $s \in \mathcal{O}$ , then

$$(\hat{x}, x^0) \in \ker A_s \iff \hat{x} = -\hat{A}_s^{-1}A^0x^0.$$

Now define  $\Phi : \mathbb{K}^r \times \mathcal{O} \rightarrow E_{\mathcal{O}} = p^{-1}(\mathcal{O})$  by the equality

$$\mathbb{K}^r \times \mathcal{O} \ni (x^0, s) \mapsto (-\hat{A}_s^{-1}A^0x^0 \oplus x^0, s) \in \ker A_s \times \{s\} \subset E_{\mathcal{O}}.$$

Then  $\Phi$  is a bijection and its inverse  $\Psi : E_{\mathcal{O}} \rightarrow \mathbb{K}^r_{\mathcal{O}}$  satisfies the local triviality condition (b) in Definition 3.1.1.

To see this general construction at work, consider the unit sphere

$$S^n = \{\vec{x} \in \mathbb{R}^{n+1}; |\vec{x}| = 1\},$$

and define  $A : S^n \rightarrow \text{Hom}(\mathbb{R}^{n+1}, \mathbb{R})$  by setting

$$A_{\vec{x}}u = \vec{x} \bullet u, \quad \forall \vec{x} \in S^n, \quad u \in \mathbb{R}^{n+1},$$

where  $\bullet$  denotes the canonical inner product in  $\mathbb{R}^{n+1}$ . Note that  $A_{\vec{x}}$  is surjective for any unit vector  $\vec{x}$ . Hence the family  $\ker A$  of kernels of the family  $(A_{\vec{x}})_{\vec{x} \in S^n}$  defines a rank  $n$  vector bundle over  $S^n$ . It is none other than the tangent bundle of  $S^n$  since for any  $\vec{x} \in S^n$  the tangent space  $T_{\vec{x}}S^n$  can be identified with the subspace of  $\mathbb{R}^{n+1}$  perpendicular to  $\vec{x}$ .  $\square$

**Example 3.1.5.** Suppose  $U, V$  are  $\mathbb{K}$  vector spaces of dimensions  $\dim U = m$ ,  $\dim V = n$ , and  $A : S \rightarrow \text{Hom}_{\mathbb{K}}(U, V)$  is a continuous family of linear maps such that  $\dim \ker A_s$  is independent of  $s$ . We denote by  $r$  the common dimension of the vector spaces  $\ker A_s$ . Then the family  $(\ker A_s)_{s \in S}$  can be organized as a topological  $\mathbb{K}$ -vector bundle of rank  $r$ . The total space is

$$E = \{(u, s) \in U \times S; A_s u = 0\}.$$

Clearly, condition (a) of Definition 3.1.1 is satisfied. To check the local triviality condition (b) we proceed as follows. Fix inner products on  $U$  and  $V$ . (If  $\mathbb{K} = \mathbb{C}$  then we assume that these inner products are Hermitian.) Let  $s_0 \in S$ , and denote by  $P_0$  the orthogonal projection onto  $W_0 = A_{s_0}(U)$ . The linear transformation

$$B_{s_0} := P_0 A_{s_0} : U \rightarrow W_0$$

is onto. Hence, there exists an open neighborhood  $\mathcal{N}$  of  $s_0$  such that  $B_s = P_0 A_s : U \rightarrow W_0$  is onto<sup>1</sup> for any  $s \in \mathcal{N}$ . Set  $W_s = A(U_s) \subset V$ . We have  $\dim W_s = \dim W_0$  for all  $s \in S$  so that the restriction of  $P_0$  to  $W_s$  is an isomorphism  $P_s : W_s \rightarrow W_0$  for any  $s \in \mathcal{N}$ . Hence

$$\ker B_s = \ker A_s, \quad \forall s \in \mathcal{N}$$

and thus we deduce that

$$\{(u, s) \in U \times \mathcal{N}; A_s u = 0\} = \{(u, s) \in U \times \mathcal{N}; B_s u = 0\}.$$

This shows that locally we can replace the family  $A_s$  with a family of surjective linear mappings. This places us in the situation investigated in Example 3.1.4.

<sup>1</sup>Can you see why?

To see this result at work, consider a  $\mathbb{K}$ -vector space of dimension  $n$  and denote by  $\mathbf{Gr}_k(V)$  the *Grassmannian of  $\mathbb{K}$ -vector subspaces of  $V$  of dimension  $r$* . To topologize  $\mathbf{Gr}_r(V)$  we fix an inner product on  $V$  (hermitian if  $\mathbb{K} = \mathbb{C}$ )

$$\langle -, - \rangle : V \times V \rightarrow \mathbb{K}.$$

For every subspace  $U \subset V$  we denote by  $P_U$  the orthogonal projection onto  $U$ , Let  $Q_U = \mathbb{1}_V - P_U$  so that  $Q_U$  is the orthogonal projection onto  $U^\perp$  and  $\ker Q_U = U$ .

We denote by  $\text{End}_{\mathbb{K}}^+(V)$  the real vector space of *selfadjoint* endomorphisms of  $V$ , i.e.,  $\mathbb{K}$ -linear maps  $A : V \rightarrow V$  such that

$$\langle Av_0, v_1 \rangle = \langle v_0, Av_1 \rangle, \quad \forall v_0, v_1 \in V.$$

Note that  $Q_U \in \text{End}^+(V)$  so that we have an embedding

$$\mathbf{Gr}_r(V) \ni U \mapsto Q_U \in \text{End}^+(V).$$

We use this embedding to topologize  $\mathbf{Gr}_r(V)$  as a subset of  $\text{End}^+(V)$ . We now have a continuous map

$$Q : \mathbf{Gr}_k(V) \rightarrow \text{Hom}_{\mathbb{K}}(V, V), \quad U \mapsto Q_U$$

such that  $\dim_{\mathbb{K}} \ker Q_U = r$ ,  $\forall U \in \mathbf{Gr}_r(V)$ . Hence the collection of kernels  $\{\ker Q_U\}_{U \in \mathbf{Gr}_r(V)}$  defines a rank  $r$  vector bundle over  $\mathbf{Gr}_r(V)$  with total space given by the *incidence set*

$$\mathcal{U}_k = \{ (v, U) \in V \times \mathbf{Gr}_k(V); \quad u \in U \}.$$

This vector bundle is called the *tautological* or *universal vector bundle* over  $\mathbf{Gr}_r(V)$  and we denote it by  $\mathcal{U}_{r,V}$ . We also set  $\mathcal{U}_{r,n}^{\mathbb{K}} := \mathcal{U}_{r,\mathbb{K}^n}$ . □

**Example 3.1.6** (Gluing cocycles). Suppose  $B$  is a topological space., and  $\mathbb{V}$  is a  $\mathbb{K}$ -vector space of dimension  $r$ . A  $\text{GL}(\mathbb{V})$ -valued *gluing cocycle* on  $B$  consists of an open cover  $\mathcal{U} = (U_\alpha)_{\alpha \in A}$  of  $B$  and a collection of continuous maps

$$g_{\beta\alpha} : U_{\alpha\beta} \rightarrow \text{GL}(\mathbb{V}) = \text{Aut}_{\mathbb{K}}(\mathbb{V}), \quad U_{\alpha\beta} := U_\alpha \cap U_\beta$$

such that, for any  $\alpha, \beta, \gamma \in A$ , and any  $x \in U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$  we have

$$g_{\gamma\alpha}(x) = g_{\gamma\beta}(x) \cdot g_{\beta\alpha}(x). \quad (3.1.1)$$

To such a gluing cocycle we associate in a canonical fashion a  $\mathbb{K}$ -vector bundle of rank  $r$  over  $B$  defined as follows. The total space  $E$  is the quotient of the disjoint union

$$\bigsqcup_{\alpha \in A} \mathbb{V} \times U_\alpha$$

modulo the equivalence relation

$$\mathbb{V} \times U_\alpha \ni (u_\alpha, x_\alpha) \sim (u_\beta, x_\beta) \in \mathbb{V} \times U_\beta \iff x_\alpha = x_\beta = x, \quad u_\beta = g_{\beta\alpha}(x)u_\alpha.$$

The condition (3.1.1) implies that  $\sim$  is indeed an equivalence relation. We denote by  $[u_\alpha, x_\alpha]$  the equivalence class of  $(U_\alpha, x_\alpha)$ . The natural projections  $p_\alpha : \mathbb{V}_{U_\alpha} \rightarrow U_\alpha$  are compatible with the equivalence relation and thus define a continuous map

$$p : E \rightarrow B$$

Note that if  $\lambda \in \mathbb{K}$  and  $[u_\alpha, x_\alpha], [u_\beta, x_\beta] \in E$  are such that  $x_\alpha = x_\beta = x$  we define

$$\lambda \cdot [u_\alpha, x_\alpha] := [\lambda u_\alpha, x_\alpha],$$

$$[u_\alpha, x_\alpha] + [u_\beta, x_\beta] := [g_{\beta\alpha}(x)u_\alpha + u_\beta, x] = [u_\alpha + g_{\alpha\beta}(x)u_\beta, x].$$

One can verify that these operations are well defined and define a vector bundle with total space  $E$ , base  $B$  and natural projection  $p$ . We will refer to this bundle as the vector bundle defined by the gluing cocycle and we will denote it by  $E(\mathcal{U}, g_{\bullet\bullet}, \mathbb{V})$ .  $\square$

**Exercise 3.1.7.** Fill in the missing details in the above example.  $\square$

**Proposition 3.1.8.** Any vector bundle is isomorphic to a vector bundle associated to a gluing cocycle.  $\square$

**Exercise 3.1.9.** (a) Prove Proposition 3.1.8.

(b) Find a gluing cocycle description for the tautological bundle  $\mathcal{U}_1 \rightarrow \mathbb{C}\mathbb{P}^1$ .  $\square$

**Example 3.1.10** (The clutching construction). We want to describe a simple way of producing vector bundles over the sphere  $S^n$  called the *clutching construction*. Soon we will see that this techniques produces all the vector bundles over  $S^n$ . For simplicity we consider only the case of complex vector bundles.

This construction is a special case of the gluing cocycle construction. It associates to each continuous map

$$g : S^{n-1} \rightarrow \mathrm{GL}_r(\mathbb{C}), \quad S^{n-1} \ni p \mapsto g(p) \in \mathrm{GL}_r(\mathbb{C})$$

a rank  $r$  complex vector bundle  $E_g \rightarrow S^n$ . The map  $g$  is called the *clutching map*.

Consider the unit sphere

$$S^n = \{(x^0, x^1, \dots, x^n) \in \mathbb{R}^{n+1}; \sum_i |x^i|^2 = 1\}.$$

For  $\varepsilon > 0$  sufficiently small we set

$$D^+(\varepsilon) := \{(x^0, \dots, x^n) \in S^n; x^0 > -\varepsilon\}, \quad D^-(\varepsilon) = \{(x^0, \dots, x^n) \in S^n; x^0 < \varepsilon\}.$$

Note that the open sets  $D^\pm(\varepsilon)$  are fattened versions of the upper/lower hemispheres. The overlap  $\mathcal{O}_\varepsilon = D^+(\varepsilon) \cap D^-(\varepsilon)$  is homeomorphic to the (open) cylinder  $(-\varepsilon, \varepsilon) \times S^{n-1}$ , where  $S^{n-1}$  is identified with the Equator  $\{x^0 = 0\}$  of the unit sphere  $S^n$ .

The total space of  $E_g$  is obtained by gluing the bundle  $\underline{\mathbb{C}}_{D^-(\varepsilon)}^r$  to the bundle  $\underline{\mathbb{C}}_{D^+(\varepsilon)}^r$  along  $\underline{\mathbb{C}}_{\mathcal{O}_\varepsilon}^r$  using the gluing map

$$\underline{\mathbb{C}}_{D^+(\varepsilon)}^r \supset \mathbb{C}^r \times (-\varepsilon, \varepsilon) \times S^{n-1} \ni (v, t, p) \mapsto (g(p)v, t, p) \in \mathbb{C}^r \times (-\varepsilon, \varepsilon) \times S^{n-1} \subset \underline{\mathbb{C}}_{D^-(\varepsilon)}^r.$$

We see that this is a special case of the gluing construction associated to the gluing construction determined by the open cover

$$U_0 = D^+(\varepsilon), \quad U_1 = D^-(\varepsilon), \quad g_{01}(t, p) = g(p) = g_{10}(t, p)^{-1}, \quad \forall (t, p) \in (-\varepsilon, \varepsilon) \times S^{n-1} = U_{01}. \quad \square$$

**Definition 3.1.11.** (a) Suppose  $E \xrightarrow{p} B$  and  $F \xrightarrow{q} B$  are two  $\mathbb{K}$ -vector bundles over the same topological space  $B$ . Then a *morphism of vector bundles* from  $E$  to  $F$  is a continuous map  $T : E \rightarrow F$  such that, for any  $b \in B$  we have  $T(E_b) \subset F_b$ , and the induced map  $T_b : E_b \rightarrow F_b$  is a morphism of  $\mathbb{K}$ -vector spaces. We denote by  $\underline{\mathrm{Hom}}(E, F)$  the vector space of bundle morphisms  $E \rightarrow F$ .

(b) A bundle morphism  $T \in \underline{Hom}(E, F)$  is called an *isomorphism* if the map  $T$  is a homeomorphism. We denote by  $\mathbf{VB}_{\mathbb{K}}^r(B)$  the collection of isomorphism classes of  $\mathbb{K}$ -vector bundles of rank  $r$  over  $B$ .

(c) A  $\mathbb{K}$ -vector bundle  $E$  of rank  $r$  over a topological space  $X$  is called *trivializable* if there exists an isomorphism  $\Phi : E \rightarrow \underline{\mathbb{K}}_X^r$ . Such an isomorphism is called a *trivialization* of  $E$ .

(d) Suppose  $E \xrightarrow{p} B$  is a topological vector bundle. A continuous *section* of  $E$  over a subset  $S \subset B$  is a continuous map  $u : S \rightarrow E$  such that  $u(s) \in E_s, \forall s \in S$ . Equivalently  $p \circ u = \mathbb{1}_S$ . We denote by  $\Gamma(S, E)$  the space of continuous sections over  $S$  and we set  $\Gamma(E) := \Gamma(B, E)$ .  $\square$

Observe that the space of sections of a  $\mathbb{K}$ -vector bundle over  $S$  is naturally a module of the ring of continuous functions  $S \rightarrow \mathbb{K}$ .

**Example 3.1.12.** (a) Every vector bundle  $E \rightarrow B$  admits at least one section, the *zero section* which associates to each  $b \in B$  the origin of the vector space  $E_b$ . In fact, the space  $\Gamma(E)$  is very large, infinite dimensional more precisely. For example, given any  $b \in B$  there exists a section  $u$  of  $E$  such that  $u(b) \neq 0$ . To see this, fix a neighborhood  $U$  of  $b$  such that  $E|_U$  is isomorphic to a trivial bundle. Thus we can find a section  $s$  of  $E$  over  $U$ . Next, Tietze's extension theorem implies that there exists a continuous function  $f : B \rightarrow \mathbb{R}$  such that

$$\text{supp } f \subset U, \quad f(b) = 1.$$

Then the section  $f \cdot s$  over  $U$  extends by zero to a section of  $E$  over  $B$ .

(b) A section of the trivial vector bundle  $\underline{\mathbb{R}}_B^k$  is a continuous map  $u : B \rightarrow \mathbb{R}^k$ . A bundle morphism  $T : \underline{\mathbb{R}}_B^m \rightarrow \underline{\mathbb{R}}_B^n$  is a continuous map  $T : B \rightarrow \text{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ .  $\square$

(c) Suppose  $E \rightarrow B$  is a  $\mathbb{K}$ -vector bundle. We have a natural isomorphism of  $\mathbb{K}$ -vector spaces  $\Gamma(E) \rightarrow \underline{Hom}(\underline{\mathbb{K}}_B, E), \Gamma(B) \ni u \mapsto L_u \in \underline{Hom}(\underline{\mathbb{K}}_B, E)$

$$\underline{\mathbb{K}}_B \ni (t, b) \mapsto tu(b) \in E.$$

(d) If the  $\mathbb{K}$ -vector bundle  $E \rightarrow B$  of rank  $r$  is described by a gluing cocycle  $(U_\alpha, g_{\beta\alpha})$ , then a section of  $E$  is described by a collection of continuous functions  $s_\alpha : U_\alpha \rightarrow \mathbb{K}^r$  satisfying

$$s_\beta(x) = g_{\beta\alpha}(x)s_\alpha(x), \quad \forall \alpha, \beta, \quad x \in U_{\alpha\beta}. \quad \square$$

**Definition 3.1.13.** A *vector subbundle* of the  $\mathbb{K}$ -vector bundle  $E \xrightarrow{p} B$  is a  $\mathbb{K}$ -vector bundle  $E_0 \xrightarrow{p_0} B$  with the following properties.

- The total space  $E_0$  is a subspace of  $E$  and  $p_0 = p|_{E_0}$ .
- For any  $b \in B$  the fiber  $p_0^{-1}(b)$  is a vector subspace of  $E_b$ .

$\square$

### 3.2. Functorial operations with vector bundles

We would like to explain how to generate in a natural fashion many new examples of vector bundles from a collection of given bundles. A first group of operations are essentially algebraic in nature. More precisely, all the natural operations with vector spaces (direct sum, duals, tensor products,

exterior products etc.) have a vector bundle counterpart. Note that a vector space can be viewed as a vector bundle over the topological space consisting of a single point.

Suppose  $E_i \xrightarrow{p_i} S$ ,  $i = 0, 1$  are  $\mathbb{K}$ -vector bundles of respective ranks  $r_i$ . Then the *direct sum* or *Whitney sum* of  $E_0$  and  $E_1$  is the vector bundle  $E_0 \oplus E_1 \xrightarrow{p} S$  whose fiber over  $s \in S$  is the direct sum of the fibers  $E_0(s)$  and  $E_1(s)$  of  $E_0$  and respectively  $E_1$  over  $s$ . The total space of  $E_0 \oplus E_1$  is the fibered product  $E_0 \times_S E_1$  given by the Cartesian diagram

$$\begin{array}{ccc} E_0 \times_S E_1 & \xrightarrow{\hat{p}_1} & E_0 \\ \hat{p}_0 \downarrow & \searrow p & \downarrow p_0 \\ E_1 & \xrightarrow{p_1} & S \end{array}$$

The natural projection  $p : E_0 \times_S E_1 \rightarrow S$  is given by  $p = p_1 \circ \hat{p}_0 = p_0 \circ \hat{p}_1$ . Note that  $p^{-1}(s) = E_0(s) \times E_1(s)$ . Equivalently, if  $E_i$  is described by the open cover  $\mathcal{U} = (U_\alpha)_{\alpha \in A}$  (same open cover for  $i = 0$  and  $i = 1$ ) and gluing cocycle

$$g_{\beta\alpha,i} : U_{\alpha\beta} \rightarrow \mathrm{GL}_{r_i}(\mathbb{K}),$$

then  $E_0 \oplus E_1$  is described by the same open cover  $\mathcal{U}$  and gluing cocycle

$$g_{\beta\alpha,0} \oplus g_{\beta\alpha,1} : U_{\alpha\beta} \rightarrow \mathrm{GL}_{r_0+r_1}(\mathbb{K}).$$

The *tensor product* of  $E_0$  and  $E_1$  is the vector bundle  $E_0 \otimes_{\mathbb{K}} E_1$  defined by the open cover  $\mathcal{U}$  and gluing cocycle

$$g_{\beta\alpha,0} \otimes g_{\beta\alpha,1} : U_{\alpha\beta} \rightarrow \mathrm{GL}_{r_0 r_1}(\mathbb{K}).$$

The *dual vector bundle*  $E_0^*$  is the vector bundle defined by the open cover  $\mathcal{U}$  and gluing cocycle

$$g_{\beta\alpha} : U_{\alpha\beta} \rightarrow \mathrm{GL}_{r_0}(\mathbb{K}), \quad g_{\beta\alpha}(s) = (g_{\beta\alpha,0}(s)^\dagger)^{-1},$$

where  $A^\dagger$  denotes the transpose of the matrix  $A$ . The exterior powers  $\Lambda^k E_0$  are defined in a similar fashion. The top exterior power  $\Lambda^{r_0} E_0$  is called the *determinant line bundle* associated to  $E_0$  and it is denoted by  $\det E_0$ .

**Exercise 3.2.1.** Prove that

$$\det(E_0 \oplus E_1) \cong \det E_0 \otimes \det E_1. \quad \square$$

Observe that a section  $\lambda$  of the dual bundle  $E_0^*$  defines a bundle morphism

$$L_\lambda : E_0 \rightarrow \underline{\mathbb{K}}_S, \quad E_0(s) \ni v \mapsto \lambda_s(v) \in \underline{\mathbb{K}}_s, \quad \forall s \in S.$$

In particular we have a bilinear map

$$\begin{aligned} \langle -, - \rangle &= \langle -, - \rangle_S : \Gamma(E_0^*) \times \Gamma(E_0) \rightarrow C(S, \mathbb{K}) (= \text{the space of continuous functions } S \rightarrow \mathbb{K}), \\ \langle \lambda, u \rangle_s &= \langle \lambda(s), u(s) \rangle \in \mathbb{K}, \quad \forall s \in S, \quad \lambda \in \Gamma(E_0^*), \quad u \in \Gamma(E_0). \end{aligned}$$

More generally, a section  $u$  of  $E_0^* \otimes_{\mathbb{K}} E_1^*$  defines a  $\Gamma(\underline{\mathbb{K}}_S)$ -bilinear map

$$u : \Gamma(E_0) \times \Gamma(E_1) \rightarrow C(S, \mathbb{K}) = \Gamma(\underline{\mathbb{K}}_S).$$

The bundle  $\mathrm{Hom}(E_0, E_1)$  is the bundle defined by the open cover  $\mathcal{U}$  and gluing cocycle

$$\begin{aligned} g_{\beta\alpha} &: U_{\alpha\beta} \rightarrow \mathrm{GL}(\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{r_0}, \mathbb{K}^{r_1})), \\ g_{\beta\alpha}(s)T &= g_{\beta\alpha,1}(s)Tg_{\beta\alpha,0}(s)^{-1}, \quad \forall s \in S, \quad T \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{r_0}, \mathbb{K}^{r_1}). \end{aligned}$$

The sections of  $\text{Hom}_{\mathbb{K}}(E_0, E_1)$  is a bundle morphism  $E_0 \rightarrow E_1$ , i.e.,

$$\underline{\text{Hom}}(E_0, E_1) = \Gamma(\text{Hom}(E_0, E_1)).$$

A special important case of these construction is the bundle  $E_0^* \otimes_{\mathbb{K}} E_1$ . Observe that we have a canonical isomorphism, called the *adjunction isomorphism*

$$\mathfrak{a} : \Gamma(E_0^* \otimes E_1) \rightarrow \underline{\text{Hom}}(E_0, E_1),$$

uniquely defined by the following requirement.

For any  $\lambda \in \Gamma(E_0^*)$  and any  $u \in \Gamma(E_1)$ , then  $\mathfrak{a}(\lambda \otimes u)$  is the bundle morphism  $E_0 \rightarrow E_1$  such that, for  $s \in S$ , and  $v \in E_0(s)$  then

$$\mathfrak{a}(\lambda \otimes u)(v) = \langle \lambda(s), u(s) \rangle \cdot v \in E_1(s).$$

**Exercise 3.2.2.** Prove that the adjunction map  $\mathfrak{a}$  is well defined and is indeed an isomorphism.  $\square$

**Proposition 3.2.3.** For any topological space  $X$  we set  $\text{Pic}_{\text{top}}(X) := \mathbf{V} \mathbf{B}_{\mathbb{C}}^1(X)$  = the set of isomorphism classes of complex line bundles over  $X$ . Then the tensor product operation

$$\otimes : \text{Pic}_{\text{top}}(X) \times \text{Pic}_{\text{top}}(X) \rightarrow \text{Pic}_{\text{top}}(X)$$

defines an Abelian group structure on  $\text{Pic}_{\text{top}}(X)$  in which the trivial line bundle is the identity element, and the inverse of a line bundle  $L$  is its dual  $L^* = \text{Hom}(L, \underline{\mathbb{C}})$ . The group  $\text{Pic}_{\text{top}}(X)$  is called the topological Picard group of  $X$ .  $\square$

**Exercise 3.2.4.** Prove Proposition 3.2.3.  $\square$

For every complex vector bundle  $E \rightarrow S$  we denote by  $\overline{E} \rightarrow S$  the complex vector bundle with the same total space and canonical projection as  $E$ , but such that, for every  $s \in S$  the complex vector space  $\overline{E}_s$  is the complex conjugate of  $E_s$ . We will refer to  $\overline{E}$  as the complex vector bundle *conjugate* to  $E$ . This means that in  $\overline{E}_s$  the multiplication by a complex scalar  $\lambda$  coincides with the multiplication by  $\bar{\lambda}$  in  $E_s$ .

**Definition 3.2.5.** If  $E \rightarrow S$  is a real vector bundle, then a metric on  $E$  is a section  $g \in \Gamma(E^* \otimes_{\mathbb{R}} E^*)$  such that, for every  $s \in S$ , the bilinear form  $g_s$  defines an Euclidean product on  $E_s$ .

If  $E \rightarrow S$  is a complex vector bundle, then a *hermitian metric* on  $E$  is a section  $h \in \Gamma(E^* \otimes_{\mathbb{C}} \overline{E}^*)$  such that, for every  $s \in S$ , the form  $h_s$  defines a Hermitian inner product on  $E_s$ . (The form  $h_s(-, -)$  is linear in the first variable and conjugate linear in the second variable.)  $\square$

**Exercise 3.2.6.** (a) Prove that any vector bundle  $E \rightarrow S$  over a paracompact vector space admits metrics.

(b) Suppose  $E \rightarrow S$  is a real (respectively complex) vector bundle of rank  $n$ . Prove that  $S$  can be described using an open cover  $\mathcal{U} = (U_{\alpha})_{\alpha \in A}$  and a gluing cocycle

$$g_{\beta\alpha} : U_{\alpha\beta} \rightarrow O(n), \quad (\text{respectively } g_{\beta\alpha} : U_{\alpha\beta} \rightarrow U(n)). \quad \square$$

**Proposition 3.2.7.** Suppose  $E \xrightarrow{\pi} S$  is a  $\mathbb{K}$ -vector bundle equipped with a metric (hermitian if  $\mathbb{K} = \mathbb{C}$ ). For any vector sub-bundle  $F \hookrightarrow S$  we set

$$F^{\perp} := \{ v \in E; \quad v \in F_{\pi(v)}^{\perp} \subset E_{\pi(v)} \}.$$

Then  $F^{\perp}$  is also a sub-bundle of  $E$  of rank  $\text{rank } F^{\perp} = \text{rank } E - \text{rank } F$  and  $F \oplus F^{\perp} \cong E$ .  $\square$

Another versatile operation is the *pullback operation*. It is topological in nature.

Suppose we are given a  $\mathbb{K}$ -vector bundle of rank  $r$  over  $S$ ,  $E \xrightarrow{p} S$  and a continuous map  $f : T \rightarrow S$ . Then we can form a  $\mathbb{K}$ -vector bundle of rank  $r$  over  $T$ ,  $f^*E \xrightarrow{q} T$  uniquely determined by the fibered product  $E \times_S T$  with Cartesian diagram

$$\begin{array}{ccc} f^*E = E \times_S T & \xrightarrow{q} & T \\ \hat{f} \downarrow & & \downarrow f \\ E & \xrightarrow{p} & S \end{array}$$

The total space is  $T \times_S E$  and the canonical projection is  $q$ . The map  $\hat{f}$  induces an isomorphism between the fiber  $(f^*E)_t = q^{-1}(t)$  and the fiber  $E_{f(t)}$ .

If the bundle  $E \xrightarrow{p} S$  is described by the gluing cocycle  $(U_\alpha; g_{\beta\alpha})_{\alpha, \beta \in A}$ , then  $f^*E$  is described by the gluing cocycle  $(V_\alpha = f^{-1}(U_\alpha), f^*g_{\beta\alpha} = g_{\beta\alpha} \circ f)$ .

**Example 3.2.8.** (a) Suppose  $M$  is a smooth  $n$ -dimensional manifold of a real Euclidean space  $V$ . Then for every  $x \in M$  the tangent space  $T_x M$  can be canonically identified with a  $n$ -dimensional subspace of  $V$ . We obtain in this fashion a map

$$M \rightarrow \mathbf{Gr}_n(V), \quad M \ni x \mapsto T_x M \in \mathbf{Gr}_n(V)$$

called the *Gauss map* of the embedding, and it is denoted by  $\gamma_M$ . This map is clearly continuous (even smooth) and we have a bundle isomorphism

$$T^*M \cong \gamma_M^* \mathcal{U}_n,$$

where  $\mathcal{U}_n \rightarrow \mathbf{Gr}_n(V)$  is the tautological vector bundle over  $\mathbf{Gr}_n(V)$ .

(b) Suppose that  $V$  is a vector space and  $U \subset V$  is a vector subspace. For  $n \geq \dim U$  we denote by  $\mathcal{U}_{n,V}$  (respectively  $\mathcal{U}_{n,U}$ ) the tautological vector bundle over  $\mathbf{Gr}_n(V)$  (respectively  $\mathbf{Gr}_n(U)$ ). We have a natural inclusion

$$i = i_n : \mathbf{Gr}_n(U) \hookrightarrow \mathbf{Gr}_n(V).$$

Clearly  $i_n$  is continuous. Note that  $i_n^* \mathcal{U}_{n,V} \cong \mathcal{U}_{n,U}$ . □

**Proposition 3.2.9.** *Suppose  $f_0, f_1 : Y \rightarrow X$  are homotopic continuous maps. Then for every  $\mathbb{K}$ -vector bundle  $E \xrightarrow{p} X$  of rank  $n$  the pullbacks  $E_0 = f_0^*E$ ,  $E_1 = f_1^*E$  are isomorphic vector bundles.*

**Proof.** We follow the strategy in [Hatch2, §1.2] and [Hu, §3.4]. For simplicity we consider only the special case when  $Y$  is compact. We need two auxiliary results.

**Lemma 3.2.10.** *Suppose that  $E \rightarrow [a, b] \times Z$  is a  $\mathbb{K}$ -vector bundle of rank  $n$  over the topological space  $Z$ . Then the following statements are equivalent.*

- (a) *The vector bundle  $E$  is trivializable.*
- (b) *There exists  $c \in [a, b]$  such that the restrictions of  $E$  to  $[a, c] \times Z$  and  $[c, b] \times Z$  are trivializable.*

**Proof.** The implication (a)  $\implies$  (b) is obvious. To prove (b)  $\implies$  (a) we fix trivializations

$$\Psi_- : E|_{[a,c] \times Z} \rightarrow \mathbb{K}_{[a,c] \times Z}^n, \quad \Psi_+ : E|_{[c,b] \times Z} \rightarrow \mathbb{K}_{[c,b] \times Z}^n,$$

and set

$$\Phi : \mathbb{K}_{c \times Z}^n \rightarrow \mathbb{K}_{c \times Z}^n, \quad \Phi = \Psi_-|_{c \times Z} \circ \Psi_+^{-1}|_{c \times Z}.$$

We can regard  $\Phi$  as a continuous map  $\Phi : Z \rightarrow \mathrm{GL}_n(\mathbb{K})$ . We extend it as a continuous map

$$\hat{\Phi} : [c, b] \times Z \rightarrow \mathrm{GL}_n(\mathbb{K}), \quad \hat{\Phi}(t, z) = \Phi(z).$$

We regard  $\hat{\Phi}$  as a bundle isomorphism  $\hat{\Phi} : \mathbb{K}_{[c,b] \times Y}^n \rightarrow \mathbb{K}_{[c,b] \times Z}^n$ . Now define  $\Psi : E \rightarrow \mathbb{K}_{[a,b] \times Z}^n$  to be

$$\Psi|_{[a,c] \times Z} = \Psi_-, \quad \Psi|_{[c,b] \times Z} = \hat{\Phi} \circ \Psi_+.$$

□

**Lemma 3.2.11.** *Suppose that  $Z$  is a paracompact space, and  $E \rightarrow [0, 1] \times Z$  is a  $\mathbb{K}$ -vector bundle of rank  $n$ . Then for every point  $z \in Z$  there exists an open neighborhood  $U_z$  of  $z$  in  $Z$  such that  $E|_{[0,1] \times U_z}$  is trivializable.*

**Proof.** For every  $t \in [0, 1]$  there exists open interval  $I_t \subset \mathbb{R}$  centered at  $t$ , and an open neighborhood  $U_t$  of  $z$  in  $Z$  such that  $I_t \times U_t$  is trivializable. The collection  $(I_t)_{t \in [0,1]}$  is an open cover of the unit interval so we can find a sufficiently large positive integer  $\nu$  such that any subinterval of  $[0, 1]$  of length  $\leq \frac{1}{\nu}$  is contained in one of the intervals  $I_t$ . We can find  $t_1, \dots, t_\nu \in [0, 1]$  such that

$$\left[ \frac{k-1}{\nu}, \frac{k}{\nu} \right] \subset I_{t_k}, \quad \forall k = 1, \dots, \nu.$$

We set

$$U_z := \bigcap_{k=1}^{\nu} U_{t_k}.$$

We deduce that the restriction of  $E$  to any of the cylinders  $[\frac{k-1}{\nu}, \frac{k}{\nu}] \times U_z$ ,  $1 \leq k \leq \nu$ , is trivializable. Lemma 3.2.10 implies that the restriction of  $E$  to  $[0, 1] \times U_z$  is trivializable as well.

□

We can now finish the proof of Proposition 3.2.9. Consider a homotopy  $F : [0, 1] \times Y \rightarrow X$  between  $f_0$  and  $f_1$ , i.e.,  $f_k(y) = F(k, y)$ ,  $\forall y \in Y$ ,  $k = 0, 1$ . Denote by  $\hat{E}$  the bundle  $F^*E$  over  $[0, 1] \times X$ . For  $t \in [0, 1]$  we denote by  $i_t$  the inclusion

$$i_t : Y \rightarrow [0, 1] \times Y, \quad x \mapsto (t, y).$$

Then

$$E_0 = i_0^* \hat{E}, \quad E_1 = i_1^* \hat{E}.$$

It suffices to prove that  $i_0^* \hat{E}$  and  $i_1^* \hat{E}$  are isomorphic.

Using Lemma 3.2.11 we deduce that there exists an open cover  $\mathcal{U} = (U_i)_{i \in I}$  of  $Y$  and trivializations

$$\Psi_\alpha : \hat{E}|_{[0,1] \times U_i} \rightarrow \mathbb{K}_{[0,1] \times U_i}^n$$

Since  $Y$  is compact, we can assume  $I$  is finite,  $I = \{1, 2, \dots, \nu\}$ . Observe that the trivialization  $\Psi_i$  defines linear isomorphism

$$\Psi_i(t, y) : \hat{E}_{(t,y)} \rightarrow \mathbb{K}^n, \quad \forall t \in [0, 1], \quad y \in U_i.$$



Fix a partition of unity subordinated to the cover  $\mathcal{U}$ , that is, continuous functions  $\rho_i : Y \rightarrow [0, 1]$ ,  $1 \leq i \leq \nu$  such that

$$\text{supp } \rho_i \subset U_i, \quad \sum_{i=1}^{\nu} \rho_i = 1.$$

We set

$$t_0(y) = 0, \quad t_i(y) = \sum_{j \leq i} \rho_j(y), \quad \forall i = 1, \dots, \nu.$$

Note that  $t_0(y) = 0$  and  $t_\nu(y) = 1$ ,  $\forall y \in Y$ . For any  $i \in I$  and any  $y \in Y$  we define

$$H_i(y) : \widehat{E}_{t_{i-1}(y), y} \rightarrow \widehat{E}_{t_i(y), y},$$

as follows. If  $\rho_i(y) = 0$ , so that  $t_{i-1}(y) = t_i(y)$ , then we set  $H_i(y) = \mathbb{1}$ . If  $\rho_i(y) > 0$ , then  $y \in U_i$  and we define  $H_i(y)$  as the composition

$$\widehat{E}_{(t_{i-1}(y), y)} \xrightarrow{\Psi_i(t_{i-1}, y)} \mathbb{K}^n \xrightarrow{\Psi_i(t_i, y)^{-1}} \widehat{E}_{t_i, y}.$$

We let  $T_y : \widehat{E}_{(0, y)} \rightarrow \widehat{E}_{(1, y)}$  denote the composition  $H_\nu(y) \circ \dots \circ H_1(y)$ . The collection  $(T_y)_{y \in Y}$  defines a vector bundle isomorphism  $T : E_0 \rightarrow E_1$ .  $\square$

**Corollary 3.2.12.** *Any vector bundle  $E$  over a contractible space  $X$  is trivializable.*

**Proof.** Since  $X$  is contractible, there exists  $x_0 \in X$  such that the constant map  $c_{x_0} : X \rightarrow X$ ,  $c_{x_0}(x) = x_0$ ,  $\forall x \in X$  is homotopic to the identity map  $\mathbb{1}_X$ . Hence

$$E \cong c_{x_0}^* E \cong \underline{E}_{x_0, X}.$$

$\square$

**Exercise 3.2.13.** Recall the clutching construction in Example 3.1.10 which associates to each continuous map  $g : S^{n-1} \rightarrow \text{GL}_N(\mathbb{C})$  a rank  $N$  complex vector bundle  $E_g \rightarrow S^n$ . Prove that any rank  $N$  complex vector bundle over  $S^n$  can be obtained via the clutching construction.  $\square$

### 3.3. The classification of vector bundles

In this section we want to show that the situations described in Example 3.2.8 are special manifestations of a general phenomenon. The following is the main result of this section.

**Theorem 3.3.1.** (a) *For any  $\mathbb{K}$ -vector bundle  $E \xrightarrow{p} X$  of rank  $n$  over a compact space  $X$  there exists a finite dimensional vector space  $V$  and a continuous map*

$$\gamma : X \rightarrow \mathbf{Gr}_n(V)$$

*such that the vector bundle  $\gamma^* \mathcal{U}_n^V$  is isomorphic to  $E$ .*

(b) *If  $V_0, V_1$  are two finite dimensional vector spaces and  $\gamma_k : X \rightarrow \mathbf{Gr}_n(V_k)$ ,  $k = 0, 1$ , are two continuous maps. Then the following statements are equivalent.*

(b1) *Vector bundles  $\gamma_k^* \mathcal{U}_n^{V_k}$  are isomorphic.*

(b2) *there exists a finite dimensional vector space  $V$  containing both  $V_0$  and  $V_1$  as subspaces such that if  $j_k : \mathbf{Gr}_n(V_k) \rightarrow \mathbf{Gr}_n(V)$  denotes the canonical inclusion, then  $j_0 \circ \gamma_0$  is homotopic to  $j_1 \circ \gamma_1$  as maps  $X \rightarrow \mathbf{Gr}_n(V)$ .*

**Proof.** (a) Fix a *finite* open cover  $(U_\alpha)_{\alpha \in A}$  such that  $E$  is trivializable over  $U_\alpha$ . Next, choose a partition of unity  $(\rho_\alpha)_{\alpha \in A}$ , i.e., continuous functions

$$\rho_\alpha : X \rightarrow [0, 1], \quad \text{supp } \rho_\alpha \subset U_\alpha, \quad \sum_{\alpha \in A} \rho_\alpha(x) = 1, \quad \forall x \in X$$

Since  $E$  is trivializable over  $U_\alpha$  we can find continuous sections of  $E|_{U_\alpha}$

$$e_1^\alpha, \dots, e_n^\alpha : U_\alpha \rightarrow E$$

such that for any  $x \in U_\alpha$  the collection  $(e_i^\alpha(x))_{1 \leq i \leq n}$  is a basis of  $E_x$ . The section  $\rho_\alpha e_i^\alpha$  over  $U_\alpha$  extends by 0 to a section  $u_{\alpha,i}$  of  $E$  over  $X$ . Note that for every  $x \in X$  the vectors  $u_{\alpha,i}(x) \in E_x$  span the fiber  $E_x$ . Denote by  $V$  the finite dimensional subspace of  $\Gamma(X)$  spanned by the sections  $u_{\alpha,i}$ ,  $\alpha \in A$ ,  $1 \leq i \leq n$ .

We have a canonical bundle morphism

$$ev : \underline{V}_X \rightarrow E, \quad V \times X \ni (v, x) \mapsto v(x) \in E_x.$$

This map is surjective. If we fix a metric on  $\underline{V}$  then the map  $ev$  induces an isomorphism between the bundle  $(\ker ev)^\perp$  and  $E$ . Now observe that

$$(\ker ev)^\perp = \Gamma^* \mathcal{U}_n$$

where  $\Gamma : X \rightarrow \mathbf{Gr}_n(V)$  is the map  $x \mapsto (\ker ev_x)^\perp \in \mathbf{Gr}_n(V)$ .

(b) The implication (b2)  $\Rightarrow$  (b1) is obvious because  $j_k^* \mathcal{U}_n^V \cong \mathcal{U}_n^{V_k}$ . We only need to prove the implication (b1)  $\Rightarrow$  (b2).

Suppose  $\gamma_k : X \rightarrow \mathbf{Gr}_n(V_k)$ ,  $k = 0, 1$  are two continuous maps such that the bundles  $E_0 = \gamma_0^* \mathcal{U}_n^{V_0}$  and  $E_1 = \gamma_1^* \mathcal{U}_n^{V_1}$  are isomorphic. Then we can regard  $E_k$  as a subbundle of  $\underline{V}_k|_X$ . In particular, for every  $x \in X$ , the fiber  $E_k(x)$  of  $E_k$  over  $x$  can be identified with an  $n$ -dimensional subspace of  $V_k$ . We set  $V := V_0 \oplus V_1$ .

Suppose  $A : E_0 \rightarrow E_1$  is a bundle isomorphism. Then for every  $x \in X$ , the graph of  $A_x$  is a subspace of  $E_0(x) \oplus E_1(x)$

$$\Gamma_{A_x} = \{(v, A_x v); v \in E_0(x)\} \subset V.$$

We obtain in this fashion a continuous map  $\Gamma_A : X \rightarrow \mathbf{Gr}_n(V)$ . Consider the homeomorphism

$$s : [0, 1) \rightarrow [0, \infty), \quad s(t) = \frac{t}{1-t}.$$

For every  $x \in X$  and  $t \in [0, 1)$  we set

$$L_{t,x} := \Gamma_{s(t)A_x} \in \mathbf{Gr}_k(V).$$

Observe that  $L_{0,x} = E_0(x)$  and

$$\lim_{t \nearrow 1} L_{t,x} = \lim_{s \rightarrow \infty} \Gamma_{sA_x} = E_1(x) \quad \text{uniformly with respect to } x \in X. \quad (3.3.1)$$

We obtain in this fashion a continuous function  $L : [0, 1] \times X \rightarrow \mathbf{Gr}_k(V)$  such that

$$L_{k,x} = E_k(x), \quad \forall x \in X, \quad k = 0, 1.$$

Now observe that the map  $L_k : X \rightarrow \mathbf{Gr}_n(V)$  can be identified with the composition

$$X \xrightarrow{\gamma_k} \mathbf{Gr}_n(V_k) \xrightarrow{j_k} \mathbf{Gr}_n(V).$$

□

**Exercise 3.3.2.** Prove the claim (3.3.1).  $\square$

We can rephrase the statements in Theorem 3.3.1 in a more concise form using the *infinite dimensional Grassmannian*. Let  $H$  be a separable  $\mathbb{K}$ -Hilbert space. We denote by  $\mathbf{Gr}_m = \mathbf{Gr}_m^{\mathbb{K}}$  the collection of  $m$ -dimensional  $\mathbb{K}$ -subspaces of  $H$ . We regard it as a subset of the space of bounded self-adjoint operators on  $H$  by associating to a subspace in  $H$  the orthogonal projection onto that subspace. As such,  $\mathbf{Gr}_m$  is equipped with a natural topology, called the *projector topology*. Note that, for any subspace  $V \subset H$  we obtain a tautological continuous embedding

$$\mathbf{Gr}_m(V) \hookrightarrow \mathbf{Gr}_m(H).$$

**Exercise 3.3.3.** (a) Suppose  $(U_\nu)_{\nu \geq 1}$  is a sequence of  $m$ -dimensional subspaces of the separable Hilbert space  $H$ . Then  $U_n$  converges to  $U \in \mathbf{Gr}_m(H)$  in the projector topology if and only if there exists a basis  $\{e_1, \dots, e_m\}$  of  $U$  and bases  $\{e_1(\nu), \dots, e_m(\nu)\}$  of  $U_\nu$  such that

$$\lim_{\nu \rightarrow \infty} |e_i(\nu) - e_i| = 0, \quad \forall i = 1, \dots, m.$$

(b) Suppose  $f : X \rightarrow \mathbf{Gr}_m(H)$  is a continuous map. Show that if  $X$  is compact then there exists a finite dimensional subspace  $V \subset H$  such that  $f$  and a map  $g : X \rightarrow \mathbf{Gr}_m(H)$  such that  $f$  is homotopic to  $g$  and  $g(X) \subset \mathbf{Gr}_m(V)$ .  $\square$

For any topological spaces  $S, T$  we denote by  $[S, T]$  the set of homotopy classes of continuous maps  $S \rightarrow T$ . From Theorem 3.3.1 and Exercise 3.3.3 we obtain the following result.

**Corollary 3.3.4.** For any compact<sup>2</sup> space  $X$  the pullback construction

$$[X, \mathbf{Gr}_m^{\mathbb{K}}] \ni f \mapsto f^* \mathcal{U}_m \in \mathbf{VB}_{\mathbb{K}}^m(X)$$

defines a bijection  $[X, \mathbf{Gr}_m^{\mathbb{K}}] \rightarrow \mathbf{VB}_{\mathbb{K}}^m(X)$ .  $\square$

Observe that the correspondence  $X \mapsto \mathbf{VB}_{\mathbb{K}}^m(X)$  is a contravariant functor from the category of topological spaces to the category of sets, where for every continuous map  $f : X \rightarrow Y$  the induced morphisms  $\mathbf{VB}_{\mathbb{K}}^m(Y) \rightarrow \mathbf{VB}_{\mathbb{K}}^m(X)$  is the pullback map. Because of the above corollary we will often refer to the infinite Grassmannian  $\mathbf{Gr}_m^{\mathbb{K}}$  as the *classifying space* for the functor  $\mathbf{VB}_{\mathbb{K}}^m$ .

**Exercise 3.3.5.** Prove that the clutching construction of Example 3.1.10 defines a bijection

$$[S^{n-1}, \mathrm{GL}_N(\mathbb{C})] \ni g \mapsto E_g \in \mathbf{VB}_{\mathbb{C}}^N(S^n).$$

In particular, this shows that

$$\pi_{n-1}(\mathrm{GL}_N(\mathbb{C})) \cong \pi_n(\mathbf{Gr}_N^{\mathbb{C}}). \quad \square$$

<sup>2</sup>Corollary 3.3.4 is true for any paracompact space  $X$ .



# The Thom isomorphism, the Euler class and the Gysin sequence

This chapter is devoted to a result central to all of our future considerations, namely the Thom isomorphism theorem. Our presentation is greatly inspired by [MS].

## 4.1. The Thom isomorphism

Suppose  $R$  is a convenient ring. For every closed subset  $S$  of a topological space  $X$  we denote by  $H_S^\bullet(X, R)$  the *local cohomology* of  $X$  along  $S$ ,

$$H_S^\bullet(X, R) := H^\bullet(X, X \setminus S; R)$$

We have a natural cup product

$$\cup : H_S^n(X, R) \times H^k(X, R) \rightarrow H_S^{n+k}(X, R),$$

a cap product

$$H_S^k(X, R) \times H_n^S(X, R) \rightarrow H_{n-k}(X, R)$$

and a Mayer-Vietoris sequence

$$\cdots \rightarrow H_{S_1 \cap S_2}^n(X, R) \rightarrow H_{S_1}^n(X, R) \oplus H_{S_2}^n(X, R) \rightarrow H_{S_1 \cup S_2}^n(X, R) \rightarrow H_{S_1 \cap S_2}^{n+1}(X, R) \rightarrow \cdots \quad (4.1.1)$$

For a vector bundle  $p : E \rightarrow B$  we regard the base  $B$  as a closed subset of the total space  $E$  via the embedding given by the zero section  $\zeta : B \rightarrow E$ . Observe that we have isomorphisms

$$p_* : H_\bullet(E, R) \rightarrow H_\bullet(B, R), \quad p^* : H^\bullet(B, R) \rightarrow H^\bullet(E, R).$$

**Definition 4.1.1.** (a) A real vector bundle  $E \xrightarrow{p} B$  of rank  $n$  over  $B$  is called *homologically  $R$ -orientable* if there exists a cohomology class  $\Phi_E \in H_B^n(E, R)$  such that for every  $b \in B$  the class  $i_b^* \Phi_E \in H_{\{0\}}^n(E_b, R)$  is a generator of the free  $R$ -module  $H_{\{0\}}^n(E_b, R)$ . Here, the map  $i_b$  is the natural inclusion  $E_b \rightarrow E$ . The class  $\Phi_E$  is called a *Thom class* (with coefficients in  $R$ ).

(b) A real vector bundle  $E \xrightarrow{p} B$  of rank  $n$  over  $B$  is called *orientable* if the determinant line bundle  $\det E = \Lambda^n E \rightarrow B$  is trivializable.  $\square$

**Theorem 4.1.2** (Thom isomorphism theorem). *Suppose  $E \xrightarrow{p} B$  is an  $R$ -orientable rank  $n$  real vector bundle over  $B$ , and  $\Phi_E \in H_B^n(E, R)$  is a Thom class with coefficients in  $R$ . Then, for every  $k \geq 0$  the maps*

$$\Phi_E \cap : H_k^B(E, R) \rightarrow H_{k-n}(E, R) \quad \text{and} \quad \Phi_E \cup : H^k(E, R) \rightarrow H_B^{k+n}(E, R)$$

are isomorphisms of  $R$ -modules. The resulting isomorphism

$$\mathcal{T}_E : H_k^B(E, R) \xrightarrow{\Phi_E \cap} H_{k-n}(E, R) \xrightarrow{p_*} H_{k-n}(B, R)$$

and

$$\mathcal{T}^E : H^k(B, R) \xrightarrow{p^*} H^k(E, R) \xrightarrow{\Phi_E \cup} H_B^{k+n}(E, R)$$

are called the homological (respectively cohomological) Thom morphisms.

**Outline of the proof.** The proof is carried in several conceptually distinct steps. We outline them below and we refer to [MS, §10] for more details.

**Step 1.** A direct computation proves the theorem is true for trivial bundles.

**Step 2.** An inductive argument, based on the Mayer-Vietoris sequence proves that the theorem is true if  $B$  can be covered by finitely many open sets  $U_1, \dots, U_\nu$  such that the restriction to  $E$  to any  $U_i$  is trivializable. In particular, the theorem is true for any compact base  $B$ .

**Step 3.** For any space  $X$  we have

$$H_\bullet(X, R) \cong \varinjlim_K H_\bullet(K, R),$$

where the inductive limit is over all the compact subsets of  $X$  ordered by inclusion. This implies that the homological Thom isomorphism is true for any  $R$  and any base  $B$ .

**Step 4.** If  $R$  is a field,  $R = \mathbb{F}_p, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ , then the cohomological Thom map is an isomorphism. This follows by passing to limits over compacts as in Step 3 and using the universal coefficients isomorphism

$$H^i(X, R) \cong \text{Hom}_R(H_i(X, R), R),$$

which is due to the fact that  $R$  is a field.

**Step 5.** Assume  $B$  is connected. If  $E$  is  $\mathbb{Z}$ -orientable, then  $E$  is  $R$ -orientable for any field  $R$ . At this point one needs to use the homological Thom isomorphism over  $\mathbb{Z}$  to conclude that  $H_{n-1}^B(E, \mathbb{Z}) = 0$ . The universal coefficients theorem implies that

$$H_B^n(E) = \text{Hom}(H_n^B(E), \mathbb{Z}) \cong \text{Hom}(H_0(B), \mathbb{Z}) \cong \mathbb{Z}.$$

This implies that the natural morphism  $H_B^n(E, \mathbb{Z}) \rightarrow H_B^n(E, R)$  maps a  $\mathbb{Z}$ -Thom class  $\Phi_E$  to an  $R$ -Thom class, for any field  $\mathbb{Z}$ .

