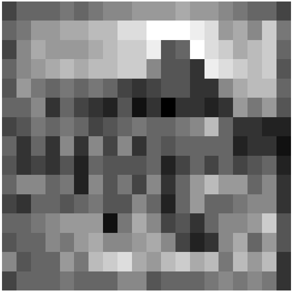