**Step 6.** Conclusion. Suppose that  $E$  is  $\mathbb{Z}$ -orientable. One shows that the fact that if the cohomological Thom morphism is isomorphism for any field  $R$ , then it is an isomorphism for  $R = \mathbb{Z}$ . This is a purely algebraic result, based on the fact that the singular homology with integral coefficients of a space  $X$  is the homology of a chain complex of free Abelian groups.  $\square$

We need to have simple criteria for recognizing when a vector bundle is homologically  $R$ -orientable. Here is a first result.

**Theorem 4.1.3.** *Any real bundle is homologically  $\mathbb{F}_2$ -orientable.*

**Idea of Proof.** Follow Steps 1,2,4, in the proof of Theorem 4.1.2.  $\square$

To investigate the geometric meaning of the  $\mathbb{Z}$ -orientability condition we first need to elucidate the geometric orientability condition. First some terminology.

We define a *frame* of a finite dimensional real vector space  $V$  to be a linearly ordered basis  $\underline{e} := (e_1, \dots, e_n)$ ,  $n = \dim V$ . We denote by  $\mathcal{F}(V)$  the set of frames of  $V$ . We say that two frames  $\underline{e}, \underline{f} \in \mathcal{F}(V)$  are *identically oriented*, and we write this  $\underline{e} \sim \underline{f}$  if there exists  $T \in \text{GL}(V)$  such that

$$\det T > 0, \quad (\underline{f}_1, \dots, \underline{f}_n) = T\underline{e} = (Te_1, \dots, Te_n).$$

The relation “ $\sim$ ” is an equivalence relation. An equivalence class of “ $\sim$ ” is called an *orientation* of  $V$ . We denote by  $\text{Or}(V)$  the set of orientations on  $V$ . Observe that  $\text{Or}(V)$  is a set consisting of two elements.

Recall that  $\det V = \Lambda^n V$ . We define a map  $\det : \mathcal{F}(V) \rightarrow \mathcal{F}(\det V)$  by setting

$$\mathcal{F} \ni (e_1, \dots, e_n) \mapsto \det(e_1, \dots, e_n) := e_1 \wedge \dots \wedge e_n.$$

Observe that this map is surjective and satisfies the equivariance condition

$$\det T\underline{e} = \det T \cdot \det \underline{e}, \quad \forall \underline{e} \in \mathcal{F}(V), \quad T \in \text{GL}(V).$$

This shows that the set of orientations on  $V$  can be identified with the set of orientations on the determinant line  $\det V$ . The latter, can be identified with the set of path components of the punctured line  $\det V \setminus \{0\}$ . Thus, we can specify an orientation on  $V$  by specifying a nonzero vector in  $\det V$ .

**Proposition 4.1.4.** *Suppose  $E \rightarrow X$  is a rank  $n$ -real vector bundle over a paracompact space  $X$ . Then the following statements are equivalent.*

- (a) *The vector bundle is geometrically orientable.*
- (b) *The vector bundle  $E$  can be described by an open cover  $(U_\alpha)_{\alpha \in \mathcal{A}}$  and a gluing cocycle*

$$g_{\beta\alpha} : U_{\alpha\beta} \rightarrow \text{GL}_{\mathbb{R}}(n)$$

such that

$$\det g_{\beta\alpha}(x) > 0, \quad \forall \alpha, \beta \in \mathcal{A}, \quad x \in U_{\alpha\beta}. \quad (4.1.2)$$

**Proof.** We denote by  $\underline{e}$  the canonical basis of  $\mathbb{R}^n$ , and by  $\mathcal{S}_n$  the group of permutations of  $\{1, \dots, n\}$ . For every permutation  $\varphi \in \mathcal{S}_n$  we define  $T_\varphi \in \text{GL}_{\mathbb{R}}(n)$  by

$$T_\varphi(\underline{e}_i) = \underline{e}_{\varphi(i)}, \quad \forall i = 1, \dots, n.$$

Note that

$$\det T_\varphi = \epsilon(\varphi)x,$$

where  $\epsilon(\varphi) = \pm 1$  is the signature of the permutation  $\varphi$ .

- (a)  $\implies$  (b). Fix a trivializing open cover  $(U_\alpha)_{\alpha \in \mathcal{A}}$  and local trivializations

$$\Psi_\alpha : E_{U_\alpha} \rightarrow \mathbb{R}^n \times U_\alpha.$$

We obtain in this fashion a gluing cocycle  $g_{\beta\alpha} = \Psi_\beta \circ \Psi_\alpha^{-1}$ . A section of  $\det E$  is then described by a collection of continuous functions

$$d_\alpha : U_\alpha \rightarrow \mathbb{R}$$

satisfying the compatibility conditions

$$d_\beta(x) = \det g_{\beta\alpha}(x) d_\alpha(x), \quad \forall \alpha, \beta \in \mathcal{A}, \quad x \in U_{\alpha\beta}.$$

Since  $E$  is geometrically orientable we can find such a collection satisfying

$$d_\alpha(x) \neq 0, \quad \forall \alpha \in \mathcal{A}, \quad x \in U_\alpha.$$

Now we can find locally constant maps  $\varphi_\alpha : U_\alpha \rightarrow \mathcal{S}_n$  such that

$$\epsilon(\varphi_\alpha(x)) d_\alpha(x) > 0$$

Using these maps we obtain new trivializations

$$\Phi_\alpha : E_{U_\alpha} \rightarrow \mathbb{R}^n \times U_\alpha, \quad \Phi_\alpha = T_{\varphi_\alpha} \circ \Psi_\alpha.$$

Clearly the gluing cocycle  $\Phi_\beta \circ \Phi_\alpha^{-1}$  satisfies the positivity constraint (4.1.2).

(b)  $\implies$  (a) Choose a trivializing locally finite cover  $(U_\alpha)$  with trivializations  $\Psi_\alpha : E_{U_\alpha} \rightarrow \mathbb{R}^n \times U_\alpha$  such that the resulting gluing cocycle  $g_{\beta\alpha}$  satisfies the positivity condition (4.1.2). Now choose a partition of unity  $(\eta_\alpha)_{\alpha \in \mathcal{A}}$  subordinated to the cover  $(U_\alpha)_{\alpha \in \mathcal{A}}$ . Thus each  $\eta_\alpha$  is a continuous nonnegative function on  $X$  such that  $\text{supp } \eta_\alpha \subset U_\alpha, \forall \alpha \in \mathcal{A}$  and  $\sum_\alpha \eta_\alpha = 1$ . Define

$$\omega_\alpha \in \Gamma(U_\alpha, \det E), \quad \omega_\alpha = \Psi_\alpha^{-1}(x) \det \underline{e}.$$

Note that on the overlap  $U_{\alpha\beta}$  these two nowhere vanishing sections differ by a multiplicative factor

$$\omega_\alpha = \lambda_{\alpha\beta} \omega_\beta.$$

Observe that

$$\det \underline{e} = \lambda_{\alpha\beta} \det(\Psi_\alpha \circ \Psi_\beta^{-1})(\underline{e})$$

so that

$$\lambda_{\alpha\beta}(x) = \det g_{\beta\alpha}(x) > 0, \quad \forall x \in U_{\alpha\beta}$$

Now define  $\omega \in \Gamma(X, \det E)$  by setting

$$\omega := \sum_\alpha \eta_\alpha \omega_\alpha.$$

This is a nowhere vanishing section because of the above positivity.  $\square$

If  $E \rightarrow X$  is a geometrically orientable real vector bundle of rank  $r$ , then we can define an equivalence relation on the set of nowhere vanishing sections of  $\det E$  by declaring two sections  $\omega_0, \omega_1 \in \Gamma(\det E)$  equivalent if and only if there exists a continuous function  $\lambda : X \rightarrow (0, \infty)$  such that

$$\omega_1 = \lambda \omega_0.$$

An equivalence class of nowhere vanishing sections of  $\det E$  is called a *geometric orientation* of  $E$ . We denote by  $\mathbf{Or}(E)$  the set of geometric orientations of  $E$ . Observe that any geometric orientation  $\omega$  on  $E$  defines an orientation  $\omega_x$  in each fiber  $E_x$ , and in particular, a canonical generator of  $H_{\{0\}}^r(E_x, \mathbb{Z})$ , which for simplicity we continue to denote by  $\omega_x$ .

Note that if we fix a metric  $g$  on  $E$  we obtain a metric on  $\det E$ . We denote by  $\tilde{X}_E$  the closed subset of  $\det E$  consisting of all the vectors of length one in all the fibers of  $\det E$ . The natural projection  $\tilde{X}_E \rightarrow X$  is a double cover of  $X$ , called the *orientation double cover* determined by



$E$ . We see that  $E$  is geometrically orientable if and only if the orientation double cover is trivial. Moreover, in this case, a choice of orientation on  $E$  is equivalent to a choice of a section of the orientation cover.

**Theorem 4.1.5.** *If  $E \rightarrow X$  is a rank  $r$  geometrically orientable real vector bundle over the paracompact vector space  $X$ , then  $E$  is homologically  $\mathbb{R}$ -orientable for any convenient ring  $R$ . Moreover, any geometric orientation  $\omega$  on  $E$  determines a canonical Thom class  $\Phi_\omega \in H_X^r(E, \mathbb{Z})$  uniquely defined by the requirement that the restriction of  $\Phi_\omega$  to the fiber  $E_x$  is the generator  $\omega_x$  of  $H_{\{0\}}^r(E_x, \mathbb{Z})$ .  $\square$*

The proof of this theorem follows closely the proof of Theorem 4.1.2. For more details we refer [MS, §10].

**Remark 4.1.6.** Theorem 4.1.5 has a sort converse: any homologically  $\mathbb{Z}$ -orientable vector bundle is geometrically orientable, and the above correspondence between geometric orientations and Thom classes with integral coefficients is a bijection.  $\square$

**Example 4.1.7.** Suppose  $E \rightarrow X$  is a complex vector bundle of (complex) rank  $r$  described by the open cover  $\mathcal{U} = (U_\alpha)_{\alpha \in A}$  and the gluing cocycle

$$g_{\beta\alpha} : U_{\alpha\beta} \rightarrow \mathrm{GL}_r(\mathbb{C}).$$

We can regard it in a tautological fashion as a real vector bundle of (real) rank  $2r$ . We denote this real vector bundle by  $E_{\mathbb{R}}$ . Let  $i : \mathrm{GL}_r(\mathbb{C}) \rightarrow \mathrm{GL}_{2r}(\mathbb{R})$  be the tautological inclusion obtained by thinking of a complex linear map as a real linear map. Then the real vector bundle  $E_{\mathbb{R}}$  is described by the open cover  $\mathcal{U}$  and the gluing cocycle

$$g_{\beta\alpha}^{\mathbb{R}} = i \circ g_{\beta\alpha} : U_{\alpha\beta} \rightarrow \mathrm{GL}_{2r}(\mathbb{R}).$$

Since  $\mathrm{GL}_r(\mathbb{C})$  is connected, we deduce that  $i(\mathrm{GL}_r(\mathbb{C}))$  lies in the component of  $\mathrm{GL}_{2r}(\mathbb{R})$  containing 1. The linear automorphisms in this component have positive determinant so that

$$\det g_{\beta\alpha}^{\mathbb{R}}(x) > 0, \quad \forall \alpha, \beta \in A, \quad x \in U_{\alpha\beta}.$$

This proves that the real vector bundle tautologically determined by a complex vector bundle is geometrically orientable.

In fact, this real vector bundle carries a natural geometric orientation. To describe it, observe that if  $V$  is a complex vector space of dimension  $r$ , then for any complex frame  $\underline{e} = (e_1, \dots, e_r)$  we obtain a real basis

$$\underline{e}_{\mathbb{R}} = (e_1, \mathbf{f}_1 = ie_1, \dots, e_r, \mathbf{f}_r = ie_r), \quad \mathbf{i} = \sqrt{-1}.$$

We set

$$\det_{\mathbb{R}} \underline{e} := \det \underline{e}_{\mathbb{R}}.$$

For any complex linear automorphism  $T$  of  $V$  we have  $\det_{\mathbb{R}} T \underline{e} = |\det T|^2 \det_{\mathbb{R}} \underline{e}$ , where  $\det T \in \mathbb{C}$  is the determinant of the automorphism  $T$ . This shows that a complex structure on  $V$  canonically determines an equivalence class of real frames on  $V$ , and thus a canonical orientation.  $\square$

Suppose that  $E_i \xrightarrow{p_i} X$ ,  $i = 0, 1$  are two geometrically oriented real vector bundles over  $X$  of ranks  $r_i$ . We denote by  $\Phi_{E_i} \in H_X^{r_i}(E_i)$  the Thom classes determined by the orientations. We fix nowhere vanishing sections  $\omega_i \in \gamma(\det E_i)$  defining the orientations of these bundles. Then the

direct sum  $E_0 \oplus E_1$  admits a natural orientation given by the nowhere vanishing sect  $\omega \in \Gamma(\det(E_0 \oplus E_1)) \cong \Gamma(\det E_0 \otimes \det E_1)$  given by

$$\omega = \omega_0 \otimes \omega_1$$

We denote by  $\Phi_{E_0 \oplus E_1}$  the Thom class determined by this orientation on  $E_0 \oplus E_1$ . To relate this Thom class to the Thom classes  $\Phi_{E_i}$  we need to recall that  $E_0 \oplus E_1$  is defined by the Cartesian diagram

$$\begin{array}{ccc} E_0 \oplus E_1 = E_0 \times_X E_1 & \xrightarrow{q_1} & E_1 \\ q_0 \downarrow & & \downarrow p_0 \\ E_0 & \xrightarrow{p_1} & X \end{array}$$

Then

$$\Phi_{E_0 \oplus E_1} = q_0^* \Phi_{E_0} \cup q_1^* \Phi_{E_1}. \quad (4.1.3)$$

To see this, note first that the above result is valid when  $X$  consists of a single point. This means that for a general  $X$ , and any  $x \in X$ , the restriction of  $q_0^* \Phi_{E_0} \cup q_1^* \Phi_{E_1}$  to the fiber  $E_0(x) \oplus E_1(x)$  is the generator of  $H_{\{0\}}^{r_0 \oplus r_1}(E_0(x) \oplus E_1(x))$  defined by  $\omega(x)$ . This is precisely the meaning of (4.1.3).

## 4.2. Fiber bundles

**Definition 4.2.1.** (a) A *fiber bundle* over the topological space  $X$  with *model fiber* a topological space  $F$  is a triplet  $(E, X, p)$ , where  $p : E \rightarrow X$  is a continuous map such that for any point  $x \in X$  there exists a neighborhood  $U$  and a homeomorphism  $\Psi : E_U := p^{-1}(U) \rightarrow F \times U$  such that the diagram below is commutative.

$$\begin{array}{ccc} E_U & \xrightarrow{\Psi} & F \times U \\ p \searrow & & \swarrow \pi \\ & U & \end{array}$$

Above,  $\pi$  denotes the natural projection  $F \times U \rightarrow U$ . The map  $\Psi$  with the above property is called a *local trivialization*. The *fiber over*  $x \in X$  is the subspace  $E_x := p^{-1}(\{x\})$ . The space  $E$  is called the *total space*. Often we will use the more refined notation

$$F \hookrightarrow E \xrightarrow{p} X$$

to denote a fiber bundle with model fiber  $F$ , total space  $E$  and base  $X$ .

(b) A *section* of a fiber bundle  $E \xrightarrow{p} X$  over a subset  $A \subset X$  is a continuous maps  $u : A \rightarrow E$  such that  $u(a) \in E_a, \forall a \in A$ .

(c) Two fiber bundles over the same topological space  $X$ ,  $E_i \xrightarrow{p_i} X, i = 0, 1$ , are called *isomorphic* if there exists a homeomorphism  $\Phi : E_0 \rightarrow E_1$  such that the diagram below is commutative.

$$\begin{array}{ccc} E_0 & \xrightarrow{\Phi} & E_1 \\ p_0 \searrow & & \swarrow p_1 \\ & X & \end{array} \quad \square$$

**Example 4.2.2.** For every topological spaces  $F$  and  $X$  we denote by  $\underline{F}_X$  the *trivial bundle*

$$F \times X \xrightarrow{\pi} X.$$

A bundle is called *trivializable* if it is isomorphic to a trivial bundle. □

**Example 4.2.3.** (a) Any vector bundle is a fiber bundle with model fiber homeomorphic to a vector space.

(b) Suppose  $E \xrightarrow{p} X$  is a real vector bundle of rank  $r$ . Then any metric  $g$  on  $E$  defines a bundle  $D(E, g) \rightarrow X$  with standard fiber a closed  $r$ -dimensional disk. The total space  $D(E, g)$  is the closed subset of  $E$

$$D(E, g) = \bigcup_{x \in X} D(E_x, g)$$

where  $D(E_x, g)$  denotes the closed unit disk in the Euclidean vector space  $(E_x, g)$ . If  $g_0$  and  $g_1$  are two metrics on  $E$  then the corresponding bundles  $D(E, g_i)$  are isomorphic. The isomorphism class of these bundles is called the *unit disk bundle* associated to the vector bundle  $E$ , and we denote it by  $D(E)$ .

Similarly, we can define the *unit sphere bundle* associated to  $E$ , and we denote it by  $S(E)$ . We will find convenient to think of the total space of  $S(E)$  as closed subset of  $D(E)$ .

(c) Suppose  $E \xrightarrow{p} X$  is a  $\mathbb{K}$ -vector bundle of rank  $r$ . We denote by  $\mathbb{P}(E) \rightarrow X$  the fiber bundle whose fiber over  $x \in X$  is the projective space  $\mathbb{P}(E_x)$  of one dimensional  $\mathbb{K}$ -subspaces of  $E_x$ . Equivalently, the total space is the quotient of the  $S(E)$  modulo the equivalence relation

$$p \sim q \iff \exists x \in X, \lambda \in \mathbb{K}^*; p, q \in S(E_x), p = \lambda q.$$

We can define in a similar fashion the bundle  $\mathbf{Gr}_m(E)$  whose fiber over  $x \in X$  the Grassmannian  $\mathbf{Gr}_m(E_x)$  of  $m$ -dimensional  $\mathbb{K}$ -subspaces of  $E_x$ .  $\square$

The Thom isomorphism is closely related to the Poincaré duality. Suppose  $M$  is a compact smooth orientable manifold of dimension  $n$ , and  $E \rightarrow M$  is a (geometrically) orientable smooth real vector bundle of rank  $r$  over  $M$ . We fix an orientation on  $M$ , with orientation class  $\mu_M \in H_n(M)$ , and a geometric orientation of the vector bundle  $E$ , with associated Thom class  $\Phi_E$ .

The unit disk bundle  $D(E)$  associated to  $E$  is a smooth manifold with boundary,  $\partial D(E) \cong S(M)$ . The total space of  $D(E)$  has an orientation induced from the orientation on  $M$  and the orientation of  $E$  via the fiber-first orientation convention. We denote by  $\mu_E \in H_{n+r}(D(E), \partial D(E))$  the fundamental class determined by the orientation of  $D(E)$ . We have a Poincaré duality isomorphism

$$H^k(D(E), \partial D(E)) \rightarrow H_{n+r-k}(D(E)).$$

We denote by

$$PD_E : H_{n+r-k}(E) \rightarrow H^k(D(E), \partial D(E))$$

the inverse of the above isomorphism. For simplicity, we set

$$c^\dagger := PD_E(c), \quad \forall c \in H_\bullet(E) \iff c^\dagger \cap \mu_E = c.$$

Let  $\zeta : M \rightarrow D(E)$  be the natural inclusion of  $M$  in  $E$  as zero section. We define

$$\eta_E := (\zeta_* \mu_M)^\dagger \in H^r(D(E), \partial D(E)) \iff \eta_E \cap \mu_E = \zeta_* \mu_M.$$

Observe that the inclusion  $(D(E), \partial D(E)) \hookrightarrow (E, E \setminus \zeta(E))$  induces an isomorphism

$$H^\bullet(D(E), \partial D(E)) \cong H^\bullet(E, E \setminus \zeta(M)) = H_M^\bullet(E).$$

We can thus also regard  $\eta_M$  as a local cohomology class,  $\eta_M \in H_M^r(E)$ .

**Proposition 4.2.4.** *The morphism  $\tau : H^k(E) \rightarrow H^{k+r}(D(E), \partial D(E))$  defined by*

$$H^k(D(E)) \ni \alpha \mapsto \eta_E \cup \alpha \in H^{k+r}(D(E), \partial D(E)),$$

*is an isomorphism.*

**Proof.** We have

$$\begin{aligned} \eta_E \cup \alpha &= \pm \alpha \cup \eta_E \\ &= \pm PD_E((\alpha \cup \eta_E) \cap \mu_E) = \pm PD_E(\alpha \cap (\eta_E \cap \mu_E)) = \pm PD_E(\alpha \cap \zeta_* \mu_M) \\ &\text{(use the projection formula (1.2.4))} \\ &= \pm PD_E \zeta_*(\zeta^* \alpha \cap \mu_M) \end{aligned}$$

This shows that  $\tau$  coincides with the composition of morphisms

$$H^k(D(E)) \xrightarrow{\zeta^*} H_{n-k}(M) \xrightarrow{\cap \mu_M} H_{n-k}(M) \xrightarrow{\zeta_*} H_{n-k}(E) \xrightarrow{PD_E} H^{k+r}(D(E), \partial D(E)).$$

the morphisms  $\zeta_*$  and  $\zeta^*$  are isomorphisms since  $\zeta$  is a homotopy inverse of the natural projection  $\pi : D(E) \rightarrow M$ , and the morphisms  $\cap \mu_M$  and  $PD_E$  are isomorphisms by the Poincaré duality theorem. □

By composing the isomorphism  $\tau$  with the isomorphism  $\pi^* : H^k(M) \rightarrow H^k(E)$  we obtain an isomorphism

$$H^k(M) \rightarrow H^{k+r}(D(E), \partial D(E)) \cong H_M^{k+r}(E), \quad \alpha \mapsto \eta_E \cup \pi^* \alpha$$

This resembles very much the Thom isomorphism  $\mathcal{T}^E$ . The next result pretty much explains this coincidence.

**Proposition 4.2.5.** *The class  $\eta_E \in H_M^r(E)$  is a Thom class for  $E$  with integral coefficients.* □

For a proof of this fact we refer to [Bre, Cor.VI.11.6].

**Remark 4.2.6.** (a) Under appropriate sign conventions for the cap and cup product the class  $\eta_E$  coincides with the class  $\Phi_E$ . In other words, if we think of the Thom class as a cohomology class  $\Phi_E \in H^r(D(E), \partial D(E))$ , then it can be identified with  $\eta_E$ , the Poincaré dual of the the homology class determined by the inclusion of  $M$  in  $D(E)$  as zero section.

(b) If  $f : M_0 \rightarrow M_1$  is a map between compact oriented manifolds, possibly with boundary, such that  $f(\partial M_0) \subset \partial M_1$  then we have two *umkehr* (or Gysin) maps

$$f_! : H^{m_0-k}(M_0) \rightarrow H^{m_1-k}(M_1), \quad f^! : H_{m_1-k}(M_1) \rightarrow H_{m_0-k}(M_0)$$

defined by the commutative diagrams

$$\begin{array}{ccc} H_k(M_0, \partial M_0) & \xrightarrow{f_*} & H_k(M_1, \partial M_1) \\ \uparrow \cap \mu_{M_0} & & \uparrow \cap \mu_{M_1} \\ H^{m_0-k}(M_0) & \xrightarrow{f_!} & H_{m_1-k}(M_1) \end{array}$$

and

$$\begin{array}{ccc} H_{m_1-k}(M_1) & \xrightarrow{f^!} & H_{m_0-k}(M_0) \\ \uparrow \cap \mu_{M_1} & & \uparrow \cap \mu_{M_0} \\ H^k(M_1, \partial M_1) & \xrightarrow{f^*} & H^k(M_0, \partial M_0) \end{array}$$

The proof of Proposition 4.2.4 shows that if  $E \rightarrow M$  is a rank  $r$  geometrically oriented real vector bundle over a compact, oriented smooth manifold  $M$  then the Thom isomorphism can be identified with the Gysin map  $\zeta_! : H^k(M) \rightarrow H^{k+r}(D(E), \partial D(E))$ , where  $\zeta : M \rightarrow E$  is the zero section.  $\square$

### 4.3. The Gysin sequence and the Euler class

Let  $R$  be a convenient ring, and suppose that  $E \xrightarrow{\pi} X$  is homologically  $R$ -oriented real vector bundle of rank  $r$  with Thom class  $\Phi_E$ . We denote by  $D(E)$  and respectively  $S(E)$  the associated unit disk and respectively unit sphere bundles. Let  $\zeta : X \rightarrow E$  denote the zero section, and by  $\Phi_E$  the Thom class determined by geometric orientation of  $E$ . We regard  $\Phi_E$  as a relative cohomology class  $\Phi_E \in H^r(D(E), \partial D(E); R)$  and the Thom morphism as a morphism  $\mathcal{T}^E : H^k(X) \rightarrow H^{k+r}(D(E), \partial D(E),)$

$$\mathcal{T}^E : H^k(X, R) \rightarrow H^{k+r}(D(E), \partial D(E); R), \quad \alpha \mapsto \Phi_E \cup \pi^* \alpha.$$

Observe that we have a natural morphism

$$\pi_! : H^k(S(E), R) \rightarrow H^{k-(r-1)}(X, R)$$

defined by the composition

$$H^k(S(E), R) \xrightarrow{\delta} H^{k+1}(D(E), \partial D(E); R) \xrightarrow{(\mathcal{T}^E)^{-1}} H^{k+1-r}(X, R),$$

where  $\delta$  is the connecting morphism in the cohomological long exact sequence of the pair  $(D(E), \partial D(E))$ . The morphism  $\pi_!$  is usually referred to as the *Gysin morphism* or the *integration along fibers* morphism.

Observe that  $H^\bullet(S(E), R)$  has a natural structure of *right* module over the cohomology ring  $H^\bullet(X, R)$  given by

$$H^\bullet(S(E), R) \times H^\bullet(X, R) \ni (\alpha, \beta) \mapsto \alpha \cup \pi^* \beta \in H^\bullet(S(E), R).$$

**Lemma 4.3.1.** *The morphisms*

$$\pi^* : H^\bullet(X, R) \rightarrow H^\bullet(S(E), R), \quad \pi_! : H^\bullet(S(E), R) \rightarrow H^{\bullet-(r-1)}(X, R)$$

*are morphisms of  $H^\bullet(X, R)$ -modules.*  $\square$

**Proof.** The fact that  $\pi^*$  is a morphism of  $H^\bullet(X, R)$ -modules follows from the fact that  $\pi^*$  is a morphism of cohomology rings. To prove that  $\pi_!$  is a morphism of modules we consider  $\beta \in H^k(X, R)$ ,  $\alpha \in H^\ell(S(E), R)$ . Then

$$\pi_!(\alpha \cdot \beta) = \pi_!(\alpha \cup \pi^* \beta) = (\mathcal{T}^E)^{-1} \delta(\alpha \cup \pi^* \beta).$$

The connecting morphism is a derivation for the cup product, i.e.,

$$\delta(\alpha \cup \pi^* \beta) = \delta(\alpha) \cup \pi^* \beta \pm \alpha \cup (\delta \pi^* \beta).$$

We now observe that  $\delta\pi^*\beta = 0$  because  $\pi^*\beta \in H^k(D(E), R)$  and the composition

$$H^k(D(E), R) \rightarrow H^k(S(E), R) \xrightarrow{\delta} H^{k+1}(D(E), S(E); R)$$

is trivial. Hence

$$\pi_!(\alpha \cdot \beta) = (\mathcal{J}^E)^{-1}((\delta\alpha) \cup \pi^*\beta).$$

If we write  $\delta\alpha = \Phi_E \cup \pi^*\gamma$ ,  $\gamma = \pi_!(\alpha) \in H^{\ell+1-r}(S, R)$  then we get

$$\pi_!(\alpha \cdot \beta) = \gamma \cup \beta = (\pi_!\alpha) \cup \beta.$$

□

**Definition 4.3.2.** The *Euler class* of the homologically  $R$ -oriented real vector bundle  $E \xrightarrow{\pi} X$  of rank  $r$  with Thom class  $\Phi_E$  is the cohomology class

$$e(E) \in H^r(X, R), \quad e(E) = \zeta^*\Phi_E,$$

where  $\zeta : X \rightarrow E$  denotes the zero section. □

**Proposition 4.3.3.** Suppose  $E \rightarrow X$  is a geometrically oriented real vector bundle of odd rank  $r$ . Then

$$2e(E) = 0 \in H^r(X, \mathbb{Z}).$$

**Proof.** Observe that any section  $u$  of  $E$  is homotopic to the zero section. Hence

$$e(E) = \zeta^*\Phi_E = u^*\Phi_E = (-u)^*\Phi_E.$$

On the other hand, since  $E$  has odd rank, the automorphisms  $one_E$  of  $E$  is orientation reversing so that

$$(-u)^*\Phi_E = u^*(-1)^*\Phi_E = -u^*\Phi_E = -e(E)$$

□

**Theorem 4.3.4** (Gysin). If  $R$  is a convenient ring and  $E \xrightarrow{\pi} X$  is a homologically  $R$ -oriented real vector bundle of rank  $r$  then we have a long exact sequence of right  $H^\bullet(X, R)$ -modules

$$\cdots \rightarrow H^k(X, R) \xrightarrow{e^{(E) \cup}} H^{r+k}(X, R) \xrightarrow{\pi^*} H^{r+k}(S(E), R) \xrightarrow{\pi_!} H^{k+1}(X, R) \rightarrow \cdots$$

**Proof.** Consider the following diagram (where for simplicity we dropped the ring  $R$  from our notations)

$$\begin{array}{ccccccc} H^{k+r}(D(E), S(E)) & \xrightarrow{i^*} & H^{k+r}(D(E)) & \xrightarrow{j^*} & H^{k+r}(S(E)) & \xrightarrow{\delta} & H^{k+r+1}(D(E), S(E)) \\ \uparrow \mathcal{J}^E & & \uparrow \pi^* & & \uparrow \mathbb{1} & & \uparrow \mathcal{J}^E \\ H^k(X) & \xrightarrow{a} & H^{k+r}(X) & \xrightarrow{b} & H^{k+r}(S(E)) & \xrightarrow{c} & H^{k+1}(X) \end{array}$$

The top row is a segment of the long exact sequence of the pair  $(D(E), S(E))$ . The morphism  $\pi^*$  is an isomorphism with inverse  $\zeta^*$ . The morphisms  $\alpha, \beta, \gamma$  are defined so that the diagram is commutative. More precisely

$$\mathbf{a} = (\pi^*)^{-1} \circ i^* \circ \mathcal{J}^E, \quad \mathbf{b} = j^* \circ \pi^*, \quad \mathbf{c} = (\mathcal{J}^E)^{-1} \circ \delta = \pi_!$$

Observe that for any  $\alpha \in H^k(X)$  we have

$$\mathbf{a}(\alpha) = \zeta^*(\Phi_E \cup \pi^* \alpha) = \mathbf{e}(E) \cup \zeta^* \pi^*(\alpha) = \mathbf{e}(E) \cup \alpha.$$

□

**Example 4.3.5** (The integral cohomology ring of  $\mathbb{C}\mathbb{P}^n$ ,  $1 \leq n \leq \infty$ ). We consider first the case  $n < \infty$ . Consider the universal complex line bundle  $\mathcal{U} \rightarrow \mathbb{C}\mathbb{P}^n$ . Its total space is the incidence relation

$$\mathcal{U} = \{(v, L) \in (\mathbb{C}^{n+1} \setminus 0) \times \mathbb{C}\mathbb{P}^n; v \in L\}$$

The associated sphere bundle has total space

$$S(\mathcal{U}) = \{(v, L) \in \mathcal{U}; |v| = 1\}.$$

We have natural map  $S(\mathcal{U}) \rightarrow S^{2n+1} = \text{unit sphere in } \mathbb{C}^{n+1}$  given by  $(u, L) \mapsto u$ . This map is continuous and bijective and thus a homeomorphism since both  $S(\mathcal{U})$  and  $S^{2n+1}$  are compact.

We now regard  $\mathcal{U}$  as an oriented real vector bundle  $\mathcal{U}_\mathbb{R}$  of rank 2 over  $\mathbb{C}\mathbb{P}^n$  and we denote by  $e$  its Euler class

$$e = e(\mathcal{U}_\mathbb{R}) \in H^2(\mathbb{C}\mathbb{P}^n, \mathbb{Z}).$$

The Gysin sequence of the sphere bundle  $S(\mathcal{U}) \rightarrow \mathbb{C}\mathbb{P}^n$  takes the form

$$\dots \rightarrow H^{k+1}(S^{2n+1}) \xrightarrow{\pi_!} H^k(\mathbb{C}\mathbb{P}^n) \xrightarrow{e \cup} H^{k+2}(\mathbb{C}\mathbb{P}^n) \xrightarrow{\pi^*} H^{k+2}(S^{2n+1}) \xrightarrow{\pi_!} \dots$$

We deduce that if  $0 \leq k < 2n - 1$  then we have isomorphism

$$e \cup : H^k(\mathbb{C}\mathbb{P}^n) \rightarrow H^{k+2}(\mathbb{C}\mathbb{P}^n).$$

This proves that

$$H^{2k}(\mathbb{C}\mathbb{P}^n) \cong H^0(\mathbb{C}\mathbb{P}^n) \cup e^k, \quad 1 \leq k \leq n.$$

Arguing similarly we deduce that  $H^{2k+1}(\mathbb{C}\mathbb{P}^n) = 0$ , for any  $k$ . We deduce that the cohomology ring of  $\mathbb{C}\mathbb{P}^n$  is a commutative ring with a single generator  $e$  of degree 2 satisfying the conditions  $e^{n+1} = 0$ , i.e.,

$$H^\bullet(\mathbb{C}\mathbb{P}^n) = \mathbb{Z}[e]/(e^{n+1}), \quad \deg e = 2.$$

When  $n = \infty$  we have a similar circle bundle  $S^1 \hookrightarrow S^\infty \rightarrow \mathbb{C}\mathbb{P}^\infty$  but in this case the total space is contractible. Arguing as above we deduce that the cohomology ring of  $\mathbb{C}\mathbb{P}^\infty$  is

$$H^\bullet(\mathbb{C}\mathbb{P}^\infty, \mathbb{Z}) = \mathbb{Z}[u], \quad \deg u = 2. \quad \square$$

The Euler class has several functoriality properties that will come in handy later.

Suppose  $E \xrightarrow{p} X$  is a homologically  $R$ -oriented real vector bundle of rank  $r$  with Thom class  $\Phi_E$ . Then for any continuous maps  $f : Y \rightarrow X$  we have a Cartesian diagram

$$\begin{array}{ccc} E \times_X Y = f^*E & \xrightarrow{p} & Y \\ \hat{f} \downarrow & & \downarrow f \\ E & \xrightarrow{\pi} & X \end{array}$$

Then  $\hat{f}^*\Phi_E$  is a Thom class for  $f^*E$  and induces a homology  $R$ -orientation. We denote this Thom class by  $\Phi_{f^*E}$ . If  $\zeta$  denotes the zero section of  $E$  and  $\zeta_f$  denotes the zero section of  $f^*E$  then

$$\hat{f} \circ \zeta_f = \zeta \circ f \implies f^* \circ \zeta^* = \zeta_f^* \circ \hat{f}^*.$$

We deduce that

$$e(f^*E) = f^*e(E) \in H^r(Y, R). \quad (4.3.1)$$

Suppose that  $E_i \xrightarrow{p_i} X$  are homologically  $R$ -oriented real vector bundles of ranks  $r_i$ ,  $i = 0, 1$ . Denote by  $\Phi_{E_i}$  the corresponding Thom classes. Then, as we have seen before, we have a Cartesian diagram

$$\begin{array}{ccc} E_0 \oplus E_1 = E_0 \times_X E_1 & \xrightarrow{q_1} & E_1 \\ q_0 \downarrow & & \downarrow p_0 \\ E_1 & \xrightarrow{p_1} & X \end{array}$$

Then  $q_0^*\Phi_{E_0} \cup q_1^*\Phi_{E_1}$  is a Thom class for  $E_0 \oplus E_1$  which we denote by  $\Phi_{E_0 \oplus E_1}$ . From the equality (4.1.3) we deduce that

$$e(E_0 \oplus E_1) = e(E_0) \cup e(E_1) \in H^{r_0+r_1}(X, R). \quad (4.3.2)$$

We present last the most important feature of the Euler class, namely its interpretation as a measure of nontriviality of a vector bundle. We begin by introducing a relative version of Euler class.

Suppose  $E \rightarrow X$  is a geometrically oriented real vector bundle of rank  $r$  and  $\Phi_E \in H^r(E, E \setminus X)$  the associated Thom class. Suppose  $u : X \rightarrow E$  is a section of  $E$  and  $A \subset X$  is a subset such that  $u(a) \neq 0, \forall a \in A$ . Then  $u$  defines a continuous map of pairs

$$u : (X, A) \rightarrow (E, E \setminus X)$$

We set

$$e(E, u|_A) := u^*\Phi_E \in H^r(X, A),$$

and we refer to it as the *relative Euler class* (modulo the nowhere vanishing section  $u|_A \in \Gamma(A, E)$ ).

One can see easily that if  $v : X \rightarrow E$  is a nother section of  $E$  that is nowhere vanishing on  $A$  and it is homotopic to  $u|_A$  in the space of such sections, then

$$e(E, u|_A) = e(E, v|_A).$$

If  $\mathcal{E} : H^\bullet(X, A) \rightarrow H^\bullet(X)$  is the natural (extension by zero) morphism then

$$\mathcal{E}e(E, u|_A) = e(E).$$

**Proposition 4.3.6.** *Suppose  $E \rightarrow X$  is an oriented real vector bundle of rank  $r$  with Thom class  $\Phi_E$  which admits a nowhere vanishing section  $u : X \rightarrow E$ . Then  $e(E) = 0 \in H^r(X, \mathbb{Z})$ .*



**Proof.** The section  $u$  defines a relative Euler class

$$e(E, u) \in H^r(X, X) = 0$$

which extends to  $e(E) \in H^r(X)$ . This extension is therefore trivial. Hence  $e(E) = 0$ .  $\square$

The above result implies that if an oriented real vector bundle has nontrivial Euler class, then the bundle cannot admit nowhere vanishing sections, and in particular, it cannot be trivial. The computations in Example 4.3.5 imply that the universal line bundle over a complex projective space is not trivial.

#### 4.4. The Leray-Hirsch isomorphism

In this brief section we formulate a generalization of Thom's isomorphism theorem that we will need later. Let us first observe that for any fiber bundle  $E \xrightarrow{\pi} B$  and any convenient ring  $R$  the cohomology  $H^\bullet(E, R)$  is naturally a right module over the cohomology ring  $H^\bullet(B, R)$  via the rule

$$H^\bullet(E, R) \times H^\bullet(B, R) \ni (\alpha, \beta) \mapsto \alpha \cup \pi^* \beta \in H^\bullet(E, R).$$

**Theorem 4.4.1** (Leray-Hirsch). *Let  $R$  be a convenient ring and  $F \hookrightarrow E \rightarrow B$  a fiber bundle with model fiber  $F$  satisfying the following conditions.*

- (a) *The cohomology  $H^\bullet(F, R)$  of the model fiber  $F$  is a finitely generated free  $R$ -module.*
- (b) *There exists cohomology classes  $u_1, \dots, u_N \in H^\bullet(E, R)$  such that for any  $x \in X$ , there restrictions to the fiber  $E_x$  define an  $R$ -basis of the cohomology  $H^\bullet(E_x, R)$ .*

*Then  $H^\bullet(E, R)$  is a free  $H^\bullet(B, R)$ -module, and  $u_1, \dots, u_N$  is a basis of this module.  $\square$*

When the bundle  $E$  is of finite type, i.e., there exists a finite open cover  $(U_i)_{1 \leq i \leq n}$  such that  $E_{U_i}$  is trivializable, then the above theorem follows by a simple application of the Mayer-Vietoris theorem. For example, when  $B$  is compact, then any fiber bundle over  $B$  is of finite type. The general result is obtained from the finite type case by a limiting process. For more details we refer to [Hatch1, Thm. 4D.1].

**Example 4.4.2.** Suppose  $E \rightarrow X$  is a complex vector bundle of rank  $r$ . We denote by  $\mathbb{P}(E) \xrightarrow{\pi} X$  the associated bundle of projective spaces, whose fiber over  $x$  is the projective space  $\mathbb{P}(E_x)$  consisting of one-dimensional subspaces of  $E_x$ . We denote by  $\widehat{E}$  the pullback of  $E$  to  $\mathbb{P}(E)$ ,  $\widehat{E} = \pi^* E$ .

The pullback  $\widehat{E}$  is a bundle over  $\mathbb{P}(E)$  such that for every  $x \in X$  and every line  $L_x \subset E_x$  the fiber of  $\widehat{E}$  over  $L_x$  is the space  $E_x$ . We denote by  $\mathcal{U}_E$  the complex line subbundle of  $\widehat{E}$  over  $\mathbb{P}(E)$  whose fiber over  $L_x \in \mathbb{P}(E)_x$  is the one dimensional subspace  $L_x$  of  $\widehat{E}_{L_x}$ . The restriction of  $\mathcal{U}_E$  to the fiber  $\mathbb{P}(E_x)$  is none other than the universal line bundle over the projective space  $\mathbb{P}(E_x)$ . Let  $u \in H^2(\mathbb{P}(E))$  denote the Euler class of  $\mathcal{U}_E$  viewed as a rank 2 oriented real vector bundle.

The computations in Example 4.3.5 show that the restrictions of the cohomology classes  $\mathbf{1}, u, \dots, u^{r-1}$  to any fiber  $\mathbb{P}(E_x)$  of  $\mathbb{P}(E)$  form an integral basis of the the cohomology of  $\mathbb{P}(E_x)$ . From the Leray-Hirsch theorem we deduce that the morphism

$$\underbrace{H^\bullet(X) \oplus \dots \oplus H^\bullet(X)}_r \rightarrow H^\bullet(\mathbb{P}(E)),$$

$$(\beta_0, \dots, \beta_{r-1}) \mapsto \pi^* \beta_0 + \mathbf{u} \cup \pi^* \beta_1 + \dots + \mathbf{u}^{r-1} \cup \pi^* \beta_{r-1}$$

is an isomorphism of  $H^\bullet(X)$ -modules.

□

# The construction of Chern classes à la Grothendieck

## 5.1. The first Chern class

Every complex line bundle  $L \rightarrow X$  can be regarded as an *oriented* rank two bundle. As such it has a well defined Euler class which is an element of  $H^2(X)$ . This element is called the *first Chern class* of the complex line bundle  $L$  and it is denoted by  $c_1(L)$ .

**Proposition 5.1.1.** *The first Chern class is a function  $c_1$  that associates to each complex line bundle  $L \rightarrow X$  a cohomology class  $c_1(L) \in H^2(X, \mathbb{Z})$  such that the following hold.*

(a) *For any continuous map  $f : X \rightarrow Y$ , and any complex line bundle  $L \rightarrow Y$  we have*

$$c_1(f^*L) = f^*c_1(L).$$

(b) *For any  $1 \leq n \leq \infty$  the first Chern class of the universal line bundle  $\mathcal{U}_{1,n} \rightarrow \mathbb{C}\mathbb{P}^n$  is a generator of  $H^2(\mathbb{C}\mathbb{P}^n, \mathbb{Z})$ .*

**Proof.** Property (a) follows from (4.3.1) while (b) follows from the computations in Example 4.3.5. □

We would like to give a homotopic description of the first Chern class. Note first that the space  $\mathbb{C}\mathbb{P}^\infty$  is an Eilenberg-MacLane space  $K(\mathbb{Z}, 2)$ . Indeed this follows from the long exact homotopy sequence of the fibre bundle  $S^1 \hookrightarrow S^\infty \rightarrow \mathbb{C}\mathbb{P}^\infty$  in which the total space  $S^\infty$  is contractible. Thus

$$\pi_{i+1}(\mathbb{C}\mathbb{P}^\infty) \cong \pi_i(S^1) = \begin{cases} \mathbb{Z} & i+1 = 2 \\ 0 & \text{otherwise} \end{cases}$$

Denote by  $\mathbf{u} \in H^2(\mathbb{C}\mathbb{P}^\infty)$  the first Chern class of  $\mathcal{U}_{1,\infty}$ . We know that  $\mathbf{u}$  is a generator of  $H^2(\mathbb{C}\mathbb{P}^\infty)$ . Since  $\mathbb{C}\mathbb{P}^\infty$  is a  $K(\mathbb{Z}, 2)$  we deduce that for any  $CW$ -complex  $X$  the map

$$[X, \mathbb{C}\mathbb{P}^\infty] \ni f \mapsto f^*\mathbf{u} \in H^2(X, \mathbb{Z})$$

is a bijection.

On the other hand, according to Corollary 3.3.4 the map

$$[X, \mathbb{C}\mathbb{P}^\infty] \ni f \mapsto f^*\mathcal{U}_{1,\infty} \in \mathbf{VB}^1_{\mathbb{C}}(X)$$

is also a bijection. Since  $c_1(f^*\mathcal{U}_{1,\infty}) = f^*\mathbf{u}$ , we deduce that the map

$$c_1 : \mathbf{VB}^1_{\mathbb{C}}(X) \rightarrow H^2(X, \mathbb{Z}), \quad L \mapsto c_1(L)$$

is a bijection. On the other hand, the tensor product introduces a group structure on  $\mathbf{VB}^1_{\mathbb{C}}(X)$ . We denoted this group  $\text{Pic}_{\text{top}}(X)$  and we referred to it as the topological Picard group.

**Proposition 5.1.2.** *For any paracompact space  $X$  the first Chern class  $c_1 : \mathbf{VB}^1_{\mathbb{C}}(X) \rightarrow H^2(X, \mathbb{Z})$  is a morphism of groups. In particular, if  $X$  is a CW-complex then the Chern class defines a group isomorphism*

$$c_1 : \text{Pic}_{\text{top}}(X) \rightarrow H^2(X, \mathbb{Z}),$$

i.e.,

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2), \quad \forall L_1, L_2 \in \mathbf{VB}^1_{\mathbb{C}}(X). \quad (5.1.1)$$

In particular, if  $X$  is homotopy equivalent to a CW-complex then the first Chern class defines an isomorphism  $c_1 : \text{Pic}_{\text{top}}(X) \rightarrow H^2(X, \mathbb{Z})$ .

**Proof.** We follow the proof in [Hatch2, Prop. 3.10] and we begin by proving a special case of (5.1.1).

For simplicity we set  $\mathcal{U} := \mathcal{U}_{1,\infty}$ . We let  $G, G_1$  and  $G_2$  be three copies of  $\mathbb{C}\mathbb{P}^\infty$  and we denote by  $\pi_i, i = 1, 2$  the canonical projection

$$G_1 \times G_2 \rightarrow G, \quad (x_1, x_2) \mapsto x_i.$$

We set  $\mathcal{U}_i = \pi_i^*\mathcal{U} \in \mathbf{VB}^1(G_1 \times G_2)$ ,  $\mathbf{u}_i = \pi_i^*\mathbf{u} \in H^2(G_1 \times G_2)$ ,  $i = 1, 2$ . We want to prove that

$$c_1(\mathcal{U}_1 \otimes \mathcal{U}_2) = \mathbf{u}_1 + \mathbf{u}_2. \quad (5.1.2)$$

Fix  $p \in G$  and consider the wedge

$$G_1 \vee G_2 = G_1 \times \{p\} \cup \{p\} \times G_2 \subset G_1 \times G_2.$$

The inclusion  $i : G_1 \vee G_2 \rightarrow G_1 \times G_2$  induces an injection  $i^*H^2(G_1 \times G_2) \rightarrow H^2(G_1 \vee G_2)$  so it suffices to prove

$$c_1(i^*\mathcal{U}_1 \otimes i^*\mathcal{U}_2) = i^*c_1(\mathcal{U}_1 \otimes \mathcal{U}_2) = i^*\mathbf{u}_1 + i^*\mathbf{u}_2. \quad (5.1.3)$$

If we denote by  $j_k$  the inclusion  $G_k \hookrightarrow G_1 \vee G_2$ , then

$$\forall x, y \in H^2(G_1 \vee G_2) : \quad x = y \iff j_1^*x = j_1^*y, \quad j_2^*x = j_2^*y.$$

Now observe that

$$j_k^*(i^*\mathbf{u}_1 + i^*\mathbf{u}_2) = \mathbf{u}, \quad \forall k = 1, 2$$

and

$$j_1^*i^*\mathcal{U}_2 \cong \underline{\mathbb{C}}, \quad j_2^*i^*\mathcal{U}_1 \cong \underline{\mathbb{C}}.$$

Hence

$$j_k^*(i^*\mathcal{U}_1 \otimes i^*\mathcal{U}_2) \cong \mathcal{U}, \quad j_k^*c_1(i^*\mathcal{U}_1 \otimes i^*\mathcal{U}_2) = \mathbf{u}, \quad \forall k = 1, 2.$$

This concludes the proof of (5.1.2).

Suppose now that  $L_1, L_2$  are two complex line bundles over the paracompact space  $X$ . Then there exist continuous maps  $f_k : X \rightarrow G, k = 1, 2$  such that  $L_k \cong f_k^*\mathcal{U}$ . We denote by  $\Delta$  the diagonal map

$$\Delta : X \rightarrow X \times X, \quad x \mapsto (x, x).$$

Then  $L_1 \otimes L_2$  the pullback of the complex line bundle  $\mathcal{U}_1 \otimes \mathcal{U}_2 \rightarrow G_1 \times G_2$  via the map

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f_1 \times f_2} G_1 \times G_2.$$

For  $k = 1, 2$  we denote by  $p_k : X \times X \rightarrow X$  the projection  $(x_1, x_2) \mapsto x_k$ . Using (5.1.2) we deduce that

$$c_1((f_1 \times f_2)^*(\mathcal{U}_1 \otimes \mathcal{U}_2)) = p_1^* f_1^*(\mathbf{u}) + p_2^* f_2^*(\mathbf{u}) = p_1^* c_1(L_1) + p_2^* c_1(L_2).$$

Hence

$$c_1(L_1 \otimes L_2) = \Delta^*(p_1^* c_1(L_1) + p_2^* c_1(L_2)) = c_1(L_1) + c_2(L_2)$$

because  $p_k \circ \Delta = \mathbb{1}_X$ . This completes the proof of (5.1.1) and of the proposition as well.  $\square$

From (5.1.1) we deduce that for any paracompact space  $X$  we have

$$c_1(L^*) = -c_1(L), \quad \forall L \in \mathbf{VB}_{top}^1(X). \quad (5.1.4)$$

Indeed, this follows from the isomorphism  $L \otimes L^* \cong \underline{\mathbb{C}}_X$ .

## 5.2. The Chern classes: uniqueness and a priori investigations

**Theorem 5.2.1.** *There exists at most one sequences of functions  $\mathcal{C} = \{c_1, c_2, \dots, c_n, \dots\}$  that associate to any paracompact space  $X$  and any isomorphism class of complex vector bundles  $E \rightarrow X$  a sequence of cohomology classes*

$$c_k(E) \in H^{2k}(X, \mathbb{Z}), \quad k = 1, 2, \dots$$

with the following properties.

- (a) The class  $c_1(E)$  is the first Chern class of  $E$  when  $E$  is a complex line bundle.
- (b)  $c_k(E) = 0$ ,  $\forall k > \text{rank}(E)$ .
- (c) For any continuous map  $f : Y \rightarrow X$  and any vector bundle  $E \rightarrow X$  we have

$$c_k(f^*(E)) = f^* c_k(E), \quad \forall k \geq 1.$$

- (d) For any paracompact space  $X$  and any complex vector bundles  $E_0, E_1 \rightarrow X$  we have

$$c_k(E_0 \oplus E_1) = \sum_{i+j=k} c_i(E_0) \cup c_j(E_1), \quad \forall k \geq 1.$$

**Proof.** Before we present the proof, we need to introduce some simplifying notations. Observe that all the classes  $c_k$  belong to the subring

$$H^{even}(X) = \bigoplus_{k \geq 0} H^{2k}(X) \subset H^\bullet(X)$$

This is a commutative ring with respect to the cup product, and for simplicity we will denote the multiplication by the traditional  $\cdot$ .

For any sequence  $\mathcal{C}$  as in the statement of the theorem, and any complex vector bundle  $E \rightarrow X$  we denote by  $\mathcal{C}_E(t)$  the polynomial

$$\mathcal{C}_E(t) := 1 + c_1(E)t + c_2(E)t^2 + \dots \in H^{even}(X)[t].$$

The identity (d) can be rewritten in the more compact form

$$\mathcal{C}_{E_0 \oplus E_1}(t) = \mathcal{C}_{E_0}(t) \cdot \mathcal{C}_{E_1}(t).$$

Suppose two sequences  $\mathcal{C}$  and  $\mathcal{C}'$  with the properties (a)-(d). We have to prove that  $\mathcal{C}_E(t) = \mathcal{C}'_E(t)$  for any complex vector bundle  $E$ .

The properties (a) and (b) imply that

$$\mathcal{C}_L(t) = \mathcal{C}'_L(t), \text{ for any complex line bundle } L.$$

Property (d) implies that if  $E$  is a direct sum of line bundles,

$$E = L_1 \oplus \cdots \oplus L_r, \quad x_i = c_1(L_i), \quad 1 \leq i \leq r,$$

then

$$\mathcal{C}_E(t) = \prod_{i=1}^r (1 + tx_i) = \mathcal{C}'_E(t).$$

In particular, we deduce that

$$c_k(E) = \sigma_k(x_1, \dots, x_r) := \sum_{1 \leq i_1 < \cdots < i_k \leq r} x_{i_1} \cdots x_{i_k}. \quad (5.2.1)$$

Thus,  $c_k(E)$  is the  $k$ -th elementary symmetric polynomial in the variables  $x_i$ . To conclude we need to use the following elegant and very useful trick.

**Lemma 5.2.2** (Splitting Principle). *For any complex vector bundle  $E \rightarrow X$  there exists a continuous map  $f : Y \rightarrow X$  with the following properties.*

- The induced map  $f^* : H^\bullet(X, \mathbb{Z}) \rightarrow H^\bullet(Y)$  is injective.
- the pulled back bundle  $f^*E$  splits as a direct sum of line bundles

$$f^*E = L_1 \oplus \cdots \oplus L_r, \quad r = \text{rank}(E).$$

A map  $f : Y \rightarrow X$  with the above properties is called a splitting map for the vector bundle  $E \rightarrow X$ .

Let us complete the proof of Theorem 5.2.1 assuming the Splitting Principle. If  $E \rightarrow X$  is a complex vector bundle, we choose a splitting map  $f : Y \rightarrow X$ . Then

$$\mathcal{C}_{f^*E}(t) = \mathcal{C}'_{f^*E}(t)$$

because  $f^*E$  is a direct sum of vector bundle. From (c) we deduce that

$$f^* \mathcal{C}_Y(t) = \mathcal{C}_{f^*E}(t) = \mathcal{C}'_{f^*E}(t) = f^* \mathcal{C}'_E(t),$$

and the equality  $\mathcal{C}_E(t) = e\mathcal{C}'_E(t)$  now follows from the injectivity of  $f^*$ .

□

**Proof of the Splitting Principle** We argue inductively on the rank of the vector bundle. The Splitting Principle is clearly true for all line bundles (just take  $\mathbb{1}_X : X \rightarrow X$  as splitting space). We assume the statement is true for all vector bundles of rank  $\leq r$  and we prove it for vector bundles of rank  $r + 1$ .

Let  $E \rightarrow X$  be a complex vector bundle of rank  $r + 1$  and consider the associate bundle of projective spaces

$$\mathbb{C}P^r \hookrightarrow \mathbb{P}(E) \xrightarrow{\pi} X$$

We fix a Hermitian metric on  $E$ . This induces a metric on  $\widehat{E} = \pi^*E$ . As explained in Example 4.4.2 the projection  $\pi$  induces an injection in cohomology and moreover, the universal line bundle  $\mathcal{U}_E$  is a sub-bundle of  $\widehat{E}$ . We have a direct sum decomposition  $\widehat{E} = \mathcal{U}_E \oplus \mathcal{U}_E^\perp$ , where  $\mathcal{U}_E^\perp$  has rank  $r$ . The inductive assumption implies the existence of a splitting map  $g : Y \rightarrow \mathbb{P}(E)$  for  $\mathcal{U}_E^\perp$ . Then the map  $f = \pi \circ g : Y \rightarrow X$  is a splitting map for  $E$ .  $\square$

**Definition 5.2.3.** A sequence of maps  $(c_k)$  as in Theorem 5.2.1 is called a *system Chern classes*. The mapping  $c_k$  is called the  $k$ -th Chern class. The polynomial

$$\mathcal{C}_E(t) := \sum_{k \geq 0} c_k(E)t^k \in H^{\text{even}}(E)[t], \quad c_0(E) = 1,$$

is called the *Chern polynomial* of  $E$  determined by the system of Chern classes.  $\square$

In the remainder of this section we will work under the assumption that Chern classes do exist, and try to extract as much information about them as possible.

**Proposition 5.2.4.** *If they exist, the Chern class satisfy the following additional properties.*

- (a) *If the vector bundle  $E$  is trivializable then  $c_k(E) = 0, \forall k \geq 1$ .*
- (b) *For any complex vector bundle  $E \rightarrow X$  of complex rank  $r$  we have*

$$c_r(E) = e(E_{\mathbb{R}})$$

where  $E_{\mathbb{R}}$  denotes the bundle  $E$  viewed in the canonical way as an oriented real vector bundle.

- (c) *For any complex line bundle  $L \rightarrow X$  and any complex vector bundle  $E \rightarrow X$  of rank  $r$  we have*

$$c_r(L \otimes E) = \sum_{k=0}^r c_k(E) \mathbf{x}^{r-k}, \quad (5.2.2)$$

where  $\mathbf{x} = c_1(L), c_0(E) = 1 \in H^0(X)$ .

- (d) *For any complex vector bundle  $E \rightarrow X$  of rank  $r$  we have*

$$c_1(E) = c_1(\det E)$$

where  $\det E$  is the determinant line bundle of  $E$ ,  $\det E = \Lambda^r E$ .

**Proof.** (a) If the vector bundle  $E$  of rank  $r$  is trivializable then it is a direct sum of  $r$  trivial line bundles. Hence

$$\mathcal{C}_E(t) = \mathcal{C}_{\mathbb{C}_X}(t)^r = 1,$$

where at the last step we invoked Proposition 4.3.6 which implies that  $\mathcal{C}_{\mathbb{C}}(t) = 1$ .

- (b) If  $E$  is a direct sum of complex line bundles,  $E = L_1 \oplus \cdots \oplus L_r$  then

$$c_r(E) = c_1(L_1) \cdots c_1(L_r)$$

and, on the other hand,

$$e(E_{\mathbb{R}}) \stackrel{(4.3.2)}{=} e((L_1)_{\mathbb{R}}) \cdots e((L_r)_{\mathbb{R}}) = c_1(L_1) \cdots c_1(L_r).$$

In general, we choose a splitting map  $Y \xrightarrow{f} X$  for  $E$ . Then  $f^*E$  splits as a direct sum of complex line bundles and by the above argument we deduce

$$f^*c_r(E) = c_r(f^*E) = e(f^*E_{\mathbb{R}}) = f^*e(E_{\mathbb{R}}).$$

The desired conclusion now follows from the fact that  $f^* : H^\bullet(X) \rightarrow H^\bullet(Y)$  is injective.

(c) We first prove the equality (5.2.2) under the simplifying assumption that  $E$  is a direct sum of line bundles

$$E = L_1 \oplus \cdots \oplus L_r, \quad \mathbf{y}_k := c_1(L_k).$$

Then

$$L \otimes E \cong L \otimes L_1 \oplus \cdots \oplus L \otimes L_r$$

and thus

$$\mathcal{C}_{L \otimes E} = \prod_{k=1}^r \mathcal{C}_{L \otimes L_k}(t), \quad \mathcal{C}_{L \otimes L_k}(t) = 1 + tc_1(L \otimes L_k) = 1 + (\mathbf{x} + \mathbf{y}_k)t.$$

Hence

$$c_r(L \otimes E) = \prod_{k=1}^r (\mathbf{x} + \mathbf{y}_k) = \sum_{k=0}^r \sigma_k(\mathbf{y}_1, \dots, \mathbf{y}_r) \mathbf{x}^{r-k} \stackrel{(5.2.1)}{=} \sum_{k=0}^r c_k(E) \mathbf{x}^{r-k}.$$

To prove (5.2.2) in general we proceed as in (b) using a splitting map  $Y \xrightarrow{f} X$  for  $E$ .

(d) Observe that if  $E$  is a direct sum of line bundles

$$E = L_1 \oplus \cdots \oplus L_r, \quad \text{and } x_i := c_1(L_i), \quad \forall 1 \leq i \leq r$$

then  $\det E \cong L_1 \otimes \cdots \otimes L_r$  so that

$$c_1(\det E) = \sigma_1(x_1, \dots, x_r) = x_1 + \cdots + x_r = c_1(E).$$

The general case follows as before by using a splitting map for  $E$ . □

**Remark 5.2.5.** Property (a) shows that the Chern classes provide a measure of the nontriviality of a vector bundle. □

**Corollary 5.2.6.** For any paracompact space  $X$ , any rank  $r$  complex vector bundle  $E \rightarrow X$  and any complex line bundle  $L \rightarrow X$  such that  $\mathbf{y} = c_1(L)$  we have

$$c_r(\text{Hom}(L, E)) = \sum_{k=0}^r (-1)^k c_{r-k}(E) \mathbf{y}^k$$

**Proof.** Set  $\mathbf{x} = c_1(L^*) = -\mathbf{y}$ . Then

$$c_r(\text{Hom}(L, E)) = c_r(L^* \otimes E) \stackrel{(5.2.2)}{=} \sum_{k=0}^r c_{r-k}(E) \mathbf{x}^k.$$

□



**Corollary 5.2.7.** *Suppose  $E \rightarrow X$  is a rank  $r$  complex vector bundle and  $L \rightarrow X$  is a complex line sub-bundle of  $E$ . If  $\mathbf{y} = c_1(L)$  then*

$$\sum_{k=0}^r (-1)^k c_{r-k}(E) \mathbf{y}^k = 0.$$

**Proof.** The inclusion  $i : L \hookrightarrow E$  is a nowhere vanishing section of the complex vector bundle  $F = \text{Hom}(L, E)$ . From Corollary 5.2.6 and Proposition 4.3.6 we deduce

$$\sum_{k=0}^r (-1)^k c_{r-k}(E) \mathbf{y}^k = c_r(F) = e(F_{\mathbb{R}}) = 0.$$

□

**Remark 5.2.8.** The above result has an amusing consequence. Suppose  $X$  is a  $CW$ -complex. We denote by  $\mathcal{R}_X$  the commutative ring  $H^{\text{even}}(X, \mathbb{Z})$ . For any complex vector bundle  $E$  we have a polynomial

$$\mathcal{C}_E^*(t) = \sum_{k=0}^r c_{r-k}(E) t^k \in \mathcal{R}_X[t], \quad r = \text{rank}_{\mathbb{C}}(E).$$

The above corollary implies that if the equation  $\mathcal{C}_E^*(\mathbf{y}) = 0$  has no solution  $\mathbf{y} \in \mathcal{R}_X^2 := H^2(X, \mathbb{Z}) \subset \mathcal{R}_X$ , then the vector bundle  $E$  has no line sub-bundles.

If we set  $\mathcal{R}_E = \mathcal{R}_{\mathbb{P}(E)}$ , then we can regard  $\mathcal{R}_X$  as a subring of  $\mathcal{R}_E$ , or equivalently,  $\mathcal{R}_E$  as an extension of  $\mathcal{R}_X$ . The proof of the splitting principle implies that  $\mathcal{C}_E^*(t)$  has a root  $\mathbf{y}$  in the extension  $\mathcal{R}_E$ , namely  $\mathbf{y} = -c_1(\mathcal{U}_E) \in \mathcal{R}_E^2$ . □

**Corollary 5.2.9.** *Suppose  $E \rightarrow X$  is a complex vector bundle of rank  $r$ , and  $\mathbb{P}(E) \xrightarrow{\pi} X$  is its projectivization. If  $\mathcal{U}_E \rightarrow \mathbb{P}(E)$  is the universal line bundle and  $\mathbf{u} = c_1(\mathcal{U}_E)$  then*

$$\sum_{k=0}^r (-1)^k \pi^* c_{r-k}(E) \mathbf{u}^k = 0.$$

**Proof.** Observe that  $\mathcal{U}_E$  is a line sub-bundle of  $\pi^*E$ . The conclusion now follows from Corollary 5.2.7. □

### 5.3. Chern classes: existence

**Theorem 5.3.1.** *There exists a system of Chern classes.*

**Proof.** The starting point of our construction is Corollary 5.2.9. Suppose  $E \rightarrow X$  is a complex vector bundle of rank  $r$  over  $X$ . We denote by  $\mathbb{P}(E) \xrightarrow{\pi} X$  the projectivization of  $E$ , by  $\mathcal{U}_E$  the tautological line bundle over  $\mathbb{P}(E)$  and by  $\mathbf{u} \in H^2(\mathbb{P}(E))$  its first Chern class.

As shown in Example 4.4.2, for any cohomology class  $c \in H^\bullet(\mathbb{P}(E))$  there exist classes  $\beta_i = \beta_i(c) \in H^\bullet(X)$ ,  $0 \leq i \leq r-1$ , uniquely determined by the equality

$$c = \sum_{k=0}^{r-1} \pi^* \beta_k \mathbf{u}^k.$$

If we let  $c = \mathbf{u}^r$  in the above, and use Corollary 5.2.9 as guide, we define the Chern classes of  $E$  to be the unique cohomology classes  $c_k(E) \in H^{2k}(X)$ , such that

$$c_0(E) = 1, \quad \sum_{k=0}^r (-1)^k \pi^* c_{r-k}(E) \mathbf{u}^k = 0, \quad c_k(E) = 0, \quad \forall k > r.$$

The properties (a)-(c) Theorem 5.2.1 are immediate. Only property (d) requires some work.

Suppose  $E_0 \xrightarrow{\pi_0} X, E_1 \xrightarrow{\pi_1} X$  are complex vector bundles of ranks  $r_0$  and  $r_1$ . We set  $E = E_0 \oplus E_1$  and  $r = r_0 + r_1$ . We continue to denote by  $\pi_i$  the canonical projections  $\mathbb{P}(E_i) \rightarrow X$ . Note that we have inclusions

$$\mathbb{P}(E_0) \xrightarrow{j_0} \mathbb{P}(E), \quad \mathbb{P}(E_1) \xrightarrow{j_1} \mathbb{P}(E).$$

Moreover  $\mathbb{P}(E_0) \cap \mathbb{P}(E_1) = \emptyset$ . Tautologically

$$j_i^* \mathcal{U}_E = \mathcal{U}_{E_i}, \quad i = 0, 1.$$

Set  $\mathbf{u} = c_1(\mathcal{U}_E)$ ,  $\mathcal{O}_0 = \mathbb{P}(E) \setminus \mathbb{P}(E_1)$ ,  $\mathcal{O}_1 = \mathbb{P}(E) \setminus \mathbb{P}(E_0)$ . Thus  $\mathcal{O}_i$  is an open neighborhood of  $\mathbb{P}(E_i)$  and  $\mathcal{O}_0 \cup \mathcal{O}_1 = \mathbb{P}(E)$ .

Define

$$\omega_i = \sum_{k=0}^{r_i} (-1)^k \pi_* c_{r_i-k}(E_i) \mathbf{u}^k \in H^{2r_i}(\mathbb{P}(E)), \quad i = 0, 1.$$

Note that

$$\omega_0 \omega_1 = \sum_{\ell=0}^r (-1)^\ell \left( \sum_{i+j=r-\ell} \pi^* c_i(E_0) \pi^* c_j(E_1) \right) \mathbf{u}^\ell.$$

The equality

$$\mathcal{C}_{E_0 \oplus E_1}(t) = \mathcal{C}_0(t) \mathcal{C}_1(t)$$

is then equivalent to the equality

$$\omega_0 \omega_1 = 0.$$

To prove this we will need the following topological fact whose proof will be given a bit later.

**Lemma 5.3.2.**  $\mathbb{P}(E_i)$  is a deformation retract of  $\mathcal{O}_i$ ,  $i = 0, 1$ . □

Observe that  $j_i^* \omega_i = 0$  due to the definition of the Chern classes of  $E_i$ . Hence  $\omega_i$  is the image of a relative class  $\hat{\omega}_i \in H^{2r_i}(\mathbb{P}(E), \mathbb{P}(E_i))$ . Invoking Lemma 5.3.2, we can regard  $\hat{\omega}_i$  as a class  $\hat{\omega}_i \in H^{2r_i}(\mathbb{P}(E), \mathcal{O}_i)$ . Thus  $\omega_0 \omega_1$  is the image of the cohomology class

$$\hat{\omega}_0 \cup \hat{\omega}_1 \in H^{2r}(\mathbb{P}(E), \mathcal{O}_0 \cup \mathcal{O}_1)$$

via the natural morphism  $H^{2r}(\mathbb{P}(E), \mathcal{O}_0 \cup \mathcal{O}_1) \rightarrow H^{2r}(\mathbb{P}(E))$ . We now observe that

$$H^{2r}(\mathbb{P}(E), \mathcal{O}_0 \cup \mathcal{O}_1) = 0$$

since  $\mathcal{O}_0 \cup \mathcal{O}_1 = \mathbb{P}(E)$ . □

**Proof of Lemma 5.3.2.** Denote by  $P_i$  the canonical projection  $P_i : E_0 \oplus E_1 \rightarrow E_i$ . For every  $x \in X$  we set  $\mathcal{O}_i(x) = \mathbb{P}(E_x) \cap \mathcal{O}_i$ . For every line  $\ell(x) \in \mathcal{O}_0(x)$  we set  $\ell_0(x) = P_0 \ell(x) \subset E_0(x)$ . The line  $\ell(x)$  intersects  $E_1(x)$  trivially and thus we can find a linear map  $T = T_{\ell(x)} : \ell_0(x) \rightarrow E_1(x)$  such that  $\ell(x)$  is the graph of  $T$  viewed as a subset of  $\ell_0(x) \oplus E_1(x)$ . The correspondence

$$\mathcal{O}_0(x) \ni \ell(x) \mapsto T_{\ell(x)} \in \text{Hom}(\ell_0(x), E_1(x))$$

determines a homeomorphism between  $\mathcal{O}_0$  and the total space of the fibration

$$\text{Hom}(\mathcal{U}_{E_0}, \pi_0^* E_1) \rightarrow X$$

Under this homeomorphism the subset  $\mathbb{P}(E_0) \subset \mathcal{O}_0$  corresponds to the zero section of this vector bundle.  $\square$

**Exercise 5.3.3.** (a) Suppose  $L \rightarrow X$  is a complex line bundle. Prove that its conjugate  $\bar{L}$  is isomorphic as a *complex* line bundle to the dual line bundle  $L^*$ .

(b) Prove that for every complex vector bundle  $E \rightarrow X$  we have

$$c_k(\bar{E}) = (-1)^k c_k(E), \quad \forall k \geq 1.$$

Above,  $\bar{E}$  denotes the conjugate complex vector bundle.  $\square$

**Remark 5.3.4.** There exists a real version of Theorem 5.3.1 More precisely there exists a unique sequence of functions

$$\mathcal{W} = \{w_1, w_2, \dots, w_n, \dots\}$$

that associate to any paracompact space  $X$ , and any isomorphism class of real vector bundles  $E \rightarrow X$  the sequence of cohomology classes

$$w_k(E) \in H^k(X, \mathbb{F}_2), \quad k = 1, 2, \dots$$

with the following properties.

(a) The class  $w_1(E)$  is the Euler class of  $E$  with  $\mathbb{F}_2$ -coefficients if  $E$  is a real line bundle.

(b)  $w_k(E) = 0, \forall k > \text{rank}(E)$ .

(c) For any continuous map  $f : Y \rightarrow X$ , and any vector bundle  $E \rightarrow X$  we have

$$w_k(f^* E) = f^* w_k(E), \quad \forall k \geq 1.$$

(d) For any paracompact space  $X$  and any real vector bundles  $E_0, E_1 \rightarrow X$  we have

$$w_k(E_0 \oplus E_1) = \sum_{i+j=k} w_i(E_0) \cup w_j(E_1).$$

The functions  $w_k$  are called the *Stieffel-Whitney classes* of a real vector bundle. The proof of this theorem is identical to the proof of Theorem 5.2.1 + 5.3.1.  $\square$

## 5.4. The localization formula and some of its applications

Suppose  $M$  is a smooth compact, oriented  $n$ -dimensional manifold and  $E \rightarrow M$  is a (geometrically) oriented smooth real vector bundle of rank  $c \leq n$ . We would like to give a more geometric interpretation of the Euler class  $e(E)$  in terms of zero loci of sections of  $E$ .

Suppose  $u$  is a smooth section of this vector bundle and  $x_0 \in M$  is a zero of  $u, u(x_0) = 0 \in E_{x_0}$ . We fix a local trivialization of  $E$  on a neighborhood  $\mathcal{O}$  of  $x_0, \Psi : E|_{\mathcal{O}} \rightarrow \underline{E}_{x_0, \mathcal{O}}$ , and we define

$$\mathcal{A}_{u, x_0} : T_{x_0} M \rightarrow E_{x_0}, \quad T_{x_0} M \ni X \mapsto D\Psi \circ u(X), \quad (5.4.1)$$

where  $D\Psi \circ u : T_{x_0} M \rightarrow E_{x_0}$  is the differential of the map  $\Psi \circ u : \mathcal{O} \rightarrow E_{x_0}$ . The condition  $u(x_0) = 0$  implies that the map  $\mathcal{A}_{u, x_0}$  is independent of the choice of local trivialization  $\Psi$ . We will

refer to it as the *differential of  $u$  at  $x_0$* , or the *adjunction morphism at  $x_0$* . We want to emphasize that this definition works only for zeros of  $u$ . We obtain in this fashion a bundle morphism

$$\mathcal{A}_u : TM|_{\{u=0\}} \rightarrow E|_{\{u=0\}} \quad (5.4.2)$$

called the *adjunction morphism*

We say that  $u$  is a *regular section* if for any  $x \in u^{-1}(0)$  the differential of  $u$  at  $x$  is a surjective linear map  $T_x M \rightarrow E_x$ .

If  $u$  is a regular section, then the implicit function theorem implies that zero set of  $u$  is a smooth submanifold  $Z_u \subset M$ ,  $\text{codim}_M Z_u = c$ . Moreover  $\forall x \in Z_u$  we have

$$T_x Z_u = \ker(\mathcal{A}_{u,x} : T_x M \rightarrow E_x).$$

Now some terminology. For any smooth submanifold  $Z \subset M$  the tangent bundle  $TZ$  is a subbundle of  $TM|_Z$ , and dually, the cotangent bundle  $T^*Z$  is a quotient bundle of  $T^*M|_Z$ . We denote by  $T_Z M$  the cokernel of the inclusion  $TZ \hookrightarrow TM|_Z$ , and by  $T^*Z M$  the kernel of the surjection  $T^*M|_Z \rightarrow T^*Z$ . We thus have two short exact sequences of vector bundles

$$0 \rightarrow TZ \rightarrow TM|_Z \rightarrow T_Z M \rightarrow 0 \quad (5.4.3a)$$

$$0 \rightarrow T_Z^* M \rightarrow T^*M|_Z \rightarrow T^*Z \rightarrow 0. \quad (5.4.3b)$$

The (quotient) bundle  $T_Z M$  is called the *normal bundle* of the embedding  $Z \hookrightarrow M$ , while the subbundle  $T_Z^* M$  is called the *conormal bundle* of the embedding  $Z \hookrightarrow M$ .

Returning to our case at hand, we deduce that the differentials  $(\mathcal{A}_{u,x})_{u(x)=0}$  determine a bundle isomorphism

$$\mathcal{A}_u : T_Z M \rightarrow E|_M,$$

and their duals  $(\mathcal{A}_{u,x}^*)_{u(x)=0}$  define a bundle isomorphism  $\mathcal{A}_u^\dagger : E^*|_Z \rightarrow T_Z^* M$ .

In the sequence (5.4.3b) the bundle  $T^*M|_Z$  has an orientation induced by the orientation of  $M$ , and the bundle  $T_Z^* M$  has an orientation induced via  $\mathcal{A}_u^*$  from the orientation of  $E^*$ . In particular, by duality, we also get an orientation on the normal bundle  $T_Z M$ , and we deduce that  $T^*Z$  has an induced orientation given by the fiber-first convention

$$\text{or}(T^*M|_Z) = \text{or}(T_Z^* M) \wedge \text{or}(T^*Z).$$

Thus  $Z$  is equipped with a natural orientation  $\mu_u \in H_{n-c}(Z)$ .<sup>1</sup> If we denote by  $i$  the natural inclusion  $Z \hookrightarrow M$  then we obtain a homology class  $i_* \mu_u \in H_{n-c}(M)$ . We denote this class by  $\langle u = 0 \rangle$ . We will refer to it as the *homology class determined by the zero locus of  $u$* .

**Theorem 5.4.1** (Localization formula). *Suppose that  $E \rightarrow M$  is a smooth, real, oriented vector bundle of rank  $c$  over the smooth compact oriented manifold  $M$  of dimension  $n$ . Then for every non-degenerate smooth section  $u$  of  $E$ , the homology class determined by the zero locus of  $u$  is Poincaré dual to the Euler class of  $E$ . (We say that the Euler class localizes along the zero locus of  $u$ .)*

**Proof.** Denote by  $\Phi_E \in H^c(E, E \setminus M)$  the Thom class of  $E$  so that  $e(E) = u^* \Phi_E$ . We fix a Riemannian metric  $g$  on  $M$ . Set as above  $Z := u^{-1}(0)$  and define

$$U_\varepsilon := \{p \in M; \text{dist}(p, Z) \leq \varepsilon\}, \quad M_\varepsilon := M \setminus U_\varepsilon.$$

The section  $u$  does not vanish on  $M_\varepsilon$  and we obtain a relative Euler class

$$\eta_\varepsilon := e(E, u|_{M_\varepsilon}) \in H^c(M, M_\varepsilon)$$

<sup>1</sup>This orientation depends on the choice of  $u$ . For example,  $\mu_{-u} = (-1)^c \mu_u$ .

satisfying

$$\mathcal{E}\eta_\varepsilon = e(E) \in H^c(M), \quad (5.4.4)$$

where  $\mathcal{E} : H^c(M, M_\varepsilon) \rightarrow H^c(M)$  is the natural extension morphism.

The metric  $g$  induces a metric on the normal bundle  $T_Z M$  so that we have an orthogonal decomposition

$$TM|_Z = TZ \oplus T_Z M.$$

We denote by  $D_\varepsilon(T_Z M) \rightarrow Z$  the  $\varepsilon$ -disk bundle over  $Z$  determined by this induced metric. We can now regard  $T_Z M$  as a subbundle of  $TM|_Z$  and the exponential map of the metric defines a smooth map

$$\exp_g : D_\varepsilon(T_Z M) \rightarrow M.$$

For  $\varepsilon > 0$  sufficiently small, this induces a diffeomorphism of manifolds with boundary

$$\exp_g : D_\varepsilon(T_Z M) \rightarrow U_\varepsilon.$$

The projection  $\pi : D_\varepsilon(T_Z M) \rightarrow Z$  defines a projection  $\pi_Z : U_\varepsilon \rightarrow Z$ . Moreover we can identify the bundle  $E|_{U_\varepsilon}$  with the pullback  $\pi_Z^*(E_Z) \cong E|_{U_\varepsilon}$ .

The bundle morphism  $\mathcal{A}_u : T_Z M \rightarrow E_Z$  defines a diffeomorphism from  $D_\varepsilon(T_Z M)$  to a closed neighborhood of  $Z$  in  $E_Z$ . We denote this neighborhood  $D_{Z,\varepsilon}(E)$ . This neighborhood is canonically a disk bundle over  $Z$ . Using the projection  $\pi_Z : U_\varepsilon \rightarrow Z$  we can pullback the disk bundle  $D_{Z,\varepsilon}(E)$  to a disk bundle  $D_{U_\varepsilon} \hookrightarrow \pi_Z^* E_Z$ .

If we denote by  $\Phi_{E_Z}$  the Thom class of  $E_Z$ , then we can represent it as an element

$$\Phi_{E_Z,\varepsilon} \in H^c(D_{Z,\varepsilon}(E), \partial D_{Z,\varepsilon}(E)) \cong H^c(E_Z, E_Z \setminus Z).$$

From our orientation conventions we deduce that

$$\mathcal{A}_u^* \Phi_{E_Z,\varepsilon} = \Phi_{T_Z M} \in H^c(D_\varepsilon(T_Z M), \partial D_\varepsilon(T_Z M)) = H^c(U_\varepsilon, \partial U_\varepsilon), \quad (5.4.5)$$

where  $\Phi_{T_Z M}$  is the Thom class of  $T_Z M$  equipped with the orientation induced by  $\mathcal{A}_u$  from the orientation of  $E$ .

On the other hand, we can view the section  $u|_{U_\varepsilon} : U_\varepsilon \rightarrow E$  as a section of  $\pi_Z^* E_Z \cong E|_{U_\varepsilon}$ . Identifying  $U_\varepsilon$  with  $D_\varepsilon(T_Z M)$  via the exponential map we can also view the restriction to  $D_\varepsilon(T_Z M)$  of the bundle morphism  $\mathcal{A}_u$  as a section

$$\hat{\mathcal{A}}_u : U_\varepsilon \rightarrow \pi_Z^* E_Z.$$

The map  $\mathcal{A}_u$  is essentially the differential of  $u$  so that, if  $\varepsilon$  is sufficiently small, the sections  $u|_{U_\varepsilon}$  and  $\hat{\mathcal{A}}_u|_{U_\varepsilon}$  are homotopic as sections of  $\pi_Z^* E_Z$  that do not vanish along  $\partial U_\varepsilon$ .

Since the Thom class of  $\pi_Z^* E_Z$  is  $\pi_Z^* \Phi_{E_Z,\varepsilon}$  we conclude that

$$e(E)|_{U_\varepsilon} = u^* \pi_Z^* \Phi_{E_Z,\varepsilon} = \hat{\mathcal{A}}_u^* \pi_Z^* \Phi_{E_Z,\varepsilon} = \mathcal{A}_u^* \Phi_{E_Z,\varepsilon} \in H^c(U_\varepsilon, \partial U_\varepsilon). \quad (5.4.6)$$

In (5.4.4) we can choose  $\eta_\varepsilon = u^* \pi_Z^* \Phi_{E_Z,\varepsilon}$ . Using the equality (5.4.5) we deduce that  $\eta_\varepsilon \in H^c(U_\varepsilon, \partial U_\varepsilon)$  can be identified with the Thom class of  $T_Z M$ .

On the other hand, Proposition 4.2.5 implies that the Thom class of  $T_Z M$  can be identified with the Poincaré dual of the class  $\langle u = 0 \rangle \in H_{n-c}(U_\varepsilon)$  relative to the Poincaré duality on the manifold

with boundary  $(U_\varepsilon, \partial U_\varepsilon)$ . The localization formula now follows from the commutative diagram

$$\begin{array}{ccc} H^r(M) \ni e(E) & \xrightarrow{\cap \mu_M} & \in H_{n-r}(M) \\ \varepsilon \uparrow & & \uparrow i_* \\ H^c(M, M_\varepsilon) \cong H^r(U_\varepsilon, \partial U_\varepsilon) \ni \eta_\varepsilon & \xrightarrow{\cap \mu_{U_\varepsilon}} & \langle u = 0 \rangle \in H_{n-r}(U_\varepsilon) \end{array}$$

□

**Corollary 5.4.2.** *Suppose  $E \rightarrow M$  is a complex vector bundle of rank  $r$  over the smooth, compact oriented  $n$ -dimensional manifold  $M$ . Then for any nondegenerate section  $u : M \rightarrow E$  the homology class determined by its zero locus is Poincaré dual to the top Chern class  $c_r(E)$ .* □

The above simple localization formula is quite useful for concrete computations, and it is one of the main reasons why Chern classes are computationally friendlier than the Pontryagin classes that we will define in the next chapter. We close this section with some important applications of this formula.

**Example 5.4.3** (The hyperplane line bundle). Consider the tautological line bundle  $\mathcal{U} = \mathcal{U}_n \rightarrow \mathbb{C}\mathbb{P}^n$ . We denote by  $\mathcal{H}$  its dual,  $\mathcal{H} = \mathcal{H}_n := \mathcal{U}^*$ . The line bundle  $\mathcal{H}$  is usually referred to as the *hyperplane line bundle*.

Let us observe that any linear function  $\alpha : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  induces a linear function  $\alpha_L$  on any one-dimensional subspace  $L \subset \mathbb{C}^{n+1}$ . We obtain in this fashion a correspondence

$$\mathbb{C}\mathbb{P}^n \ni L \mapsto \alpha_L \in \text{Hom}(L, \mathbb{C}) = \mathcal{U}_L^* = \mathcal{H}_L$$

that defines a smooth section  $u_\alpha$  of  $\mathcal{H}$ .

We choose linear coordinates  $z_1, \dots, z_n$  on  $\mathbb{C}^{n+1}$  such that

$$\alpha(z_1, \dots, z_{n+1}) = z_{n+1}$$

Then the zero set of  $u_\alpha$  is

$$Z_\alpha := \{[z_1, \dots, z_n, 0] \in \mathbb{C}\mathbb{P}^n\}.$$

We claim that  $u_\alpha$  is a nondegenerate section. Let  $\ell_0 \in Z_\alpha$ . Assume for simplicity that  $z_1(\ell_0) \neq 0$ . Then, in the neighborhood  $\mathcal{O}_1 = \{z_1 \neq 0\}$  of  $\ell_0$  we can use as coordinates the functions

$$\zeta_2(\ell) = \frac{z_2(\ell)}{z_1(\ell)}, \dots, \zeta_{n+1}(\ell) = \frac{z_{n+1}(\ell)}{z_1(\ell)}.$$

The line  $\ell \subset \mathbb{C}^{n+1}$  is spanned by the vector

$$\vec{v}_\ell = (1, \zeta_2(\ell), \dots, \zeta_{n+1}(\ell)).$$

Hence, the collection  $(\vec{v}_\ell)_{\ell \in \mathcal{O}_1}$  defines a framing of  $\mathcal{U}_n|_{\mathcal{O}_1}$ . Define  $\omega_\ell \in \mathcal{H}_\ell$  to be the linear map  $\ell \rightarrow \mathbb{C}$  uniquely determined by the equality  $\omega_\ell(\vec{v}_\ell) = 1$ . The collection  $(\omega_\ell)_{\ell \in \mathcal{O}_1}$  defines a framing of  $\mathcal{H}|_{\mathcal{O}_1}$ . Moreover

$$u_\alpha(\vec{v}_\ell) = \zeta_{n+1}(\ell) = \zeta_{n+1} \omega_\ell(\vec{v}_\ell).$$

This shows that the section  $u_\alpha$  can be identified with the smooth function

$$\mathcal{O}_1 \ni \ell \mapsto \alpha(\vec{v}_\ell) = \zeta_{n+1}(\ell) \in \mathbb{C}.$$

This function is a *linear* submersion so that  $u_\alpha$  is nondegenerate. The orientation on the zero set  $Z_\alpha$  is precisely the orientation as a complex submanifold. In fact,  $Z_\alpha$  is a projective hyperplane in  $\mathbb{C}\mathbb{P}^n$ . The homology class carried by  $Z_\alpha$  is independent of the nonzero section  $\alpha$ . It's Poincaré dual is denoted by  $[H]$  and it is called the *hyperplane class*. For this reason the line bundle  $\mathcal{H}$  is called the *hyperplane line bundle*. From the localization formula we deduce

$$c_1(\mathcal{H}) = -c_1(\mathcal{U}) = [H] \in H^2(\mathbb{C}\mathbb{P}^n, \mathbb{Z}).$$

If we consider the vector bundle

$$E = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_n$$

then  $c_n(E) = c_1(\mathcal{H})^n$ . On the other hand, if we take the linearly independent linear functions

$$\alpha_1, \dots, \alpha_n : \mathbb{C}^{n+1} \rightarrow \mathbb{C}, \quad \alpha_i(z_1, \dots, z_{n+1}) = z_i, \quad i = 1, \dots, n,$$

We obtain a section  $u : u_{\alpha_1} \oplus \cdots \oplus u_{\alpha_n} : \mathbb{C}\mathbb{P}^n \rightarrow E$ . Its zero set consists of a single point

$$\ell_0 = [z_1 = 0, \dots, z_n = 0, z_{n+1} \neq 0] = [0, \dots, 0, 1]$$

A simple computation as above shows that  $u$  is a nondegenerate section. The homology class associated to its zero locus is represented by the 0-cycle  $[\ell_0]$ . This proves that the class  $[H]^n \in H^{2n}(\mathbb{C}\mathbb{P}^n, \mathbb{Z})$  is the generator given by the complex orientation of  $\mathbb{C}\mathbb{P}^n$ .  $\square$

**Corollary 5.4.4.** *The signature of the complex projective space  $\mathbb{C}\mathbb{P}^{2n}$  equipped with the complex orientation is  $\tau_{\mathbb{C}\mathbb{P}^{2n}} = 1$ .*

**Proof.** The middle homology is generated by the class  $[H]^n$  and  $[H]^n \cup [H]^n = H^{2n}$  is the canonical generator of the top cohomology group determined by the orientation. Thus, in the basis  $[H]^n$  of  $H^{2n}(\mathbb{C}\mathbb{P}^{2n})$  the intersection form is represented by the  $1 \times 1$  matrix  $[1]$ .  $\square$

**Example 5.4.5** (The Chern classes of  $T\mathbb{C}\mathbb{P}^n$ ). Since  $\mathbb{C}\mathbb{P}^n$  is a complex manifold, the tangent bundle has a natural complex structure and thus we can speak of Chern classes. We regard  $\mathbb{C}\mathbb{P}^n$  as the space of complex lines in the Euclidean space  $\mathbb{C}\mathbb{P}^n$ . As such, the trivial vector bundle  $\underline{\mathbb{C}}_{\mathbb{C}\mathbb{P}^n}^{n+1}$  has a canonical metric. The universal line bundle  $\mathcal{U}$  is naturally a subbundle of this trivial vector bundle. We denote by  $\mathcal{U}^\perp$  its orthogonal complement in  $\underline{\mathbb{C}}^{n+1}$ . Let us first observe that we have a canonical isomorphism of complex vector bundles

$$\dot{\Gamma} : \text{Hom}_{\mathbb{C}}(\mathcal{U}, \mathcal{U}^\perp) \cong T\mathbb{C}\mathbb{P}^n. \tag{5.4.7}$$

To understand this isomorphism we fix  $L \in \mathbb{C}\mathbb{P}^n$ . Then any complex linear map  $A : L \rightarrow L^\perp$  determines a 1-dimensional subspace

$$\Gamma_L(A) := \{x \oplus Ax \in L \oplus L^\perp; x \in A\}.$$

Observe that  $\Gamma_L(0) = L$ . The map  $A \mapsto \Gamma_L(A)$  is a diffeomorphism from the vector space  $\text{Hom}_{\mathbb{C}}(L, L^\perp)$  onto an open neighborhood of  $L$  in  $\mathbb{C}\mathbb{P}^n$ . We get a map

$$\text{Hom}_{\mathbb{C}}(L, L^\perp) \ni A \mapsto \dot{\Gamma}_L(A) \in T_L\mathbb{C}\mathbb{P}^n, \quad \dot{\Gamma}_L(A) := \left. \frac{d}{dt} \right|_{t=0} \Gamma_L(tA).$$

The map  $\dot{\Gamma}_L$  is  $\mathbb{C}$ -linear and bijective since the map  $A \mapsto \Gamma_A(L)$  is a diffeomorphism onto its image. The collection  $(\dot{\Gamma}_L)_{L \in \mathbb{C}\mathbb{P}^n}$  defines the bundle isomorphism (5.4.7). We conclude that

$$T\mathbb{C}\mathbb{P}^n \cong \mathcal{U}^* \otimes \mathcal{U}^\perp = \mathcal{H} \otimes \mathcal{U}^\perp.$$

From the equality  $\underline{\mathbb{C}}^{n+1} = \mathcal{U} \oplus \mathcal{U}^\perp$  we deduce that

$$\mathcal{H}^{n+1} \cong \mathcal{H} \otimes \underline{\mathbb{C}}^{n+1} \cong \mathcal{H} \otimes (\mathcal{U} \oplus \mathcal{U}^\perp) \cong (\mathcal{H} \otimes \mathcal{U}) \oplus (\mathcal{H} \otimes \mathcal{U}^\perp) \cong \underline{\mathbb{C}} \oplus T\mathbb{C}\mathbb{P}^n.$$

so that

$$\mathcal{C}_{\mathcal{H}^{n+1}}(t) = \mathcal{C}_{T\mathbb{C}\mathbb{P}^n}(t).$$

Hence

$$\mathcal{C}_{T\mathbb{C}\mathbb{P}^n}(t) = \mathcal{C}_{\mathcal{H}}(t)^{n+1} = (1 + [H]t)^{n+1} = \sum_{k=0}^n \binom{n+1}{k} [H]^k t^k, \quad (5.4.8a)$$

$$c_k(T\mathbb{C}\mathbb{P}^n) = \binom{n+1}{k} [H]^k, \quad \forall k = 1, \dots, n. \quad (5.4.8b)$$

□

**Example 5.4.6** (Oriented vector bundles over  $S^4$ ). We want to investigate two oriented vector bundles of rank 4 over  $S^4$ .

(a) The first such bundle is the tangent bundle  $E = TS^4$ . We describe  $S^4$  as the subset

$$S^4 = \{ \mathbf{x} = (x_0, \dots, x_4) \in \mathbb{R}^5; |\mathbf{x}| = 1 \}.$$

The tangent space to  $S^4$  at  $\mathbf{x}$  can be identified with the subspace

$$\langle \mathbf{x} \rangle^\perp := \{ \mathbf{u} \in \mathbb{R}^5; \mathbf{u} \perp \mathbf{x} \}.$$

We denote by  $P_{\mathbf{x}}$  the orthogonal projection onto  $\langle \mathbf{x} \rangle^\perp$ . Observe that

$$P_{\mathbf{x}} \mathbf{y} = \mathbf{y} - (\mathbf{y}, \mathbf{x}) \mathbf{x}, \quad \forall \mathbf{y} \in \mathbb{R}^5,$$

where  $(-, -)$  denotes the canonical inner product on  $\mathbb{R}^5$ .

We denote by  $\mathbf{e}_0, \dots, \mathbf{e}_4$  the canonical basis of  $\mathbb{R}^5$  and we form the vector field

$$V : S^4 \rightarrow TS^4, \quad \mathbf{x} \mapsto P_{\mathbf{x}} \mathbf{e}_0.$$

We will prove that  $V$  is a nondegenerate smooth section of  $TS^4$  and then we will compute the cycle determined by the zero locus of this section.

Note first that the zero locus of this section consists of two points, the North pole  $\mathbf{x}_+ = (1, 0, \dots, 0)$  and the South Pole  $\mathbf{x}_- = (-1, 0, \dots, 0)$ .

Near  $\mathbf{x}_+$  we can use  $\mathbf{x}' = (x_1, \dots, x_4)$  as local coordinates. In these coordinates  $\mathbf{x}_+$  is identified with the origin of  $\mathbb{R}^4$ . Then, near  $\mathbf{x}_+$  we have

$$x_0 = \sqrt{1 - r^2}, \quad r(\mathbf{x}') := |\mathbf{x}'| = \sqrt{x_1^2 + \dots + x_4^2}.$$

Moreover, the canonical orientation<sup>2</sup> of  $S^4$  near  $\mathbf{x}_+$  is given by the frame  $(\partial_{x_1}, \dots, \partial_{x_4})$ . These tangent vectors can be identified with the vectors in  $\mathbb{R}^5$  given by

$$\partial_{x_i} \rightarrow \frac{\partial}{\partial x_i} \mathbf{x} = \frac{\partial}{\partial x_i} \sum_{i=0}^4 x_i \mathbf{e}_i = \frac{\partial x_0}{\partial x_i} \mathbf{e}_0 + \mathbf{e}_i, \quad i = 1, \dots, 4.$$

More explicitly

$$\partial_{x_i} = -\frac{x_i}{\sqrt{1 - r^2}} \mathbf{e}_0 + \mathbf{e}_i, \quad i = 1, \dots, 4. \quad (5.4.9)$$

<sup>2</sup>This is the orientation of  $S^4$  as boundary of the unit ball in  $\mathbb{R}^4$ .



We can now find four smooth functions  $v_1(\mathbf{x}'), \dots, v_4(\mathbf{x}')$  defined in a neighborhood of  $0 \in \mathbb{R}^4$  such that near  $\mathbf{x}_+$  we have

$$V(\mathbf{x}) = \sum_{i=1}^4 v_i \partial_{x_i} \iff \mathbf{e}_0 - \sum_{i=1}^4 v_i \partial_{x_i} \perp \partial_{x_j}, \quad \forall j = 1, \dots, 4.$$

Using (5.4.9) we deduce that

$$-\frac{x_j}{\sqrt{1-r^2}} = \sum_{i=1}^4 v_i(\partial_{x_i}, \partial_{x_j}), \quad \forall j = 1, \dots, 4.$$

Now observe that

$$(\partial_{x_i}, \partial_{x_j}) = \frac{x_i x_j}{1-r^2} + \delta_{ij},$$

where  $\delta_{ij}$  is the Kronecker symbol

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

If we now introduce the  $4 \times 4$ -matrix  $H$  with entries  $H_{ij} = x_i x_j$  we deduce that

$$\begin{aligned} \left( \mathbb{1} + \frac{1}{1-r^2} H \right) \begin{bmatrix} v_1 \\ \vdots \\ v_4 \end{bmatrix} &= -\frac{1}{\sqrt{1-r^2}} \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \\ \iff \left( (\mathbb{1} + r^2 + O(3)) H \right) \begin{bmatrix} v_1 \\ \vdots \\ v_4 \end{bmatrix} &= -(1 + r^2 + O(3)) \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix}, \end{aligned}$$

where  $O(k)$  denotes a term which in norm is  $\leq \text{const} \cdot r^k$  near  $\mathbf{x}_+$ . Observe that  $H = O(2)$ ,  $r^2 H = O(4)$  so that

$$\left( (\mathbb{1} + r^2 + O(3)) H \right)^{-1} = \mathbb{1} + O(2).$$

If we set

$$\vec{v} := \begin{bmatrix} v_1 \\ \vdots \\ v_4 \end{bmatrix}$$

then we deduce

$$\vec{v} = -(\mathbb{1} + O(2)) \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix}.$$

We deduce that the differential at  $\mathbf{x}_+ = (0, 0, 0, 0)$  of the map

$$\mathbb{R}^4 \ni \mathbf{x}' \mapsto \vec{v}(\mathbf{x}') \in \mathbb{R}^4$$

is  $-\mathbb{1}$ . This is precisely the adjunction morphism  $\mathcal{A}_{V, \mathbf{x}_+} : T_{\mathbf{x}_+} S^4 \rightarrow T_{\mathbf{x}_+} S^4$  in (5.4.1). In particular the map  $\mathcal{A}_{V, \mathbf{x}_+}$  is orientation preserving. Arguing similarly, we deduce that the adjunction map  $\mathcal{A}_{V, \mathbf{x}_-} : T_{\mathbf{x}_-} S^4 \rightarrow T_{\mathbf{x}_-} S^4$  is the identity map  $T_{\mathbf{x}_-} S^4 \rightarrow T_{\mathbf{x}_-} S^4$ . This shows that the cycle determined by the zero locus of  $V$  is

$$\langle V = 0 \rangle = [\mathbf{x}_+] + [\mathbf{x}_-] \in H_0(S^4).$$

If we denote by  $\omega_4 \in H^4(S^4)$  the generator of the cyclic group  $H^4(S^4)$  determined by the canonical orientation, we deduce from the localization formula that

$$e(TS^4) = 2\omega_4. \quad (5.4.10)$$

(b) Denote by  $\mathbb{H}$  the (skew) field of quaternions. We consider the direct sum  $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$  which we regard as a *right*  $\mathbb{H}$ -vector space. We denote by  $\mathbb{H}\mathbb{P}^1$  the set of one dimensional (right)  $\mathbb{H}$ -subspaces. We can identify such a subspace with a pair  $[q_0; q_1]$ ,  $q_0, q_1 \in \mathbb{H}$ ,  $|q_0| + |q_1| \neq 0$ , where

$$[q_0, q_1] = [q_0t, qt], \quad \forall t \in \mathbb{H} \setminus 0.$$

The quaternionic projective line  $\mathbb{H}\mathbb{P}^1$  is a smooth 4-manifold that admits a coordinate atlas consisting of two charts

$$U_i = \{ [q_0, q_1]; q_i \neq 0 \}, \quad i = 0, 1.$$

The map  $\mathbb{H} \rightarrow \mathbb{H}\mathbb{P}^1$  given by  $q \mapsto [1, q]$  extends to a diffeomorphism  $S^4 \rightarrow \mathbb{H}\mathbb{P}^1$ . We have a tautological quaternionic line bundle  $\mathcal{U}_{\mathbb{H}} \rightarrow \mathbb{H}\mathbb{P}^1$  with total space

$$\{ (q_0, q_1, [q_0, q_1]) \in \mathbb{H}^2 \times \mathbb{H}\mathbb{P}^1 \}.$$

We denote by  $\mathcal{U}_{\mathbb{H}}^*$  the quaternionic dual of  $\mathcal{U}_{\mathbb{H}}$ . The fiber of this line bundle over the quaternionic line  $L \in \mathbb{H}\mathbb{P}^1$  is the space  $\text{Hom}_{\mathbb{H}}(L, \mathbb{H})_r$  of morphisms of right  $\mathbb{H}$ -spaces.

Note that we have a natural isomorphism

$$\mathbb{H}^2 \rightarrow \text{Hom}_{\mathbb{H}}(\mathbb{H}^2, \mathbb{H})_r, \quad (q_0, q_1) \mapsto L_{q_0, q_1}$$

where for any  $x_0, x_1 \in \mathbb{H}$  we have

$$L_{q_0, q_1}(x_0, x_1) = q_0x_0 + q_1x_1.$$

As in Example 5.4.3 we deduce that every element  $\vec{q} \in \mathbb{H}^2$  determines a smooth section  $u_{\vec{q}}$  of  $\mathcal{U}_{\mathbb{H}}^*$ . If we choose  $\vec{q} = (1, 0)$  we deduce exactly as in Example 5.4.3 that the section  $u_{\vec{q}} : \mathbb{H}\mathbb{P}^1 \rightarrow \mathcal{U}_{\mathbb{H}}^*$  has a unique *nondegenerate* zero  $L_0 = [0, 1] \in \mathbb{H}\mathbb{P}^1$ . The quaternionic line bundle  $\mathcal{U}_{\mathbb{H}}$  is naturally an orientable rank 4 real vector bundle over  $S^4$ . It carries a natural orientation induced by the quaternionic structure. We denote by  $e(\mathcal{U}_{\mathbb{H}}^*) \in H^4(S^4)$  its Euler class. From the localization formula we conclude that<sup>3</sup>

$$e(\mathcal{U}_{\mathbb{H}}^*) = \pm\omega_4 \quad (5.4.11)$$

□

Let us point out that the computation in Example 5.4.6(a) proves something more general.

**Corollary 5.4.7.** *Denote by  $\omega_n$  the canonical generator of  $H^n(S^n, \mathbb{Z})$  induced by the canonical orientation on  $S^n$ . Then*

$$e(TS^n) = (1 + (-1)^n)\omega_n = \begin{cases} 2\omega_n & n \equiv 0 \pmod{2} \\ 0 & n \equiv 1 \pmod{2}. \end{cases} \quad \square$$

<sup>3</sup>One can prove that the correct sign in (5.4.11) is +.

# Pontryagin classes and Pontryagin numbers

In this chapter, as in the previous ones, the topological spaces will be assumed paracompact.

## 6.1. The Pontryagin classes of a real vector bundle

For a real vector bundle  $E \rightarrow X$  we denote by  $E_{\mathbb{C}} \rightarrow X$  its complexification,  $E_{\mathbb{C}} := \underline{\mathbb{C}}_X \otimes_{\mathbb{R}} E$ . It becomes in an obvious fashion a complex vector bundle.

Equivalently, we can define  $E_{\mathbb{C}}$  as the direct sum  $E \oplus E$ , where for every  $x \in X$  the action of  $i$  on  $E_{\mathbb{C}}(x) \cong E(x) \oplus E(x)$  is given by

$$E(x) \oplus E(x) \ni (v_1, v_2) \mapsto (-v_2, v_1) \in E(x) \oplus E(x).$$

If we denote by  $(E_{\mathbb{C}})_{\mathbb{R}}$  the bundle  $E_{\mathbb{C}}$  viewed as a *real* vector bundle, then we have an isomorphism of real vector bundles

$$(E_{\mathbb{C}})_{\mathbb{R}} \cong E \oplus E.$$

**Definition 6.1.1.** For every real vector bundle  $E \rightarrow X$  and for every  $k \geq 1$  we define the  $k$ -th *Pontryagin class* of  $E$  the cohomology class

$$p_k(E) := (-1)^k c_{2k}(E_{\mathbb{C}}) \in H^{4k}(X, \mathbb{Z}). \quad \square$$

**Proposition 6.1.2.** For every oriented real vector bundle  $E \rightarrow X$  of rank  $2r$ . Then

$$p_r(E) = e(E)^2,$$

where  $e(E) \in H^{2r}(X)$  denotes the Euler class of  $E$ .

**Proof.** Denote by  $\mathbf{or}_c$  the geometric orientation of  $(E_{\mathbb{C}})_{\mathbb{R}}$  as a *complex* vector bundle, and by  $\mathbf{or}^2$  the geometric orientation of  $(E_{\mathbb{C}})_{\mathbb{R}}$  induced from the orientation of  $E$  via the isomorphism  $(E_{\mathbb{C}})_{\mathbb{R}} = E \oplus E$ . Since  $e((E_{\mathbb{C}})_{\mathbb{R}}, \mathbf{or}_c) = c_{2r}(E_{\mathbb{C}})$  we need to prove that

$$e((E_{\mathbb{C}})_{\mathbb{R}}, \mathbf{or}_c) = (-1)^r e(E)^2.$$

On the other hand, we have

$$e((E_{\mathbb{C}})_{\mathbb{R}}, \mathbf{or}^2) = e(E)^2$$

so it suffices to show that  $\mathbf{or}_c = (-1)^r \mathbf{or}^2$ .

Let  $x \in X$ , choose a positively oriented frame  $e_1, \dots, e_{2r}$  of  $E(x)$ . We regard  $E(x)$  as a subspace of  $E_{\mathbb{C}}(x)$  and set

$$f_k := i e_k \in E_{\mathbb{C}}(x), \quad k = 1, \dots, 2r.$$

Then  $\mathbf{or}_c$  is defined by  $e_1 \wedge f_1 \cdots e_{2r} \wedge f_{2r}$ , while  $\mathbf{or}^2$  is defined by  $e_1 \wedge \cdots \wedge e_{2r} \wedge f_1 \wedge \cdots \wedge f_{2r}$ . Now observe that

$$e_1 \wedge f_1 \cdots e_{2r} \wedge f_{2r} = (-1)^{1+2+\cdots+(2r-1)} e_1 \wedge \cdots \wedge e_{2r} \wedge f_1 \wedge \cdots \wedge f_{2r},$$

and

$$1 + 2 + \cdots + (2r - 1) = r(2r - 1) \equiv r \pmod{2}.$$

□

From the naturality properties of the Chern classes we deduce that for every continuous map  $f : Y \rightarrow X$ , and every real vector bundle  $E \rightarrow X$  we have

$$p_k(f^* E) = f^* p_k(E), \quad \forall k \geq 1. \quad (6.1.1)$$

For any topological space  $X$  we set

$$\overline{H}^{\bullet}(X) := H^{\bullet}(X, \mathbb{Z})/\text{Torsion}.$$

Note that the cup product on  $H^{\bullet}(X)$  induces a cup product on  $\overline{H}^{\bullet}(X)$ .

For every real vector bundle  $E \rightarrow X$  we denote by  $\bar{p}_k(E) \in \overline{H}^{4k}(X)$  the image of the  $k$ -th Pontryagin class of  $E$  via the natural surjection  $H^{4k}(X, \mathbb{Z}) \rightarrow \overline{H}^{4k}(X)$ . We will refer to  $\bar{p}_k$  as the  $k$ -th *reduced Pontryagin class* and we set

$$\bar{\mathcal{P}}_E(t) = \sum_{k \geq 0} \bar{p}_k(E) t^{2k} \in \overline{H}^{even}(X), \quad p_0(E) = 1.$$

We say that  $\bar{\mathcal{P}}_E$  is the (*reduced*) *Pontryagin polynomial* of  $E$ .

**Proposition 6.1.3.** *For any pair of real vector bundles  $E_0, E_1 \rightarrow X$  we have*

$$\bar{\mathcal{P}}_{E_0 \oplus E_1}(t) = \bar{\mathcal{P}}_{E_0}(t) \cdot \bar{\mathcal{P}}_{E_1}(t). \quad (6.1.2)$$

**Proof.** For any complex vector bundle  $E \rightarrow X$  we denote by  $\bar{c}_k(E)$  the image of its  $k$ -th Chern class in  $\overline{H}^{2k}(X)$ , and we set

$$\bar{\mathcal{C}}_E(t) := \sum_{k \geq 0} \bar{c}_k(E) t^k \in \overline{H}^{even}(X).$$

The proposition is a consequence of the following identity.

**Lemma 6.1.4.** *If  $E$  is a real vector bundle then*

$$\bar{\mathcal{P}}_E(t) = \bar{\mathcal{C}}_{E_{\mathbb{C}}}(\mathbf{i}t) \in \mathbb{Z}[i] \otimes_{\mathbb{Z}} \overline{H}^{even}(X)[t] \quad (6.1.3)$$

**Proof.** From the equality

$$\bar{\mathcal{C}}_{E_{\mathbb{C}}}(\mathbf{i}t) = \sum_{k \geq 0} (-1)^k \bar{c}_{2k}(E_{\mathbb{C}}) t^{2k} + \mathbf{i} \sum_{k \geq 0} (-1)^k \bar{c}_{2k+1}(E_{\mathbb{C}}) t^{2k+1}$$

we deduce that the equality (6.1.3) is equivalent with the equalities  $\bar{c}_{2k+1}(E_{\mathbb{C}}) = 0$ , i.e., the odd Chern classes of  $E_{\mathbb{C}}$  are torsion elements of  $H^{even}(X, \mathbb{Z})$ . To see this we observe that we have

an isomorphism of *complex* vector bundles  $E_{\mathbb{C}} \rightarrow \overline{E_{\mathbb{C}}}$  induced from the tautological isomorphism  $\underline{\mathbb{C}}_X \rightarrow \overline{\underline{\mathbb{C}}}_X$  given by the conjugation map  $\mathbb{C} \rightarrow \mathbb{C}$ ,  $z \mapsto \bar{z}$ . From Exercise 5.3.3 we deduce

$$-c_{2k+1}(E_{\mathbb{C}}) = c_{2k+1}(\overline{E_{\mathbb{C}}}) = c_{2k+1}(E_{\mathbb{C}}) \text{ so that } 2c_{2k+1}(E_{\mathbb{C}}) = 0.$$

□

If  $E_0, E_1$  are two real bundles then

$$\bar{\mathcal{P}}_{E_0 \oplus E_1}(t) = \bar{\mathcal{C}}_{(E_0)_{\mathbb{C}} \oplus (E_1)_{\mathbb{C}}}(\mathbf{i}t) = \bar{\mathcal{C}}_{(E_0)_{\mathbb{C}}}(\mathbf{i}t) \cdot \bar{\mathcal{C}}_{(E_1)_{\mathbb{C}}}(\mathbf{i}t) = \bar{\mathcal{P}}_{E_0}(t) \cdot \bar{\mathcal{P}}_{E_1}(t).$$

□

**Corollary 6.1.5.** *The reduced Pontryagin classes are stable, i.e., for any real vector bundle  $E \rightarrow X$ , and any  $n > 0$  we have*

$$\bar{p}_k(E \oplus \mathbb{R}_X^n) = \bar{p}_k(E), \quad \forall k \geq 1.$$

**Proof.** Use the equality (6.1.3).

□

**Corollary 6.1.6.**  $\bar{\mathcal{P}}_{TS^n}(t) = 1$ .

**Proof.** The equality follows from the isomorphisms

$$\mathbb{R}_{S^n}^{n+1} = (T\mathbb{R}^{n+1})|_{S^n} \cong TS^n \oplus T_{S^n}\mathbb{R}^{n+1} \cong TS^n \oplus \mathbb{R}_{S^n}$$

and the stability of the reduced Pontryagin classes.

□

**Proposition 6.1.7.** *If  $E \rightarrow X$  is a complex vector bundle and  $E_{\mathbb{R}}$  denotes the same bundle but viewed as a real vector bundle then*

$$\begin{aligned} \bar{\mathcal{P}}_{E_{\mathbb{R}}}(t) &= \bar{\mathcal{C}}_E(\mathbf{i}t)\bar{\mathcal{C}}_{\overline{E}}(-\mathbf{i}t) \\ &= 1 - (c_1(E)^2 - 2c_2(E))t^2 + (2c_4(E) + c_2(E)^2 - 2c_1(E)c_3(E))t^4 + \dots \end{aligned} \quad (6.1.4)$$

**Proof.** The identity (6.1.4) is an immediate consequence of the bundle isomorphism

$$(E_{\mathbb{R}})_{\mathbb{C}} \cong E \oplus \overline{E}. \quad (6.1.5)$$

Indeed, we have

$$\bar{\mathcal{P}}_{E_{\mathbb{R}}}(s) = \bar{\mathcal{C}}_E(\mathbf{i}s)\bar{\mathcal{C}}_{\overline{E}}(\mathbf{i}s) = \bar{\mathcal{C}}_E(\mathbf{i}s)\bar{\mathcal{C}}(-\mathbf{i}s)$$

To prove the isomorphism (6.1.5) we observe that the multiplication by  $\mathbf{i}$  in the complex vector bundle  $E$  defines a linear endomorphism on the real vector bundle  $E_{\mathbb{R}}$  that we denote by  $J$ . The endomorphism  $J$  satisfies the identity  $J^2 = -\mathbb{1}$  and extends to a *complex* linear endomorphism  $J_c$  of  $(E_{\mathbb{R}})_{\mathbb{C}}$ . For every  $x \in X$ , the linear map  $J_c(x) : E_{\mathbb{R}}(x) \otimes \mathbb{C} \rightarrow E_{\mathbb{R}}(x) \otimes \mathbb{C}$  has two eigenvalues  $\pm \mathbf{i}$ . The corresponding eigenspaces

$$\ker(\mathbf{i} - J_c(x)), \quad \ker(\mathbf{i} + J_c(x))$$

are isomorphic as complex vector spaces with  $E(x)$  and respectively  $\overline{E}(x)$ . We obtain in this fashion two subbundles

$$E^{1,0} := \ker(\mathbf{i} - J_c) \hookrightarrow (E_{\mathbb{R}})_{\mathbb{C}}, \quad E^{0,1} := \ker(\mathbf{i} + J_c) \hookrightarrow (E_{\mathbb{R}})_{\mathbb{C}}.$$

We have

$$E^{1,0} \oplus E^{0,1} \cong (E_{\mathbb{R}})_{\mathbb{C}}, \quad E^{1,0} \cong_{\mathbb{C}} E, \quad E^{0,1} \cong_{\mathbb{C}} \bar{E}.$$

□

**Corollary 6.1.8.**

$$\bar{\mathcal{P}}_{T\mathbb{C}P^n}(t) = (1 + [H]^2 t^2)^{n+1}, \quad [H]^{n+1} = 0. \quad (6.1.6)$$

**Proof.** From (5.4.8a) we deduce that

$$\bar{\mathcal{C}}_{T\mathbb{C}P^n}(t) = (1 + [H]t)^{n+1}$$

so that

$$\bar{\mathcal{P}}_{T\mathbb{C}P^n}(t) = \bar{\mathcal{C}}_{T\mathbb{C}P^n}(it) \cdot \bar{\mathcal{C}}_{T\mathbb{C}P^n}(-it) = (1 + [H]^2 t^2)^{n+1}.$$

□

## 6.2. Pontryagin numbers

For every smooth manifold  $M$  we set

$$\bar{p}_k(M) := \bar{p}_k(TM), \quad \bar{\mathcal{P}}_M(t) := \bar{\mathcal{P}}_{TM}(t).$$

We define a partition to be a finite weakly decreasing sequence  $I$  of positive integers

$$I := \{i_1 \geq i_2 \geq \dots \geq i_\ell \geq 1\}.$$

We say that  $\ell$  is the *length* of the partition and  $w(I) := i_1 + \dots + i_\ell$  is the *weight* of the partition. We denote by  $\text{Part}$  the set of all partitions, by  $\text{Part}(k)$  the set of partitions of weight  $k$ , and we denote by  $p(k)$  its cardinality. Observe that to any finite sequence of positive integers  $(i_1, \dots, i_\ell)$  (no monotonicity is assumed) we can associate a canonical partition  $(i_1, \dots, i_\ell)_{\searrow}$  obtained by rearranging the terms of the sequence in decreasing order. Using this point of view we can define an operation

$$* : \text{Part} \times \text{Part} \rightarrow \text{Part}, \quad (i_1, \dots, i_\ell) * (j_1, \dots, j_m) = (i_1, \dots, i_\ell, j_1, \dots, j_m)_{\searrow} \in \text{Part}.$$

Note that  $w(I * J) = w(I) + w(J)$ ,  $\forall I, J \in \text{Part}$ .

For any smooth, compact oriented manifold  $M$  of dimension  $m$ , and any partition  $I = (i_1, \dots, i_\ell) \in \text{Part}(k)$  we set

$$P_I(M) := \begin{cases} 0 & m \neq 4w(I) \\ \langle p_{i_1}(M) \cdots p_{i_\ell}(M), [M] \rangle & m = 4w(I), \end{cases} \quad (6.2.1)$$

where  $\langle -, - \rangle : H^m(M) \times H_m(M) \rightarrow \mathbb{Z}$  denotes the Kronecker pairing and  $[M] \in H_m(M)$  denotes the orientation class. The integers  $P_I(M)$  are called the *Pontryagin numbers* of  $M$ . Observe that if the dimension of  $M$  is not divisible by 4 then all the Pontryagin numbers are trivial.

**Proposition 6.2.1.** *If the smooth compact oriented manifold  $M$  is the boundary of a smooth compact manifold with boundary  $\widehat{M}$ , then all the Pontryagin numbers of  $M$  are trivial.*

**Proof.** Clearly we can assume  $\dim M = 4k$ . Choose an orientation class  $[\widehat{M}] \in H_{4k+1}(\widehat{M}, \partial\widehat{M})$  that induces the orientation  $[M]$  on  $M$ , i.e.,  $\partial[\widehat{M}] = [M]$ . Let  $\widetilde{M}$  be a neck extension of  $\widehat{M}$ . Observe that the normal bundle  $T_M \widetilde{M}$  is a trivial real line bundle so that

$$(T\widetilde{M})_M \cong T_M \widetilde{M} \oplus TM.$$

From the stability of the reduced Pontryagin classes we deduce

$$\bar{p}_i(M) = j^* p_i(\widetilde{M}), \quad \forall i > 0$$

where  $j : M \hookrightarrow \widetilde{M}$  denotes the natural inclusion. If  $I = (i_1, \dots, i_\ell) \in \text{Part}(k)$  then

$$\begin{aligned} P_I(M) &= \langle \bar{p}_{i_1}(M) \cdots \bar{p}_{i_\ell}(M), [M] \rangle = \langle \bar{p}_{i_1}(M) \cdots \bar{p}_{i_\ell}(M), \partial[\widehat{M}] \rangle \\ &= \langle j^*(\bar{p}_{i_1}(\widetilde{M}) \cdots \bar{p}_{i_\ell}(\widetilde{M})), \partial[\widehat{M}] \rangle = \langle \partial^\dagger j^*(\bar{p}_{i_1}(\widetilde{M}) \cdots \bar{p}_{i_\ell}(\widetilde{M})), [\widehat{M}] \rangle, \end{aligned}$$

where  $\partial^\dagger : H^{4k}(M) \rightarrow H^{4k+1}(\widehat{M}, \partial\widehat{M})$  is the connecting morphism in the long exact sequence of the pair  $(\widehat{M}, \partial\widehat{M})$ . Now observe that  $\partial^\dagger j^* = 0$  due to the exactness of the sequence

$$H^{4k}(\widehat{M}) \xrightarrow{j^*} H^{4k}(M) \xrightarrow{\partial^\dagger} H^{4k+1}(\widehat{M}, \partial\widehat{M}).$$

□

The above result shows that we can regard the Pontryagin numbers as morphism of groups

$$P_I : \Omega_{4k}^+ \rightarrow \mathbb{Z}, \quad I \in \text{Part}(k).$$

Note that  $P_I$  vanishes on the torsion elements of  $\Omega_{4k}^+$  and thus  $P_I$  is uniquely determined by the induced morphism

$$P_I : \Omega_{4k}^+ \otimes \mathbb{Q} \rightarrow \mathbb{Q}.$$

**Proposition 6.2.2.** *The Pontryagin numbers  $(P_I)_{I \in \text{Part}(k)}$  form a linearly independent subfamily of the  $\mathbb{Q}$ -vector space*

$$\Xi_k := \text{Hom}_{\mathbb{Q}}(\Omega_{4k}^+ \otimes \mathbb{Q}, \mathbb{Q}).$$

In particular,  $\dim \Omega_{4k}^+ \otimes \mathbb{Q} \geq p(k)$ .

**Proof.** To understand the origin of the main construction in the proof it is convenient to think of the Pontryagin polynomial of a manifold  $M$  of dimension  $k$  as a product of elementary polynomials

$$\bar{p}_0(M) + \bar{p}_1(M)z + \cdots + \bar{p}_k(M)z^k = (1 + r_1z) \cdots (1 + r_kz), \quad z = t^2.$$

Thus, we would like to think of the Pontryagin classes as elementary symmetric polynomials in the (possibly nonexistent) variables  $r_i$ . Any other symmetric polynomial in these variables is then a polynomial combination of the elementary ones. We use this idea to construct a new basis in the vector space generated by the morphism  $P_I$  and the conclusion of the proposition will much more transparent in this basis. Here are the details.

We denote by  $\sigma_i$  the  $i$ -th elementary symmetric polynomial in the variables  $r_1, \dots, r_k$ . For every partition  $I = (i_1, i_2, \dots, i_\ell) \in \text{Part}$  we define

$$s_I \in \mathbb{Z}[\sigma_1, \dots, \sigma_k]$$

the symmetric polynomial

$$s_I(\sigma_1, \dots, \sigma_k) = \sum r_1^{i_1} \cdots r_\ell^{i_\ell}.$$

Above, the sum is over all monomials equivalent to  $r_1^{i_1} \cdots r_\ell^{i_\ell}$  where two monomials are declared equivalent if one can be obtained from the other by a permutation of the variables  $r_1, \dots, r_k$ .

$$s_{2,1} = \sum_{i \neq j} r_i^2 r_j = \sigma_1 \sigma_2 - 3\sigma_3, \quad s_{1,1} = \sum_{i < j} r_i r_j = \sigma_2.$$

The following result should be obvious.

**Lemma 6.2.3.** *The polynomials  $(s_I)_{I \in \text{Part}(k)}$  form a basis of the free Abelian group space of symmetric polynomials of degree  $d \leq k$  in the variables  $r_1, \dots, r_k$  with integral coefficients.  $\square$*

We denote by  $\mathcal{S}_k$  the ring of symmetric polynomials with integral coefficients, and by  $\mathcal{S}_k^d$  the subgroup consisting of homogeneous polynomials of degree  $d$ . For any smooth manifold  $M$  of dimension  $4k$  and any partition  $I$  of weight  $k$  we set

$$s_I(M) := s_I(\bar{p}_1(M), \dots, \bar{p}_k(M)), \quad S_I(M) := \langle s_I(M), \mu_M \rangle,$$

where  $\mu_M \in H_{4k}(M)$  denotes the orientation class of  $M$ . For every  $I \in \text{Part}(k)$  we obtain in this fashion morphisms of groups

$$S_I : \Omega_{4k}^+ \rightarrow \mathbb{Z}$$

and thus  $\mathbb{Q}$ -linear maps  $S_I : \Omega_{4k}^+ \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ . The above lemma shows that the linear subspace of  $\Xi_k$  spanned by  $(S_I)_{I \in \text{Part}(k)}$  is equal to the linear subspace spanned by the linear maps  $P_I : \Omega_{4k}^+ \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ . Thus it suffices to show that the functionals  $S_I$  are linearly independent. We will prove something stronger. For every  $I = (i_1, \dots, i_\ell) \in \text{Part}(k)$  we set

$$\mathbb{C}\mathbb{P}^{2I} = \mathbb{C}\mathbb{P}^{2i_1} \times \dots \times \mathbb{C}\mathbb{P}^{2i_\ell}$$

and we denote by  $\widehat{\Omega}_{4k}(\mathbb{Q})$  the subspace of  $\Omega_{4k}^+$  spanned by the cobordism classes of  $\mathbb{C}\mathbb{P}^{2I}$ ,  $I \in \text{Part}(k)$ . We will show that the vector space  $\widehat{\Omega}_{4k}(\mathbb{Q})$  has dimension  $p(k)$  and that the functionals  $S_I$ ,  $I \in \text{Part}(k)$ , form a basis of

$$\widehat{\Xi}_k := \text{Hom}_{\mathbb{Q}}(\widehat{\Omega}_{4k}(\mathbb{Q}), \mathbb{Q}).$$

More precisely we will prove that

$$\text{the } p(k) \times p(k) \text{ matrix } [S_I(\mathbb{C}\mathbb{P}^J)]_{I, J \in \text{Part}(k)} \text{ is nonsingular.} \quad (6.2.2)$$

**Lemma 6.2.4** (Thom). *Suppose  $M_0$  and  $M_1$  are smooth, compact oriented manifolds of dimensions  $4k_0$  and respectively  $4k_1$ . Then for every  $I \in \text{Part}(k_0 + k_1)$  we have*

$$S_I(M_0 \times M_1) = \sum_{(I_0, I_1) \in \text{Part}(k_0) \times \text{Part}(k_1), I_0 * I_1 = I} S_{I_0}(M_0) \cdot S_{I_1}(M_1) \quad (6.2.3)$$

**Proof.** The above equality is a consequence of an universal identity involving symmetric polynomials. To state this universal identity we need to introduce some more notation. We denote by  $\sigma_i$  the  $i$ -th elementary symmetric polynomial in the variables  $r_1, \dots, r_n$ , by  $\sigma'_i$  the  $i$ -th symmetric polynomial in the variables  $r'_1, \dots, r'_n$ , and by  $\sigma''_i$  the  $i$ -th elementary symmetric polynomial in the variables

$$r''_j = \begin{cases} r_j & j \leq n \\ r'_{j-n} & j > n. \end{cases}$$

From the equalities

$$\sum_{i=0}^n \sigma_i t^i = \prod_{i=0}^n (1 + r_i t), \quad \sum_{j=0}^n \sigma'_j t^j = \prod_{j=1}^n (1 + r'_j t)$$

we deduce

$$\sigma''_\ell = \sum_{i+j=\ell} \sigma_i \sigma'_j \quad (6.2.4)$$

The following is the universal identity alluded to above.



**Sublemma 6.2.5.**

$$s_I(\sigma''_I) = \sum_{J*K=I} s_J(\sigma_i) s_K(\sigma'_j), \quad \forall I \in \text{Part}(2n). \quad (6.2.5)$$

Let us first show that Lemma 6.2.4 is a consequence of (6.2.5). Indeed, let  $n = k_0 + k_1$ . We denote by  $\pi_i$  the natural projection  $M_0 \times M_1 \rightarrow M_i$ ,  $i = 0, 1$ . We set

$$p''_j := \bar{p}_j(M_0 \times M_1), \quad p_i = \pi_0^* \bar{p}_i(M_0), \quad p'_i = \pi_1^* \bar{p}_i(M_1).$$

Then

$$p''_\ell = \sum_{i+j=\ell} p_i p'_j,$$

and since the variables  $\sigma_i, \sigma'_j$  are algebraically independent we deduce from (6.2.5) that

$$s_I(M_0 \times M_1) = s_I(p''_i) = \sum_{J*K=I} s_J(p_j) s_K(p'_k)$$

We deduce

$$\begin{aligned} S_I(M_0 \times M_1) &= \sum_{J*K=I} \langle s_J(p_j) s_K(p'_k), \mu_{M_0} \times \mu_{M_1} \rangle \\ &= \sum_{J*K=I} \langle s_J(p_j), \mu_{M_0} \rangle \cdot \langle s_K(p'_k), \mu_{M_1} \rangle = \sum_{J*K=I} S_J(M_0) \cdot S_J(M_1). \end{aligned}$$

This proves the equality in Lemma 6.2.4. As for (6.2.5), it follows from the simple observation that any monomial in the variables  $r''_i$  can be written in a unique fashion as a product between a monomial in the variables  $r_i$  and a monomial in the variables  $r'_j$ . □

We can now prove (6.2.2). We define a partial order  $\succeq$  on  $\text{Part}(k)$  by declaring  $I \succeq J$

$$I = (i_1 \geq \dots \geq i_\ell > 0) \succeq J = (j_1 \geq \dots \geq j_m > 0) \in \text{Part}(k),$$

if

$$I = I_1 * \dots * I_m, \quad I_\alpha \in \text{Part}(j_\alpha), \quad \forall \alpha = 1, \dots, m.$$

From the product formula (6.2.3) we deduce that  $S_I(\mathbb{C}\mathbb{P}^{2J}) = 0$  if  $J \not\succeq I$ . This shows that the matrix  $[S_I(\mathbb{C}\mathbb{P}^{2J})]_{I, J \in \text{Part}(k)}$  is upper triangular with respect to the above partial order on  $\text{Part}(k)$ . The diagonal elements are  $S_I(\mathbb{C}\mathbb{P}^{2I})$ . An inductive application of (6.2.3) shows that

$$S_{(i_1, \dots, i_\ell)}(\mathbb{C}\mathbb{P}^{2i_1} \times \dots \times \mathbb{C}\mathbb{P}^{2i_\ell}) = \prod_{\nu=1}^{\ell} S_{i_\nu}(\mathbb{C}\mathbb{P}^{2i_\nu})$$

Thus the matrix  $[S_I(\mathbb{C}\mathbb{P}^{2J})]_{I, J \in \text{Part}(k)}$  is nonsingular if and only if  $S_n(\mathbb{C}\mathbb{P}^{2n}) \neq 0, \forall n$ . This is a consequence of the following more precise result.

**Lemma 6.2.6.**

$$S_n(\mathbb{C}\mathbb{P}^{2n}) = 2n + 1, \quad \forall n \geq 1.$$

**Proof.** Let  $\sigma_i$  be the  $i$ -th elementary symmetric polynomial in the variables  $r - 1, \dots, r_N$ ,  $N \geq n$ . Set  $f(t) = \sum_{i=0}^N \sigma_i t^i$ ,  $\sigma_0 = 1$  so that

$$f(z) = \prod_{i=1}^N (1 + r_i z).$$

We then have the equality<sup>1</sup>

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^N \frac{r_i}{1 + r_i z} = \sum_{m \geq 0} (-1)^m s_{m+1} z^m. \quad (6.2.6)$$

If we let

$$f(z) = \sum_{i \geq 0} \bar{p}_i(\mathbb{C}\mathbb{P}^{2n}) z^i = (1 + [H]^2 z)^{2n+1}, \quad [H]^{2n+1} = 0.$$

We deduce

$$\frac{f'(z)}{f(z)} = (2n + 1)[H]^2 (1 + [H]^2 z)^{-1} = \sum_{m \geq 0} (-1)^m s_{m+1}(M) z^m.$$

We deduce that  $s_n(M) = (2n + 1)[H]^{2n}$ . □

The proof of Proposition 6.2.2 is complete. □

**Corollary 6.2.7.** *The cobordism classes of the manifold  $\mathbb{C}\mathbb{P}^{2I}$ ,  $I \in \text{Part}(k)$  are linearly independent in the  $\mathbb{Q}$ -vector space  $\Omega_{4k}^+ \otimes \mathbb{Q}$ .* □

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<sup>1</sup>This is often referred to as *Newton's formula*.

# The Hirzebruch signature formula

We have now developed the topological language needed to state Hirzebruch's signature theorem. There are still two missing pieces needed in the proof. One is of a combinatorial nature and will be described in this section, while the other, more serious, is the award winning work of R. Thom on cobordisms. We will deal with this in Chapter 9.

## 7.1. Multiplicative sequences

To any commutative, graded  $\mathbb{Q}$ -algebra

$$A^\bullet = \bigoplus_{n \geq 0} A^n, \quad A^n \cdot A^m \subset A^{n+m}$$

we associate the ring of formal power series  $A^\bullet[[z]]$ ,

$$p \in A^\bullet[[z]] \iff p(z) = \sum_{n \geq 0} p_n z^n, \quad p_n \in A^n.$$

We think of the elements of  $A^n$  as having degree  $n$ . The ring  $A^\bullet[[z]]$  contains a multiplicative semi-group with 1,

$$A^\bullet[[z]]^\# := \left\{ \sum_{n \geq 0} p_n z^n \in A^\bullet[[z]]; \quad p_0 = 1 \right\}.$$

Consider the commutative, graded  $\mathbb{Q}$ -algebra  $\mathfrak{S} := \mathbb{Q}[p_1, p_2, \dots]$ <sup>1</sup>, where the variable  $p_n$  has degree  $n$ . An element of

$$K = K((p_n)_{n \geq 1}; z) \in \mathfrak{S}[[z]]$$

can be written as

$$K = \sum_{n \geq 0} K_n((p_i)_{i \geq 1}) z^n, \quad K_n \text{ is homogeneous of degree } n.$$

<sup>1</sup>One should think of  $\mathfrak{S}$  as the ring of symmetric polynomials in an indefinite number of variables

In particular, we deduce that  $K_0$  must be a constant and that  $K_n$  only depends on the variables  $p_1, \dots, p_n, \forall n \geq 1$ .

**Definition 7.1.1.** A multiplicative sequence (or  $m$ -sequence for brevity) is an element

$$K = 1 + \sum_{n \geq 1} K_n(p_1, \dots, p_n)z^n \in \mathfrak{S}[[z]]^\#$$

such that, for any graded  $\mathbb{Q}$ -algebra  $A^\bullet$  the map  $\widehat{K} : A^\bullet[[z]]^\# \rightarrow A^\bullet[[z]]^\#$  given by

$$A^\bullet[[z]]^\# \ni a(z) = 1 + \sum_{n \geq 1} a_n z^n \mapsto \widehat{K}(a(z)) := 1 + \sum_{n \geq 1} K_n(a_1, \dots, a_n)z^n \in A^\bullet[[z]]^\#,$$

is a morphism of semigroups with 1. □

For example, the above definition implies that if  $K$  is an  $m$ -sequence then for any commutative  $\mathbb{Q}$ -algebra  $R$  with 1, and any sequences  $(r_n)_{n \geq 0}, (r'_i)_{i \geq 0}$  and  $(r''_j)_{j \geq 0}$  in  $R$  such that

$$r_0 = r'_0 = r''_0 = 1 \text{ and } r_n = \sum_{i+j=n} r'_i r''_j,$$

we have

$$K_n(r_1, \dots, r_n) = \sum_{i+j=n} K_i(r'_1, \dots, r'_i) K_j(r''_1, \dots, r''_j), \quad \forall n.$$

Suppose

$$K = 1 + \sum_{n \geq 1} K_n(p_1, \dots, p_n)z^n$$

is a multiplicative sequence. For every  $n \geq 1$  the polynomial  $K_n(\xi, 0, \dots, 0) \in \mathbb{Q}[\xi]$  is homogeneous of degree  $n$  and thus it has the form

$$K_n(\xi, 0, \dots, 0) = k_n \xi^n, \quad k_n \in \mathbb{Q}.$$

We can now form the power series

$$a_K(\xi) := 1 + \sum_{n \geq 1} k_n \xi^n \in \mathbb{Q}[\xi].$$

We will refer to  $a_K(\xi)$  as the *symbol* of the multiplicative sequence  $K$ . Note that

$$a_K(p_1 z) = \widehat{K}(1 + p_1 z).$$

**Proposition 7.1.2.** A multiplicative sequence  $K$  is uniquely determined by its symbol  $a_K(\xi)$ .

**Proof.** Consider the graded ring

$$A^\bullet = \mathbb{Q}[t_1, \dots, t_n].$$

We denote by  $\sigma_k$  the  $k$ -th elementary symmetric function in the variables  $t_1, \dots, t_n$ . Then we have the following equality in  $A^\bullet$

$$1 + \sum_{k=1}^n \sigma_k z^k = \prod_{k=1}^n (1 + t_k z)$$

and we deduce

$$1 + \sum_{j=1}^n K_j(\sigma_1, \dots, \sigma_j) z^j = \widehat{K}\left(1 + \sum_{j=1}^n \sigma_j z^j\right) = \prod_{j=1}^n \widehat{K}(1 + t_j z) = \prod_{j=1}^n a_K(t_j z).$$

This uniquely determines the polynomials  $K_j$  since the polynomials  $\sigma_k$  are algebraically independent.  $\square$

**Proposition 7.1.3.** *For any power series*

$$a(\xi) = 1 + \sum_{n \geq 1} a_n \xi^n \in \mathbb{Q}[\xi]$$

*there exists a multiplicative sequence  $K = K^a$  such that  $a_K(\xi) = a(\xi)$ .*

**Proof.** The proof of the uniqueness result has in it the seeds of the proof of the existence result. Consider the product

$$\prod_{j=1}^n a(t_j z) = 1 + \sum_{j=1}^n A_{j;n}(t_1, \dots, t_n) z^j + \sum_{N > n} A_{N;n}(t_1, \dots, t_n) z^N.$$

Observe that the polynomials  $A_{j;n}$  are homogeneous of degree  $j$  and symmetric in the variables  $t_1, \dots, t_n$  so we can express them as polynomials in the elementary symmetric functions,

$$A_{j;n}(t_1, \dots, t_n) = K_{j;n}(\sigma_1, \dots, \sigma_j), \quad j \leq n. \quad (7.1.1)$$

Moreover, for  $j \leq n < N$  we have

$$K_{j;n}(\sigma_1, \dots, \sigma_j) = K_{j;N}(\sigma_1, \dots, \sigma_j)$$

For this reason we will denote by  $K_j(\sigma_1, \dots, \sigma_j)$  any of the polynomials  $K_{j;n}$ ,  $j \leq n$ .

If instead of the elementary symmetric polynomials  $\sigma_i$  we use the polynomials  $s_I$ ,  $I \in \text{Part}(j)$  introduced in the previous chapter, then we have

$$K_j = \sum_{I \in \text{Part}(j)} a_I s_I(\sigma_1, \dots, \sigma_j), \quad a_{i_1, \dots, i_\ell} := a_{i_1} \cdots a_{i_\ell}.$$

Now define

$$K((p_i)_{i \geq 1}, z) := 1 + \sum_{j \geq 1} K_j(p_1, \dots, p_j) z^j,$$

where  $K_j$  are the polynomials defined by (7.1.1), or equivalently,

$$K_j(p_1, \dots, p_j) := \sum_{I \in \text{Part}(j)} a_I s_I(p_1, \dots, p_n).$$

If in some ring  $A^\bullet[[z]]$  we have an equality of the form

$$1 + \sum_{n \geq 1} p_n'' z^n = \left(1 + \sum_{i \geq 1} p_i z^i\right) \cdot \left(1 + \sum_{j \geq 1} p_j' z^j\right),$$

then

$$p_n'' = \sum_{i+j=n} p_i p_j',$$

and from (6.2.5) we deduce

$$s_I(p'') = \sum_{J * K = I} s_J(p) s_K(p').$$

Coupled with the identity

$$a_{I * J} = a_I a_J, \quad \forall I, J \in \text{Part}$$

this implies

$$K_n(p'') = \sum_{i+j=n} K_i(p.)K_j(p')$$

so that

$$\widehat{K}\left(1 + \sum_{n \geq 1} p''_n z^n\right) = \widehat{K}\left(1 + \sum_{i \geq 1} p_i z^i\right) \cdot \widehat{K}\left(1 + \sum_{j \geq 1} p'_j z^j\right).$$

□

**Example 7.1.4.** (a) The set  $\text{Part}(1)$  consists of single partition  $I = 1$  and we have

$$s_1(t_1, \dots, n) = t_1 + \dots + t_n = \sigma_1. \quad (7.1.2)$$

(b) The set  $\text{Part}(2)$  consists of two partitions, 2 and  $(1, 1)$ . Then

$$\begin{aligned} s_2 &= \sum t_i^2 = \sigma_1^2 - 2\sigma_2, \\ s_{1,1} &= \sum_{i < j} t_i t_j = \sigma_2 \end{aligned}$$

(c) The set  $\text{Part}(3)$  consists of three partitions 3,  $(2, 1)$ ,  $1^3 := (1, 1, 1)$  and we have

$$\begin{aligned} s_{1,1,1} &= \sigma_3, \quad s_3 = \sum_i t_i^3, \\ s_{2,1} &= \sum_{i \neq j} t_i^2 t_j = \sum_j t_j \sum_{i \neq j} t_i^2 = \sum_j t_j (s_2 - t_j^2) \\ &= s_2 \sigma_1 - s_3 = \sigma_1^3 - 2\sigma_1 \sigma_2 - s_3. \end{aligned}$$

To express  $s_3$  in terms of the elementary symmetric functions we use Newton's formulæ (6.2.6)

$$\sum_{j=1}^n j \sigma_j z^{j-1} = \left(1 + \sum_{j=1}^n \sigma_j z^j\right) \cdot \left(\sum_{m \geq 0} (-1)^m s_{m+1} z^m\right).$$

We deduce

$$\sigma_1 = s_1, \quad 2\sigma_2 = -s_2 + \sigma_1 s_1, \quad \sigma_3 = s_3 - s_1 s_2 + \sigma_2 s_1$$

so that

$$s_3 = \sigma_1^3 - 3\sigma_1 \sigma_2 + 3\sigma_3 \quad (7.1.3)$$

and

$$s_{2,1} = \sigma_1 \sigma_2 - 3\sigma_3. \quad (7.1.4)$$

□

**Example 7.1.5.** (a) Let  $a(z) = 1 + \xi$ . Then the multiplicative sequence with symbol  $a$  is

$$K = 1 + p_1 z + p_2 z^2 + \dots$$

Indeed,

$$1 + \sum_{j=1}^n K_j(\sigma_1, \dots, \sigma_j) z^j = \prod_{j=1}^n (1 + t_j z) = 1 + \sum_{j=1}^n \sigma_j z^j.$$

(b) Consider the series

$$\ell(\xi) := \frac{\sqrt{\xi}}{\tanh \sqrt{\xi}} = 1 + \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} b_{2k} \xi^k,$$

where  $b_n$  denote the Bernoulli numbers. They are defined by the following recurrence relation

$$\sum_{j=1}^n \binom{n}{j} b_{n-j} = 0$$

where  $b_0 = 1, b_1 = -\frac{1}{2}$ . We have  $b_{2j+1} = 0$  for all  $j \geq 1$ . Equivalently

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} b_n \frac{t^n}{n!}.$$

Here are the values of  $b_n$  for  $n \leq 18$ .

$n$	0	1	2	4	6	8	10	12	14	16	18
$b_n$	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3615}{510}$	$\frac{43867}{798}$

We have

$$\ell(z) = 1 + \frac{1}{3}z - \frac{1}{45}z^2 + \frac{2}{945}z^3 - \frac{1}{4725}z^4 + \frac{2}{93555}z^5 + O(z^6).$$

We denote by  $\ell_n$  the coefficient of  $z^n$  in the above expansion. If we denote by  $L$  the multiplicative sequence with symbol  $\ell(\xi)$ , then we have

$$L_n = \sum_{I \in \text{Part}(n)} \ell_I s_I(p_1, \dots, p_n).$$

Here is the explicit description of  $L_n$  for  $n \leq 3$ . We have

$$L_1 = \ell_1 s_1(p_1) \stackrel{(7.1.2)}{=} \frac{1}{3}p_1,$$

$$L_2 = \ell_2 s_2 + \ell_1^2 s_{1,1} = -\frac{1}{45}(p_1^2 - 2p_2) + \frac{1}{9}p_2 = \frac{7}{45}p_2 - \frac{1}{45}p_1^2.$$

$$L_3 = \ell_1^3 s_{1,1,1} + \ell_2 \ell_1 s_{2,1} + \ell_3 s_3$$

$$\stackrel{(7.1.4),(7.1.3)}{=} \frac{1}{27}p_3 - \frac{1}{135}(p_1 p_2 - 3p_3) + \frac{2}{945}(p_1^3 - 3p_1 p_2 + 3p_3) = \frac{62}{945}p_3 - \frac{13}{945}p_2 p_1 + \frac{2}{945}p_1^3. \quad \square$$

## 7.2. Genera

**Definition 7.2.1.** A *genus* (or *multiplicative genus*) is a ring morphism  $\gamma : \Omega_{\bullet}^+ \rightarrow \mathbb{Q}$ , where  $\Omega_{\bullet}^+$  denotes the oriented cobordism ring. □

For example, the signature function is a genus.

**Proposition 7.2.2.** Any multiplicative sequence  $K = (K_n)_{n \geq 1}$  defines a genus  $\gamma = \gamma_K$  such that, for any compact, oriented manifold  $M$  we have

$$\gamma(M) = 0, \quad \dim M \not\equiv 0 \pmod{4},$$

$$\gamma_K(M) = \langle K(\bar{p}_1(M), \dots, \bar{p}_n(M)), [M] \rangle,$$

if  $\dim M = 4n$ . Above,  $[M]$  denotes the orientation class of  $M$ . Moreover, if  $a(\xi)$  is the symbol of  $K$ , then

$$\gamma(\mathbb{C}P^{2n}) = \text{the coefficient of } \xi^{2n} \text{ in the formal power series } a(\xi^2)^{2n+1}. \quad (7.2.1)$$

**Proof.** Suppose  $M_0, M_1$  are two compact, oriented smooth manifolds. We set  $M = M_0 \times M_1$  and we denote by  $\pi_i$  the natural projection  $M \rightarrow M_i, i = 0, 1$ . We set

$$\mathcal{P}_i(z) = 1 + \sum_{k \geq 1} \bar{p}_k(M_i) z^k, \quad \bar{\mathcal{P}}(z) = 1 + \sum_{k \geq 1} \bar{p}_k(M) z^k, \quad i = 0, 1.$$

Then

$$TM \cong \pi_0^* TM_0 \oplus \pi_1^* TM_1,$$

From the product formula (6.1.2) we deduce

$$\bar{\mathcal{P}}(z) = \pi_0^* \bar{\mathcal{P}}_0(z) \cdot \pi_1^* \bar{\mathcal{P}}_1(z).$$

Hence

$$\begin{aligned} \langle K(\bar{\mathcal{P}}(z)), [M] \rangle &= \langle K(\bar{\mathcal{P}}(z)), [M_0] \times [M_1] \rangle = \langle \pi_0^* K(\bar{\mathcal{P}}_0(z)) \cdot \pi_1^* K(\bar{\mathcal{P}}_1(z)), [M_0] \times [M_1] \rangle \\ &= \langle K(\bar{\mathcal{P}}_0(z)), [M_0] \rangle \cdot \langle K(\bar{\mathcal{P}}_1(z)), [M_1] \rangle \end{aligned}$$

If we let  $z = 1$  we deduce  $\gamma_K(M_0 \times M_1) = \gamma_K(M_0) \cdot \gamma_K(M_1)$ . The linearity of  $\gamma_K$  is obvious.

Observe that

$$\gamma_K(\mathbb{C}\mathbb{P}^{2n}) = \langle K_n(p_1(\mathbb{C}\mathbb{P}^{2n}), \dots, p_n(\mathbb{C}\mathbb{P}^{2n})), [\mathbb{C}\mathbb{P}^{2n}] \rangle$$

To estimate this number we need to introduce a notation. For any formal power series

$$u = u_0 + u_z + \dots + u_n z^n + \dots$$

we denote by  $\mathbf{j}_z^n u$  its  $n$ -th jet

$$\mathbf{j}_z^n u := \sum_{k \leq n} u_k z^k$$

and by  $[z^n]u$  the coefficient of  $z^n$  in the expansion of  $u$ ,  $[z^n]u := u_n$ . If we set

$$\mathcal{P}_n(z) = 1 + \sum_{k \geq 1} p_k(\mathbb{C}\mathbb{P}^{2n}) z^k$$

then

$$K_n = [z]^n \widehat{K}(\mathcal{P}_n(z)) = [z]^n \widehat{K}(\mathbf{j}_z^n \mathcal{P}_n(z))$$

On the other hand we deduce from (6.1.6) that

$$\mathcal{P}_n(z) = \mathbf{j}_z^n (1 + [H]^2 z)^{2n+1}$$

so that

$$\begin{aligned} K_n &= [z]^n \widehat{K}(\mathbf{j}_z^n (1 + [H]^2 z)^{2n+1}) = [z]^n \widehat{K}((1 + [H]^2 z)^{2n+1}) \\ &= [z]^n \left( \widehat{K}(1 + [H]^2 z) \right)^{2n+1} = ([z]^n a(z)^{2n+1}) [H]^{2n} = [\xi^{2n}](a(\xi^2))^{2n+1} [H]^{2n} \end{aligned}$$

Then

$$\gamma_K(\mathbb{C}\mathbb{P}^{2n}) = [\xi^{2n}](a(\xi^2))^{2n+1} \langle [H]^{2n}, [\mathbb{C}\mathbb{P}^{2n}] \rangle.$$

□



A genus  $\gamma$  extends to a ring morphism  $\gamma : \Omega_{\bullet}^+ \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ . Consider the subring  $\widehat{\Omega}_{\bullet}^+$  of  $\Omega_{\bullet}^+$  generated by the cobordism classes  $\mathbb{C}\mathbb{P}^{2n}$ ,  $n \geq 1$ . To any ring morphism  $\gamma : \widehat{\Omega}_{\bullet}^+ \otimes \mathbb{Q} \rightarrow \mathbb{Q}$  we form the generating series

$$r^\gamma(t) = 1 + \sum_{n \geq 1} r_{2n}^\gamma t^{2n}, \quad r_{2n}^\gamma := \gamma(\mathbb{C}\mathbb{P}^{2n}).$$

We also set

$$\mathbf{R}^\gamma := \int_0^t r^\gamma = \sum_{n \geq 0} \frac{r_{2n}^\gamma}{2n+1} t^{2n+1}, \quad \gamma_0 = 1.$$

Since the collection  $\{\mathbb{C}\mathbb{P}^{2i_1} \times \cdots \times \mathbb{C}\mathbb{P}^{2i_\ell}; (i_1, \dots, i_\ell) \in \text{Part}(n)\}$  forms a rational basis of  $\Omega_{4n}^+ \otimes \mathbb{Q}$  we deduce that the sequence  $r_{2n}^\gamma$  completely determines the restriction to  $\widehat{\Omega}_{\bullet}^+ \otimes \mathbb{Q}$  of the genus  $\gamma$ . Conversely, any sequence of rational numbers  $(r_{2n})_{n \geq 1}$  determines a unique morphism  $\gamma : \widehat{\Omega}_{\bullet}^+ \otimes \mathbb{Q} \rightarrow \mathbb{Q}$  such that

$$\rho(\mathbb{C}\mathbb{P}^{2n}) = r_{2n}, \quad \forall n \geq 1.$$

We have thus established a bijection between formal power series

$$r(t) = 1 + \sum_{n \geq 1} r_{2n} t^{2n} \in \mathbb{Q}[[z]]^\#.$$

and ring morphisms  $\widehat{\Omega}_{\bullet}^+ \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ .

Note that to every power series  $a \in \mathbb{Q}[[\xi]]^\#$  we can associate a multiplicative sequence  $K = K^a$  whose symbol is  $a$ . We denote by  $\gamma^a$  the genus determined by  $K^a$ , and we set

$$r^a(t) := 1 + \sum_{n \geq 0} \gamma^a(\mathbb{C}\mathbb{P}^{2n}) t^{2n} \in \mathbb{Q}[[t^2]]^\#.$$

We obtain in this fashion a bijection

$$\mathbb{Q}[[\xi]]^\# \ni a(\xi) \mapsto r^a(t) \in \mathbb{Q}[[t^2]]^\#.$$

We would like to give a description of the inverse of this correspondence. This will require a detour in classical combinatorics.

For any formal power series  $u = \sum_{n \geq 0} u_n t^n \in \mathbb{R}[[t]]$  we set

$$\int_0^t u := \sum_{n \geq 0} \frac{u_n}{n+1} t^{n+1}.$$

We can define in an obvious way the composition of two formal power series,  $u \circ v$ ,  $[t^0]v = 0$ ,

$$u \circ v(t) = u_0 + u_1 \left( \sum_{n \geq 1} v_n t^n \right) + u_2 \left( \sum_{n \geq 1} v_n t^n \right)^2 + \cdots.$$

A formal power series  $u$  is called *formally invertible* if it has an expansion of the form

$$u(t) = u_1 t + u_2 t^2 + \cdots, \quad u_1 \neq 0, \quad \text{i.e., } \frac{du}{dt} \Big|_{t=0} \neq 0.$$

In this case, a version of the implicit function theorem for formal series implies that we can find a formal power series  $v = v_1 t + v_2 t^2 + \cdots$  such that

$$u \circ v = t = v \circ u.$$

The series  $v$  is called the *formal inverse* of  $u$ , and we denote it  $u^{[-1]}$ . The coefficients of  $v$  are found via *Lagrange inversion formula*. We write  $u$  as a ratio  $u = \frac{t}{q(t)}$  and then we have

$$[t^n]v = \frac{1}{n} [t^{n-1}]q(t)^n. \quad (7.2.2)$$

To keep the flow of arguments uninterrupted we will present the proof of this classical identity later on.

**Proposition 7.2.3.** *For any morphism  $\gamma : \widehat{\Omega}_\bullet^+ \otimes \mathbb{Q} \rightarrow \mathbb{Q}$  there exists a unique multiplicative sequence  $K = K^\gamma$  such that  $\gamma = \gamma_K$  on  $\widehat{\Omega}_\bullet^+ \otimes \mathbb{Q}$ . More precisely,  $K^\gamma$  is the unique multiplicative sequence whose symbol  $a_\gamma$  satisfies the equation*

$$\mathbf{R}^\gamma = \left( \frac{t}{a_\gamma(t^2)} \right)^{[-1]}. \quad (7.2.3)$$

Equivalently,

$$a_\gamma(t^2) = \frac{t}{(\mathbf{R}^\gamma)^{[-1]}}. \quad (7.2.4)$$

**Proof.** Observe first that  $\mathbf{R}^\gamma$  is formally invertible because  $[t^0]\mathbf{R}^\gamma = 0$  and  $[t]\mathbf{R}^\gamma = 1$ . From (7.2.1) we deduce

$$r_{2n}^\gamma = [t^{2n}]a(t^2)^{2n+1} \implies [t^{2n+1}]\mathbf{R}^\gamma = \frac{1}{2n+1} [t^{2n}]a(t^2)^{2n+1}.$$

Observing that

$$[t^{2n}]\mathbf{R}^\gamma = 0 = \frac{1}{2n} [t^{2n-1}]a(t^2)^{2n}$$

we deduce that

$$[t^n]\mathbf{R}^\gamma = \frac{1}{n} [t^{n-1}]a(t^2)^n, \quad \forall n \geq 1.$$

The Lagrange inversion formula implies that  $\mathbf{R}^\gamma$  must be the formal inverse of  $\frac{t}{a(t^2)}$ . □

We have thus produced several bijections

$$\mathbb{Q}[[\xi]]^\# \ni a \longrightarrow K_a \in \text{Multiplicative sequences}; \quad \text{inverse, symbol map : } K \mapsto a_K.$$

$$\text{Hom}\left(\widehat{\Omega}_\bullet^+ \otimes \mathbb{Q}, \mathbb{Q}\right) \ni \gamma \longmapsto r^\gamma = 1 + \sum_{n \geq 1} \gamma(\mathbb{C}\mathbb{P}^{2n})t^{2n} \in \mathbb{Q}[[t^2]]^\#.$$

$$\text{Multiplicative sequences} \ni K \mapsto \gamma_K \in \text{Hom}\left(\widehat{\Omega}_\bullet^+ \otimes \mathbb{Q}, \mathbb{Q}\right); \quad \text{inverse : } \gamma \mapsto K^\gamma.$$

At this point we want to invoke R.Thom's results on the structure of the oriented cobordism ring. We will discuss its proof in Chapter 9.

**Theorem 7.2.4** (Thom cobordism theorem). (a) *If  $n \neq 0$  then  $\Omega_n^+$  is a finite group.*

(b) *The group  $\Omega_{4n}^+/\text{Tors}$  is a finitely generated free Abelian group of rank  $p(n) = |\text{Part}(n)|$ . In particular, we have*

$$\widehat{\Omega}_\bullet^+ \otimes \mathbb{Q} = \Omega_\bullet^+ \otimes \mathbb{Q}. \quad \square$$

Putting together all the facts established so far we obtain the following result.

**Corollary 7.2.5.** *For any genus  $\gamma : \Omega_{\bullet}^+ \rightarrow \mathbb{Q}$  there exists a unique multiplicative sequence  $K$  such that, for any smooth, compact oriented manifold  $M$  of dimension  $4n$  we have*

$$\gamma(M) = \langle K_n(p_1(M), \dots, p_n(M)), [M] \rangle.$$

The symbol  $a(\xi)$  of  $K$  is related to the generating series

$$r^\gamma(t) = 1 + \sum_{n \geq 1} \gamma(\mathbb{C}\mathbb{P}^{2n}) t^{2n}$$

via the equality

$$a(t^2) = \frac{t}{\mathbf{R}^{[-1]}}, \quad \text{where } \mathbf{R} = \int_0^t r^\gamma.$$

### 7.3. The signature formula

The signature function defines a morphism

$$\tau : \Omega_{\bullet}^+ \otimes \mathbb{Q} \rightarrow \mathbb{Q}$$

with generating function

$$r^\tau(t) = 1 + \sum_{n \geq 1} \tau(\mathbb{C}\mathbb{P}^{2n}) t^{2n} = \frac{1}{1-t^2} = \frac{1}{2} \left( \frac{1}{t+1} - \frac{1}{t-1} \right)$$

so that

$$\mathbf{R}^\tau(t) = \frac{1}{2} \log \left( \frac{t+1}{t-1} \right)$$

The formal inverse of  $\mathbf{R}^\tau$  can be obtained by solving for  $t$  the equation

$$u = \frac{1}{2} \log \left( \frac{t+1}{t-1} \right) \iff \frac{t+1}{t-1} = e^{2u} \iff t = \frac{e^{2u} - 1}{e^{2u} + 1} = \frac{e^u - e^{-u}}{e^u + e^{-u}} = \tanh(u)$$

There exists a multiplicative sequence  $K = K^\tau$  such that  $\tau = \gamma_K$  on  $\widehat{\Omega}_{\bullet}^+ \otimes \mathbb{Q}$ . Using (7.2.4) we deduce that the symbol  $a_\tau$  of  $K_\tau$  satisfies

$$a(\xi^2) = \frac{\xi}{\tanh(\xi)} \Rightarrow a(\xi) = \frac{\sqrt{\xi}}{\tanh(\sqrt{\xi})} = \ell(\xi),$$

where  $\ell(\xi)$  is the power series investigated in Example 7.1.5(b). We denoted by  $\mathbf{L}$  the associated multiplicative sequence, and we will refer to the associated genus as the  $\mathbf{L}$ -genus. We have thus obtained the celebrated *signature formula* due to F. Hirzebruch.

**Corollary 7.3.1** (Signature formula). *For any compact oriented, smooth manifold  $M$  of dimension  $4n$  we have*

$$\tau_M = \langle \mathbf{L}_n(p_1(M), \dots, p_n(M)), [M] \rangle,$$

where  $\mathbf{L}$  is the multiplicative sequence with symbol  $\ell(\xi) = \frac{\sqrt{\xi}}{\tanh(\sqrt{\xi})}$ , and  $[M] \in H_{4n}(M, \mathbb{Z})$  denotes the generator determined by the orientation of  $M$ . For example,

$$\tau_M = \frac{1}{3} \langle p_1(M), [M] \rangle, \quad \text{if } \dim M = 4,$$

$$\tau_M = \frac{1}{45} \langle p_2(M) - 7p_1(M)^2, [M] \rangle \quad \text{if } \dim M = 8,$$

$$\tau_M = \frac{1}{945} \langle 62p_3(M) - 13p_2(M)p_1(M) + 2p_1(M)^3, [M] \rangle. \quad \square$$

**Example 7.3.2.** We want to present a simple amusing application of the signature formula. Later on we will discuss more complicated ones. In the sequel, for every topological space  $X$  we define its  $k$ -th Betti number to be

$$b_k(X) = \dim_{\mathbb{Q}} H_k(X, \mathbb{Q}).$$

If all the Betti number of  $X$  are finite, then we define the *Poincaré series of  $X$*  to be the formal power series

$$\sum_{k \geq 0} b_k(X) t^k \in \mathbb{Z}[t]$$

Given any sequence of nonnegative integers  $a_1, \dots, a_n, \dots$  finitely generated groups  $(G_n)_{n \geq 1}$  we can find a compact, connected,  $CW$ -complex  $X$  such that<sup>2</sup>

$$b_k(X) = a_k, \quad \forall k \geq 1$$

Given a sequence of nonnegative integers  $(a_k)_{k \geq 1}$ , we can ask if can we find a smooth compact, connected, orientable manifold  $M$ ,  $\dim M = m$  which such that

$$b_k(M) = a_k, \quad \forall k \geq 1.$$

Clearly, we need to impose some restrictions on the numbers  $a_k$  groups  $G_n$ . First, we need to require that there exists  $m > 0$  such that  $a_n = 0, \forall n > m$ . Secondly, Poincaré duality requires that

$$a_m = 1, \quad a_i = a_{m-i}, \quad \forall 0 < i < m.$$

We want to show that there is no, smooth, connected, orientable manifold  $M$  whose Poincaré polynomial is

$$P_M(t) = 1 + t^6 + t^{12}.$$

We argue by contradiction. If such an  $M$  existed, then  $\dim M = 12$ . The middle Betti number is 1 so that the signature of  $M$  can only be  $\pm 1$ . Since  $b_p(M) = b_8(M) = 0$  we deduce that the reduced Pontryagin classes  $\bar{p}_1(M)$  and  $\bar{p}_2(M)$  are trivial. From the signature formula we deduce

$$\pm 1 = \frac{62}{945} \langle \bar{p}_3(M), [M] \rangle \Rightarrow \pm 945 = 62 \langle \bar{p}_3(M), [M] \rangle.$$

This is impossible since 945 is an odd integer. □

## 7.4. The Lagrange inversion formula

In this last section, we give the promised proof of the Lagrange inversion formula.

Consider the ring of formal power series  $\mathbb{C}[[t]]$  and the ring of formal Laurent series  $\mathbb{C}[t^{-1}, t]$ , where

$$a \in \mathbb{C}[t^{-1}, t] \iff a = t^N b(t), \quad N \in \mathbb{Z}, \quad b(t) \in \mathbb{C}[[t]].$$

For  $a \in \mathbb{C}[t^{-1}, t]$  and  $n \in \mathbb{Z}$  we denote by  $[t^n]a$  the coefficient of  $t^n$  in the expansion of  $a$ . The coefficient  $[t^{-1}]a$  is called the *residue of  $a$*  and it is denoted by  $\text{Res } a$ . Let us observe that  $\mathbb{C}[t^{-1}, t]$  is a field, i.e., any nonzero Laurent series  $u$  has a multiplicative inverse  $u^{-1}$ . We denote by  $D$  the formal differentiation operator

$$\mathbb{C}[t^{-1}, t] \longrightarrow \mathbb{C}[t^{-1}, t], \quad a \mapsto Da := \frac{d}{dt} a.$$

<sup>2</sup>Can you prove this?

**Lemma 7.4.1** (Formal residue theorem). (a) For any Laurent series  $u \in \mathbb{C}[t^{-1}, t]$  we have

$$\operatorname{Res} Du = 0.$$

(b) If  $u = u_1t + u_2t^2 + \cdots \in \mathbb{C}[[t]]$  is formally invertible, i.e.,  $u_1 \neq 0$  then

$$\operatorname{Res} u^{-1} Du = 1. \quad \square$$

**Exercise 7.4.2.** Prove the formal residue theorem. □

Suppose  $u = u_1t + u_2t^2 + \cdots$  is a formally invertible power series and  $v = v_1t + v_2t^2 + \cdots$  is its formal inverse, i.e.,  $v(u(t)) = t$ . Then

$$t = v_1u + v_2u^2 + \cdots$$

so that,

$$1 = \sum_{k \geq 1} kv_k u^{k-1} Du.$$

Multiplying both sides by  $u^{-n}$  we deduce

$$u^{-n} = \sum_{k \geq 1} kv_k u^{k-n-1} Du = nv_n u^{-1} Du + \sum_{k \neq n} \frac{k}{k-n} Du^{k-n}.$$

Using the formal residue theorem we deduce

$$\operatorname{Res} u^{-n} = nv_n \operatorname{Res} u^{-1} Du + \sum_{k \neq n} \frac{k}{k-n} \operatorname{Res} Du^{k-n} = nv_n$$

so that,

$$[t^n]v = [t^{-1}]u^{-n}.$$

This clearly implies the Lagrange inversion formula (7.2.2).



# Milnor's exotic spheres

In this chapter we want to describe one of Milnor's methods of constructing exotic smooth structure on the 7-sphere. In proving that these structures are indeed exotic Hirzebruch signature formula will play an important role. Our presentation follows closely Milnor's history making paper [Mi56].

## 8.1. An invariant of smooth rational homology 7-spheres

Denote by  $\mathcal{X}_7$  the collection of orientation preserving diffeomorphism classes of oriented 7-dimensional manifolds  $X$  satisfying the following conditions

- (a) The manifold  $X$  is a  $\mathbb{Q}$ -homology 7-sphere, i.e.,  $H^\bullet(X, \mathbb{Q}) \cong H^\bullet(S^7, \mathbb{Q})$ .
- (b) There exists a compact, connected oriented 8-dimensional manifold with boundary  $\widehat{X}$  such that  $\partial\widehat{X}$  is orientation preserving diffeomorphic to  $X$ .<sup>1</sup>

Suppose  $X \in \mathcal{X}_7$  and  $\widehat{X}$  is a compact, connected oriented 8-dimensional manifold bounding  $X$ . Let  $[\widehat{X}] \in H_8(\widehat{X}, X, \mathbb{Z})$  the element determined by the orientation of  $\widehat{X}$ . We get a symmetric bilinear form

$$Q_{\widehat{X}} : H^4(\widehat{X}, X; \mathbb{Q}) \times H^4(\widehat{X}, X; \mathbb{Q}) \rightarrow \mathbb{Q}, \quad Q_{\widehat{X}}(\alpha, \beta) = \langle \alpha \cup \beta, [\widehat{X}] \rangle,$$

and we denote by  $\tau_{\widehat{X}}$  its signature.

Since  $X$  is a  $\mathbb{Q}$ -homology 7-sphere the inclusion  $j : (\widehat{X}, \emptyset) \hookrightarrow (\widehat{X}, X)$  induces isomorphisms

$$j^* : H^k(\widehat{X}, X; \mathbb{Q}) \rightarrow H^k(\widehat{X}, \mathbb{Q})$$

for any  $1 \leq j \leq 6$ , so we can regard the Pontryagin class  $\bar{p}_1(\widehat{X}) \in H^4(\widehat{X}, \mathbb{Q})$  as an element in  $\hat{p}_1(\widehat{X}) = (j^*)^{-1}p_1(\widehat{X}) \in H^4(\widehat{X}, X; \mathbb{Q})$ .

Recall that the second  $L$ -polynomial is

$$L_2(p_1, p_2) = \frac{7}{45}p_2 - \frac{1}{45}p_1^2.$$

<sup>1</sup>The condition (b) is automatically satisfied because the oriented cobordism group  $\Omega_7^+ = 0$ .

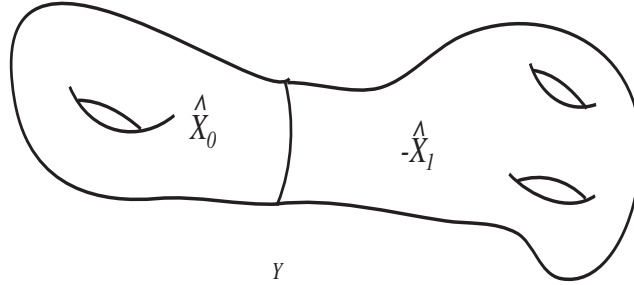
Define  $\lambda(\widehat{X}, X) \in \mathbb{Q}/\mathbb{Z}$  by setting.

$$\lambda(\widehat{X}, X) := \frac{45}{7} \left( \tau_{\widehat{X}} - \langle L_2(\widehat{p}_1(\widehat{X}), 0), [\widehat{X}] \rangle \right) \bmod \mathbb{Z} = \frac{1}{7} \left( 45\tau_{\widehat{X}} + \langle \widehat{p}_1(\widehat{X})^2, [\widehat{X}] \rangle \right) \bmod \mathbb{Z}.$$

**Lemma 8.1.1.** *If  $X \in \mathcal{X}_7$  and  $\widehat{X}_0, \widehat{X}_1$  are two compact, connected 8-dimensional manifolds bounding  $X$  then*

$$\lambda(\widehat{X}_0, X) = \lambda(\widehat{X}_1, X) \in \mathbb{Q}/\mathbb{Z}.$$

**Proof.** Denote by  $-\widehat{X}_1$  the manifold  $\widehat{X}_1$  equipped with the opposite orientation. By gluing  $\widehat{X}_0$  to  $-\widehat{X}_1$  along the common boundary  $X$  we obtain a smooth, compact oriented 8-dimensional manifold  $Y = \widehat{X}_0 \cup_X -\widehat{X}_1$ ; see Figure 8.1.



**Figure 8.1.** *Gluing two manifolds with identical boundary.*

Using the relative Mayer-Vietoris long exact cohomological sequences (1.1.4) and (1.1.6) associated to the decomposition

$$(Y, Y) = (Y, \widehat{X}_0) \cup (Y, \widehat{X}_1), \quad (Y, X) = (Y, \widehat{X}_0) \cap (Y, \widehat{X}_1),$$

the fact  $X$  is a  $\mathbb{Q}$ -homology sphere, and the long exact sequences of the pair  $(Y, X)$  we deduce

$$H^4(Y, \mathbb{Q}) \cong H^4(Y, X; \mathbb{Q}) \cong H^4(\widehat{X}_0, X; \mathbb{Q}) \oplus H^4(\widehat{X}_1, X; \mathbb{Q}).$$

On the other hand, from the Mayer-Vietoris sequences we obtain an isomorphism

$$H_8(Y, X; \mathbb{Z}) \rightarrow H_8(\widehat{X}_0, X; \mathbb{Z}) \oplus H_8(\widehat{X}_1, X; \mathbb{Z}) \cong \mathbb{Z}\langle [\widehat{X}_0] \rangle \oplus \mathbb{Z}\langle [\widehat{X}_1] \rangle, \quad (8.1.1)$$

while the long exact sequence of the pair  $(Y, X)$  yields the exact sequence

$$0 \rightarrow H_8(Y, \mathbb{Z}) \rightarrow H_8(Y, X, \mathbb{Z}) \xrightarrow{\partial} H_7(X, \mathbb{Z}) \cong \mathbb{Z}\langle [X] \rangle.$$

Using the isomorphism (8.1.1) we can express the boundary operator  $\partial$  by

$$\partial(m_0[\widehat{X}_0] + m_1[\widehat{X}_1]) = m_0\partial[\widehat{X}_0] + m_1\partial[\widehat{X}_1] = (m_0 + m_1)[X].$$

We deduce that the orientation class  $[Y] \in H_8(Y, \mathbb{Z})$  can be identified with  $[\widehat{X}_0] - [\widehat{X}_1] \in H_8(Y, X; \mathbb{Z})$ . This implies that, over  $\mathbb{Q}$ , the intersection form of  $Y$  is isomorphic to the direct sum

$$Q_Y = Q_{\widehat{X}_0} \oplus Q_{-\widehat{X}_1} = Q_{\widehat{X}_0} \oplus (-Q_{\widehat{X}_1}),$$

so that,

$$\tau_Y = \tau_{X_0} - \tau_{X_1}.$$



Using the signature formula we deduce

$$\tau(Y) = \frac{1}{45} \langle 7p_2(Y) - p_1(Y)^2, [Y] \rangle$$

so that

$$\frac{45}{7}(\tau_{X_0} - \tau_{X_1}) = -\frac{1}{7} \langle p_1(Y)^2, [Y] \rangle + \langle p_2(Y), [Y] \rangle = -\frac{1}{7} \langle p_1(Y)^2, [Y] \rangle \pmod{\mathbb{Z}} \quad (8.1.2)$$

since  $p_2(Y) \in H^8(Y, \mathbb{Z})$ .

From the isomorphism

$$H^4(Y, \mathbb{Q}) \cong H^4(Y, X; \mathbb{Q}) \cong H^4(\widehat{X}_0, X; \mathbb{Q}) \oplus H^4(\widehat{X}_1, X; \mathbb{Q}),$$

and the equalities

$$p_1(Y)|_{\widehat{X}_i} = p_1(\widehat{X}_i),$$

we deduce that

$$\langle p_1(Y)^2, [Y] \rangle = \langle \widehat{p}_1(\widehat{X}_0)^2, [\widehat{X}_0] \rangle - \langle \widehat{p}_1(\widehat{X}_1)^2, [\widehat{X}_1] \rangle.$$

Using this in (8.1.2) we deduce

$$\frac{45}{7}(\tau_{X_0} - \tau_{X_1}) = -\frac{1}{7} (\langle \widehat{p}_1(\widehat{X}_0)^2, [\widehat{X}_0] \rangle - \langle \widehat{p}_1(\widehat{X}_1)^2, [\widehat{X}_1] \rangle) \pmod{\mathbb{Z}},$$

so that

$$\lambda(\widehat{X}_0, X) = \lambda(\widehat{X}_1, X) \in \mathbb{Q}/\mathbb{Z}.$$

□

Lemma 8.1.1 shows that  $\lambda$  induces a well defined map  $\lambda : \mathcal{X}_7 \rightarrow \mathbb{Q}/\mathbb{Z}$

$$\lambda(X) = \frac{1}{7} (45\tau_{\widehat{X}} + \langle \widehat{p}_1(\widehat{X}), [\widehat{X}] \rangle) \pmod{\mathbb{Z}} = \frac{1}{7} (\langle \widehat{p}_1(\widehat{X}), [\widehat{X}] \rangle - 4\tau_{\widehat{X}}) \pmod{\mathbb{Z}},$$

where  $\widehat{X}$  is any oriented 8-manifold that bounds  $X$ . Note  $\lambda(S^7) = \lambda(D^8, S^7) = 0$ . We obtain the following consequence.

**Corollary 8.1.2.** *If  $X \in \mathcal{X}_7$  and  $\lambda(X) \neq 0$  then  $X$  is not diffeomorphic to the standard sphere  $S^7$ .* □

We can now describe Milnor's strategy for detecting exotic 7-spheres. More precisely, we will construct a manifold  $X \in \mathcal{X}_7$  such that  $X$  is homeomorphic to  $S^7$ , but such that  $\lambda(X) \neq 0$ .

## 8.2. Disk bundles over the 4-sphere

We will seek our examples of exotic 7-spheres within a rather restricted class of 7-manifolds, namely the total spaces of sphere bundles of certain rank 4 real vector bundles over the 4-sphere.

Consider the 4-sphere

$$S^4 : \{(x^0, \dots, x^4) \in \mathbb{R}^5; \sum_{i=0}^4 |x^i|^2 = 1\}.$$

For every  $\varepsilon > 0$  we consider the regions

$$D^+ = \{(x^0, \dots, x^4); x^0 > -1\}, \quad D^- = \{(x^0, \dots, x^4); x^0 < 1\},$$

and set  $C := D^+ \cap D^-$ .

We denote by  $E$  the equator  $\{x^0 = 0\} \subset S^4$  oriented as boundary of the upper hemisphere. If we identify the hyperplane  $\{x^0 = 0\} \subset \mathbb{R}^5$  with the space  $\mathbb{H}$  of quaternions, then we can identify  $E$  with the set of unit quaternions.

Via the stereographic projection from the South pole (respectively North Pole) we obtain a diffeomorphism

$$u_+ : D^+ \rightarrow \mathbb{H} \quad (\text{respectively } u_- : D^- \rightarrow \mathbb{H}).$$

The maps  $u_{\pm}$  are related by the equalities

$$u_+ = \frac{1}{|u_-|^2} u_-, \quad u_- = \frac{1}{|u_+|^2} u_+.$$

For every continuous map  $g : C \rightarrow SO(4)$  we obtain a rank 4 oriented real vector bundle  $E(g) \rightarrow S^4$  obtained by the clutching construction. More precisely, we glue  $\mathbb{R}_{D^+}^4 \rightarrow \mathbb{R}_{D^-}^4$  over  $C$  via the gluing map

$$\mathbb{R}^4 \times D^+ \ni (v_+, x) \mapsto (v_-, x) = (g(x)v_+, x) \in \mathbb{R}^4 \times D^-, \quad x \in C = D^+ \cap D^-.$$

For any  $k, j \in \mathbb{Z}$  we define  $g_{k,j} : C \rightarrow SO(4)$  by

$$g_{k,j}(x)v := \frac{1}{|u_+(x)|^{k-j}} u_+(x)^k v u_+(x)^{-j}, \quad \forall x \in C, \quad v \in \mathbb{H} (\cong \mathbb{R}^4). \quad (8.2.1)$$

We set  $E_{k,j} = E(g_{k,j})$  and we set

$$e_{k,j} = \langle e(E_{k,j}), [S^4] \rangle \in \mathbb{Z}, \quad p_{k,j} = \langle p_1(E_{k,j}), [S^4] \rangle \in \mathbb{Z}.$$

The topological type of the bundle  $E_{k,j}$  is uniquely determined by the homotopy class of the restriction to  $E$  of  $g_{k,j}$  and we have (see Exercise A.4.3)

$$e_{k,j} = (k - j), \quad p_{k,j} = -2(k + j). \quad (8.2.2)$$

We denote by  $\widehat{X}_{k,j}$  the disk bundle of  $E_{k,j}$  and by  $X_{k,j}$  the sphere bundle of  $E_{k,j}$ . Then  $\widehat{X}_{k,j}$  is an oriented 8-dimensional manifold with boundary  $X_{k,j}$ . From Exercise A.3.1 we know that if  $k - j = \pm 1$  then  $X_{k,j}$  has the same homology as the unit sphere  $S^7$ . We set

$$E_k := E_{k,k-1}, \quad \widehat{X}_k := \widehat{X}_{k,k-1}, \quad X_k := X_{k,k-1}.$$

so that  $H^\bullet(X_k, \mathbb{Z}) \cong H^\bullet(S^7, \mathbb{Z})$ . In fact we can be much more precise.

**Proposition 8.2.1.** *The 7-dimensional manifold  $X_k$  is homeomorphic to the unit sphere  $S^7$ .  $\square$*

The proof is not very hard, but it relies on some elementary Morse theory. Since the arguments in the proof bare no relevance to our future considerations we will present the proof of this proposition in a later section.

**Proposition 8.2.2.**

$$\lambda(X_k) = \frac{4}{7} \left( (2k - 1)^2 - 1 \right) \text{ mod } \mathbb{Z}.$$

**Proof.** We denote by  $\Phi_k$  the Thom class of  $E_k \rightarrow S^4$ . We regard it as a relative class  $\Phi_k \in H^4(\widehat{X}_k, X_{k-1})$ . If  $\zeta : S^4 \rightarrow E_k$  denotes the zero section then we deduce from Proposition 4.2.5 and Remark 4.2.6(a) that  $\zeta_*[S^4]$  is the Poincaré dual of  $\Phi_k$ , i.e.,

$$\zeta_*[S^4] = \Phi_k \cap [\widehat{X}_k]. \quad (8.2.3)$$

Denote by  $\pi$  the natural projection  $\widehat{X}_k \rightarrow S^4$ . Then the image of  $\Phi_k$  under the morphism

$$j^* : H^4(\widehat{X}_k, X_k; \mathbb{Z}) \rightarrow H^4(\widehat{X}_4, \mathbb{Z})$$

is  $\pi^*e(E_k)$ . Indeed, since the zero section  $\zeta : S^4 \rightarrow \widehat{X}_k$  is a homotopy equivalence with homotopy inverse the natural projection  $\pi : \widehat{X}_k \rightarrow S^4$ , we deduce

$$\zeta^*j^*\Phi_k = \zeta^*\Phi_k = e(E_k) \Rightarrow j^*\Phi_k = \pi^*\zeta^*j^*\Phi_k = \pi^*e(E_k).$$

The intersection form  $Q$  can be computed as follows

$$\begin{aligned} Q(\Phi_k, \Phi_k) &= \langle \Phi_k \cup \Phi_k, [\widehat{X}_k] \rangle = \langle j^*\Phi_k \cup \Phi_k, [\widehat{X}_k] \rangle = \langle j^*\Phi_k, \Phi_k \cap [\widehat{X}_k] \rangle \\ &\stackrel{(8.2.3)}{=} \langle j^*\Phi_k, \zeta^*[S^4] \rangle = \langle \pi^*e(E_k), \zeta^*[S^4] \rangle = \langle e(E_k), [S^4] \rangle = e_{k,k-1} = 1. \end{aligned}$$

Hence the signature of  $\widehat{X}_k$  is

$$\tau_{\widehat{X}_k} = 1.$$

Now observe that the tangent bundle of  $\widehat{X}_k$  splits as a direct sum of two sub-bundles:

- The vertical sub-bundle  $VT\widehat{X}_k$  consisting of vectors tangent to the fibers of  $E_k \rightarrow S^4$ .
- The horizontal sub-bundle  $HT\widehat{X}_k$  consisting of vector tangent to  $\widehat{X}_k$  and perpendicular to the vertical vectors with respect to a fixed Riemann metric on  $\widehat{X}_k$ .

Observing that

$$VT\widehat{X}_k \cong \pi^*E_k \quad \text{and} \quad HT\widehat{X}_k \cong \pi^*TS^4,$$

we deduce that

$$p_1(\widehat{X}_k) = \pi^*p_1(E_k) + \pi^*p_1(TS^4) = \pi^*p_1(E_k),$$

where  $p_1(TS^4) = 0$  since the tangent bundle of  $S^4$  is stably trivial. If we denote by  $\omega_4$  the generator of  $H^4(S^4, \mathbb{Z})$  determined by the orientation,  $\omega_4 \cap [S^4] = 1 \in H_0(S^4)$ , then (8.2.2) implies

$$p_1(E_k) = -2(2k-1)\omega_4, \quad e(E_k) = \omega_4.$$

We regard  $\omega_4$  as a class in  $H^4(\widehat{X}_4)$  via the isomorphism  $\pi^*$ . Since  $(j^*)^{-1}\pi^*e(E_k) = \Phi_k$  we deduce

$$\hat{p}_1(\widehat{X}_k) = (j^*)^{-1}\pi^*p_1(E_k) = 2(2k-1)\Phi_k$$

and thus,

$$\langle \hat{p}_1(\widehat{X}_k)^2, [\widehat{X}_k] \rangle = 4(2k-1)^2 Q(\Phi_k, \Phi_k) = 4(2k-1)^2.$$

We deduce that

$$\lambda(X_k) = \frac{1}{7} \left( \langle \hat{p}_1(\widehat{X}_k)^2, [\widehat{X}_k] \rangle - 4\tau_{\widehat{X}_k} \right) \bmod \mathbb{Z} = \frac{4}{7} \left( (2k-1)^2 - 1 \right) \bmod \mathbb{Z}.$$

□

**Corollary 8.2.3.** *If  $k \not\equiv 0, 1 \pmod{7}$  then the manifold  $X_k$  is homeomorphic but not diffeomorphic to the unit sphere  $S^7$ .*

**Proof.** Note that

$$\lambda(X_k) \neq 0 \iff (2k-1)^2 \not\equiv 1 \pmod{7} \iff 2k-1 \not\equiv \pm 1 \pmod{7} \iff k \not\equiv 0, 1 \pmod{7}.$$

□

**Remark 8.2.4.** Observe that  $\lambda(X_k) = \lambda(X_{k+7})$  and

$$\lambda(X_0) = \lambda(X_1) = 0, \quad \lambda(X_2) = \lambda(X_6) = \frac{3}{7}, \quad \lambda(X_3) = \lambda(X_5) = \frac{2}{7}, \quad \lambda(X_4) = \frac{4}{7}$$

This shows that amongst the smooth manifolds homeomorphic to  $S^7$  there are at least 4 different diffeomorphism types. In fact Kervaire and Milnor have shown in [KM63] that there are precisely 28 diffeomorphism types. More precisely for every  $k \geq 1$  consider the complex hypersurface  $Z_k$  in  $\mathbb{C}^5$  given by the polynomial equation

$$z_0^2 + z_1^2 + z_2^2 + z_3^3 + z_4^{6k-1} = 0.$$

Denote by  $Y_k(\varepsilon)$  the intersection of  $Z_k$  with the sphere of radius  $\varepsilon$  centered at the origin of  $\mathbb{C}^5$ . Then the diffeomorphism type of the 7-manifold  $Y_k(\varepsilon)$  is independent of  $\varepsilon$  small. Moreover  $Y_k$  is homeomorphic to  $S^7$  for any  $k$  and  $Y_k$  is diffeomorphic to  $Y_\ell$  if and only if  $k \equiv \ell \pmod{28}$ . As a curiosity, let us mention other conclusions of [KM63]. For example, there exist exactly 992 exotic 11-spheres and exactly 16256 exotic 15-spheres, while there exist only 8 exotic 9-spheres and 3 exotic 13-spheres.  $\square$

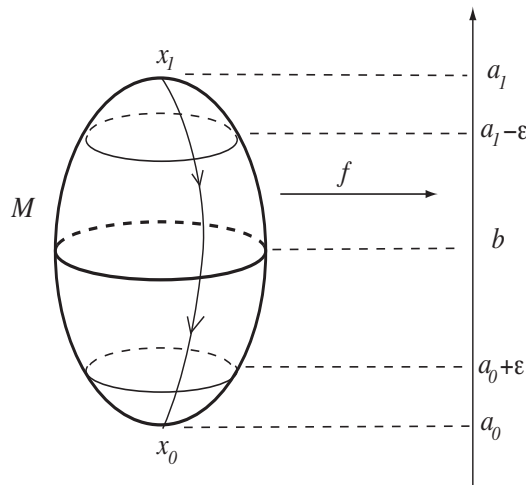
**Proof of Proposition 8.2.1.** The proof relies on the following result of G. Reeb.

**Proposition 8.2.5.** *Suppose  $M$  is a compact, smooth manifold and  $f : M \rightarrow \mathbb{R}$  is a smooth function that has only two critical points: a minimum  $x_0$  and a maximum  $x_1$  such that the Hessians of  $f$  at these points are nondegenerate. Then  $M$  is homeomorphic to  $S^n$ .*

**Proof.** Let  $a_0 := f(x_0)$  and  $a_1 = f(x_1)$  so that  $a_0 \leq f(x) \leq a_1$  for all  $x \in M$ . We denote by  $b$  the midpoint of the interval  $[a_0, a_1]$ ,  $b = \frac{1}{2}(a_0 + a_1)$ . For any  $c \in \mathbb{R}$  we set

$$\{f \leq c\} := \{x \in M; f(x) \leq c\}, \quad \{f \geq c\} := \{x \in M; f(x) \geq c\},$$

$$M_c := \{x \in M; f(x) = c\}.$$



**Figure 8.2.** A negative gradient flow with only two stationary points.

Now we need to invoke a classical result usually referred to as the *Morse Lemma*. For a proof we refer to [N2, Thm. 1.12].

**Lemma 8.2.6.** *If  $x_0$  is a critical point of the smooth function  $f : X \rightarrow \mathbb{R}$ , and the Hessian of  $f$  at  $x_0$  is nondegenerate, then there exist coordinates  $(u^1, \dots, u^n)$ ,  $n = \dim X$ , defined in a neighborhood  $U$  of  $x_0$  such that the following hold.*

- $u^i(x_0) = 0, \forall i = 1, \dots, n.$
- $f|_U = f(x_0) - \frac{1}{2} \sum_{i=1}^m |u^i|^2 + \frac{1}{2} \sum_{j=m+1}^n |u^j|^2$ , where  $m$  denotes the Morse index of  $x_0$ , i.e., the number of negative eigenvalues of the Hessian of  $f$  at  $x_0$ .

□

Using Morse Lemma we can find neighborhood  $U_0$  and  $U_1$  of  $x_0$ , and respectively  $x_1$ , coordinates  $(u^i)$  on  $U_0$ , and  $(v^j)$  on  $U_1$  such that

$$f|_{U_0} = a_0 + \frac{1}{2} \sum_{i=1}^n |u^i|^2, \quad f|_{U_1} = a_1 - \frac{1}{2} \sum_{j=1}^n |v^j|^2.$$

This shows for  $\varepsilon > 0$  sufficiently small we have diffeomorphisms

$$\begin{aligned} \Psi^+ &= \Psi_\varepsilon^+ : \{f \geq a_1 - \varepsilon\} \rightarrow D^n, \quad (v^1, \dots, v^n) \mapsto \frac{1}{\sqrt{2\varepsilon}}(v^1, \dots, v^n), \\ \Psi^- &= \Psi_\varepsilon^- : \{f \leq a_0 + \varepsilon\} \rightarrow D^n, \quad (u^1, \dots, u^n) \mapsto \frac{1}{\sqrt{2\varepsilon}}(u^1, \dots, u^n). \end{aligned}$$

where  $D^n$  denotes the closed unit disk in  $\mathbb{R}^n$ . Set

$$M^* = M \setminus \{x_0, x_1\}.$$

Now fix a Riemann metric  $g$  on  $M$ , such that

$$g|_{U_0} = \sum_i (du^i)^2, \quad g|_{U_1} = (dv^j)^2.$$

Denote by  $\nabla f$  the gradient of  $f$  with respect to the metric  $g$ . In other words,  $\nabla f$  is the unique vector field on  $M$  such that

$$g(\nabla f, X) = df(X), \quad \forall X \in \text{Vect}(M).$$

We denote by  $\Phi : \mathbb{R} \times M \rightarrow M$ ,  $(t, x) \mapsto \Phi^t(x)$ , the flow determined by  $-\nabla f$ . This flow has only two stationary points,  $x_0, x_1$ , the function  $f$  decreases strictly along the nonconstant trajectories and for any  $x \in M^*$  we have

$$\lim_{t \rightarrow \infty} \Phi^t(x) = x_0, \quad \lim_{t \rightarrow -\infty} \Phi^t(x) = x_1.$$

In fact, if  $x \in U_0$  has coordinates  $(u^i)$  then,  $\Phi^t(x) \in U_0, \forall t \geq 0$  and its coordinates are  $(e^{-t}u^i)$ . Similarly, if  $x \in U_1$  has coordinates  $(v^j)$ ,  $\Phi^t(x) \in U_1, \forall t \leq 0$  and its coordinates are  $(e^t v^j)$ . I

Now consider the vector field  $X \in \text{Vect}(M^*)$ ,

$$X = -\frac{1}{|\nabla f|_g^2} \nabla f$$

where for any vector field  $Y$  we denoted by  $|Y|_g$  the pointwise length of  $Y$ ,  $|Y|_g = \sqrt{g(Y, Y)}$ . If  $\gamma(t)$  is an integral curve of  $X$  then

$$\frac{d}{dt} f(\gamma(t)) = -1.$$

Indeed,

$$\frac{d}{dt}f(\gamma(t)) = df(\dot{\gamma}(t)) = df(X(\gamma(t))) = g(\nabla f, X) = -\frac{1}{|\nabla f|_g^2}g(\nabla f, \nabla f) = -1.$$

If we denote by  $\Gamma^t$  the (partial flow) determined by  $X$  then we conclude that for any  $x \in M$ , and any  $t$  such that  $\Gamma^t$  is defined we have

$$f(\Gamma^t x) = f(x) - t.$$

We deduce that for any  $x \in M^*$  the integral line  $t \rightarrow \Gamma^t x$  exists for any  $t \in (f(x) - a_1, f(x) - a_0)$ . The flow lines of  $\Gamma^t$  are perpendicular to the level sets  $\{f = c\}$  of  $f$ . In fact,  $\Gamma^t$  induces diffeomorphisms

$$\Gamma^t : \{f = c\} \rightarrow \{f = c - t\}, \quad \forall t < c - a_0.$$

Let

$$\mathcal{R} := \{a_0 + \varepsilon \leq f \leq a_1 - \varepsilon\}.$$

Observe that we have a diffeomorphism  $\Psi : \mathcal{R} \rightarrow [a_0 + \varepsilon, a_1 - \varepsilon] \times M_{a_1 - \varepsilon}$ , (see Figure 8.2)

$$\mathcal{R} \ni x \mapsto (f(x), \Gamma^{f(x) - a_1 + \varepsilon} x) \in [a_0 + \varepsilon, a_1 - \varepsilon] \times M_{a_1 - \varepsilon}$$

As we have explained above, the level set  $M_{a_1 - \varepsilon}$  is diffeomorphic to a sphere  $S^{n-1}$ . We deduce that  $M$  is homeomorphic to a cylinder  $[a_0 + \varepsilon, a_1 - \varepsilon] \times S^{n-1}$  with the two ends coned off by two  $n$ -disks. In other words,  $M$  is homeomorphic to  $S^n$ . □

To prove that  $X_k$  is homeomorphic to  $S^7$  we use Proposition 8.2.5 so it suffices to construct a smooth function on  $X_k$  that has only two nondegenerate critical points. To achieve this we need to recall the construction of  $X_k$ . We identifies the regions  $D^\pm$  of the sphere  $S^4$  with the vector space  $\mathbb{H}$  via stereographic projections

$$u_\pm : D^\pm \rightarrow \mathbb{H},$$

and then we glue the sphere bundle  $S^3 \times D^+$  to the sphere bundle  $S^3 \times D^-$  via the gluing map

$$S^3 \times (\mathbb{H} \setminus 0) \ni (v_+, u_+) \mapsto (v_-, u_-) = \left( \frac{1}{|u_+|} u_+^k v_+ u_+^{-(k-1)}, \frac{1}{|u_+|^2} u_+ \right) \in S^3 \times (\mathbb{H} \setminus 0).$$

Choose new coordinates  $(w_-, q_-)$  on  $S^3 \times D^-$  by setting  $w_- = v_-$ ,  $q_- = u_-(v_-)^{-1}$ . Then

$$\frac{\mathbf{Re} v_+}{(1 + |u_+|^2)^{1/2}} = \frac{\mathbf{Re} q_-}{(1 + |q_-|^2)^{1/2}}. \quad (8.2.4)$$

Indeed

$$q_- = \mathbf{Re} u_-(v_-)^{-1} = \frac{1}{|u_+|^2} u_+ \cdot (|u_+| u_+^{k-1} v_+^{-1} u_+^{-k}) = \frac{1}{|u_+|} u_+^k v_+^{-1} u_+^{-k}.$$

Since  $|v_+| = 1$  we deduce  $\mathbf{Re} v_+ = \mathbf{Re} v_+^{-1}$  so that

$$\mathbf{Re} q_- = \frac{1}{|u_+|} \mathbf{Re} v_+.$$

One the other hand,  $|q_-| = \frac{1}{|u_+|}$ . This proves (8.2.4). Hence, the function  $f : X_k \rightarrow \mathbb{R}$  given by

$$f|_{S^3 \times D^+} = \frac{\mathbf{Re} v_+}{(1 + |u_+|^2)^{1/2}}, \quad f|_{S^3 \times D^-} = \frac{\mathbf{Re} q_-}{(1 + |q_-|^2)^{1/2}}$$

is well defined and smooth. Moreover, a simple computation shows that this function has only two critical points  $(v_+, u_+) = (\pm 1, 0)$  and they are both nondegenerate.

□

**Remark 8.2.7.** From the proof of Proposition 8.2.5 one can conclude easily that any smooth  $n$ -dimensional manifold can be obtained by gluing two closed  $n$ -dimensional disks using an orientation preserving diffeomorphism  $\varphi$  of  $S^{n-1} = \partial D^n$ . We denote by  $\Sigma_\varphi^n$  the manifold obtained in this fashion. In particular,  $\Sigma_{\mathbb{1}}^n$  is the standard  $n$ -sphere  $S^n$ .

If we denote by  $\text{Diff}^+(M)$  the group of orientation preserving diffeomorphism of a compact, connected, oriented manifold  $M$ , we see that if  $\varphi, \psi \in \text{Diff}^+(S^{n-1})$  are in the same the path component of 1 then  $\Sigma_\varphi$  is diffeomorphic to  $S^n$ .

The fact that there exist exotic 7-sphere implies that  $\text{Diff}^+(S^6)$  is not connected, and in fact, it has at least 28. components. This should be contrasted with low dimensional situations. For example S. Smale has shown that  $\text{Diff}^+(S^2)$  is homotopy equivalent with the (connected) group  $SO(3)$ , while A. Hatcher has shown that  $\text{Diff}^+(S^3)$  is homotopy equivalent with the (connected) group  $SO(4)$ . For  $n \geq 7$  it is shown in [ABK] that  $\text{Diff}^+(S^n)$  is not dominated by any finite dimensional  $CW$ -complex. Recall that a space  $X$  is dominated by a space  $Y$  if there exist maps  $Y \xrightarrow{r} X \xrightarrow{i} Y$  such that  $r \circ i \simeq \mathbb{1}_X$ . In particular,  $\text{Diff}^+(S^n)$  is not homotopy equivalent to a finite  $CW$ -complex if  $n \geq 7$ . □





# Thom's work on cobordisms

The goal of this chapter is to give outline a proof of Thom's cobordism theorem 7.2.4. His strategy can be summarized as follows. Via a geometric method (the Pontryagin-Thom construction) he identifies the cobordism group  $\Omega_{4n}^+$  with the a homotopy group of a certain space (Thom space). To compute this homotopy group, modulo torsion, he uses some techniques of Cartan and Serre<sup>1</sup> to reduce it to a much simpler calculation of the homology groups with rational coefficients of certain Grassmannians. This leads to Corollary 9.2.5 that states

$$\dim_{\mathbb{Q}} \Omega_n^+ \otimes \mathbb{Q} = 0, \quad \text{if } n \not\equiv 0 \pmod{4},$$

$$\dim_{\mathbb{Q}} \Omega_{4k}^+ \otimes \mathbb{Q} \leq p(k) = |\text{Part}(k)|.$$

The opposite inequality is proved in Proposition 6.2.2. Thom's work is considerably broader than the slice we describe in this chapter. For more details we refer the reader to the very readable and very rich classic source [Th54].

## 9.1. The Pontryagin-Thom construction

Consider the real Grassmannian  $\mathbf{Gr}_k(\mathbb{R}^n)$  of  $k$ -dimensional subspaces of  $\mathbb{R}^n$ . The universal vector bundle  $\mathcal{U}_{k,n}$  is not (geometrically) orientable. We form the determinant line bundle  $\det \mathcal{U}_{k,n} \rightarrow \mathbf{Gr}_k(\mathbb{R}^n)$ . The sphere bundle  $S(\det \mathcal{U}_k)$  is a two-to-one cover of  $\mathbf{Gr}_k(\mathbb{R}^n)$  that we denote by  $\widetilde{\mathbf{Gr}}_k(\mathbb{R}^n)$ . We can identify the points of  $\widetilde{\mathbf{Gr}}_k(\mathbb{R}^n)$  with the *oriented*  $k$ -dimensional subspaces of  $\mathbb{R}^n$ , and we will refer to  $\widetilde{\mathbf{Gr}}_k(\mathbb{R}^n)$  as the *Grassmannian of oriented  $k$ -planes in  $\mathbb{R}^n$* .

The Grassmannian  $\mathbf{Gr}_k(\mathbb{R}^n)$  is a smooth manifold of dimension  $k(n - k)$  (see [N1, Example 1.2.20]) and  $\mathcal{U}_{k,n}$  is a smooth vector bundle. The oriented Grassmannian  $\widetilde{\mathbf{Gr}}_k(\mathbb{R}^n)$ , being a double cover, is also a smooth manifold. We denote by  $\widetilde{\mathcal{U}}_{k,n}$  the pullback of  $\mathcal{U}_{k,n}$  to  $\widetilde{\mathbf{Gr}}_k(\mathbb{R}^n)$ . The vector bundle  $\widetilde{\mathcal{U}}_{k,n}$  is *canonically oriented* and we will refer to it as the *universal* or *tautological* oriented  $k$ -plane bundle over  $\widetilde{\mathbf{Gr}}_k(\mathbb{R}^n)$ . For simplicity we set  $\widetilde{\mathbf{Gr}}_{k,n} := \widetilde{\mathbf{Gr}}_k(\mathbb{R}^n)$

<sup>1</sup>We won't present the details of this beautiful theory of Serre, but we will mention some facts relevant to our specific goal.

We denote by  $\tilde{D}_{k,n}$  and respectively  $\tilde{S}_{k,n}$  the disk and respectively sphere bundle associated to  $\tilde{U}_{k,n}$ . The total space  $\tilde{D}_{k,n}$  is a compact oriented manifold with boundary  $\partial\tilde{D}_{k,n} = \tilde{S}_{k,n}$ .

**Definition 9.1.1.** The *Thom space* of a real vector bundle  $E \rightarrow X$  is the one-point compactification of the total space of  $E$ , or equivalently, the quotient  $\mathbf{Th}(E) := D(E)/S(E)$ . This space has a distinguished point that we denote by  $*$ .<sup>2</sup> We denote by  $\widetilde{\mathbf{Th}}_{k,n}$  the Thom space of  $\tilde{U}_{k,n}$ .  $\square$

The Pontryagin-Thom construction yields explicit isomorphisms

$$\pi_{k+n}(\widetilde{\mathbf{Th}}_{k,N}, *) \rightarrow \Omega_n^+, \quad \forall N > k + n, \quad k > n + 1,$$

where  $\pi_m$  denotes the  $m$ -th homotopy group.

To describe this map we need to recall some facts of differential topology. For more details and proofs we refer to [DFN-vol.2, Hir, Kos, W36].

Recall that a smooth map  $f : M \rightarrow N$  is called an *embedding* if it is a homeomorphism onto its image and its differentials  $df : T_x M \rightarrow T_f(x)N$  are injective for any  $x \in M$ . Two embeddings  $f, g : M \rightarrow N$  are said to be *isotopic* if there exists a smooth map

$$F : [0, 1] \times M \rightarrow N, \quad (t, x) \mapsto F_t(x) \in N$$

such that  $F_0 = f$ ,  $F_1 = g$  and for any  $t \in (0, 1)$  the smooth map  $F_t : M \rightarrow N$  is also an embedding. The map  $F$  as above is called an *isotopy*.

An isotopy  $F : [0, 1] \times M \rightarrow N$ ,  $(t, x) \mapsto F_t(x)$  is called an *ambient isotopy* if there exists an isotopy  $H : [0, 1] \times N \rightarrow N$ ,  $(t, x) \mapsto H_t(x)$  such that

$$H_0 = \mathbb{1}_N, \quad F_t = H_t \circ F_0, \quad \forall t \in [0, 1].$$

We say that the maps  $F_0$  and  $F_1$  are *ambiently isotopic*.

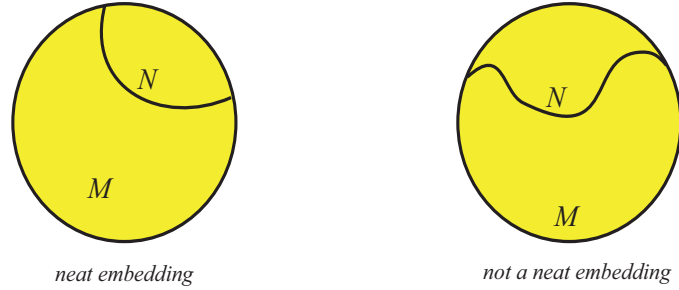
We define a *tubular neighborhood* of a submanifold  $N \hookrightarrow M$  to be a triplet  $(U, E, \Psi)$ , where  $U$  is an open neighborhood of  $N$  in  $M$ ,  $E$  is a smooth vector bundle over  $N$  and  $\Psi$  is a diffeomorphism from the total space of the bundle  $E$  diffeomorphically onto  $U$ , that maps the zero section of  $E$  diffeomorphically onto the submanifold  $N$ . Observe that the differential of  $\Psi$  induces a bundle isomorphism between  $E$  and the normal bundle  $T_N M$ . We will denote this isomorphism by  $\Psi_*$ . Often when referring to tubular neighborhoods we will omit the reference to the bundle  $E$  and the diffeomorphism  $\Psi$ .

**Theorem 9.1.2** (Whitney-Thom). *Suppose  $M$  is a compact  $n$  dimensional manifold. Then for any  $\nu \geq 2n + 1$  there exists embeddings  $f : M \rightarrow \mathbb{R}^\nu$ . Moreover, if  $\nu \geq 2n + 2$  any two embeddings  $f, g : M \rightarrow \mathbb{R}^\nu$  are ambiently isotopic.*  $\square$

Suppose  $M$  and  $N$  are smooth manifolds with boundary. Then a *neat embedding* of  $N$  into  $M$  is an embedding  $f : \widehat{N} \rightarrow \widehat{M}$ , where  $\widehat{N}$  and  $\widehat{M}$  are neck extensions of  $N$  and  $M$ , such that the following hold (see Figure 9.1).

- $f(\text{int}(N)) \subset \text{int}(M)$ ,
- The boundary  $\partial M \subset \widehat{M}$  intersects  $f(\widehat{N})$  transversally and  $f(\partial N) = \partial M \cap f(\widehat{N})$ .

<sup>2</sup>Observe that the complement of  $X$  in  $\mathbf{Th}(E)$  is a contractible neighborhood of  $*$ .



**Figure 9.1.** A neat and a non-neat embedding of a 1-dimensional manifold with boundary.

If  $f : \widehat{N} \rightarrow \widehat{M}$  is a neat embedding of the manifold with boundary  $N$  into the manifold with boundary  $M$ , then a *neat tubular neighborhood* is a tubular neighborhood  $(\widehat{U}, \widehat{E}, \widehat{\Psi})$  of  $\widehat{N}$  in  $\widehat{M}$  such that, if we set

$$U_{\partial} := \widehat{U} \cap \partial M, \quad E_{\partial} := \widehat{E}|_{\partial N},$$

then  $(U_{\partial}, E_{\partial}, \widehat{\Psi}|_{E_{\partial}})$  is a tubular neighborhood of  $\partial N$  in  $\partial M$ . We know that the bundle  $\widehat{E}$  is isomorphic to the normal bundle  $T_{\widehat{N}}\widehat{M}$ , and we denote by  $T_N M$  the isomorphism class of the restriction of  $\widehat{E}$  to  $N$ .

Every compact, neatly embedded submanifold of a manifold with boundary admits neat tubular neighborhoods. The Whitney embedding theorem has a version for manifolds with boundary.

**Theorem 9.1.3.** Denote by  $H^{\nu}$  the half-space

$$H^{\nu} := \{(x_1, \dots, x_{\nu}) \in \mathbb{R}^{\nu}; x_1 \leq 0\}.$$

We regard  $H^{\nu}$  as a manifold with boundary. Suppose  $M$  is a smooth, compact  $n$ -dimensional manifold with boundary. Then for any  $\nu \geq 2n + 1$  the manifold  $M$  admits a neat embedding in the half-space  $H^{\nu}$ .  $\square$

In the smooth context we can improve a bit the results about the classification of vector bundles stated in Theorem 3.3.1. More precisely we have the following result, [Hir, Thm. 4.3.4] or [St, §19].

**Theorem 9.1.4.** Suppose  $M$  is a smooth, compact manifold of dimension  $m$ .

(a) If  $E \rightarrow M$  is an oriented real rank  $k$  vector bundle, then for every  $N \geq k + m$  there exists a smooth map  $f : M \rightarrow \widetilde{\mathbf{Gr}}_{k,N}$  such that  $f^*\widetilde{\mathcal{U}}_{k,N} \cong E$ .

(b) If  $f, g : M \rightarrow \widetilde{\mathbf{Gr}}_{k,N}$  are two smooth maps such that  $N \geq k + m + 1$  and

$$f^*\widetilde{\mathcal{U}}_{k,N} \cong g^*\widetilde{\mathcal{U}}_{k,N},$$

then the maps  $f$  and  $g$  are smoothly homotopic.  $\square$

We can now describe the Pontryagin-Thom map  $\pi_{\nu}(\widetilde{\mathbf{Th}}_{k,N}, *) \rightarrow \Omega_{\nu-k}^+$ ,  $N \geq \nu$ . To keep the geometric ideas as transparent as possible we decided to omit several technical details. However, a motivated reader should not have any trouble filling these gaps.

We regard the sphere  $S^{\nu}$  as the one-point compactification of the vector space  $\mathbb{R}^{\nu}$ . We let  $\infty$  denote the point at infinity of  $S^{\nu}$ . Also we regard  $\mathbb{R}^{\nu}$  as embedded in  $\mathbb{R}^N$  via the linear embedding

$$(x_1, \dots, x_{\nu}) \mapsto (x_1, \dots, x_{\nu}, 0, \dots, 0).$$

Suppose  $f : (S^\nu, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$  is a continuous map. Then, we can slightly deform<sup>3</sup>  $f$  to a new map  $\hat{f} : (S^\nu, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$  such that the restriction of  $\hat{f}$  to  $S^\nu \setminus \{f^{-1}(*)\} = \hat{f}^{-1}(\tilde{D}_{k,N})$  is smooth and transversal to the submanifold  $\widetilde{\mathbf{Gr}}_{k,N} \hookrightarrow \tilde{D}_{k,N}$ . We will say that such maps are *convenient*. The map  $\hat{f}$  is homotopic to  $f$  and thus every homotopy class  $u \in \pi_\nu(\widetilde{\mathbf{Th}}_{k,N}, *)$  can be represented by convenient maps.

If  $f : (S^\nu, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$  is a convenient map, the preimage

$$M = M_f := f^{-1}(\widetilde{\mathbf{Gr}}_{k,N}) \quad (9.1.1)$$

is a smooth submanifold of  $S^\nu$ , and its normal bundle  $T_M S^\nu$  is isomorphic to the pullback by  $f$  of the normal bundle of  $\widetilde{\mathbf{Gr}}_{k,N}$  in  $\tilde{D}_{k,N}$ . This is precisely the universal vector bundle  $\tilde{U}_{k,N}$  so that

$$T_M S^\nu \cong f^* \tilde{U}_{k,N}.$$

Since the bundle  $\tilde{U}_{k,N}$  is naturally oriented, we deduce that  $T_M S^\nu$  is naturally oriented and thus  $TM$  is equipped with a natural orientation uniquely determined by the requirement

$$\mathbf{or}(T_M S^\nu) \wedge \mathbf{or}(TM) = \mathbf{or}(TS^\nu|_M).$$

If  $f_0, f_1 : (S^\nu, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$  are two convenient maps representing the same element  $u \in \pi_\nu(\widetilde{\mathbf{Th}}_{k,N}, *)$ , then we can find a homotopy connecting them

$$F : [0, 1] \times (S^\nu, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$$

such that  $F$  is smooth on  $F^{-1}(*)$  and it is transversal to  $\widetilde{\mathbf{Gr}}_{k,N}$ . The preimage  $M_F = F^{-1}(\widetilde{\mathbf{Gr}}_{k,N})$  is a compact oriented manifold with boundary that produces an oriented cobordism between  $M_{f_0}$  and  $M_{f_1}$ . We have thus produced a map

$$PT : \pi_\nu(\widetilde{\mathbf{Th}}_{k,N}, *) \rightarrow \Omega_{\nu-k}^+, \quad \pi_\nu(\widetilde{\mathbf{Th}}_{k,N}, *) \ni u \mapsto PT(u) := [M_F]_+ \in \Omega_{\nu-k}^+,$$

where  $f$  is a convenient map in the homotopy class  $u$ , and  $[M_F]_+$  is the oriented cobordism class of  $f^{-1}(\widetilde{\mathbf{Gr}}_{k,N})$ . This is the *Pontryagin-Thom construction*. We set  $n := \nu - k$ , so that  $\nu = k + n$ .

**Lemma 9.1.5.** *The Pontryagin-Thom map*

$$PT : \pi_{k+n}(\widetilde{\mathbf{Th}}_{k,N}, *) \rightarrow \Omega_n^+$$

*is a morphism of groups.*

**Proof.** We can think of the elements in the homotopy group as represented by continuous maps  $(D^{k+n}, \partial D^{k+n}) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$  or by continuous maps  $(S^{k+n}, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$ .

Suppose that  $u_\pm \in \pi_{k+n}(\widetilde{\mathbf{Th}}_{k,N}, *)$  are represented by continuous maps

$$f_\pm : (D^{n+k}, \partial D^{n+k}) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *).$$

We can also assume that the restriction of  $f_\pm$  to  $f_\pm^{-1}(*)$  is smooth and transversal to the zero section of  $\tilde{U}_{k,N}$ . As in (9.1.1) we set

$$M_\pm = f_\pm^{-1}(\widetilde{\mathbf{Gr}}_{k,N}).$$

Then

$$PT(u_\pm) = [M_\pm] \in \Omega_n^+.$$

<sup>3</sup>The existence of such a deformation relies on standard transversality techniques pioneered by Whitney in the 30s and 40s; see [Hir].

The sum  $u_+ + u_- \in \pi_{k+n}(\widetilde{\mathbf{Th}}_{k,N}, *)$  is represented by the map  $f : (S^{k+n}, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$  defined as follows.

Pick an equator on  $S^{k+n}$  that contains the point  $\infty$ . (Think of this equator as the one-point compactification of the hyperplane  $\{x_1 = 0\} \subset \mathbb{R}^{n+k}$ . This equator defines two hemispheres  $S_{\pm}^{k+n}$ . Fix orientation preserving diffeomorphism  $D^{n+k} \rightarrow S_{\pm}^{n+k}$  and we let  $f$  be equal  $f_{\pm}$  on  $S_{\pm}^{n+k}$ . The map  $f$  is convenient and the manifold  $M_f$  defined as in (9.1.1) is equal to the disjoint union of  $M_+$  and  $M_-$ . We thus have the following equality in  $\Omega_n^+$

$$PT(u_+ + u_-) = [M_f] = [M_+] + [M_-] = PT(u_+) + PT(u_-).$$

□

To investigate the surjectivity of the Pontryagin-Thom map we need to have a way of producing many examples of convenient maps  $S^{\nu} \rightarrow \widetilde{\mathbf{Th}}_{k,N}$ . Fortunately, there is one simple way of generating such maps.

Start with a compact, oriented submanifold  $M \hookrightarrow \mathbb{R}^{\nu} \hookrightarrow S^{\nu}$  of codimension  $k$  that does not contain the pole  $\infty$ . Recall that we view  $\mathbb{R}^{\nu}$  as a linear subspace of  $\mathbb{R}^N$ .

Using the metric on  $\mathbb{R}^{\nu}$  we can identify the normal bundle  $T_M \mathbb{R}^{\nu}$  as a subbundle of the trivial bundle  $\underline{\mathbb{R}}'_M$  which is a subbundle of the trivial bundle  $\underline{\mathbb{R}}_M^N$ . We obtain a Gauss map

$$\gamma : M \ni x \mapsto (T_M \mathbb{R}^{\nu})_x \in \widetilde{\mathbf{Gr}}_{k,N}$$

This induces a smooth map  $\hat{\gamma} : T_M \mathbb{R}^{\nu} \rightarrow \widetilde{\mathcal{U}}_{k,N}$  such that the diagram below is commutative

$$\begin{array}{ccc} T_M \mathbb{R}^{\nu} & \xrightarrow{\hat{\gamma}} & \widetilde{\mathcal{U}}_{k,N} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\gamma} & \widetilde{\mathbf{Gr}}_{k,N} \end{array}$$

Above, the vertical arrows are the natural projections, and the map  $\hat{\gamma}$  is transversal to the zero section and a linear isomorphism along the fibers of  $T_M \mathbb{R}^{\nu}$ .

If  $(U, E, \Psi)$  is a tubular neighborhood of  $M$ , then we have a bundle isomorphism  $\Psi_* : E \rightarrow T_M \mathbb{R}^{\nu}$  and a diffeomorphism  $\Psi : E \rightarrow U$ . We obtain in this fashion a *proper* map  $\Phi_{M,U}$

$$U \xrightarrow{\Psi^{-1}} E \xrightarrow{\Psi_*} T_M \mathbb{R}^{\nu} \xrightarrow{\hat{\gamma}} \widetilde{\mathcal{U}}_{k,N}.$$

We can now extend  $\Phi_{M,U}$  to a map

$$\Phi = \Phi_{M,U} : (S^{\nu}, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *), \quad \Phi(x) = \begin{cases} \hat{Q}(x) & x \in U \\ * & x \in S^{\nu} \setminus U. \end{cases}$$

We will refer to the map  $\Phi = \Phi_{M,U}$  as the *nice map* associated to a codimension  $k$ -submanifold  $M$  of  $\mathbb{R}^{\nu}$  and a tubular neighborhood  $U$  of that submanifold. Observe that  $\Phi_{M,U}$  is a convenient map, and if  $[\Phi_{M,U}] \in \pi_{n+k}(\widetilde{\mathbf{Th}}_{k,N}, *)$  denotes its homotopy class, then

$$PT([\Phi_{M,U}]) = [M]_+.$$

Thus, the image of the Pontryagin-Thom morphism contains all the cobordism classes of oriented  $n$ -dimensional manifolds that can be embedded in  $\mathbb{R}^{n+k}$ . If  $k \geq n + 1$ , then any smooth compact,  $n$ -dimensional manifold can be embedded in  $\mathbb{R}^{n+k}$  so that the Pontryagin-Thom map gives a surjection

$$PT : \pi_{k+n}(\widetilde{\mathbf{Th}}_{k,N}, *) \rightarrow \Omega_n^+, \quad k \geq n + 1, \quad N \geq n + k.$$

If  $M$  is a compact oriented  $n$ -dimensional submanifold of  $S^{n+k}$  then the homotopy class of the nice map  $\Phi_{M,U}$  is independent of the choice of tubular neighborhood and for this reason we will use the simpler notation  $\Phi_M$  when referring to a nice map. Using Theorem 9.1.4 and the homotopy extension property of a  $CW$ -pair [Hatch1, Chap.0] we deduce immediately the following result.

**Lemma 9.1.6.** *Suppose  $k \geq n + 1$ ,  $N \geq k + n + 1$  and  $f : S^{k+n} \rightarrow \widetilde{\mathbf{Th}}_{k,N}$  is a convenient map. Set  $M := M_f = f^{-1}(\widetilde{\mathbf{Gr}}_{k,N})$ . Then  $f$  has the same homotopy type as a nice map  $\Phi_M$ .  $\square$*

The above result implies that when  $k \geq n + 1$  and  $N \geq k + n + 1$  every homotopy class  $u \in \pi_{n+k}(\widetilde{\mathbf{Th}}_{k,N}, *)$  can be represented by a nice map  $\Phi_M$ , where  $M$  is a compact oriented  $n$ -dimensional submanifold of  $S^{k+n}$  whose image in the cobordism group  $\Omega_n^+$  is  $PT(u)$ . If  $k \geq n + 2$ , then any two embeddings in  $\mathbb{R}^{k+n}$  of a compact oriented  $n$ -dimensional manifold  $M$  are ambiently isotopic, so that the homotopy type of  $\Phi_M$  is independent of the embedding of  $M$ . In other words, if  $k \geq n + 2$  then the homotopy type of  $\Phi_M$  depends only on the oriented diffeomorphism type of  $M$ . In fact, much more is true.

**Proposition 9.1.7.** *Let  $M$  be a smooth, compact, oriented  $n$ -dimensional manifold. If  $k \geq n + 2$ ,  $N \geq n + k + 1$ , and  $M$  defines the trivial element in the oriented cobordism group  $\Omega_n^+$  then the nice map  $\Phi_M : S^{k+n} \rightarrow \widetilde{\mathbf{Th}}_{k,N}$ , is homotopically trivial. In other words, the Pontryagin-Thom morphism  $PT : \pi_{n+k}(\widetilde{\mathbf{Th}}_{k,N}, *) \rightarrow \Omega_n^+$  is an isomorphism if  $k \geq n + 2$  and  $N \geq n + k + 1$ .*

**Proof.** We regard the closed disk  $D^{n+k+1}$  as the one point compactification of the half-space  $H^{n+k+1}$ . Its boundary can be identified with the one-point compactification of the hyperplane  $x_1 = 0$  in  $\mathbb{R}^{k+n+1} \hookrightarrow \mathbb{R}^N$ .

Suppose  $k \geq n + 2$  and that  $W$  is an oriented  $(n + 1)$ -dimensional manifold with boundary such that  $\partial W = M$ . Observe that  $n + k + 1 \geq 2 \dim W + 1$  so Theorem 9.1.3 implies that  $W$  admits a neat embedding in the half-space  $H^{n+k+1} \subset \mathbb{R}^{n+k+1}$ .

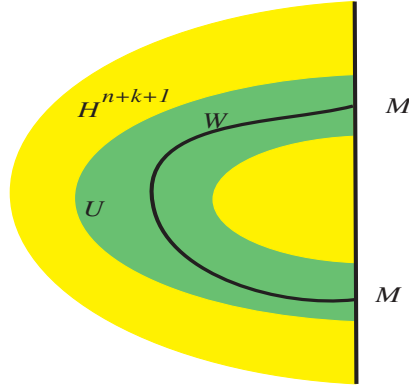
Choose a neat tubular neighborhood  $U$  of  $W$  in  $H^{n+k+1}$  (see Figure 9.2). In other words, we have a tubular neighborhood  $(\widehat{U}, \widehat{E}, \widehat{\Psi})$  of an embedding in the open half-space  $\{x_1 < \varepsilon\}$  of a neck extension  $\widehat{W}$  of  $W$ . Then  $W = \widehat{W} \cap \{x_1 \leq 0\}$  and  $\partial W = \widehat{W} \cap \{x_1 = 0\}$ . We can even choose this embedding so that  $\widehat{W}$  intersects the hyperplane  $\{x_1 = 0\}$  orthogonally. Thus, for every  $x \in \widehat{W} \cap \{x_1 = 0\}$ , the orthogonal complement of  $T_x \widehat{W}$  in  $\mathbb{R}^{n+k+1}$  can be identified with the orthogonal complement of  $T_x \partial W$  in the subspace  $\{x_1 = 0\}$ . We set  $U_0 = \widehat{U} \cap H^{n+k+1}$ .

Imitating the construction of a nice map we obtain a smooth proper map

$$\widehat{U} \cap H^{n+k+1} \rightarrow \widetilde{\mathcal{U}}_{k,N}$$

that maps  $W$  to  $\widetilde{\mathbf{Gr}}_{k,N}$ , the zero section of  $\widetilde{\mathcal{U}}_{k,N}$ . This determines a continuous map between one-point compactifications

$$(D^{n+k+1}, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$$



**Figure 9.2.** A neat embedding of the bordism  $W$  in a half-space.

which extends the nice map  $\Phi_{M,U_0} = \Phi_{\partial W,U_0} : (S^{n+k}, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$ . Thus, the nice map  $\Phi_M$  is homotopically trivial. □

## 9.2. The cohomology of the Grassmannians of oriented subspaces

To compute the homotopy groups  $\pi_{n+k}(\widetilde{\mathbf{Th}}_{k,N})$  modulo torsion we want to invoke the following deep result of Cartan-Serre. Its proof is based on the award winning work of Serre on the homotopy theory of loop spaces and Serre classes of Abelian groups. For a particularly readable presentation of the basic facts of this theory we refer to [DFN-vol.3].

**Theorem 9.2.1** (Cartan-Serre). *Suppose  $X$  is a connected CW-complex and  $n \geq 2$  is an integer such that*

$$\pi_k(X) = 0, \quad \forall k < n.$$

*Then the natural Hurewicz morphism*

$$\pi_q(X) \otimes \mathbb{Q} \rightarrow H_q(X, \mathbb{Q})$$

*is an isomorphism for any  $q < 2n - 1$ .* □

The next result is an immediate consequence of the Cartan-Serre theorem and some elementary transversality results.

**Proposition 9.2.2.** *We have*

$$\pi_q(\widetilde{\mathbf{Th}}_{k,N}, *) = 0, \quad \forall q < k. \quad (9.2.1)$$

*and*

$$\pi_q(\widetilde{\mathbf{Th}}_{k,H}, *) \otimes \mathbb{Q} \cong H^{q-k}(\widetilde{\mathbf{Gr}}_{k,N}, \mathbb{Q}), \quad \forall q \leq 2k - 2. \quad (9.2.2)$$

**Proof.** Suppose  $f : (S^q, \infty) \rightarrow (\widetilde{\mathbf{Th}}_{k,N}, *)$  is a continuous map. Fix a neighborhood  $\mathcal{O}$  of  $*$  in  $\widetilde{\mathbf{Th}}_{k,N}$ . We can assume that  $f$  is smooth outside  $f^{-1}(\mathcal{O})$  and that the restriction of  $f$  to  $S^q \setminus f^{-1}(\mathcal{O})$  is transversal to  $\widetilde{\mathbf{Gr}}_{k,N}$ , the zero section of  $\widetilde{\mathcal{U}}_{k,\nu}$ . Since  $\widetilde{\mathbf{Gr}}_{k,N}$  has codimension  $k$  in  $\widetilde{\mathcal{U}}_{k,\nu}$  we deduce

from the transversality of  $f$  that  $f(S^q)$  does not intersect  $\widetilde{\mathbf{Gr}}_{k,N}$  if  $q < k$ . Thus, if  $q < k$  any homotopy class  $u \in \pi_q(\widetilde{\mathbf{Th}}_{k,N}, *)$  can be represented by a map into the contractible set  $\widetilde{\mathbf{Th}}_{k,N} \setminus \widetilde{\mathbf{Gr}}_{k,N}$ . This proves (9.2.1).

Observe that for  $q > 0$  we have

$$H_q(\widetilde{\mathbf{Th}}_{k,N}, \mathbb{Q}) \cong H_q(\widetilde{\mathbf{Th}}_{k,N}, *; \mathbb{Q}) \cong H_q(D(\tilde{\mathcal{U}}_{k,N}), \partial D(\tilde{\mathcal{U}}_{k,N}); \mathbb{Q}).$$

On the other hand, using the homological Thom isomorphism we conclude that

$$H_q(D(\tilde{\mathcal{U}}_{k,N}), \partial D(\tilde{\mathcal{U}}_{k,N}); \mathbb{Q}) \cong H_{q-k}(\widetilde{\mathbf{Gr}}_{k,N}, \mathbb{Q}).$$

Finally, using the universal coefficients theorem we deduce

$$H_{q-k}(\widetilde{\mathbf{Gr}}_{k,N}, \mathbb{Q}) \cong H^{q-k}(\widetilde{\mathbf{Gr}}_{k,N}, \mathbb{Q}).$$

The equality (9.2.2) follows from (9.2.1) and the Cartan-Serre theorem.  $\square$

**Corollary 9.2.3.** *If  $k \geq n + 2$  and  $N \geq n + k + 1$  then*

$$\Omega_n^+ \otimes \mathbb{Q} \cong H^n(\widetilde{\mathbf{Gr}}_{k,N}, \mathbb{Q}).$$

**Proof.** For  $k \geq n + 2$  we have  $n + k \leq 2k - 2$  so that Proposition 9.1.7 implies

$$\Omega_n^+ \otimes \mathbb{Q} \cong \pi_{n+k}(\widetilde{\mathbf{Th}}_{k,N}, *) \otimes \mathbb{Q} \stackrel{(9.2.2)}{\cong} H^n(\widetilde{\mathbf{Gr}}_{k,N}, \mathbb{Q}).$$

$\square$

To complete the proof of the cobordism theorem (Theorem 7.2.4) it suffices to compute the rational cohomology groups  $H^n(\widetilde{\mathbf{Gr}}_{k,N}, \mathbb{Q})$ , for any  $n$  and some choices of  $N$ , and  $k$  such that  $N \geq n + k + 1 \geq 2n + 3$ . We will achieve this by an induction over  $k$  aided by the Gysin theorem.

Let us first observe that the map that associates to a subspace of  $\mathbb{R}^N$  its orthogonal complement induces a diffeomorphism  $\widetilde{\mathbf{Gr}}_{k,N} \rightarrow \widetilde{\mathbf{Gr}}_{N-k,N}$ . To emphasize this symmetry we will use the notation

$$\Gamma_{k,m} := \widetilde{\mathbf{Gr}}_{k,k+m} \cong \widetilde{\mathbf{Gr}}_{m,m+k} =: \Gamma_{m,k}.$$

The conditions  $N \geq k + n + 1 \geq 2n + 3$  become  $m \geq n + 1$ ,  $k \geq n + 2$ . In particular, these conditions are satisfied when  $m \geq k \geq n + 2$ .

We will denote by  $\mathcal{T}_k$  or  $\mathcal{T}_{k,m}$  the tautological vector bundle over  $\Gamma_{k,m}$ . To formulate our next result we need to introduce a certain graded ring  $\mathcal{R}_k$ .

If  $k$  is odd,  $k = 2\ell + 1$ , then  $\mathcal{R}_k$  is the quotient of the polynomial ring  $\mathbb{Q}[x_1, \dots, x_\ell, u]$ ,  $\deg x_i = 4i$ ,  $\deg u = k$ , modulo the ideal generated homogeneous polynomial  $u$ . In other words,

$$\mathcal{R}_{2\ell+1} = \mathbb{Q}[x_1, \dots, x_\ell]$$

If  $k$  is even,  $k = 2\ell$ , then  $\mathcal{R}_k$  is the quotient of the polynomial ring  $\mathbb{Q}[x_1, \dots, x_\ell, u]$ ,  $\deg x_i = 4i$ ,  $\deg u = k$ , modulo the homogeneous ideal generated by the homogeneous polynomial  $u^2 - x_\ell$ . In this case any homogeneous element  $f$  of degree  $q$  of  $\mathcal{R}_k$  can be expressed uniquely in the form

$$f = A_q(x_1, \dots, x_\ell) + B_q(x_1, \dots, x_\ell)u,$$

where  $A_q$  and  $B_{q-k}$  are homogeneous polynomials of degrees  $q$  and respectively  $q - k$  in the variables  $x_i$ ,  $1 \leq i \leq \ell$ .



If  $E \rightarrow X$  is an oriented rank  $k$  real vector bundle over the topological space  $X$  then the correspondences

$$x_i \mapsto p_i(E) \in H^{4i}(X), \quad 1 \leq i \leq \ell, \quad u \mapsto e(E) \in H^k(X, \mathbb{Q})$$

induce a morphism of graded rings

$$\Upsilon_E : \mathcal{R}_k \rightarrow H^\bullet(X, \mathbb{Q}).$$

(When the rank  $k$  is odd then, according to Proposition 4.3.3, the rational Euler class is trivial. When the rank  $k$  is even  $r = 2\ell$  then Proposition 6.1.2 implies that  $e(E)^2 = p_\ell(E)$ .)

**Proposition 9.2.4.** *Let  $k, m$  be positive integers such that  $m \geq k$ . Denote by  $\Upsilon_k$  the ring morphism*

$$\Upsilon_k = \Upsilon_{\mathcal{T}_k} : \mathcal{R}_k \rightarrow H^\bullet(\Gamma_{k,m}, \mathbb{Q}).$$

*Then  $\Upsilon_k$  induces isomorphisms in degrees  $< m$ .*

**Proof.** We prove this by induction on  $k$ . The result is obviously true for  $k = 1$  since  $\Gamma_{1,m} \cong S^m$ .

Consider the sphere bundle  $\Sigma_{k,m} = S(\mathcal{T}_k)$  associated to the tautological vector bundle  $\mathcal{T}_k$ . A point in  $\Sigma_{k,m}$  is a unit vector  $\vec{v}$  in some oriented  $k$ -dimensional subspace  $F \subset \mathbb{R}^{k+m}$ . This defines a codimension one oriented subspace  $F_{\vec{v}} \subset F$  consisting of all the vectors orthogonal to  $\vec{v}$ . The correspondence  $\vec{v} \rightarrow F_{\vec{v}}$  defines a smooth map

$$\rho : \Sigma_{k,m} \rightarrow \Gamma_{k-1,m+1}.$$

This is a fiber bundle, and in fact it is the sphere bundle associated to the tautological vector bundle

$$\mathcal{T}_{m+1,k-1} \rightarrow \Gamma_{m+1,k-1} = \Gamma_{k-1,m+1}.$$

The fibers of  $\rho$  are spheres of dimension  $m$ . We thus obtain a double fibration

$$\begin{array}{ccccc} S^{k-1} & \hookrightarrow & \Sigma_{k,m} & \hookrightarrow & S^m \\ & \searrow \pi & & \searrow \rho & \\ & & \Gamma_{k,m} & & \Gamma_{k-1,m+1} \end{array}$$

From the long exact homotopy sequence of a fibration we deduce that  $\rho$  induces isomorphisms

$$\pi_q(\Sigma_{k,m}) \rightarrow \pi_q(\Gamma_{k-1,m+1}), \quad \forall q < m.$$

and an epimorphism  $\pi_m(\Sigma_{k,m}) \rightarrow \pi_m(\Gamma_{k-1,m+1})$ . Whitehead's theorem [Spa, Chap.7, Sec. 5] implies that  $\rho$  induces isomorphisms in cohomology

$$\rho^* : H^q(\Sigma_{k,m}) \rightarrow H^q(\Gamma_{k-1,m+1}), \quad \forall q < m.$$

Along  $\Sigma_{k,m}$  we have the isomorphism of vector bundles  $\pi^*\mathcal{T}_k = \rho^*\mathcal{T}_{k-1} \oplus \mathbb{R}$ , so that

$$\pi^*p_i(k) = \rho^*p_i(k-1). \quad (9.2.3)$$

We need to discuss separately the two cases  $k$  even, and  $k$  odd.

**1.**  $k \equiv 1 \pmod{2}$ ,  $k = 2\ell + 1$ . Let  $L \in \Gamma_{k,m}$  and denote by  $\Sigma_{k,m}(L)$  the fiber of  $\pi$  over  $L$ . This fiber is a  $(k-1)$ -sphere and the restriction of  $\rho^*\mathcal{T}_{k-1}$  to  $\Sigma_{k,m}(L)$  coincides with the tangent bundle of a  $(k-1)$ -sphere. Since the Euler class of the tangent bundle of an even dimensional sphere is non zero, we deduce that for every fiber  $\Sigma_{k,m}(L)$  of  $\pi$  the restriction of 1 and  $\rho^*e(k-1)$  to the fiber form a basis of the rational cohomology of this fiber. From the Leray-Hirsch theorem we deduce that

$$H^\bullet(\Sigma_{k,\ell}, \mathbb{Q}) \text{ is a free } H^\bullet(\Gamma_{k,\ell}, \mathbb{Q})\text{-module with 1 and } \rho^*e(k-1) \text{ as basis.} \quad (9.2.4)$$

On the other hand, the induction assumption and the injectivity of  $\rho^*$  imply that if  $q < m$  then that any cohomology class  $\alpha \in H^q(\Sigma_{k,m})$  can be written uniquely in the form

$$\alpha = A_q(\rho^* p_i(k-1)) + B_{q-k}(\rho^* p_i(k-1))e(k-1),$$

where  $A_q$  and respectively  $B_{q-k}$  are homogeneous polynomials of degree  $q$  and respectively  $q-k$  in the variables  $x_i$ . Using (9.2.3) and (9.2.4) we deduce that any cohomology class in  $H^q(\Gamma_{k,m}, \mathbb{Q})$ ,  $q < m$ , can be written uniquely as a homogeneous polynomial of degree  $q$  in the Pontryagin classes of  $\mathcal{T}_k$ .

2.  $k \equiv 0 \pmod{2}$ ,  $k = 2\ell$ . We now use the Gysin sequence of the sphere bundle  $\Sigma_{k,m} \xrightarrow{\pi} \Gamma_{k,m}$ .

$$\cdots \longrightarrow H^{q-k}(\Gamma_{k,m}) \xrightarrow{e^{(k)} \cup} H^q(\Gamma_{k,m}) \xrightarrow{\pi^*} H^q(\Sigma_{k,m}) \rightarrow H^{q+1-k}(\Gamma_{k,m}) \rightarrow \cdots,$$

Let us observe that if  $q < m$ , then the induction assumption implies that any  $\alpha \in H^q(\Sigma_{k,m})$  is a homogeneous polynomial  $A_q$  of degree  $q$  in the Pontryagin classes  $\rho^* p_i(k-1)$ ,  $1 \leq i \leq \ell-1$ , of  $\rho^* \mathcal{T}_{k-1}$ . Hence

$$\alpha = A_q(\rho^* p_i(k-1)) \stackrel{(9.2.3)}{=} A_q(\pi^* p_i(k)) = \pi^* A_q(p_i(k)),$$

so that  $\pi^* : H^q(\Gamma_{k,m}) \rightarrow H^q(\Sigma_{k,m})$  is surjective. We obtain short exact sequences

$$0 \rightarrow H^{q-k}(\Gamma_{k,m}) \xrightarrow{e^{(k)} \cup} H^q(\Gamma_{k,m}) \xrightarrow{\pi^*} H^q(\Sigma_{k,m}) \rightarrow 0, \quad q < m.$$

This mirrors the short exact sequence

$$0 \rightarrow \mathcal{R}_k^{(q-k)} \xrightarrow{u} \mathcal{R}_k^{(q)} \rightarrow \mathbb{Q}[x_1, \dots, x_{\ell-1}]^{(q)} = \mathcal{R}_{k-1}^{(q)} \rightarrow 0,$$

where for any graded ring  $R$  we denote by  $R^{(q)}$  the additive subgroup consisting of homogeneous elements of degree  $q$ . Arguing inductively on  $q$  using the isomorphisms

$$\Upsilon_{\pi^* \mathcal{T}_k} : \mathcal{R}_{k-1}^{(q)} \rightarrow H^q(\Sigma_{k,m}), \quad q < m,$$

we deduce that  $H^q(\Gamma_{k,m}) \cong \mathcal{R}_k^{(q)}$ . □

Thom's cobordism theorem is now an easy consequence of the above computation.

**Corollary 9.2.5.**

$$\dim_{\mathbb{Q}} \Omega_n^+ \otimes \mathbb{Q} = \begin{cases} 0, & n \not\equiv 0 \pmod{4} \\ p(\ell), & n = 4\ell, \end{cases}$$

where we recall that  $p(\ell)$  denotes the number of partitions of  $\ell$ .

**Proof.** For any integers  $k, n, N$  such that  $k \geq n+2$ ,  $N \geq n+k$  we have

$$\dim_{\mathbb{Q}} \Omega_n^+ \otimes \mathbb{Q} = \dim_{\mathbb{Q}} H^n(\widetilde{\mathbf{Gr}}_{k,N}, \mathbb{Q}) = \dim_{\mathbb{Q}} H^n(\Gamma_{k,m}, \mathbb{Q}), \quad m \geq k.$$

Hence  $n < m$  and Proposition 9.2.4 implies

$$\dim_{\mathbb{Q}} \Omega_n^+ = \dim_{\mathbb{Q}} \mathcal{R}_k^{(n)} = \begin{cases} 0, & n \not\equiv 0 \pmod{4} \\ p(\ell), & n = 4\ell. \end{cases}$$

□

# The Chern-Weil construction

In this last chapter we would like to present a differential geometric construction of the Chern classes. The procedure is known as the *Chern-Weil construction*. It is applicable only to smooth vector bundles over smooth manifolds, and it recovers the Chern classes modulo torsion. This procedure has proved to be useful in many geometric problems.

## 10.1. Principal bundles

Fix a Lie group  $G$ . For simplicity, we will assume that it is a matrix Lie group<sup>1</sup>, i.e. it is a closed subgroup of some  $\mathrm{GL}_n(\mathbb{K})$ . A *principal  $G$ -bundle over a smooth manifold  $B$*  is a triple  $(P, \pi, B)$  satisfying the following conditions.

- The map  $P \xrightarrow{\pi} B$  is a surjective submersion. We set  $P_b := \pi^{-1}b$ .
- There is a right free action

$$P \times G \rightarrow P, \quad (p, g) \mapsto pg$$

such that for every  $p \in P$  the  $G$ -orbit containing  $p$  coincides with the fiber of  $\pi$  containing  $p$ .

- The map  $\pi$  is locally trivial, i.e., every point  $b \in B$  has an open neighborhood  $U$  and a diffeomorphism  $\Psi_U : \pi^{-1}(U) \rightarrow G \times U$  such that the diagram below is commutative

$$\begin{array}{ccc}
 \pi^{-1}(U) & \xrightarrow{\Psi} & G \times U \\
 \pi \searrow & & \swarrow \text{proj} \\
 & U &
 \end{array}$$

and

$$\Psi(pg) = \Psi(p)g, \quad \forall p \in \pi^{-1}(U), \quad g \in G,$$

---

<sup>1</sup>Any compact Lie group is a matrix Lie group

where the right action of  $G$  on  $G \times U$ .

Any principal bundle can be obtained by gluing trivial ones. Suppose we are given an open cover  $\mathcal{U} = (U_\alpha)_{\alpha \in A}$  of  $M$  and for every  $\alpha, \beta \in A$  smooth maps

$$g_{\alpha\beta} : U_{\alpha\beta} \rightarrow G$$

satisfying the cocycle condition

$$g_{\gamma\alpha}(u) = g_{\gamma\beta}(u) \cdot g_{\beta\alpha}(u), \quad \forall u \in U_{\alpha\beta\gamma}$$

Then, exactly as in the case of vector bundles we can obtain a principal bundle by gluing the trivial bundles  $P_\alpha = G \times U_\alpha$ . More precisely we consider the disjoint union

$$X = \bigcup_{\alpha} P_\alpha \times \{\alpha\}$$

and the equivalence relation

$$G \times U_\alpha \times \{\alpha\} \ni (g, u, \alpha) \sim (h, v, \beta) \in G \times U_\beta \times \{\beta\} \iff u = v \in U_{\alpha\beta}, \quad h = g_{\beta\alpha}(u)g.$$

Then  $P = X / \sim$  is the total space of a principal  $G$ -bundle. We will denote this bundle by  $(B, \mathcal{U}, g_{\bullet\bullet}, G)$ .

**Example 10.1.1** (Fundamental example). Suppose  $E \rightarrow M$  is a  $\mathbb{K}$ -vector bundle over  $M$  of rank  $r$ , described by the gluing data  $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{V})$ , where  $\mathbb{V}$  is a  $r$ -dimensional  $\mathbb{K}$ -vector space, and  $g_{\beta\alpha} : U_{\alpha\beta} \rightarrow \text{GL}(\mathbb{V})$  is a  $\text{GL}(\mathbb{V})$ -valued gluing cocycle. In particular, for every  $U \in \mathcal{U}$ , there exists a trivialization  $\Psi_U : E|_U \rightarrow \underline{\mathbb{V}}_U$  such that  $g_{U'U} = \Psi_{U'} \cdot \Psi_U^{-1}$ .

A *frame* of  $\mathbb{V}$  is by definition an ordered basis  $\underline{e} = (e_1, \dots, e_r)$  of  $\mathbb{V}$ . We denote by  $\mathbf{Fr}(\mathbb{V})$  the set of frames of  $\mathbb{V}$ . We have a free and transitive *left* action

$$\text{GL}(\mathbb{V}) \times \mathbf{Fr}(\mathbb{V}), \quad g \cdot (e_1, \dots, e_r) = (ge_1, \dots, ge_r).$$

We also have a free and transitive *right* action

$$\mathbf{Fr}(\mathbb{V}) \times \text{GL}_r(\mathbb{K}) \rightarrow \mathbf{Fr}(\mathbb{V}), \quad (e_1, \dots, e_r) \cdot A = \left( \sum_i a_{1i}^i e_i, \dots, \sum_i a_{ri}^i e_i \right),$$

$$\forall A = [a_{ij}^i]_{1 \leq i, j \leq r} \in \text{GL}_r(\mathbb{K}), \quad (e_1, \dots, e_r) \in \mathbf{Fr}(\mathbb{V}).$$

Note that the left action of  $\text{GL}(\mathbb{V})$  commutes with the right action of  $\text{GL}_r(\mathbb{K})$ . The set of frames is naturally a smooth manifold diffeomorphic to  $\text{GL}_r(\mathbb{K})$ .

To the bundle  $E$  we associate the fiber bundle  $\mathbf{Fr}(E)$  obtained from the disjoint union

$$\bigsqcup_{\alpha} \mathbf{Fr}(\mathbb{V}) \times U_\alpha$$

via the equivalence relation

$$\mathbf{Fr}(\mathbb{V}) \times U_\alpha \ni (\underline{e}(\alpha), x_\alpha) \sim (\underline{e}(\beta), x_\beta) \in \mathbf{Fr}(\mathbb{V}) \times U_\beta$$

$$\iff x_\alpha = x_\beta = x, \quad \underline{e}(\beta) = g_{\beta\alpha}(x)\underline{e}(\alpha).$$

The right action of  $\text{GL}_r(\mathbb{K})$  on  $\mathbf{Fr}(\mathbb{V})$  makes this bundle a principal  $\text{GL}_r(\mathbb{K})$ -bundle. For any  $U \in \mathcal{U}$ , and any  $m \in U$ , the fiber of this bundle over  $m \in U$  can be identified with the space  $\mathbf{Fr}(E_m)$  of frames in the fiber  $E_m$  via the local trivialization  $\Psi_U : E|_U \rightarrow \underline{\mathbb{V}}_U$ .  $\square$

To any principal bundle  $P = (B, \mathcal{U}, g_{\bullet\bullet}, G)$  and representation  $\rho : G \rightarrow \text{Aut}_{\mathbb{K}}(\mathbb{V})$  of  $G$  on a finite dimensional  $\mathbb{K}$ -vector space  $\mathbb{V}$  we can associate a vector bundle  $E(\mathcal{U}, \rho(g_{\bullet\bullet}))$ . We will denote it by  $P \times_{\rho} \mathbb{V}$ . Equivalently,  $P \times_{\rho} \mathbb{V}$  is the quotient of  $P \times \mathbb{V}$  via the *left*  $G$ -action

$$g(p, v) = (pg^{-1}, \rho(g)v).$$

A vector bundle  $E$  on a smooth manifold  $M$  is said to have  $(G, \rho)$ -*structure* if  $E \cong P \times_{\rho} \mathbb{V}$  for some principal  $G$ -bundle  $P$ .

We denote by  $\mathfrak{g} = T_1G$  the Lie algebra of  $G$ . We have an *adjoint representation*

$$\text{Ad} : G \rightarrow \text{End } \mathfrak{g}, \quad \text{Ad}(g)X = gXg^{-1} = \left. \frac{d}{dt} \right|_{t=0} g \exp(tX)g^{-1}, \quad \text{for all } g \in G.$$

The associated vector bundle  $P \times_{\text{Ad}} \mathfrak{g}$  is denoted by  $\text{Ad}(P)$ .

*In the sequel we will denote by  $\underline{u}(n)$ ,  $\underline{so}(n)$ ,  $\underline{o}(n)$  and respectively  $\underline{gl}_n(\mathbb{K})$  the Lie algebra of the matrix Lie groups  $U(n)$ ,  $SO(n)$ ,  $O(n)$  and respectively  $\text{GL}_n(\mathbb{K})$ .*

For any representation  $\rho : G \rightarrow \text{Aut}(V)$  we denote by  $\rho_*$  the differential of  $\rho$  at 1

$$\rho_* : \mathfrak{g} \rightarrow \text{End}(V).$$

Observe that for every  $X \in \mathfrak{g}$  we have

$$\rho_*(\text{Ad}(g)X) = \rho_*(gXg^{-1}) = \rho(g)(\rho_*X)\rho(g)^{-1}. \quad (10.1.1)$$

If we set  $\text{End}_{\rho}(V) := \rho_*(\mathfrak{g}) \subset \text{End}(V)$  we have an induced action

$$\text{Ad}_{\rho} : G \rightarrow \text{End}_{\rho}(V), \quad \text{Ad}_{\rho}(g)T := \rho(g)T\rho(g)^{-1}, \quad \forall T \in \text{End } V, g \in G.$$

If  $E = P \times_{\rho} \mathbb{V}$  then we set

$$\text{End}_{\rho}(E) := P \times_{\text{Ad}_{\rho}} \text{End}_{\rho}(V).$$

This bundle can be viewed as the bundle of infinitesimal symmetries of  $E$ .

**Example 10.1.2.** (a) Suppose  $G$  is a Lie subgroup of  $\text{GL}_m(\mathbb{K})$ . It has a tautological representation

$$\tau : G \hookrightarrow \text{GL}_m(\mathbb{K}) = \text{Aut}(\mathbb{K}^m).$$

A rank  $m$   $\mathbb{K}$ -vector bundle  $E \rightarrow M$  is said to have  $G$ -*structure* if it has a  $(G, \tau)$ -structure. This means that  $E$  can be described by a gluing cocycle  $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^m)$  with the property that the matrices  $g_{\bullet\bullet}$  belong to the subgroup  $G$ .

For example,  $SO(m), O(m) \subset \text{GL}_m(\mathbb{R})$  and we can speak of  $SO(m)$  and  $O(m)$  structures on a real vector bundle of rank  $m$ . Similarly we can speak of  $U(m)$  and  $SU(m)$  structures on a complex vector bundle of rank  $m$ .

A hermitian metric on a rank  $r$  complex vector bundle defines a  $U(r)$ -structure on  $E$ , i.e.,  $E = P \times_{\rho} \mathbb{C}^r$ , where  $P$  is the principal  $U(r)$ -bundle of unitary frames of  $E$ . In this case

$$\text{Ad } P = \text{End}_{\rho}(E) = \text{End}_h^-(E). \quad \square$$

## 10.2. Connections on vector bundles

Roughly speaking, a connection on a smooth vector bundle is a "coherent procedure" of differentiating the smooth sections. In the sequel, if  $E \rightarrow M$  is a smooth vector bundle over the smooth manifold  $M$  we denote by  $\Omega^k(M)$  the space of smooth sections of the vector bundle  $\Lambda^k T^*M \otimes_{\mathbb{R}} E$ . We will refer to the elements of  $\Omega^k(M)$  as *E-valued k-forms on M*.

**Definition 10.2.1.** Suppose  $E \rightarrow M$  is a  $\mathbb{K}$ -vector bundle. A *smooth connection* on  $E$  is a  $\mathbb{K}$ -linear operator

$$\nabla : C^\infty(E) \rightarrow C^\infty(T^*M \otimes E)$$

satisfying the product rule

$$\nabla(fs) = s \otimes df + f\nabla s, \quad \forall f \in C^\infty(M), \quad s \in C^\infty(E).$$

We say that  $\nabla s$  is the covariant derivative of  $s$  with respect to  $\nabla$ . We will denote by  $\mathcal{A}_E$  the space of smooth connections on  $E$ .  $\square$

**Remark 10.2.2.** (a) For every section  $s$  of  $E$  the covariant derivative  $\nabla s$  is a section of  $T^*M \otimes E \cong \text{Hom}(TM, E)$ . i.e.

$$\nabla s \in \underline{\text{Hom}}(TM, E).$$

As such,  $\nabla s$  associates to each vector field  $X$  on  $M$  a section of  $E$  which we denote by  $\nabla_X s$ . We say that  $\nabla_X s$  is the derivative of  $s$  in along the vector field  $X$  with respect to the connection  $\nabla$ . The product rule can be rewritten

$$\nabla_X(fs) = (L_X f)s + f\nabla s, \quad \forall X \in \text{Vect}(M), \quad f \in C^\infty(M), \quad s \in C^\infty(M),$$

where  $L_X f$  denotes the Lie derivative of  $f$  along the vector field  $X$ .

(b) Suppose  $E, F \rightarrow M$  are vector bundles and  $\Psi : E \rightarrow F$  is a bundle isomorphism. If  $\nabla$  is a connection of  $E$  then  $\Psi\nabla\Psi^{-1}$  is a connection on  $F$ .

(c) Suppose  $\nabla^0$  and  $\nabla^1$  are two connections on  $E$ . Set

$$A := \nabla^1 - \nabla^0 : C^\infty(E) \rightarrow C^\infty(T^*M \otimes E).$$

Observe that for every  $f \in C^\infty(M)$  and every  $s \in C^\infty(E)$  we have

$$A(fs) = fA(s)$$

so that

$$\begin{aligned} A &\in \underline{\text{Hom}}(E, T^*M \otimes E) \cong C^\infty(E^* \otimes T^*M \otimes E) \cong C^\infty(T^*M \otimes E^* \otimes E) \\ &\cong C^\infty(T^*M, \text{End}(E)) = \Omega^1(\text{End}(E)), \quad \text{End } E := E^* \otimes E. \end{aligned}$$

In other words, the difference between two connections is a  $\text{End } E$ -valued 1-form. Conversely, if

$$A \in \Omega^1(\text{End } E) \cong \underline{\text{Hom}}(TM \otimes E, E)$$

then for every connection  $\nabla$  on  $E$  the sum  $\nabla + A$  is a gain a connection on  $E$ . This shows that the space  $\mathcal{A}_E$ , if nonempty, is an affine space modelled by the vector space  $\Omega^1(\text{End } E)$ .  $\square$

**Example 10.2.3.** (a) Consider the trivial bundle  $\mathbb{R}_M$ . The sections of  $\mathbb{R}_M$  are smooth functions  $M \rightarrow \mathbb{R}$ . The differential

$$d : C^\infty(M) \rightarrow \Omega^1(M), \quad f \mapsto df$$

is a connection on  $\mathbb{R}_M$  called the *trivial connection*.

Observe that  $\text{End}(\mathbb{R}_M) \cong \mathbb{R}_M$  so that any other connection on  $M$  has the form

$$\nabla = d + a, \quad a \in \Omega^1(\mathbb{R}_M) = \Omega^1(M).$$

(b) Consider similarly the trivial bundle  $\mathbb{K}_M^r$ . Its smooth sections are  $r$ -uples of smooth functions

$$s = \begin{bmatrix} s^1 \\ \vdots \\ s^r \end{bmatrix} : M \rightarrow \mathbb{K}^r.$$

$\mathbb{K}^r$  is equipped with a *trivial connection*  $\nabla^0$  defined by

$$\nabla^0 \begin{bmatrix} s^1 \\ \vdots \\ s^r \end{bmatrix} = \begin{bmatrix} ds^1 \\ \vdots \\ ds^r \end{bmatrix}.$$

Any other connection on  $\mathbb{K}^r$  has the form

$$\nabla = \nabla^0 + A, \quad A \in \Omega^1(\text{End } \mathbb{K}^r).$$

More concretely,  $A$  is an  $r \times r$  matrix  $[A_b^a]_{1 \leq a, b \leq r}$ , where each entry  $A_b^a$  is a  $\mathbb{K}$ -valued 1-form. If we choose local coordinates  $(x^1, \dots, x^n)$  on  $M$  then we can describe  $A_j^i$  locally as

$$A_b^a = \sum_k A_{kb}^a dx^k.$$

We have

$$\nabla s = \begin{bmatrix} ds^1 \\ \vdots \\ ds^r \end{bmatrix} + \begin{bmatrix} \sum_b A_b^1 s^b \\ \vdots \\ \sum_b A_b^r s^b \end{bmatrix}.$$

(c) Suppose  $E \rightarrow B$  is a  $\mathbb{K}$ -vector bundle of rank  $r$  and  $\underline{e} = (e_1, \dots, e_r)$  is a local frame of  $E$  over the open set  $U$ . Suppose  $\nabla$  is a connection on  $E$ . Then for every  $1 \leq b \leq r$  we get section  $\nabla e_b$  of  $T^*M \otimes E$  over  $U$  and thus decompositions

$$\nabla e_b = \sum_a A_b^a e_a, \quad A_b^a \in \Omega^1(U), \quad \forall 1 \leq a, b \leq r. \quad (10.2.1)$$

Given a section  $s = \sum_b s^b e_b$  of  $E$  over  $U$  we have

$$\nabla s = \sum_b ds^b e_b + \sum_b s_b \sum_a A_b^a e_a = \sum_a \left( ds^a + \sum_b A_b^a s^b \right) e_a.$$

This shows that the action of  $\nabla$  on any section over  $U$  is completely determined by the action of  $\nabla$  on the local frame, i.e., by the matrix  $(A_b^a)$ . We can regard this as a 1-form whose entries are  $r \times r$  matrices. This is known as the *connection 1-form* associated to  $\nabla$  by the local frame  $\underline{e}$ . We will denote it by  $A(\underline{e})$ . We can rewrite (10.2.1) as

$$\nabla(\underline{e}) = \underline{e} \cdot A(\underline{e}).$$

Suppose  $\underline{f} = (f_1, \dots, f_r)$  is another local frames of  $E$  over  $U$  related to  $\underline{e}$  by the equalities

$$f_a = \sum_b e_b g_a^b, \quad (10.2.2)$$

where  $U \ni u \mapsto g(u) = (g_a^b(u))_{1 \leq a, b \leq r} \in \mathrm{GL}_r(\mathbb{K})$  is a smooth map. We can rewrite (10.2.2) as

$$\underline{f} = \underline{e} \cdot g.$$

Then  $A(\underline{f})$  is related to  $A(\underline{e})$  by the equality

$$A(\underline{f}) = g^{-1} A(\underline{e})g + g^{-1} dg. \quad (10.2.3)$$

Indeed

$$\underline{f} \cdot A(\underline{f}) = \nabla(\underline{f}) = \nabla(\underline{e}g) = (\nabla(\underline{e}))g + \underline{e}dg = (\underline{e}A(\underline{e}))g + \underline{f}g^{-1}dg = \underline{f}(g^{-1}A(\underline{e})g + g^{-1}dg).$$

Suppose now that  $E$  is given by the gluing cocycle  $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^r)$ . Then the canonical basis of  $\mathbb{K}^r$  induces via the natural isomorphism  $\mathbb{K}_{U_\alpha}^r \rightarrow E|_{U_\alpha}$  a local frame  $\underline{e}(\alpha)$  of  $E|_{U_\alpha}$ . We set

$$A_\alpha = A(\underline{e}(\alpha)).$$

On the overlap  $U_{\alpha\beta}$  we have the equality  $\underline{e}(\alpha) = \underline{e}(\beta)g_{\beta\alpha}$  so that on these overlaps the  $gl_r(\mathbb{K})$ -valued 1-forms  $A_\alpha$  satisfy the transition formulæ

$$A_\alpha = g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1} dg_{\beta\alpha} \iff A_\beta = g_{\beta\alpha} A_\alpha g_{\beta\alpha}^{-1} - (dg_{\beta\alpha})g_{\beta\alpha}^{-1}. \quad (10.2.4)$$

□

**Proposition 10.2.4.** *Suppose  $E$  is a rank  $r$  vector bundle over  $M$  described by the gluing cocycle  $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^r)$ . Then a collection of 1-forms*

$$A_\alpha = \Omega^1(U_\alpha) \otimes gl_r(\mathbb{K}).$$

*satisfying the gluing conditions (10.2.4) determine a connection on  $E$ .*

□

**Proposition 10.2.5.** *Suppose  $E \rightarrow M$  is a smooth vector bundle. Then there exist connections on  $E$ , i.e.,  $\mathcal{A}_E \neq \emptyset$ .*

**Proof.** Suppose that  $E$  is described by the gluing cocycle  $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^r)$ ,  $r = \mathrm{rank}(E)$ .

Denote by  $\Psi_\alpha : \mathbb{K}_{U_\alpha}^r \rightarrow E|_{U_\alpha}$  the local trivialization over  $U_\alpha$  and by  $\nabla^\alpha$  the trivial connection on  $\mathbb{K}_{U_\alpha}^r$ . Set

$$\hat{\nabla}^\alpha := \Psi_\alpha \nabla^\alpha \Psi_\alpha^{-1}.$$

Then (see Remark 10.2.2(b))  $\hat{\nabla}^\alpha$  is a connection on  $E|_{U_\alpha}$ . Fix a partition of unity  $(\theta_\alpha)$  subordinated to  $(U_\alpha)$ . Observe that for every  $\alpha$  and every  $s \in C^\infty(E)$   $\theta_\alpha s$  is a section of  $E$  with support in  $U_\alpha$ . In particular  $\hat{\nabla}^\alpha(\theta_\alpha s)$  is a section of  $T^*M \otimes E$  with support in  $U_\alpha$ . Set

$$\nabla s = \sum_{\alpha, \beta} \theta_\beta \hat{\nabla}^\alpha(\theta_\alpha s)$$

If  $f \in C^\infty(M)$  then

$$\nabla(fs) = \sum_{\alpha, \beta} \theta_\beta \hat{\nabla}^\alpha(\theta_\alpha fs) = \sum_{\beta} \theta_\beta \left( \sum_{\alpha} df \otimes (\theta_\alpha s) + f \hat{\nabla}^\alpha(\theta_\alpha s) \right)$$



$$= df \otimes s \sum_{\alpha, \beta} \theta_\alpha \theta_\beta + f \nabla s = df \otimes s \left( \underbrace{\sum_\alpha \theta_\alpha}_{=1} \right) \left( \underbrace{\sum_\beta \theta_\beta}_{=1} \right) + f \nabla s = df \otimes s + f \nabla s.$$

Hence  $\nabla$  is a connection on  $E$ .

□

**Definition 10.2.6.** Suppose  $E_i \rightarrow M$ ,  $i = 0, 1$  are two smooth vector bundles over  $M$ . Suppose also  $\nabla^i$  is a connection on  $E_i$ ,  $i = 0, 1$ . A morphism  $(E_0, \nabla^0) \rightarrow (E_1, \nabla^1)$  is a bundle morphism  $T : E_0 \rightarrow E_1$  such that for every  $X \in \text{Vect}(M)$  the diagram below is commutative.

$$\begin{array}{ccc} C^\infty(E_0) & \xrightarrow{T} & C^\infty(E_1) \\ \nabla_X^0 \downarrow & & \downarrow \nabla_X^1 \\ C^\infty(E_0) & \xrightarrow{T} & C^\infty(E_1) \end{array}$$

An isomorphism of vector bundles with connections is defined in the obvious way. We denote by  $\mathbf{VB}_{\mathbb{K}}^{\text{conn}}(M)$  the collection of isomorphism classes of  $\mathbb{K}$ -vector bundles with connections over  $M$ .

□

Observe that we have a forgetful map

$$\mathbf{VB}_{\mathbb{K}}^{\text{conn}}(M) \rightarrow \mathbf{VB}_{\mathbb{K}}(M), \quad (E, \nabla) \mapsto E.$$

The tensorial operations  $\oplus, *, \otimes, \mathfrak{S}$  and  $\Lambda^*$  on  $\mathbf{VB}(M)$  have lifts to the richer category of vector bundles with connections. We explain this construction in detail. Suppose  $(E_i, \nabla^i) \in \mathbf{VB}^{\text{conn}}(M)$ ,  $i = 0, 1$ .

- We obtain a connection  $\nabla = \nabla^0 \oplus \nabla^1$  on  $E_0 \oplus E_1$  via the equality

$$\nabla(s_0 \oplus s_1) = (\nabla^0 s_0 \oplus \nabla^1 s_1), \quad \forall s_0 \in C^\infty(E_0), \quad s_1 \in C^\infty(E_1).$$

- A connection  $\nabla$  on  $E$  induces a connection  $\nabla^\dagger$  on  $E^*$  defined by the equality

$$L_X \langle u, v \rangle = \langle \nabla_X^\dagger u, v \rangle + \langle u, \nabla_X v \rangle, \quad \forall X \in \text{Vect}(M), \quad u \in C^\infty(E^*), \quad v \in C^\infty(E),$$

where  $\langle \bullet, \bullet \rangle \in \underline{\text{Hom}}(E^* \otimes E, \underline{\mathbb{K}}_M)$  denotes the natural bilinear pairing between a bundle and its dual.

Suppose  $\underline{e} = (e_1, \dots, e_r)$  is a local frame of  $E$  and  $A_\nabla(\underline{e}) = (A_b^a)_{1 \leq a, b \leq r}$ ,  $A_b^a \in \Omega^1(U)$ , is the connection 1-form associated to  $\nabla$ ,

$$\nabla \underline{e} = \underline{e} \cdot A_\nabla(\underline{e})$$

Denote by  $\underline{e}^\dagger = (e^1, \dots, e^r)$  the dual local frame of  $E^*$  defined by

$$\langle e^a, e_b \rangle = \delta_b^a.$$

We deduce that  $\langle \nabla^\dagger e^a, e_b \rangle = -\langle e^a, \nabla e_b \rangle = -A_b^a$  so that

$$\nabla^\dagger e^a = - \sum_b A_b^a e^b.$$

We can rewrite this

$$\nabla^\dagger \underline{e}^\dagger = \underline{e}^\dagger \cdot (-A_\nabla(\underline{e})^\dagger),$$

where  $A_\nabla(\underline{e})^\dagger$  denotes the transposed of the matrix  $A_\nabla(\underline{e})$ . Hence

$$A_{\nabla^\dagger}(\underline{e}^\dagger) = -A_\nabla(\underline{e})^\dagger.$$

- We get a connection  $\nabla^0 \otimes \nabla^1$  on  $E_0 \otimes E_1$  via the equality

$$(\nabla^0 \otimes \nabla^1)(s_0 \otimes s_1) = (\nabla s_0) \otimes s_1 + s_1 \otimes (\nabla^1 s_1).$$

- We get a connection on  $\Lambda^k E_0$  via the equality

$$\begin{aligned} \nabla_X^0(s_1 \wedge \cdots \wedge s_k) &= (\nabla_X s_1) \wedge s_2 \wedge \cdots \wedge s_k + s_1 \wedge (\nabla_X^0 s_2) \wedge \cdots \wedge s_k + \cdots \\ &+ s_1 \wedge s_2 \wedge \cdots \wedge (\nabla_X^0 s_k) \end{aligned}$$

$\forall s_1, \dots, s_k \in C^\infty(M), X \in \text{Vect}(M).$

- If  $E$  is a complex vector bundle, then any connection  $\nabla$  on  $E$  induces a connection  $\bar{\nabla}$  on the conjugate bundle  $\bar{E}$  defined via the conjugation operator  $C : E \rightarrow \bar{E}$

$$\bar{\nabla} = C\nabla C^{-1}.$$

**Exercise 10.2.7.** Suppose  $\nabla^0$  and  $\nabla^1$  are connections on the vector bundles  $E_0, E_1 \rightarrow M$ . They induce a connection  $\nabla$  on  $E_1 \otimes E_0^* \cong \text{Hom}(E_0, E_1)$ . Prove that for every  $X \in \text{Vect}(M)$  and every bundle morphism  $T : E_0 \rightarrow E_1$  the covariant derivative of  $T$  along  $X$  is the bundle morphism  $\nabla_X T$  defined by

$$(\nabla_X T)s = \nabla_X^1(Ts) - T(\nabla_X^0 s), \quad \forall s \in C^\infty(E_0).$$

In particular if  $E_0 = E_1$  and  $\nabla^0 = \nabla^1$  then we have

$$\nabla_X T = [\nabla_X^0, T],$$

where  $[A, B] = AB - BA$  for any linear operators  $A$  and  $B$ . □

Suppose  $E \rightarrow N$  is a vector bundle over the smooth manifold  $N$ ,  $f : M \rightarrow N$  is a smooth map, and  $\nabla$  is a connection on  $E$ . Then  $\nabla$  induces a connection  $f^*\nabla$  on  $f^*$  defined as follows. If  $E$  is defined by the gluing cocycle  $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^r)$  and  $\nabla$  is defined by the collection  $A_\bullet \in \Omega^1(\bullet) \otimes \underline{gl}_r(\mathbb{K})$ , then  $f^*\nabla$  is defined by the collection  $f^*A_\bullet \in \Omega^1(f^{-1}(U_\bullet)) \otimes \underline{gl}_r(\mathbb{K})$ . It is the unique connection on  $f^*E$  which makes commutative the following diagram.

$$\begin{array}{ccc} C^\infty(E) & \xrightarrow{f^*} & C^\infty(f^*E) \\ \nabla \downarrow & & \downarrow f^*\nabla \\ C^\infty(T^*N \otimes E) & \xrightarrow{f^*} & C^\infty(T^*M \otimes f^*E) \end{array}$$

**Definition 10.2.8.** Suppose  $\nabla$  is a connection on the vector bundle  $E \rightarrow M$ .

- (a) A section  $s \in C^\infty(E)$  is called  $(\nabla)$ -covariant constant or parallel if

$$\nabla s = 0.$$

- (b) A section  $s \in C^\infty(E)$  is said to be parallel along the smooth path  $\gamma : [0, 1] \rightarrow M$  if the pullback section  $\gamma^*s$  of  $\gamma^*E \rightarrow [0, 1]$  is parallel with respect to the connection  $f^*\nabla$ . □

**Example 10.2.9.** Suppose  $\gamma : [0, 1] \rightarrow M$  is a smooth path whose image lies entirely in a single coordinate chart  $U$  of  $M$ . Denote the local coordinates by  $(x^1, \dots, x^n)$  so we can represent  $\gamma$  as a  $n$ -uple of functions  $(x^1(t), \dots, x^n(t))$ . Suppose  $E \rightarrow M$  is a rank  $r$  vector bundle over  $M$  which

can be trivialized over  $U$ . If  $\nabla$  is a connection on  $E$  then with respect to some trivialization of  $E|_U$  can be described as

$$\nabla = d + A = d + \sum_i dx^i \otimes A_i, \quad A_i : U \rightarrow \underline{gl}_r(\mathbb{K}).$$

The tangent vector  $\dot{\gamma}$  along  $\gamma$  can be described in the local coordinates as

$$\dot{\gamma} = \sum_i \dot{x}^i \partial_i.$$

A section  $s$  is the parallel along  $\gamma$  if  $\nabla_{\dot{\gamma}} s = 0$ . More precisely, if we regard  $s$  as a smooth function  $s : U \rightarrow \mathbb{K}^r$  then we can rewrite this condition as

$$\begin{aligned} 0 &= \frac{d}{d\dot{\gamma}} s + \sum_i dx^i(\dot{\gamma}) A_i s = \left( \sum_i \dot{x}^i \partial_i \right) s + \sum_i \dot{x}^i A_i s \\ &\frac{ds}{dt} + \sum_i \dot{x}^i A_i s = 0. \end{aligned} \quad (10.2.5)$$

Thus a section which is parallel over a path  $\gamma(t)$  satisfies a first order linear differential equation. The existence theory for such equations shows that given any initial condition  $s_0 \in E_{\gamma(0)}$  there exists a unique parallel section  $[0, 1] \ni t \mapsto S(t; s_0) \in E_{\gamma(t)}$ . We get a linear map

$$E_{\gamma(0)} \ni s_0 \rightarrow S(t; s_0)|_{t=1} \in E_{\gamma(1)}.$$

This is called the *parallel transport* along  $\gamma$  (with respect to the connection  $\nabla$ ).  $\square$

Suppose  $E$  is a real vector bundle,  $g$  is a metric on  $E$ . A connection  $\nabla$  on  $E$  is called *compatible with the metric  $g$*  (or a *metric connection*) if  $g$  is a section of  $E^* \otimes E^*$  covariant constant with respect to the connection on  $E^* \otimes E^*$  induced by  $\nabla$ . More explicitly, this means that for every sections  $u, v$  of  $E$  and every vector field  $X$  on  $M$  we have

$$L_X g(u, v) = g(\nabla_X u, v) + g(u, \nabla_X v).$$

One can define in a similar fashion the connections on a complex vector bundle compatible with a hermitian metric  $h$ .

**Proposition 10.2.10.** *Suppose  $h$  is a metric (riemannian or hermitian) on the vector bundle  $E$ . Then there exists connections compatible with  $h$ . Moreover the space  $\mathcal{A}_{E,h}$  of connections compatible with  $h$  is an affine space modelled on the vector space  $\Omega^1(\text{End}_h^-(E))$ .*  $\square$

Suppose that  $\nabla$  is a connection on a vector bundle  $E \rightarrow M$ . For any vector fields  $X, Y$  over  $M$  we get three linear operators

$$\nabla_X, \nabla_Y, \nabla_{[X,Y]} : C^\infty(E) \rightarrow C^\infty(E),$$

where  $[X, Y] \in \text{Vect}(M)$  is the Lie bracket of  $X$  and  $Y$ . Form the linear operator

$$F_\nabla(X, Y) : C^\infty(E) \rightarrow C^\infty(E), \quad F_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Observe two things. First,

$$F_\nabla(X, Y) = -F_\nabla(Y, X).$$

Second, if  $f \in C^\infty(M)$  and  $s \in C^\infty(E)$  then

$$F_\nabla(X, Y)(fs) = f F_\nabla(X, Y)s = F_\nabla(fX, Y)s = F_\nabla(X, fY)s$$

so that for every  $X, Y \in \text{Vect}(M)$  the operator  $F_{\nabla}(X, Y)$  is an endomorphism of  $E$  and the correspondence

$$\text{Vect}(M) \times \text{Vect}(M) \rightarrow \underline{\text{End}}(E), \quad (X, Y) \mapsto F_{\nabla}(X, Y)$$

is  $C^{\infty}(M)$ -bilinear and skew-symmetric. In other words  $F_{\nabla}(\bullet, \bullet)$  is a 2-form with coefficients in  $\text{End } E$ , i.e., a section of  $\Omega^2(\text{End } E)$ .

**Definition 10.2.11.** The  $\text{End } E$ -valued 2-form  $F_{\nabla}(\bullet, \bullet)$  constructed above is called the *curvature* of  $\nabla$ .  $\square$

**Example 10.2.12.** (a) Consider the trivial vector bundle  $E = \underline{\mathbb{K}}_U^r$ , where  $U$  is an open subset in  $\mathbb{R}^n$ . Denote by  $(x^1, \dots, x^n)$  the Euclidean coordinates on  $U$ . Denote by  $d$  the trivial connection on  $E$ . Any connection  $\nabla$  on  $E$  has the form

$$\nabla = d + A = d + \sum_i dx^i A_i, \quad A_i : U \rightarrow \underline{gl}_r(\mathbb{K}).$$

Set  $\partial_i := \frac{\partial}{\partial x^i}$ ,  $\nabla_i = \nabla_{\partial_i}$ . Then for every  $s : U \rightarrow \mathbb{K}^r$  we have

$$\begin{aligned} F_{\nabla}(\partial_i, \partial_j)s &= [\nabla_i, \nabla_j]s = \nabla_i(\nabla_j s) - \nabla_j(\nabla_i s) \\ &= \nabla_i(\partial_j s + A_j s) - \nabla_j(\partial_i s + A_i s) = (\partial_i + A_i)(\partial_j s + A_j s) - (\partial_j + A_j)(\partial_i s + A_i s) \\ &= \left( \partial_i A_j - \partial_j A_i + [A_i, A_j] \right) s. \end{aligned}$$

Hence

$$\sum_{i < j} F(\partial_i, \partial_j) dx^i \wedge dx^j = \left( \partial_i A_j - \partial_j A_i + [A_i, A_j] \right) dx^i \wedge dx^j.$$

We can write this formally as

$$F_{\nabla} = dA + A \wedge A = - \sum_i dx^i d(A_i) + \left( \sum_i dx^i A_i \right) \wedge \left( \sum_j dx^j A_j \right).$$

Observe that if  $r = 1$ , so that  $E$  is the trivial line bundle  $\underline{\mathbb{K}}_U$  then we can identify  $\underline{gl}_1(\mathbb{K}) \cong \mathbb{K}$  so the components  $A_i$  are scalars. In particular  $[A_i, A_j] = 0$  so that in this case

$$F_{\nabla} = dA.$$

(b) If  $E$  is a vector bundle described by a gluing cocycle  $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^r)$  and  $\nabla$  is a connection described by the collection of 1-forms  $A_{\alpha} \in \Omega^1(U_{\alpha}) \otimes \underline{gl}_r(\mathbb{K})$  satisfying (10.2.4) then the curvature of  $\nabla$  is represented by the collection of 2-forms

$$F_{\alpha} = dA_{\alpha} + A_{\alpha} \wedge A_{\alpha}$$

satisfying the compatibility conditions

$$F_{\beta} = g_{\beta\alpha} F_{\alpha} g_{\beta\alpha}^{-1} \quad \text{on } U_{\alpha\beta}. \quad (10.2.6)$$

(c) If  $\nabla$  is a connection on a complex line bundle  $L \rightarrow M$  then its curvature  $F_{\nabla}$  can be identified with a complex valued 2-form. If moreover,  $\nabla$  is compatible with a hermitian metric then  $iF_{\nabla}$  is a *real* valued 2-form.  $\square$

We define an operation

$$\wedge : \Omega^k(\text{End } E) \times \Omega^\ell(\text{End } E) \rightarrow \Omega^{k+\ell}(\text{End } E),$$

by setting

$$(\omega^k \otimes S) \wedge (\eta^\ell \otimes T) = (\omega^k \wedge \eta^\ell) \otimes (ST)$$

for any  $\Omega^k \in \Omega^k(M)$ ,  $\eta^\ell \in \Omega^\ell(M)$ ,  $S, T \in \underline{\text{End}}(E)$ .

Using a connection  $\nabla$  on  $E$  we can produce an exterior derivative

$$d^\nabla : \Omega^k(\text{End } E) \rightarrow \Omega^{k+1}(\text{End } E)$$

defined by

$$d^\nabla(\omega^k \otimes S) = (d\omega^k) \otimes S + (-1)^k(\omega \otimes \mathbb{1}_E) \wedge \nabla^{\text{End } E} S,$$

We have the following result.

**Proposition 10.2.13.** *Suppose  $\nabla', \nabla$  are two connections on the vector bundle  $E \rightarrow M$ . Their difference  $B = \nabla' - \nabla$  is an  $\text{End } E$ -valued 1-form. Then*

$$F_{\nabla'} = F_\nabla + d^\nabla B + B \wedge B.$$

**Proof.** The result is local so we can assume  $E$  is the trivial bundle over an open subset  $M \hookrightarrow \mathbb{R}^n$ . Let  $r = \text{rank } E$ . We can write

$$\nabla = d + A, \quad \nabla' = d + A', \quad A, A' \in \Omega^1(M) \otimes \underline{gl}_r(\mathbb{K}).$$

Then  $B = A' - A$ ,

$$F' = F_{\nabla'} = dA' + A' \wedge A', \quad F = F_\nabla = dA + A \wedge A$$

and thus

$$\begin{aligned} F' - F &= d(A' - A) + (A' \wedge A') - (A \wedge A) = d(A' - A) + (A + B) \wedge (A + B) - B \wedge B \\ &= dB + B \wedge A + A \wedge B + B \wedge B. \end{aligned}$$

In local coordinates  $d^\nabla$  we have (see Exercise 10.2.7)

$$\begin{aligned} d^\nabla \left( \sum_i dx^i \otimes B_i \right) &= - \sum_i dx^i \wedge \left( \sum_j dx^j \otimes \nabla_j B_i \right) \\ &= - \sum_i dx^i \wedge \left( \sum_j dx^j \otimes (\partial_j B_i + [A_j, B_i]) \right) \\ &= \sum_{i < j} dx^i \wedge dx^j \otimes (\partial_i B_j - \partial_j B_i) - \sum_{i,j} dx^i \wedge dx^j \otimes (A_j B_i - B_i A_j) \\ &= dB + \left( \sum_j dx^j \otimes A_j \right) \wedge \left( \sum_i dx^i \otimes B_i \right) + \left( \sum_i dx^i \otimes B_i \right) \wedge \left( \sum_j dx^j \otimes A_j \right) \\ &= dB + A \wedge B + B \wedge A. \end{aligned}$$

□

### 10.3. Connections on principal bundles

In the sequel we will work exclusively with *matrix Lie groups*, i.e., closed subgroups of some  $GL_r(\mathbb{K})$ .

Fix a (matrix) Lie group  $G$  and a principal  $G$ -bundle  $P = (M, \mathcal{U}, g_{\bullet\bullet})$  over the smooth manifold  $M$ . Denote by  $\mathfrak{g} = T_1G$  the Lie algebra of  $G$ . A *connection* on  $P$  is a collection

$$A = \{A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}\}$$

satisfying the following conditions

$$A_\beta(u) = g_{\beta\alpha}(u)A_\alpha(u)g_{\beta\alpha}^{-1}(u) - d(g_{\beta\alpha})g_{\beta\alpha}(u)^{-1}, \quad \forall u \in U_{\alpha\beta}. \quad (10.3.1)$$

We denote by  $\mathcal{A}_P$  the space of connections on  $P$ .

**Proposition 10.3.1.**  $\mathcal{A}_P$  is an affine space modelled on  $\Omega^1(\text{Ad } P)$ .

**Proof.** We will show that given two connections  $(A_\alpha^1), (A_\alpha^0)$  their difference  $C_\alpha = A_\alpha^1 - A_\alpha^0$  defines a global section of  $\Lambda^1 T^*M \otimes \text{Ad } P$ , i.e. on the overlaps  $U_{\beta\alpha}$  we have the equality

$$C_\beta = \text{Ad}(g_{\beta\alpha})C_\alpha = g_{\beta\alpha}C_\alpha g_{\beta\alpha}^{-1}.$$

This follows immediately by taking the difference of the transition equalities (10.3.1) for  $A_\alpha^1$  and  $A_\alpha^0$ . To finish the proof of the proposition we only need to show that  $\mathcal{A}_P \neq \emptyset$ . We refer to [N1, Ch.8] for more details. □

To formulate our next result let us introduce an operation

$$\begin{aligned} [-, -] : \Omega^k(U_\alpha) \otimes \mathfrak{g} \times \Omega^\ell(U_\alpha) \otimes \mathfrak{g} &\rightarrow \Omega^{k+\ell}(U_\alpha) \otimes \mathfrak{g}, \\ [\omega^k \otimes X, \eta^\ell \otimes Y] &:= (\omega^k \wedge \eta^\ell) \otimes [X, Y], \end{aligned}$$

where  $[X, Y]$ -denotes the Lie bracket in  $\mathfrak{g}$ , or in the case of a matrix Lie group,  $[X, Y] = XY - YX$  is the commutator of the matrices  $X, Y$ . Let us point out that if  $A, B \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$  we have

$$[A, B] = A \wedge B + B \wedge A.$$

We define

$$F_\alpha := dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha] = dA_\alpha + A_\alpha \wedge A_\alpha \in \Omega^2(U_\alpha) \otimes \mathfrak{g}.$$

For a proof of the following result we refer to [N1, Chap.8].

**Proposition 10.3.2.** (a) The collection  $F_\alpha$  defines a global section  $F(A)$  of  $\Lambda^2 T^*M \otimes \text{Ad } P$ , i.e. on the overlaps  $U_{\alpha\beta}$  it satisfies the compatibility conditions,

$$F_\beta = g_{\beta\alpha}F_\alpha g_{\beta\alpha}^{-1} = \text{Ad}(g_{\beta\alpha})F_\alpha.$$

(b) **(The Bianchi Identity)**

$$dF_\alpha + [A_\alpha, F_\alpha] = 0, \quad \forall \alpha. \quad \square$$

The 2-form  $F(A) \in \Omega^2(\text{Ad } P)$  is called the *curvature* of  $A$ .

Consider now a representation  $\rho : G \rightarrow \text{Aut}(V)$  and the vector bundle  $E = P \times_\rho V$ . Denote by  $\rho_*$  the differential of  $\rho$  at  $1 \in G$

$$\rho_* : \mathfrak{g} \rightarrow \text{End } V.$$

We recall that  $\text{End}_\rho(V) = \rho_*\mathfrak{g}$  and  $\text{End}_\rho E = P \times_{\text{Ad}_\rho} \text{End}_\rho(V)$ . The identity (10.1.1) shows that any connection  $(A_\alpha)$  on  $P$  defines a connection  $\nabla = (\rho_*A_\alpha)$  on  $E$ . We say that this connection is *compatible with the  $(G, \rho)$ -structure*. Observe that

$$F_\nabla|_{U_\alpha} = \rho_*F_\alpha.$$

In particular  $F_\nabla \in \Omega^2(\text{End}_\rho E)$ .

**Example 10.3.3.** Suppose  $E \rightarrow M$  is a complex vector bundle of rank  $r$ . A hermitian metric  $h$  on  $E$  defines a  $U(r)$ -structure. A connection  $\nabla$  is compatible with this structure if and only if it is compatible with the metric. In this case  $\text{End}_\rho E$  is the subbundle  $\text{End}_h^- E$  of  $\text{End} E$  and we have

$$F(\nabla) \in \Omega^2(\text{End}_h^- E).$$

□

**Exercise 10.3.4.** (a) Construct a connection on the tautological line bundle over  $\mathbb{C}\mathbb{P}^1$  compatible with the natural hermitian metric.

(b) The curvature of the hermitian connection  $A$  you constructed in part (a) is a purely imaginary 2-form  $F(A)$  on  $\mathbb{C}\mathbb{P}^1$ . Show that

$$\int_{\mathbb{C}\mathbb{P}^1} c_1(A) = \frac{i}{2\pi} \int_{\mathbb{C}\mathbb{P}^1} F(A) = -1.$$

(c) Prove that the tautological line bundle over  $\mathbb{C}\mathbb{P}^1$  cannot be trivialized.

□

### 10.4. The Chern-Weil construction

Suppose  $P \rightarrow M$  is a principal  $G$ -bundle over  $M$  defined by the gluing cocycle  $(\mathcal{U}, g_{\bullet\bullet})$ . To formulate the Chern-Weil construction we need to introduce first the concept of Ad-invariant polynomials on  $\mathfrak{g}$ .

The adjoint representation  $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$  induces an adjoint representation

$$\text{Ad}^k : G \rightarrow \text{GL}(\text{Sym}^k \mathfrak{g}_\mathbb{C}^*), \quad \mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}.$$

We denote by  $I_k(\mathfrak{g})$  the  $\text{Ad}^k$ -invariant elements of  $\text{Sym}^k \mathfrak{g}^*$ . Equivalently, they are  $k$ -multilinear maps

$$P : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_k \rightarrow \mathbb{C},$$

such that

$$P(X_{\varphi(1)}, \dots, X_{\varphi(k)}) = P(gX_1g^{-1}, \dots, gX_kg^{-1}) = P(X_1, \dots, X_k)$$

for any  $X_1, \dots, X_k \in \mathfrak{g}$ ,  $g \in G$  and any permutation  $\varphi$  of  $\{1, \dots, k\}$ . If in the above equality we take  $g = \exp(tY)$ ,  $Y \in \mathfrak{g}$  and then we differentiate with respect to  $t$  at  $t = 0$  we obtain

$$P([Y, X_1], X_2, \dots, X_k) + \cdots + P(X_1, \dots, X_{k-1}, [Y, X_k]) = 0, \quad \forall Y, X_1, \dots, X_k \in \mathfrak{g}. \quad (10.4.1)$$

For  $P \in I_k(\mathfrak{g})$  and  $X \in \mathfrak{g}$  we set

$$P(X) := P(\underbrace{X, \dots, X}_k).$$

We have the *polarization formula*

$$P(X_1, \dots, X_k) = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k} P(t_1 X_1 + \cdots + t_k X_k).$$

More generally, given  $P \in I_k(\mathfrak{g})$  and (not necessarily commutative)  $\mathbb{C}$ -algebra  $\mathcal{R}$  we define  $\mathcal{R}$ -multilinear map

$$P : \underbrace{\mathcal{R} \otimes \mathfrak{g} \times \cdots \times \mathcal{R} \otimes \mathfrak{g}}_k \rightarrow \mathcal{R}$$

by

$$P(r_1 \otimes X_1, \cdots, r_k \otimes X_k) = r_1 \cdots r_k P(X_1, \cdots, X_k).$$

Let us emphasize that when  $\mathcal{R}$  is not commutative the above function is *not* symmetric in its variables. For example if  $r_1 r_2 = -r_2 r_1$  then

$$P(r_1 X_1, r_2 X_2, \cdots) = -P(r_2 X_2, r_1 X_1, \cdots).$$

It will be so if  $\mathcal{R}$  is commutative. For applications to geometry  $\mathcal{R}$  will be the algebra  $\Omega^\bullet(M)$  of complex valued differential forms on a smooth manifold  $M$ . When restricted to the commutative subalgebra

$$\Omega^{even}(M) = \bigoplus_{k \geq 0} \Omega^{2k}(M) \otimes \mathbb{C}.$$

we do get a symmetric function.

Let us point out a useful identity. If  $P \in I_k(\mathfrak{g})$ ,  $U$  is an open subset of  $\mathbb{R}^n$ ,

$$F_i = \omega_i \otimes X_i \in \Omega^{d_i}(U) \otimes \mathfrak{g}, \quad A = \omega \otimes X \in \Omega^d(U) \otimes \mathfrak{g}$$

then

$$P(F_1, \cdots, F_{i-1}, [A, F_i], F_{i+1}, \cdots, F_k) = (-1)^{d(d_1 + \cdots + d_{i-1})} \omega \omega_1 \cdots \omega_k P(X_1, \cdots, [X, X_i], \cdots, X_k).$$

In particular, if  $F_1, \cdots, F_{k-1}$  have even degree we deduce that for every  $i = 1, \cdots, k$  we have

$$P(F_1, \cdots, F_{i-1}, [A, F_i], F_{i+1}, \cdots, F_k) = \omega \omega_1 \cdots \omega_k P(X_1, \cdots, [X, X_i], \cdots, X_k)$$

Summing over  $i$  and using the Ad-invariance of  $P$  we deduce

$$\sum_{i=1}^k P(F_1, \cdots, F_{i-1}, [A, F_i], F_{i+1}, \cdots, F_k) = 0, \quad (10.4.2)$$

$$\forall F_1, \cdots, F_{k-1} \in \Omega^{even}(U) \otimes \mathfrak{g}, \quad F_k, A \in \Omega^*(U) \otimes \mathfrak{g}.$$

**Theorem 10.4.1** (Chern-Weil). *Suppose  $A = (A_\bullet)$  is a connection on the principal  $G$ -bundle  $(M, \mathcal{U}, g_{\bullet\bullet})$ , with curvature  $F(A) = (F_\bullet)$ , and  $P \in I_k(\mathfrak{g})$ . Then the following hold.*

(a) *The collection of  $2k$ -forms  $P(F_\alpha) \in \Omega^{2k}(U_\alpha)$  define a global  $2k$ -form  $P(F(A))$  on  $M$ , i.e.,*

$$P(F_\alpha) = P(F_\beta) \quad \text{on } U_{\alpha\beta}.$$

(b) *The form  $P(F(A))$  is closed*

$$dP(F(A)) = 0.$$

(c) *For any two connections  $A^0, A^1 \in \mathcal{A}_P$  the closed forms  $P(F(A^0))$  and  $P(F(A^1))$  are cohomologous, i.e., their difference is an exact form.*

**Proof.** (a) On the overlap  $U_{\alpha\beta}$  we have

$$P(F_\beta) = P(\text{Ad}(g_{\beta\alpha})F_\alpha) = P(F_\alpha)$$

due to the Ad-invariance of  $P$ .



(b) Observe first that the Bianchi identity implies that  $dF_\alpha = -[A_\alpha, F_\alpha]$ . From the product formula we deduce

$$\begin{aligned} dP(F_\alpha) &= dP(\underbrace{F_\alpha, \dots, F_\alpha}_k) = P(dF_\alpha, F_\alpha, \dots, F_\alpha) + \dots + P(F_\alpha, \dots, F_\alpha, dF_\alpha) \\ &= -P([A_\alpha, F_\alpha], F_\alpha, \dots, F_\alpha) - \dots - P(F_\alpha, \dots, F_\alpha, [A_\alpha, F_\alpha]) \stackrel{(10.4.2)}{=} 0. \end{aligned}$$

(c) Consider two connections  $A^1, A^0 \in \mathcal{A}_P$ . We need to find a  $(2k-1)$  form  $\eta$  such that

$$P(F(A^1)) - P(F(A^0)) = d\eta.$$

Let  $C := A^1 - A^0 \in \Omega^1(\text{Ad } P)$ . We get a path of connections  $t \mapsto A^t = A^0 + tC$  which starts at  $A^0$  and ends at  $A^1$ . Set  $F^t := F(A^t)$  and

$$P(t) = P(F_{A^t}).$$

We want to show that  $P(1) - P(0)$  is exact. We will prove a more precise result. Define the *local transgression forms*

$$T_\alpha P(A^1, A^0) := k \int_0^1 P(F_\alpha^t, \dots, F_\alpha^t, C_\alpha) dt$$

The Ad-invariance of  $P$  implies that

$$T_\alpha P(A^1, A^0) = T_\beta P(A^1, A^0), \quad \text{on } U_{\alpha\beta}$$

so that these forms define a global form  $T(A^1, A^0) \in \Omega^{2k-1}(M)$  called the *transgression form* from  $A^0$  to  $A^1$ . We will prove that

$$P(1) - P(0) = dT(A^1, A^0).$$

We work locally on  $U_\alpha$  we have

$$P(1) - P(0) = \int_0^1 \frac{d}{dt} P(F_\alpha^t, \dots, F_\alpha^t) dt$$

$$(\dot{F}_\alpha^t = \frac{d}{dt} F_\alpha^t)$$

$$\begin{aligned} &= \int_0^1 \left( P(\dot{F}_\alpha^t, F_\alpha^t, \dots, F_\alpha^t) + \dots + P(F_\alpha^t, \dots, F_\alpha^t, \dot{F}_\alpha^t) \right) dt \\ &= k \int_0^1 P(F_\alpha^t, \dots, F_\alpha^t, \dot{F}_\alpha^t) dt. \end{aligned}$$

We have

$$F_\alpha^t = dA_\alpha^t + \frac{1}{2}[A_\alpha^t, A_\alpha^t] = F_\alpha^0 + t(dC_\alpha + [A_\alpha^0, C_\alpha]) + \frac{t^2}{2}[C_\alpha, C_\alpha]$$

so that

$$\dot{F}_\alpha^t = dC_\alpha + [A_\alpha^0, C_\alpha] + t[C_\alpha, C_\alpha] = dC_\alpha + [A_\alpha^t, C_\alpha].$$

Hence

$$P(F_\alpha^t, \dots, F_\alpha^t, \dot{F}_\alpha^t) = P(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha + [A_\alpha^t, C_\alpha]).$$

To finish the proof of the theorem it suffices to show that

$$dP(F_\alpha^t, \dots, F_\alpha^t, C_\alpha) = P(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha + [A_\alpha^t, C_\alpha]).$$

Indeed we have

$$dP(F_\alpha^t, \dots, F_\alpha^t, C_\alpha) = P(dF_\alpha^t, \dots, F_\alpha^t, C_\alpha) + \dots + P(F_\alpha^t, \dots, dF_\alpha^t, C_\alpha) + P(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha)$$

$$\begin{aligned}
(dF_\alpha^t &= -[A_\alpha^t, F_\alpha^t]) \\
&= -P([A_\alpha^t, F_\alpha^t], \dots, F_\alpha^t, C_\alpha) - \dots - P(F_\alpha^t, \dots, [A_\alpha^t, F_\alpha^t], C_\alpha) + P(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha) \\
&= P(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha + [A_\alpha^t, C_\alpha]) \\
&\quad - \left( P(F_\alpha^t, \dots, F_\alpha^t, [A_\alpha^t, C_\alpha]) + P([A_\alpha^t, F_\alpha^t], \dots, F_\alpha^t, C_\alpha) + \dots + P(F_\alpha^t, \dots, [A_\alpha^t, F_\alpha^t], C_\alpha) \right) \\
&= P(F_\alpha^t, \dots, F_\alpha^t, dC_\alpha + [A_\alpha^t, C_\alpha])
\end{aligned}$$

since the term in parentheses vanishes<sup>2</sup> due to (10.4.2). □

We set

$$\mathbb{C}[\mathfrak{g}^*]^G = \bigoplus_{k \geq 0} I_k(\mathfrak{g}), \quad \mathbb{C}[[\mathfrak{g}^*]]^G = \prod_{k \geq 0} I_k(\mathfrak{g}).$$

$\mathbb{C}[\mathfrak{g}^*]^G$  is the ring of  $Ad$ -invariant polynomials and  $\mathbb{C}[[\mathfrak{g}^*]]^G$  is the ring of  $Ad$ -invariant formal power series. We have

$$\mathbb{C}[\mathfrak{g}^*]^G \subset \mathbb{C}[[\mathfrak{g}^*]]^G$$

Suppose  $A$  is a connection on the principal  $G$ -bundle  $P \rightarrow M$ . Then for every  $f = \sum_{k \geq 0} f_k \in \mathbb{C}[[\mathfrak{g}^*]]^G$  we get an element

$$f(F(A)) = \sum_{k \geq 0} f_k(F(A))$$

Observe that  $f_k(F(A)) \in \Omega^{2k}(M)$ . In particular  $f_{2k}(A) = 0$  for  $2k > \dim M$  so that in the above sum only finitely many terms are non-zero. We obtain a well defined correspondence

$$\mathbb{C}[[\mathfrak{g}^*]]^G \times \mathcal{A}_P \rightarrow \Omega^{even}(M), \quad (f, A) \mapsto f(F(A)).$$

This is known as the *Chern-Weil correspondence*. The image of the Chern-Weil correspondence is a subspace of  $\mathcal{Z}^*(M)$ , the vector space of closed forms on  $M$ . We have also constructed a canonical map

$$T : \mathbb{C}[[\mathfrak{g}^*]]^G \times \mathcal{A}_P \times \mathcal{A}_P \rightarrow \Omega^{odd}(M), \quad (f, A_0, A_1) \mapsto Tf(A_1, A_0)$$

such that

$$f(F(A_1)) - f(F(A_0)) = dTf(A_1, A_0).$$

We will refer to it as the *Chern-Weil transgression*.

**Exercise 10.4.2.** Suppose  $E \rightarrow M$  is a rank two hermitian complex vector bundle and  $A^1, A^0$  are two hermitian connections on  $E$ . Assume  $A^0$  is flat, i.e.  $F(A^0) = 0$ . Describe the transgression  $Tc_2(A^1, A^0)$  in terms of  $C = A^1 - A^0$ . The correspondence

$$\Omega^1(\text{End}_h^- E) \ni C \mapsto Tc_2(A^0 + C, A^0)$$

is known as the *Chern-Simmons functional*. □

**Exercise 10.4.3.** Prove that the Chern classes are independent of the hermitian metric used in their definition. □

<sup>2</sup>The order in which we wrote the terms,  $F^t, \dots, F^t, C$  instead of  $C, F^t, \dots, F^t$  is **very important** in view of the asymmetric definition of

$$P : \mathcal{R} \otimes \mathfrak{g} \times \dots \times \mathcal{R} \otimes \mathfrak{g} \rightarrow \mathcal{R}.$$

The Chern-Weil construction is *natural* in the following sense. Suppose  $P = (M, \mathcal{U}, g_{\bullet\bullet}, G)$  is a principal  $G$ -bundle over  $M$  and  $f : N \rightarrow M$  is a smooth map. Then we get a pullback bundle  $f^*P$  over  $N$  described by the gluing data  $(N, f^{-1}(\mathcal{U}), f^*(g_{\bullet\bullet}), G)$ . For any connection  $A = (A_{\bullet})$  on  $P$  we get a connection  $f^*A = (f^*A_{\bullet})$  on  $f^*P$  such that

$$F(f^*A) = f^*F(A).$$

Then for every element  $h \in \mathbb{C}[[\mathfrak{g}^*]]^G$  we have

$$h(f^*F(A)) = f^*h(F(A)).$$

## 10.5. The Chern classes

We consider now the special case  $G = U(n)$ . The Lie algebra of  $U(n)$ , denoted by  $\underline{u}(n)$  is the space of skew-hermitian matrices. Observe that we have a natural identification

$$\underline{u}(1) \cong i\mathbb{R}.$$

The group  $U(n)$  acts on  $\underline{u}(n)$  by conjugation

$$U(n) \times \underline{u}(n) \ni (g, X) \mapsto gXg^{-1} \in \underline{u}(n).$$

It is a basic fact of linear algebra that for every skew-hermitian endomorphism of  $\mathbb{C}^n$  can be diagonalized, or in other words, every skew-hermitian matrix is conjugate to a diagonal one. The space of diagonal skew-hermitian matrices forms a commutative Lie subalgebra of  $\underline{u}(n)$  known as the *Cartan subalgebra* of  $\underline{u}(n)$ . We will denote it by  $\mathbf{Cartan}(\underline{u}(n))$ .

$$\mathbf{Cartan}(\underline{u}(n)) = \left\{ \text{Diag}(i\lambda_1, \dots, i\lambda_n); (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \right\}.$$

The group  $W_{U(n)}$ <sup>3</sup> of permutations of  $n$  objects acts on  $\mathbf{Cartan}(\underline{u}(n))$  in the obvious way and two diagonal matrices are conjugate if and only if we can obtain one from the other by a permutation of its entries. Thus an Ad-invariant polynomial on  $\underline{u}(n)$  is determined by its restriction to the Cartan algebra. Thus we can regard every Ad-invariant polynomial as a polynomial function  $P = P(\lambda_1, \dots, \lambda_n)$ . This polynomial is also invariant under the permutation of its variables and thus can be described as a polynomial in the elementary symmetric quantities

$$c_k = \sum_{i_1 < \dots < i_k} x_{i_1} \cdots x_{i_k}, \quad x_j = \frac{i}{2\pi}(i\lambda_j) = -\frac{\lambda_j}{2\pi}.$$

The factor  $\frac{i}{2\pi}$  appears due to historical and geometric reasons. The variables  $x_j$  are also known as the *Chern roots*. More elegantly, if we set

$$D = D(\vec{\lambda}) = \text{Diag}(i\lambda_1, \dots, i\lambda_n) \in \underline{u}(n)$$

then

$$\det(1 + \frac{it}{2\pi}D) = 1 + c_1t + c_2t^2 + \dots + c_nt^n.$$

Instead of the elementary sums we can consider the momenta

$$s_r = \sum_i x_i^r.$$

<sup>3</sup>We use the notation  $W_{U(n)}$  because this group in this case the symmetric group is isomorphic to the Weyl group of  $U(n)$ .

The elementary sums can be expressed in terms of the momenta via the *Newton relation* (6.2.6)

$$s_1 = c_1, \quad s_2 = c_1^2 - 2c_2, \quad s_3 = c_1^3 - 3c_1c_2 + 3c_3, \quad \sum_{j=1}^r (-1)^j s_{r-j} c_j = 0. \quad (10.5.1)$$

Using again the matrix  $D$  we have

$$\sum_{r \geq 0} \frac{s_r}{r!} t^r = \text{tr} \exp\left(\frac{it}{2\pi} D\right).$$

Motivated by these examples we introduce the *Chern polynomial*

$$c \in \mathbb{C}[[\underline{u}(n)^*]]^{U(n)}, \quad c(X) = \det(\mathbb{1}_{\mathbb{C}^n} + \frac{i}{2\pi} X), \quad \forall X \in \underline{u}(n).$$

Now define the *Chern character*

$$\mathbf{ch} \in \mathbb{C}[[\underline{u}(n)]]^{U(n)}, \quad \mathbf{ch}(X) = \text{tr} \exp\left(\frac{i}{2\pi} X\right).$$

Using (10.5.1)

$$\mathbf{ch} = n + c_1 + \frac{1}{2!}(c_1^2 - 2c_2) + \frac{1}{3!}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots \quad (10.5.2)$$

**Example 10.5.1.** Suppose

$$F = \begin{bmatrix} iF_1^1 & F_2^1 \\ F_1^2 & iF_2^2 \end{bmatrix} \in \underline{u}(2) \iff F_1^2 = -\bar{F}_2^1.$$

Then

$$c_1(F) = -\frac{1}{2}(F_1^1 + F_2^2), \quad c_2(F) = -\frac{1}{4\pi^2}(F_2^1 \wedge \bar{F}_2^1 - F_1^1 \wedge F_2^2).$$

□

Our construction of the Chern polynomial is a special case of the following general procedure of constructing symmetric elements in  $\mathbb{C}[[\lambda_1, \dots, \lambda_n]]$ . Consider a formal power series

$$f = a_0 + a_1x + a_2x^2 + \cdots \in \mathbb{C}[[x]], \quad a_0 = 1.$$

Then if we set  $\vec{x} = (x_1, \dots, x_n)$  the function

$$\mathbf{G}_f(\vec{x}) = f(x_1) \cdots f(x_n) \in \mathbb{C}[[x_1, \dots, x_n]]$$

is a symmetric power series in  $\vec{x}$  with leading coefficient 1. Observe that if  $D = \text{Diag}(i\vec{\lambda})$  then

$$f\left(\frac{i}{2\pi} D\right) = \text{Diag}(f(x_1), \dots, f(x_n)) \implies f(\vec{x}) = \det f\left(\frac{i}{2\pi} D\right).$$

We thus get an element  $\mathbf{G}_f \in \mathbb{C}[[\underline{u}(n)]]^{U(n)}$  defined by

$$\mathbf{G}_f(X) = \det f\left(\frac{i}{2\pi} X\right).$$

It is called the *f-genus* or the genus associated to  $f$ . When  $f(x) = 1 + x$  we obtain the Chern polynomial.

Of particular relevance in geometry is the *Todd genus*, i.e. the genus associated to the function<sup>4</sup>

$$\mathrm{td}(x) := \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \cdots = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!}x^{2k}.$$

The coefficients  $b_k$  are the same Bernoulli numbers we have encountered in Chapter 7. We set

$$\mathbf{td} := \mathbf{G}_{\mathrm{td}}.$$

Consider now a rank  $n$  complex vector bundle  $E \rightarrow M$  equipped with a hermitian metric  $h$ . We denote by  $\mathcal{A}_{E,h}$  the affine space of connections on  $E$  compatible with the metric  $h$  and by  $P_h(E)$  the principal bundle of  $h$ -orthonormal frames. Then the space of connections  $\mathcal{A}_{E,h}$  can be naturally identified with the space of connections on  $P_h(E)$ . For every  $A \in \mathcal{A}_{E,h}$  we can regard the curvature  $F(A)$  as a  $n \times n$  matrix with entries even degree forms on  $M$ . We get a non-homogeneous even degree form

$$c(A) = c(F(A)) = \det\left(\mathbb{1}_E + \frac{i}{2\pi}F(A)\right) \in \Omega^{\mathrm{even}}(M).$$

According to the Chern-Weil theorem this form is closed and its cohomology class is independent of the metric<sup>5</sup>  $h$  and the connection  $A$ . It is thus a *topological invariant* of  $E$ . We denote it by  $c(E)$  and we will call it the *total Chern class of  $E$* . It has a decomposition into homogeneous components

$$c(E) = 1 + c_1(E) + \cdots + c_n(E), \quad c_k(E) \in H^{2k}(M, \mathbb{R}).$$

We will refer to  $c_k(E)$  as the  *$k$ -th Chern class*. More generally for any  $f = 1 + a_1x + \cdots \in \mathbb{C}[[x]]$  we define  $\mathbf{G}_f(E)$  to be the cohomology class carried by the form

$$\mathbf{G}_f(A) = \det f(F(A)).$$

In particular,  $\mathbf{td}(E)$  is the cohomology class carried by the closed form

$$\mathbf{td}(A) := \det\left(\frac{\frac{i}{2\pi}F}{\exp(\frac{i}{2\pi}F) - \mathbb{1}_E}\right)$$

(see [Hirz, I.§1])

$$= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots.$$

Similarly we define the *Chern character of  $E$*  as the cohomology class  $\mathbf{ch}(E)$  carried by the form

$$\begin{aligned} \mathbf{ch}(A) &= \mathrm{tr} \exp\left(\frac{i}{2\pi}F(A)\right) \\ &= \mathrm{rank} E + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \frac{1}{3!}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)) + \cdots \end{aligned}$$

Due to the naturality of the Chern-Weil construction we deduce that for every smooth map  $f : M \rightarrow N$  and every complex vector bundle  $E \rightarrow N$  we have

$$c(f^*E) = f^*c(E). \tag{10.5.3}$$

<sup>4</sup>**Warning.** The literature is not consistent on the definition of the Todd function. We chose to work with Hirzebruch's definition in [Hirz]. This agrees with the definition in [AS3, LM], but it differs from the definitions in [BGV, Roe] where  $\mathrm{td}(x)$  is defined as  $\frac{x}{e^x - 1}$ .

<sup>5</sup>See Exercise 10.4.3.

**Example 10.5.2.** Denote by  $L_{\mathbb{P}^n}$  the tautological line bundle over  $\mathbb{C}\mathbb{P}^n$ . The natural inclusions

$$i_k : \mathbb{C}^k \hookrightarrow \mathbb{C}^{k+1}, \quad (z_1, \dots, z_k) \mapsto (z_1, \dots, z_k, 0)$$

induce inclusions  $i_k : \mathbb{C}\mathbb{P}^{k-1} \rightarrow \mathbb{C}\mathbb{P}^k$  and tautological isomorphisms

$$L_{\mathbb{P}^{k-1}} \cong i_k^* L_{\mathbb{P}^k}.$$

We deduce that

$$c_1(L_{\mathbb{P}^n})|_{\mathbb{C}\mathbb{P}^1} = c_1(L_{\mathbb{P}^1}).$$

We know that  $H^2(\mathbb{C}\mathbb{P}^n, \mathbb{R})$  is a one-dimensional space with a canonical basis, namely the cohomology class dual to homology class carried by the hyperspace  $\mathbb{C}\mathbb{P}^{n-1} \hookrightarrow \mathbb{C}\mathbb{P}^n$ . It satisfies

$$\langle H, [\mathbb{C}\mathbb{P}^1] \rangle = 1,$$

where  $[\mathbb{C}\mathbb{P}^1]$  is the homology class defined by the embedding  $\mathbb{C}\mathbb{P}^1 \hookrightarrow \mathbb{C}\mathbb{P}^n$ . We can write

$$c_1(L_{\mathbb{P}^n}) = xH,$$

where

$$x = \langle c_1(L_{\mathbb{P}^n}), [\mathbb{C}\mathbb{P}^1] \rangle = \int_{\mathbb{C}\mathbb{P}^1} c_1(L_{\mathbb{P}^1}).$$

As shown in Exercise 10.3.4 the last integral is  $-1$  so that

$$c_1(L_{\mathbb{P}^n}) = -H. \tag{10.5.4}$$

□

For a proof of the following result we refer to [N1, Chap.8].

**Proposition 10.5.3.** Suppose  $(E_i, h_i)$ ,  $i = 0, 1$  are two hermitian vector bundles,  $A_i \in A_{E_i, h_i}$  and  $f = 1 + a_1x + a_2x^2 + \dots \in \mathbb{C}[[x]]$ . We denote by  $A_0 \oplus A_1$  and  $A_0 \otimes A_1$  the induced hermitian connections on  $E_0 \oplus E_1$  and  $E_0 \otimes E_1$  respectively. Then

$$\mathbf{G}_f(A_0 \oplus A_1) = \mathbf{G}_f(A_0) \wedge \mathbf{G}_f(A_1), \quad \mathbf{ch}(A_0 \oplus A_1) = \mathbf{ch}(A_0) + \mathbf{ch}(A_1),$$

$$\mathbf{ch}(A_0 \otimes A_1) = \mathbf{ch}(A_0) \wedge \mathbf{ch}(A_1).$$

In particular, we have

$$c(E_0 \oplus E_1) = c(E_0)c(E_1), \quad \mathbf{ch}(E_0 \oplus E_1) = \mathbf{ch}(E_0) + \mathbf{ch}(E_1), \tag{10.5.5}$$

$$\mathbf{ch}(E_0 \otimes E_1) = \mathbf{ch}(E_0) \mathbf{ch}(E_1). \tag{10.5.6}$$

**Remark 10.5.4.** Arguing as in Chapter 5 we deduce that the identities (10.5.3), (10.5.4), (10.5.5) uniquely determine the Chern classes. Let us denote by  $c_k^{\text{top}}$  the Chern classes defined in that chapter, and by  $c_k^{\text{geom}}$  the Chern classes defined via the Chern-Weil procedure. If  $E \rightarrow M$  is a smooth complex vector bundle over the smooth manifold  $M$ , then  $c_k^{\text{top}} \in H^{2k}(M, \mathbb{Z})$  and  $c_k^{\text{geom}}(E) \in H^{2k}(M, \mathbb{R})$ . If  $i$  denotes the natural map

$$H^{2k}(M, \mathbb{Z}) \rightarrow H^{2k}(M, \mathbb{R}),$$

then

$$c_k^{\text{geom}}(E) = i(c_k^{\text{top}}(E)).$$

□

**Example 10.5.5.** Suppose  $L \rightarrow M$  is a hermitian line bundle. For any hermitian connection  $A$  we have

$$c(A) = 1 + \frac{i}{2\pi} F(A), \quad \mathbf{ch}(A) = \sum_{k \geq 0} \frac{1}{k!} \left( \frac{i}{2\pi} F(A) \right)^k = e^{c_1(A)}.$$

□





# Homework assignments

## A.1. Homework 1

**Exercise A.1.1.** (a) Prove that  $\mathbb{R}\mathbb{P}^n$  is orientable if and only if  $n$  is odd.

(b) Prove that  $\mathbb{C}\mathbb{P}^n$  is simply connected, and orientable. Note that the (real) dimension of  $\mathbb{C}\mathbb{P}^{2k}$  is divisible by 4 so that we can speak of intersection form. What could be the intersection form

$$Q : H_{2k}(\mathbb{C}\mathbb{P}^{2k}, \mathbb{Z})/\text{Tors} \times H_{2k}(\mathbb{C}\mathbb{P}^{2k}, \mathbb{Z})/\text{Tors} \rightarrow \mathbb{Z}. \quad \square$$

**Exercise A.1.2.** We define an operation  $+$  on the set  $\Omega_n^+$  of oriented cobordism classes of  $n$  dimensional manifolds by setting

$$[M_0, \mu_0] + [M_1, \mu_1] = [M_0 \sqcup M_1, \mu_0 \oplus \mu_1].$$

(a) Prove that  $\Omega_n^+$  is an Abelian group with neutral element  $[S^n, \mu_{S^n}]$ , where  $\mu_{S^n}$  denotes the orientation on the sphere  $S^n$  as boundary of the unit ball in  $\mathbb{R}^{n+1}$ . Moreover

$$[M, \mu_M] + [M, -\mu_M] = [S^n, \mu_{S^n}].$$

(b) Prove that  $(M_0 \# M_1, \mu_0 \# \mu_1) = [M_0, \mu_0] + [M_1, \mu_1]$ , where  $\#$  denotes the connected sum of two manifolds and  $\mu_0 \# \mu_1$  denotes the orientation on  $M_0 \# M_1$  induced by the orientations  $\mu_i$ .  $\square$

**Exercise A.1.3.** (a) Suppose  $(M_0, \mu_0), (M_1, \mu_1)$  are compact oriented manifolds whose dimensions are divisible by 4. Then

$$\tau_{(M_0 \times M_1, \mu_0 \times \mu_1)} = \tau_{(M_0, \mu_0)} \tau_{(M_1, \mu_1)}.$$

(b) Compute the intersection form of  $S^2 \times S^2$ , and the intersection form of  $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$ , where  $\mathbb{C}\mathbb{P}^2$  is equipped with the canonical orientation as a complex manifold, and  $\overline{\mathbb{C}\mathbb{P}^2}$  denotes the same manifold, but equipped with the opposite orientation. Prove that the manifolds  $S^2 \times S^2$  and  $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$  are not homeomorphic.  $\square$

## A.2. Homework 2

**Exercise A.2.1.** Suppose  $Q$  is a symmetric, bilinear, nondegenerate form on the finite dimensional real vector space  $V$ . Prove that the signature of  $Q$  is trivial if and only if there exists a subspace in  $V$  which is lagrangian with respect to  $Q$ .  $\square$

**Exercise A.2.2.** (a) Prove that any complex line bundle over a finite wedge of circles is trivializable.

(b) Suppose  $\Sigma$  is a compact, oriented surface, and  $L \rightarrow \Sigma$  is a complex line bundle. Prove that for any  $x \in \Sigma$  the restriction of  $L$  to  $\Sigma \setminus \{x\}$  is trivializable.

(c) Suppose  $\Sigma$  is a compact, oriented surface. For every point  $x \in \Sigma$  fix an open neighborhood  $U_x$  homeomorphic to an open 2-disk, and set  $V_x := \Sigma \setminus \{x\}$ . Then  $\mathcal{U}_x = \{U_x, V_x\}$  is an open cover of  $\Sigma$ , and the overlap  $\mathcal{O}_x = U_x \cap V_x = U_x \setminus \{x\}$  is homotopy equivalent to a circle.

For every continuous map  $g : \mathcal{O}_x \rightarrow \mathrm{GL}_{\mathbb{C}}(1) \cong \mathbb{C}^*$  we get a complex line bundle  $L_g \rightarrow \Sigma$  obtained by identifying over  $\mathcal{O}_x$  the trivial line bundle  $\underline{\mathbb{C}}_{U_x}$  with the trivial line bundle  $\underline{\mathbb{C}}_{V_x}$  according to the rule

$$\underline{\mathbb{C}}_{U_x} \supset \mathbb{C} \times \mathcal{O}_x \ni (z, p) \mapsto (g(p)z, p) \in \mathbb{C} \times \mathcal{O}_x \subset \underline{\mathbb{C}}_{V_x}.$$

Prove that the correspondence  $g \mapsto L_g$  induces a bijection between the set of homotopy classes of maps  $\mathcal{O}_x \rightarrow \mathbb{C}^*$  and the set of isomorphism classes of complex line bundles over  $\Sigma$ .

(d) We use the notations in (c). Observe that the space of homotopy classes of maps  $\mathcal{O}_x \rightarrow \mathbb{C}^*$  is a multiplicative group, where for any  $g_0, g_1 : \mathcal{O}_x \rightarrow \mathbb{C}^*$  we define  $g_0 \cdot g_1 : \mathcal{O}_x \rightarrow \mathbb{C}^*$  by the equality

$$(g_0 \cdot g_1)(p) = g_0(p)g_1(p), \quad \forall p \in \mathcal{O}_x.$$

Show that

$$L_{g_0 \cdot g_1} \cong L_{g_0} \otimes L_{g_1}. \quad \square$$

**Exercise A.2.3.** Prove that the space of isomorphism classes of complex line bundles over a topological space  $X$  is an Abelian group with respect to the tensor product of two complex line bundles in which the trivial line bundle  $\underline{\mathbb{C}}_X$  is the identity element, and the inverse of a line bundle  $L$  is its dual  $L^*$ . We will refer to this group as the *topological Picard group* of  $X$  and we will denote it by  $\mathrm{Pic}_{\mathrm{top}}(X)$ .  $\square$

**Remark.** Exercise A.2.2(d) shows that for any compact oriented surface  $\Sigma$  the topological Picard group is an infinite cyclic group.

### A.3. Homework 3.

**Exercise A.3.1.** (a) Suppose  $\Sigma$  is a compact oriented Riemann surface of genus  $g$ . The orientation determines a canonical generator of the infinite cyclic group  $H^2(\Sigma, \mathbb{Z})$  which we denote by  $\omega$ . Suppose  $L \rightarrow \Sigma$  is a complex line bundle of degree  $d$ , i.e.,  $c_1(L) = d\omega$ . Compute the integral cohomology of the total space of the unit sphere bundle of  $L$ .

(b) Suppose  $E \rightarrow S^4$  is a rank 4-oriented vector bundle such that its Euler class  $e(E)$  is a generator of the group  $H^4(S^4, \mathbb{Z})$ . Prove that the unit sphere of  $E$  has the same homology as a 7-sphere.  $\square$

**Exercise A.3.2.** (a) Suppose  $(U_\nu)_{\nu \geq 1}$  is a sequence of  $m$ -dimensional subspaces of the separable Hilbert space  $H$ . Then  $U_n$  converges to  $U \in \mathbf{Gr}_m(H)$  in the projector topology if and only if there exists a basis  $\{e_1, \dots, e_m\}$  of  $U$  and bases  $\{e_1(\nu), \dots, e_m(\nu)\}$  of  $U_\nu$  such that

$$\lim_{\nu \rightarrow \infty} |e_i(\nu) - e_i| = 0, \quad \forall i = 1, \dots, m.$$

(b) Suppose  $f : X \rightarrow \mathbf{Gr}_m(H)$  is a continuous map. Show that if  $X$  is compact then there exists a finite dimensional subspace  $V \subset H$  such that  $f$  and a map  $g : X \rightarrow \mathbf{Gr}_m(H)$  such that  $f$  is homotopic to  $g$  and  $g(X) \subset \mathbf{Gr}_m(V)$ .  $\square$

**Exercise A.3.3.** Suppose  $\Sigma$  is a compact oriented surface of genus  $g$ . Fix a point  $x_0 \in \Sigma$ , set  $V = \Sigma \setminus x_0$ , and choose a neighborhood  $U$  of  $x_0$  homeomorphic to the unit open disk in  $\mathbb{R}^2$  centered at the origin. Denote by  $U_{1/2}$  the closed neighborhood of  $x_0$  contained in  $U$  which corresponds to the closed disk of radius  $1/2$  centered at the origin. We regard  $U_{1/2}$  as a manifold with boundary  $C = \partial U_{1/2}$ . The boundary is a circle equipped with the orientation as boundary of  $U_{1/2}$ .

As we have seen in the previous homework, any continuous map  $g : U \cap V \rightarrow \mathbb{C}^*$  defines a complex line bundle  $L_g \rightarrow \Sigma$  obtained by the identification

$$(\mathbb{C}_U)|_{U \cap V} \xrightarrow{g} (\mathbb{C}_V)|_{U \cap V}$$

We denote by  $\deg g$  the degree of the map

$$S^1 \cong C \ni p \mapsto \frac{1}{|g(p)|} g(p) \in S^1 \subset \mathbb{C}^*.$$

If  $\mu_\Sigma \in H_2(\Sigma, \mathbb{Z})$  denotes the generator defined by the orientation of  $\Sigma$ , prove that

$$\deg g = -\langle c_1(L_g), \mu_\Sigma \rangle.$$

**Hint:** We already know that the map  $g \mapsto L_g$  gives a group isomorphism

$$[U \cap V, \mathbb{C}^*] \ni g \mapsto \mathbf{VB}_{\mathbb{C}}^1(\Sigma).$$

Note that we have two groups morphisms

$$[U \cap V, \mathbb{C}^*] \ni g \mapsto \deg g \in \mathbb{Z}, \quad \mathbf{VB}_{\mathbb{C}}^1(\Sigma) \ni L \mapsto -\langle c_1(L), \mu_\Sigma \rangle \in \mathbb{Z}$$

so you have to prove that the diagram below of morphisms of Abelian groups is commutative.

$$\begin{array}{ccc} [U \cap V, \mathbb{C}^*] & \longrightarrow & \mathbf{VB}_{\mathbb{C}}^1(\Sigma) \\ \searrow \text{deg} & & \swarrow -\langle c_1(-), \mu_\Sigma \rangle \\ & \mathbb{Z} & \end{array}$$

Start by proving that the morphism

$$\mathbf{VB}_{\mathbb{C}}^1(\Sigma) \ni L \mapsto -\langle c_1(L), \mu_\Sigma \rangle \in \mathbb{Z}$$

is an isomorphism. Then conclude using the localization formula.  $\square$

### A.4. Homework 4

**Exercise A.4.1.** Prove that if  $E \rightarrow X$  is a complex vector bundle of rank  $r$  and  $L \rightarrow X$  is a complex vector bundle with  $c_1(L) = \mathbf{u} \in H^2(X)$ , then

$$c_k(L \otimes E) = \sum_{i=0}^k \binom{r-i}{k-i} c_i(E) \mathbf{u}^{k-i}. \quad \square$$

**Exercise A.4.2.** Consider the Grassmanian  $\mathbf{Gr}_k = \mathbf{Gr}_k^{\mathbb{R}}$  of  $k$ -dimensional subspaces of the separable real Hilbert space  $H$ . We denote by  $\widetilde{\mathbf{Gr}}_k$  the orientation double cover of  $\mathbf{Gr}_k$  determined by the universal vector bundle  $\mathcal{U}_k \rightarrow \mathbf{Gr}_k$ ; see page 43. We denote by  $\widetilde{\mathcal{U}}_k$  the pullback of  $\mathcal{U}_k$  to  $\widetilde{\mathbf{Gr}}_k$ .

- (a) Prove that  $\widetilde{\mathcal{U}}_k$  is orientable, and in fact, it is equipped with a canonical orientation.  
 (b) Prove that for every compact  $CW$ -complex  $X$  and any oriented rank  $k$  real vector bundle  $E \rightarrow X$  there exists a continuous map  $f : X \rightarrow \widetilde{\mathbf{Gr}}_k$  such that  $E \cong f^*\widetilde{\mathcal{U}}_k$ . Moreover if  $f_0, f_1 \rightarrow \widetilde{\mathbf{Gr}}_k$  are continuous maps, then

$$f_0^*\widetilde{\mathcal{U}}_k \cong f_1^*\widetilde{\mathcal{U}}_k \text{ as oriented vector bundles} \iff f_0 \simeq f_1. \quad \square$$

**Exercise A.4.3.** Identify the 3-sphere with the group of unit quaternions. For  $k, j \in \mathbb{Z}$  we define

$$g_{k,j} : S^3 \rightarrow SO(4), \quad q \mapsto g_{k,j}(q) \in SO(4),$$

where

$$g_{k,j}(q)u = q^k u q^{-j}, \quad \forall q \in S^3, \quad u \in \mathbb{H}.$$

The map  $g_{k,j}$  determines an element in  $\pi_3(SO(4))$  that we denote by  $[g_{k,j}]$ .

Via the clutching construction (see Example 3.1.10 and Section 8.2 for details) we can associate to every map  $g_{k,j}$  an oriented rank 4 real vector bundle  $E_{k,j} \rightarrow S^4$ . We denote by  $e_{k,j}$  its Euler number,

$$e_{k,j} := \langle e(E_{k,j}), [S^4] \rangle \in \mathbb{Z}.$$

- (a) Show that  $e_{k,j} = 0$  if  $k = j$ . **Hint:** Construct a nowhere vanishing section of  $E_{j,j}$ .  
 (b) Show that  $e_{0,1} = -1$ . **Hint:** Use the localization formula in Theorem 5.4.1 for a cleverly chosen nondegenerate section of  $E_{1,0}$ .  
 (c) From Exercise A.4.2 above, we know that the bundle  $E_{k,j}$  determines a unique homotopy class of maps  $S^4 \rightarrow \widetilde{\mathbf{Gr}}_4$ , i.e., a unique element  $\gamma_{k,j} \in \pi_4(\widetilde{\mathbf{Gr}}_4)$ . Show that the map

$$\mathbb{Z}^2 \ni (k, j) \mapsto \gamma_{k,j} \in \pi_4(\widetilde{\mathbf{Gr}}_4)$$

is linear. **Hint:** Show that the map

$$\mathbb{Z}^2 \ni (k, j) \mapsto [g_{k,j}] \in \pi_3(SO(4))$$

is a group morphism.<sup>1</sup> Next show that the clutching construction defines a *group morphism*

$$\pi_3(SO(4)) \rightarrow \pi_4(\widetilde{\mathbf{Gr}}_4).$$

- (d) Show that the map  $\mathbb{Z}^2 \ni (k, j) \mapsto e_{k,j} \in \mathbb{Z}$  is linear and then conclude that

$$e_{k,j} = (k - j).$$

**Hint:** Consult the proof of [MS, Lemma 20.10]. □

**Exercise A.4.4.** Consider again the vector bundles  $E_{k,j} \rightarrow S^4$  in the above exercise and set

$$p_{k,j} := \langle p_1(E_{k,j}), [S^4] \rangle \in \mathbb{Z}.$$

- (a) Prove that the map

$$\mathbb{Z}^2 \ni (k, j) \mapsto p_{k,j} \in \mathbb{Z}$$

<sup>1</sup>It is in fact a group isomorphism. Can you see this?

is linear.

(b) Prove that the vector bundles  $E_{k,j}$  and  $E_{j,k}$  are isomorphic as real (unoriented) vector bundles.

(c) Show that there exists a complex rank 2 vector bundle  $F \rightarrow S^4$  such that  $F$  is isomorphic to  $E_{1,0}$  as real vector bundles and

$$\langle c_2(F), [S^4] \rangle = -1.$$

**Hint:** Construct a real endomorphism  $J : E_{1,0} \rightarrow E_{1,0}$  such that  $J^2 = -\mathbb{1}$ . Then use the localization formula to compute  $c_2(F)$ .

(d) Show that  $p_{1,0} = -2$ . **Hint:** Use part (b) and (6.1.4).

(e) Show that  $p_{k,j} = -2(k + j)$ . □



## Solutions to selected problems

### B.1. Solution to Exercise A.3.2.

We begin by proving an auxiliary result that we will use in the sequel.

**Lemma B.1.1.** *Suppose  $U$  and  $V$  are two finite dimensional subspaces of  $H$ . Denote by  $P_U$  and respectively  $P_V$  the orthogonal projections onto  $U$  and respectively  $V$ . Then the following statements are equivalent.*

- (i) *The restriction of  $P_V$  to  $U$  is 1 – 1 is injective.*
- (ii)  $U \cap V^\perp = 0$ .
- (iii)  $\|P_{V^\perp}P_U\| < 1$ , where  $P_{V^\perp} = \mathbb{1} - P_V$  is the orthogonal projection onto  $V^\perp$ .

**Proof.** The equivalence (i)  $\iff$  (ii) is obvious.

(iii)  $\implies$  (ii). We argue by contradiction. Suppose there exists  $\mathbf{u} \in U \cap V^\perp$ ,  $\mathbf{u} \neq 0$ . Observing that  $(\mathbb{1} - P_V) = P_{V^\perp}$  we deduce

$$P_{V^\perp}^{\perp} \mathbf{u} = \mathbf{u}$$

so that  $\|P_{V^\perp}P_U\mathbf{u}\| = \|\mathbf{u}\|$  contradicting the inequality  $\|P_{V^\perp}P_U\| < 1$ .

(ii)  $\implies$  (iii). Again we argue by contradiction. Suppose that  $\|(\mathbb{1} - P_V)P_U\| \geq 1$ . Note that

$$\|(\mathbb{1} - P_V)P_U\| = \|P_{V^\perp}P_U\| \leq \|P_{V^\perp}\| \cdot \|P_U\| = 1,$$

so that  $\|P_{V^\perp}P_U\| = 1$ . Hence, there exists a sequence of vectors  $\mathbf{x}_n \in H \setminus 0$  such that

$$\|P_{V^\perp}P_U\mathbf{x}_n\| = \left(1 - \frac{1}{n}\right)\|\mathbf{x}_n\| > 0.$$

In particular, if we set  $\mathbf{u}_n := P_U\mathbf{x}_n$  we deduce  $\mathbf{u}_n \neq 0$ . We now have

$$\|\mathbf{u}_n\| \geq \|P_{V^\perp}\mathbf{u}_n\| = \|P_{V^\perp}P_U\mathbf{x}_n\| = \left(1 - \frac{1}{n}\right)\|\mathbf{x}_n\| \geq \left(1 - \frac{1}{n}\right)\|\mathbf{u}_n\|. \quad (\text{B.1.1})$$

We set

$$\mathbf{w}_n := \frac{1}{\|\mathbf{u}_n\|} \mathbf{u}_n.$$

Multiplying (B.1.1) by  $\frac{1}{\|\mathbf{u}_n\|}$  we deduce that for any  $n \geq 1$  we have

$$\|\mathbf{w}_n\| = 1, \quad 1 \geq \|P_{V^\perp} \mathbf{w}_n\| \geq \left(1 - \frac{1}{n}\right). \quad (\text{B.1.2})$$

Since  $U$  is finite dimensional, the unit sphere in  $U$  is compact and we deduce that there exists a subsequence  $(\mathbf{w}_{n_k})$  of  $(\mathbf{w}_n)$  converging to the unit vector  $\mathbf{w}_\infty$  in  $U$ . If we let  $n = n_k$  in (B.1.2) and then let  $k \rightarrow \infty$  we deduce that

$$1 = \|\mathbf{w}_\infty\| = \|P_{V^\perp} \mathbf{w}_\infty\|.$$

This implies  $\mathbf{w}_\infty \in V^\perp$  so that  $U \cap V^\perp \neq \emptyset$ .

□

(a) Suppose now that  $U_\nu \in \mathbf{Gr}_m(H)$  is a sequence of  $m$ -dimensional subspaces such that for some  $U \in \mathbf{Gr}_m(H)$  we have

$$\|P_{U_\nu} - P_U\| \rightarrow 0.$$

Then

$$\|P_{U_\nu^\perp} P_U\| = \|P_U - P_{U_\nu} P_U\| = \|P_U (P_U - P_{U_\nu}) P_U\| \rightarrow 0 \quad \text{as } \nu \rightarrow \infty.$$

If we fix a basis  $\mathbf{e}_1, \dots, \mathbf{e}_m$  of  $U$ , then Lemma B.1.1 implies that the collection

$$\{\mathbf{e}_1(\nu) := P_{U_\nu} \mathbf{e}_1, \dots, \mathbf{e}_m(\nu) := P_{U_\nu} \mathbf{e}_m\}$$

is a basis of  $U_\nu$  for  $\nu$  sufficiently large. Note that  $\lim_{\nu \rightarrow \infty} \mathbf{e}_i(\nu) = \mathbf{e}_i$ .

Conversely, suppose  $\underline{\mathbf{e}}(\nu) := \{\mathbf{e}_1(\nu), \dots, \mathbf{e}_m(\nu)\}$  is a basis of  $U_\nu$  converging to the basis  $\underline{\mathbf{e}} = \{\mathbf{e}_1, \dots, \mathbf{e}_m\}$  of  $U$ . We want to prove that  $\|P_{U_\nu} - P_U\| \rightarrow 0$ .

For every  $\nu$  consider the  $m \times m$  symmetric, positive definite matrix

$$G_\nu = (g_{ij}(\nu)), \quad 1 \leq i, j \leq m, \quad g_{ij}(\nu) = (\mathbf{e}_i(\nu), \mathbf{e}_j(\nu)),$$

where  $(-, -)$  is the inner product on  $H$ . Set

$$A_\nu := G_\nu^{-1/2}, \quad A_\nu = (a_{ij}(\nu))_{1 \leq i, j \leq m}$$

Similarly, define the  $m \times m$  symmetric positive definite matrix  $G = (g_{ij})_{1 \leq i, j \leq m}$  by

$$g_{ij} = (\mathbf{e}_i, \mathbf{e}_j).$$

We set  $A = G^{-1/2}$ ,  $A = (a_{ij}(\nu))_{1 \leq i, j \leq m}$ . We now form new bases

$$\underline{\mathbf{f}}(\nu) = \{\mathbf{f}_1(\nu), \dots, \mathbf{f}_m(\nu)\} \subset U_\nu, \quad \mathbf{f}_i(\nu) = \sum_j a_{ij}(\nu) \mathbf{e}_j(\nu), \quad 1 \leq i \leq m,$$

and

$$\underline{\mathbf{f}} = \{\mathbf{f}_1, \dots, \mathbf{f}_m\}, \quad \mathbf{f}_i = \sum_j a_{ij} \mathbf{e}_j, \quad 1 \leq i \leq m.$$

The basis  $\underline{\mathbf{f}}(\nu)$  (respectively  $\underline{\mathbf{f}}$ ) is orthogonal and it is in fact the orthogonal basis obtained from  $\underline{\mathbf{e}}(\nu)$  (respectively  $\underline{\mathbf{e}}$ ) via the Gram-Schmidt procedure. Then  $\underline{\mathbf{f}}(\nu) \rightarrow \underline{\mathbf{f}}$  and

$$P_{U_\nu}(x) = \sum_i (x, \mathbf{f}_i(\nu)) \mathbf{f}_i(\nu), \quad P_U x = \sum_i (x, \mathbf{f}_i) \mathbf{f}_i$$



$$P_{U_\nu}x - P_Ux = \sum_i (x, \mathbf{f}_i(\nu) - \mathbf{f}_i) \mathbf{f}_i(\nu) + \sum_i ((x, \mathbf{f}_i) (\mathbf{f}_i(\nu) - \mathbf{f}_i))$$

so that

$$\|P_{U_\nu}x - P_Ux\| \leq 2 \left( \sum_{i=1}^m \|\mathbf{f}_i(\nu) - \mathbf{f}_i\| \right) \|x\|.$$

Hence

$$\|P_{U_\nu} - P_U\| \leq 2 \left( \sum_{i=1}^m \|\mathbf{f}_i(\nu) - \mathbf{f}_i\| \right) \xrightarrow{\nu \rightarrow \infty} 0.$$

(b) Let us first prove that there exists a finite dimensional subspace  $V \subset H$  such that  $f(x) \cap V^\perp = 0$ ,  $\forall x \in X$ .

We set

$$\mathcal{U}_x := \{U \in \mathbf{Gr}_m(H); f(x) \cap U^\perp = 0\}.$$

The set  $\mathcal{S}_x$  is nonempty because  $f(x) \in \mathcal{U}_x$ . Lemma B.1.1 implies that  $\mathcal{U}_x$  is open. Hence the collection  $(\mathcal{U}_x)_{x \in X}$  is an open cover of the compact set  $f(X) \subset \mathbf{Gr}_m(H)$ . We denote that there exist finitely many points  $x_1, \dots, x_N \in X$  such that

$$f(X) = \bigcup_{i=1}^n \mathcal{U}_{x_i}.$$

Now denote by  $V$  the sum of the subspaces  $f(x_1), \dots, f(x_N) \subset H$ .  $V$  is finite dimensional, and more precisely,

$$\dim V \leq Nm.$$

If  $x \in X$  then  $f(x) \cap \mathcal{S}_{x_i}$  for some  $i$  so that  $f(x) \cap f(x_i)^\perp = 0$ . Since  $f(x_i) \subset V$  then  $V^\perp \subset f(x_i)^\perp$  so that  $f(x) \cap V^\perp = 0$ . For any  $x \in X$  we set

$$g(x) := P_V f(x) \subset V$$

Since the restriction of  $P_V$  to  $f(x)$  is injective we deduce that  $\dim g(x) = \dim f(x)$  so that  $g(x) \in \mathbf{Gr}_m(V)$ . To prove that  $g$  is continuous choose  $x_\nu \rightarrow x$ . We need to show that  $g(x_\nu) \rightarrow g(x)$ .

Since  $f$  is continuous we deduce  $f(x_\nu) \rightarrow f(x)$ . From part (a) we deduce that there exist bases  $\underline{e}(\nu)$  of  $f(x_\nu)$  and  $\underline{e}$  of  $f(x)$  such that  $\underline{e}(\nu) \rightarrow \underline{e}$ . Then  $\underline{\mathbf{f}}(\nu) := P_V \underline{e}(\nu)$  is a basis of  $g(x_\nu)$ ,  $\underline{\mathbf{f}} = P_V \underline{e}$  is a basis of  $g(x)$  and

$$\underline{\mathbf{f}}(\nu) \rightarrow \underline{\mathbf{f}}.$$

Invoking part (a) again we deduce that  $g(x_\nu) \rightarrow g(x)$ .

Denote by  $\mathbf{GL}(H)$  the group of bounded invertible operators of  $H$ . To prove that  $g$  is homotopic to  $f$  we will construct a continuous map

$$A : [0, 1] \times X \rightarrow U(H), \quad [0, 1] \times X \ni (s, x) \mapsto A_{s,x}$$

such that

$$A_{0,x} = \mathbb{1}_H, \quad A_{1,x} f(x) = g(x).$$

From part (a) can conclude easily that the resulting map

$$F : [0, 1] \times X \rightarrow \mathbf{Gr}_m(H), \quad [0, 1] \times X \ni (s, x) \mapsto F(s, x) := A_{s,x} f(x) \in \mathbf{Gr}_m(H)$$

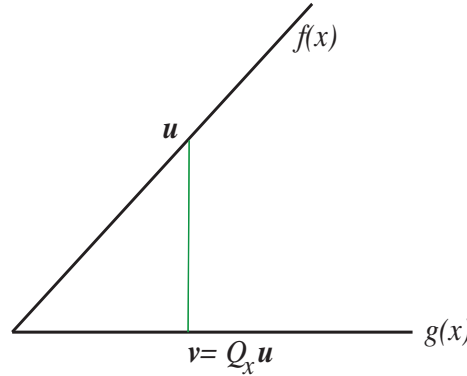
is continuous.

We denote by  $P_x$  the orthogonal projection on  $f(x)$ , by  $Q_x$  the orthogonal projection onto  $g(x)$  and by  $Q_x^\perp$  the orthogonal projection onto  $g(x)^\perp$ .

Since the restriction of  $Q_x$  to  $f(x)$  is one-to-one we deduce from Lemma B.1.1 that  $\|Q_x^\perp P_x\| < 1$ . In particular, for any  $s \in [0, 1]$  the operator

$$A_{s,x} := \mathbb{1} - sQ_x^\perp P_x : H \rightarrow H$$

is invertible, and depends continuously on  $s$  and  $x$ . We claim that  $A_{1,x}$  maps the subspace  $f(x)$  to the subspace  $g(x)$ .



**Figure B.1.** Deforming  $g(x)$  to  $f(x)$ .

Let  $u \in f(x)$  and set  $v = Q_x u$ ; see Figure B.1. Then

$$u = P_x u, \quad v = Q_x u = (\mathbb{1} - Q_x^\perp)u = u - Q_x^\perp P_x u = A_{1,x} u.$$

Hence

$$A_{1,x} f(x) \subset g(x).$$

since  $A_{1,x}$  is one-to-one and  $\dim f(x) = \dim g(x)$  we deduce that  $A_{1,x} f(x) = g(x)$ .  $\square$

## B.2. Solution to Exercise A.4.1

Using splitting principle we can reduce the problem to the special case when  $E$  splits as a direct sum of line bundles

$$E = L_1 \oplus \cdots \oplus L_r.$$

We set

$$\mathbf{x}_i = c_1(L_i)$$

so that

$$c_k(E) = \sigma_k(\mathbf{x}_1, \dots, \mathbf{x}_r) = \sum_{1 \leq i_1 < \cdots < i_k \leq r} \mathbf{x}_{i_1} \cdots \mathbf{x}_{i_k}. \quad (\text{B.2.1})$$

Then

$$L \otimes E = L \otimes L_1 \oplus \cdots \oplus L \otimes L_r,$$

$$c_1(L \otimes L_i) = \mathbf{u} + \mathbf{x}_i$$

so that

$$c_k(L \otimes E) = \sigma_k(\mathbf{u} + \mathbf{x}_1, \dots, \mathbf{u} + \mathbf{x}_r).$$

Equivalently,  $c_k(L \otimes E)$  is the coefficient of  $t^k$  in the polynomial

$$P(t, \mathbf{u}, \mathbf{x}) = \prod_{i=1}^r (1 + t(\mathbf{u} + \mathbf{x}_i)).$$

To proceed further we need to introduce a notation. For every subset  $S \subset \{1, \dots, r\}$  we set

$$(\mathbf{u} + \mathbf{x})^S := \prod_{i \in S} (\mathbf{u} + \mathbf{x}_i).$$

Then

$$c_k(L \otimes E) = \sum_{|S|=k} (\mathbf{u} + \mathbf{x})^S.$$

This shows if we express  $c_k(L \otimes E)$  as a polynomial in the variables  $\mathbf{x}_1, \dots, \mathbf{x}_r, \mathbf{u}$ , then we encounter only monomials of the form  $\mathbf{u}^n \mathbf{x}_1^{n_1} \dots \mathbf{x}_r^{n_r}$ , where  $n_i = 0, 1$ . Hence, we can write

$$c_k(L \otimes E) = \sum_{i=0}^k A_i c_i(E) \mathbf{u}^{k-i}$$

for some positive integers  $A_i$ . If we expand  $c_k(L \otimes E)$  as a polynomial in the variables  $\mathbf{x}_1, \dots, \mathbf{x}_r, \mathbf{u}$ , then we see that  $A_i$  is the coefficient of the monomial  $\mathbf{x}_1 \dots \mathbf{x}_i \mathbf{u}^{k-i}$  in this polynomial. Such a monomial appears only in products of the form  $(\mathbf{u} + \mathbf{x})^S$ , where  $S$  is a cardinality  $k$  subset of  $\{1, \dots, r\}$  containing the string  $\{1, 2, \dots, i\}$ . There are  $\binom{r-i}{k-1}$  such subsets, and for any such subset  $S$  the coefficient of  $\mathbf{x}_1 \dots \mathbf{x}_i \mathbf{u}^{k-i}$  in  $(\mathbf{u} + \mathbf{x})^S$  is 1. This shows that  $A_i = \binom{r-i}{k-i}$ .  $\square$

### B.3. Solution to Exercise A.4.2.

(a) Consider the real line bundle  $\det \mathcal{U}_k$ . Fix a metric on this line bundle. Then  $\widetilde{\mathbf{Gr}}_k$  is nothing but  $S(\det \mathcal{U}_k)$ , the unit sphere bundle of  $\det \mathcal{U}_k$ . Denote by  $\pi : \widetilde{\mathbf{Gr}}_k \rightarrow \mathbf{Gr}_k$  the natural projection. The map  $\pi$  is double cover.<sup>1</sup> The points of  $\widetilde{\mathbf{Gr}}_k$  are pairs  $(x, \delta)$ , where  $x \in \mathbf{Gr}_k$ , and  $\delta$  is a unit vector in the fiber of  $\det \mathcal{U}_k$  over  $x$ . The projection  $\pi$  is then given by  $(x, \delta) \mapsto x$ . Observe that  $\det \widetilde{\mathcal{U}}_k = \pi^* \det \mathcal{U}_k$  and thus the fiber of  $\det \widetilde{\mathcal{U}}_k$  over  $(x, \delta)$  can be identified with the fiber of  $\det \mathcal{U}_k$  over  $x$ . This line bundle has a nowhere vanishing section

$$\omega : \widetilde{\mathbf{Gr}}_k \rightarrow \det \widetilde{\mathcal{U}}_k, \quad (x, \delta) \mapsto \delta \in \det \mathcal{U}_k(x) = \det \widetilde{\mathcal{U}}_k(x, \delta).$$

This shows that  $\widetilde{\mathcal{U}}_k$  is equipped with a natural orientation.

(b) Suppose  $E \rightarrow X$  is an oriented rank  $k$  real vector bundle over the compact  $CW$ -complex. Then there exists a continuous map  $f : X \rightarrow \mathbf{Gr}_k$  such that  $E \cong f^* \mathcal{U}_k$  (as unoriented vector bundles). Note that  $\det E \cong \det f^* \mathcal{U}_k = f^*(\det \mathcal{U}_k)$ . The orientation of  $E$  defines a nowhere vanishing section  $u$  of  $f^*(\det \mathcal{U}_k)$  which we can assume has pointwise length one. This section defines continuous map

<sup>1</sup>This double cover is *nontrivial*.

$\hat{f} : X \rightarrow S(\det \mathcal{U}_k)$  such that the diagram below is commutative

$$\begin{array}{ccc} & S(\det \mathcal{U}_k) = \widetilde{\mathbf{Gr}}_k & \\ & \nearrow \hat{f} & \downarrow \pi \\ X & \xrightarrow{f} & \mathbf{Gr}_k \end{array}$$

Then clearly  $E \cong \hat{f}^* \widetilde{\mathcal{U}}_k$ .

For any continuous map  $f : X \rightarrow \widetilde{\mathbf{Gr}}_k$  we define  $\bar{f} : X \rightarrow \mathbf{Gr}_k$  by setting  $\bar{f} = \pi \circ f$ , where  $\pi$  is the natural projection  $\widetilde{\mathbf{Gr}}_k \rightarrow \mathbf{Gr}_k$ . Since  $\pi$  is a covering map we deduce that two maps  $f_0, f_1 : X \rightarrow \widetilde{\mathbf{Gr}}_k$  are homotopic if and only if the maps  $\bar{f}_0$  and  $\bar{f}_1$  are homotopic. (To see this use the homotopy lifting properties of a covering map, [Hatch1, Prop. 1.30].)

If the bundles  $f_0^* \widetilde{\mathcal{U}}_k$  and  $f_1^* \widetilde{\mathcal{U}}_k$  are isomorphic as oriented vector bundles, they are also isomorphic as unoriented vector bundles so that  $\bar{f}_0^* \mathcal{U}_k \cong \bar{f}_1^* \mathcal{U}_k$ . From the classification theorem (Theorem 3.3.1) we deduce that the maps  $\bar{f}_0$  and  $\bar{f}_1$  are homotopic. Therefore  $f_0$  and  $f_1$  must be homotopic as well.

Suppose  $f_0 \cong f_1$ . Then the bundles  $f_0^* \widetilde{\mathcal{U}}_k$  and  $f_1^* \widetilde{\mathcal{U}}_k$  are isomorphic as (unoriented) vector bundles. To prove that they are isomorphic as *oriented* vector bundles argue exactly as in the proof of Proposition 3.2.9 making sure that all the various trivializations constructed at every step are compatible with the orientations.  $\square$

#### B.4. Solution to Exercise A.4.3

(a) A section of  $E_{k,j}$  is defined by a pair of smooth functions  $s_{\pm} : D^{\pm} \rightarrow \mathbb{H}$  such that

$$s_- = g_{k,j}(q)s_+$$

For all  $q$  on the Equator of the 4-sphere. If  $k = j$  then we can choose

$$s_{\pm} : D^{\pm} \rightarrow \mathbb{H}, \quad s_{\pm} \equiv 1.$$

This defines a nonvanishing section of  $E_{j,j}$  so that  $e(E_{j,j}) = 0$ .

(b) We denote by  $u_{\pm}$  the stereographic coordinates on  $D^{\pm}$  defined as in Section 8.2. Recall that they are related by

$$u_+ = \frac{1}{|u_-|^2} u_- = \bar{u}_-^{-1}, \quad u_- = \frac{1}{|u_+|^2} u_+,$$

where  $\bar{q}$  denotes the conjugate of the quaternion  $q$ . We replace the gluing maps  $g_{kj}$  in the form described in (8.2.1) to the homotopic family

$$\hat{g}_{k,j} : D^+ \cap D^- \rightarrow \mathrm{GL}(\mathbb{R}^4),$$

$$\hat{g}_{k,j}(x)v = u_+(x)^k v u_+(x)^{-j}, \quad \forall x \in D^+ \cap D^-, \quad v \in \mathbb{R}^4 = \mathbb{H}.$$

A section of  $E_{0,1}$  is given by a pair of functions smooth  $s_{\pm} : D^{\pm} \rightarrow \mathbb{H}$  such that

$$s_-(x) = \hat{g}_{0,1}(x)s_+(x), \quad \forall x \in D^+ \cap D^-.$$

If we use the coordinates  $u_+$  on the overlap  $D^+ \cap D^-$  we can rewrite the above equality

$$s_-(x) = s_+(x)u_+^{-1}.$$

If we choose  $s_+ \equiv 1$ , then we deduce that

$$s_-|_{D^+ \cap D^-} = \bar{u}_-.$$

Thus, the pair of functions

$$s_{\pm} : D^{\pm} \rightarrow \mathbb{H}, \quad s_+(u_+) = 1, \quad s_-(u_-) = \bar{u}_-, \quad \forall u_{\pm} \in D^{\pm}$$

defines a smooth section of  $E_{0,1}$  that has a single nondegenerate zero at  $u_- = 0$ . The sign associated to this zero is the sign of the determinant of the  $\mathbb{R}$ -linear map  $\mathbb{H} \ni v \mapsto \bar{v} \in \mathbb{H}$ . This sign is negative so that  $e_{0,1} = -1$ .

(c) The space  $G = S^3 \times S^3$  is a Lie group and thus the multiplication in  $\pi_k(G)$  can be described by pointwise multiplication. More precisely, if  $f, g : S^k \rightarrow G$ , define elements  $[f]$  and  $[g]$  in  $\pi_k(G)$  then the homotopy class of the map

$$f \bullet g : S^k \rightarrow G, \quad (f \bullet g)(x) = f(x) \cdot g(x), \quad \forall x \in S^k$$

coincides with the homotopy class of  $[f] * [g]$ , where  $*$  denotes the group operation.

Denote by  $\lambda \in \pi_3(S^3 \times S^3)$  the homotopy class determined by the map

$$S^3 \rightarrow S^3 \times S^3, \quad x \mapsto (x, 1).$$

Similarly, denote by  $\rho$  the homotopy class determined by the map

$$S^3 \rightarrow S^3 \times S^3, \quad x \mapsto (1, x).$$

Then, the element of  $\pi_3(S^3 \times S^3)$  determined by the map

$$f_{k,j} : S^3 \mapsto S^3 \times S^3, \quad S^3 \ni q \mapsto (q^k, q^j) \in S^3 \times S^3$$

is  $k\lambda + j\rho$ .

The group  $\pi_3(S^3 \times S^3)$  is a free Abelian group with generator  $\lambda$  and  $\rho$  so that the map

$$\Phi : \mathbb{Z} \rightarrow \pi_3(S^3, S^3), \quad (k, j) \mapsto [f_{k,j}]$$

is a group isomorphism.

We now have a group morphism

$$T : S^3 \times S^3 \rightarrow SO(4), \quad S^3 \times S^3 \ni (q_1, q_2) \mapsto T_{q_1, q_2} \in SO(4), \quad T_{q_1, q_2} u = q_1 u q_2^{-1}$$

Note that  $T \circ f_{k,j} = g_{k,j}$  so that

$$[g_{k,j}] = T_*[f_{k,j}] \in \pi_3(SO(4))$$

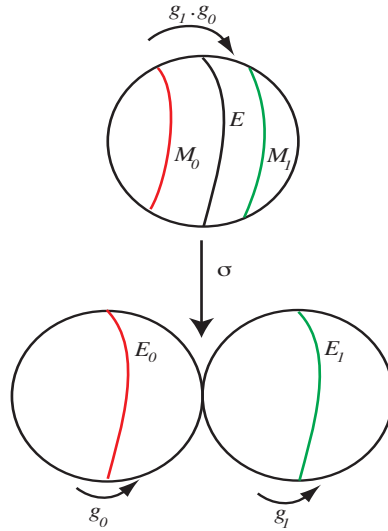
where  $T_*$  denotes the morphism in  $\pi_3$  induced by  $T$ . This proves that the correspondence

$$(k, j) \mapsto [g_{k,j}]$$

is a group morphism.

Consider the unit sphere  $S^4$  in the Euclidean space  $\mathbb{R}^5$  with coordinates  $(x^0, \dots, x^4)$ . Denote by  $E$  equator  $\{x^0 = 0\}$ , by  $M_0$  the meridian  $\{x^0 = -1/2\}$  and by  $M_1$  the meridian  $\{x^0 = 1/2\}$ ; see top of Figure B.2.

There is a continuous map  $\sigma : S^4 \rightarrow S^4 \vee S^4$  that maps  $E$  to the base point of the wedge, the meridian  $M_0$  to an Equator  $E_0$  of one of the spheres, and  $M_1$  to an equator  $E_1$  of the other sphere; see Figure B.2.



**Figure B.2.** Deforming  $g(x)$  to  $f(x)$ .

If  $c_0, c_1 \in \pi_3(SO(4), 1)$  are defined by maps  $g_0, g_1 : (S^3, *) \rightarrow (SO(4), 1)$ , then  $c_0 + c_1$  is represented by the map

$$g_1 \cdot g_0 : S^3 \rightarrow SO(4), \quad x \mapsto g_1(x) \cdot g_0(x).$$

We use  $g_0$  and  $g_1$  as clutching maps defining two oriented, real, rank 4-vector bundles  $E_0, E_1 \rightarrow S^4$ . Fix classifying maps

$$\gamma_0 = \gamma(g_0), \quad \gamma_1 = \gamma(g_1) : S^4 \rightarrow \widetilde{\mathbf{Gr}}_4.$$

for the bundles  $E_0$  and  $E_1$  such that  $\gamma_0(*) = \gamma_1(*) = L$  for some based point  $* \in S^4$ , and some fixed  $L \in \widetilde{\mathbf{Gr}}_4$ . Then the element  $[\gamma_0] + [\gamma_1]$  in  $\pi_4(\widetilde{\mathbf{Gr}}_4, L)$  is represented by the composition

$$(S^4, *) \xrightarrow{\sigma} (S^4, *) \vee (S^4, *) \xrightarrow{\gamma_0 \vee \gamma_1} (\widetilde{\mathbf{Gr}}_4, L).$$

We obtain a bundle  $E_0 \vee E_1$  on  $S^4 \vee S^4$  and by pullback a bundle  $\sigma^*(E_0 \vee E_1)$  on  $S^4$ . This bundle can be described by the the open cover

$$\mathcal{U} = \{U_-, U_0, U_+\}, \quad U_- = \{x^0 < 0\}, \quad U_0 = \{|x^0| < 1/2\}, \quad U_+ = \{x^0 > 0\}.$$

and transition maps

$$g_{0,-} = g_0, \quad g_{+,0} = g_1.$$

(In Figure B.2  $U_-$  is the region to the left of the equator  $E$ ,  $U_0$  is the region between the meridians  $M_0$   $M_1$  and  $U_+$  is the region to the right of the equator  $E$ .)

This bundle is isomorphic to the bundle given by the open cover

$$\mathcal{V} = \{V_-, V_+\}, \quad V_- = \{x^0 < 1/2\}, \quad V_+ = \{x^0 > -1/2\},$$

and gluing map  $g_1 \cdot g_0$ . (In Figure B.2,  $V_-$  is the region to the left of  $M_1$ , and  $V_+$  is the region to the right of  $M_0$ .) This bundle is the same as the bundle given by the clutching construction with clutching map  $g_1 \cdot g_0$ . This shows that the homotopy type of the classifying map  $\gamma(g_1 \cdot g_0)$  of the bundle described by the clutching  $g_1 \cdot g_0$  is the sum of the homotopy types of  $\gamma(g_1), \gamma(g_0)$

$$[\gamma(g_1 \cdot g_0)] = [\gamma(g_1)] + [\gamma(g_0)].$$

(d) Denote by Let  $\gamma_{k,j} : S^4 \rightarrow \widetilde{\mathbf{Gr}}_4$  denote a smooth map classifying the bundle  $E_{k,j}$ . Denote by  $[S^4]$  the generator of  $H_4(S^4, \mathbb{Z})$  determined by the canonical orientation of  $S^4$ . Then

$$e_{k,j} = \langle e(E_{k,j}), [S^4] \rangle = \langle \gamma_{k,j}^* e(\tilde{U}_4), [S^4] \rangle = \langle e(\tilde{U}_4), (\gamma_{k,j})_* [S^4] \rangle$$

Now consider the Hurewicz morphism

$$h : \pi_4(\widetilde{\mathbf{Gr}}_4) \rightarrow H_4(\widetilde{\mathbf{Gr}}_4).$$

Then

$$(\gamma_{k,j})_* [S^4] = h([\gamma_{k,j}]),$$

where  $[\gamma_{k,j}]$  denotes the element of  $\pi_4(\widetilde{\mathbf{Gr}}_4)$  determined by the classifying map  $\gamma_{k,j}$ . Hence

$$e_{k,j} = \langle e(\tilde{U}_4), h([\gamma_{k,j}]) \rangle$$

Since the map  $(j, k) \mapsto [\gamma_{k,j}]$  is linear and  $h$  is a morphism, we deduce that the map  $\mathbb{Z}^2 \ni (j, k) \mapsto e_{k,j} \in \mathbb{Z}$  is linear. Thus, there exist integers  $m, n$  such that

$$e_{k,j} = mj + nk, \quad \forall j, k.$$

Since  $e_{k,j} = 0$  if  $j = k$  (by (a)) we deduce  $m + n = 0$ . Finally,  $e_{0,1} = -1$  so that  $m = 1, n = -1$  and  $e_{k,j} = (k - j)$ .  $\square$

### B.5. Solution to Exercise A.4.4.

(a) Argue exactly as in the proof of the linearity of  $e_{k,j}$ . In particular we deduce that there exist integers  $m, n$  such that, for any  $k, j$  we have

$$p_{k,j} = mk + nj. \tag{B.5.1}$$

(b) A bundle isomorphism  $E_{k,j} \rightarrow E_{j,k}$  is described by a pair of smooth maps

$$T_{\pm} : D^{\pm} \rightarrow GL_4(\mathbb{R})$$

such that for any  $x \in D^+ \cap D^-$  we have a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^4 & \xrightarrow{T_+(x)} & \mathbb{R}^4 \\ g_{k,j}(x) \downarrow & & \downarrow g_{j,k}(x) \\ \mathbb{R}^4 & \xrightarrow{T_-(x)} & \mathbb{R}^4 \end{array}$$

Assume for definiteness that  $k \geq j$ . We identify  $\mathbb{R}^4$  with  $\mathbb{H}$  and use the coordinates  $u_{\pm}$  on  $D^{\pm}$ . We seek  $T_{\pm}$  of the form

$$T_+(u_+)v = |u_+|^a \bar{v}, \quad T_-(u_-)v = |u_-|^b \bar{v}$$

where  $a, b$  are two nonnegative numbers to be determined later, and  $v \mapsto \bar{v}$  denotes the conjugation in  $\mathbb{H}$ .

Observe that on the overlap  $D^{\pm}$  we can use either of the two coordinates  $u_{\pm}$  and we have

$$T_-(u_+)v = |u_+|^{-b} \bar{v}.$$

We determine  $a, b$  from the equality

$$g_{j,k}(u_+) \cdot T_+(u_+) = T_-(u_+)g_{k,j}(u_+), \quad \forall u_+ \in D^+ \cap D^-.$$

Thus we need to verify that for every  $v \in \mathbb{H}$  and every  $u_+$  we have the equality

$$|u_+|^{a-(j-k)} u_+^j \bar{v} u_+^{-k} = |u_+|^{b-(k-j)} \overline{u_+^k v u_+^{-j}}$$

For simplicity, we write  $u$  instead of  $u_+$ . Thus we need to verify

$$|u|^{a-(j-k)} u^j \bar{v} u^{-k} = |u|^{b-(k-j)} \overline{u^k v u^{-j}}.$$

Note that

$$\overline{u^k v u^{-j}} = \bar{u}^{-j} \bar{v} \bar{u}^k.$$

Now use the equality  $\bar{u} = |u|^2 u^{-1}$  to deduce

$$\bar{u}^{-j} \bar{v} \bar{u}^k = |u|^{2(k-j)} u^j \bar{v} u^{-k}.$$

Thus we need to find  $a, b$  so that

$$|u|^{a+(k-j)} u^j \bar{v} u^{-k} = |u|^{b+(k-j)} u^j \bar{v} u^{-k}$$

for any  $u \in \mathbb{H} \setminus 0, v \in \mathbb{H}$ . Thus if  $a = b = 0$  we obtain a bundle isomorphism  $E_{k,j} \rightarrow E_{j,k}$ . This bundle isomorphism is not orientation preserving, but it still implies  $p_{k,j} = p_{j,k}$  using this in (B.5.1) we deduce that  $m = n$  so that

$$p_{k,j} = m(k+j), \quad \forall k, j. \quad (\text{B.5.2})$$

(c) An automorphism of  $E_{1,0}$  is given by a pair of maps

$$J_{\pm} : D^{\pm} \rightarrow \text{GL}_4(\mathbb{R})$$

such that for every  $x \in D^{\pm}$  the diagram below is commutative

$$\begin{array}{ccc} \mathbb{R}^4 & \xrightarrow{J_+(x)} & \mathbb{R}^4 \\ g_{1,0}(x) \downarrow & & \downarrow g_{1,0}(x) \\ \mathbb{R}^4 & \xrightarrow{J_-(x)} & \mathbb{R}^4 \end{array}$$

Define  $J_{\pm} : D^{\pm} \rightarrow \text{GL}_4(\mathbb{R})$  by setting

$$T_{\pm}(x)v = v \cdot \mathbf{i}, \quad \forall x \in D^{\pm}, \quad v \in \mathbb{H}.$$

These maps define an automorphism of  $J : E_{1,0} \rightarrow E_{1,0}$  satisfying  $J^2 = -1$ . Using (6.1.4) we deduce

$$p_1(E_{1,0}) = 2c_2(E_{1,0}, J). \quad (\text{B.5.3})$$

The complex structure  $J$  induces an orientation  $\mathbf{or}_J$  on  $E_{1,0}$  and we have

$$c_2(E_{1,0}, J) = e(E_{1,0}, \mathbf{or}_J).$$

The orientation  $\mathbf{or}_J$  is described by the oriented frame

$$(1, \mathbf{i}, \mathbf{j}, \mathbf{j} \cdot \mathbf{i}) = (1, \mathbf{i}, \mathbf{j}, -\mathbf{k}).$$

This is the opposite of the canonical orientation  $\mathbf{or}_{1,0}$  of  $E_{1,0}$  which is given by the oriented frame  $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ .

Thus

$$e(E_{1,0}, \mathbf{or}_J) = e(E_{1,0}, -\mathbf{or}_{1,0}) = -e(E_{1,0}, \mathbf{or}_{1,0}).$$

Using (B.5.3) we deduce

$$p_{1,0} = -2e_{1,0} = -2.$$



Using this last equality in (B.5.2) we deduce

$$p_{k,j} = -2(k + j). \quad \square$$

**Remark B.5.1.** The computations in the previous exercises show that the tangent bundle of  $S^4$  is isomorphic as oriented bundle to the bundle  $E_{1,-1}$ .  $\square$



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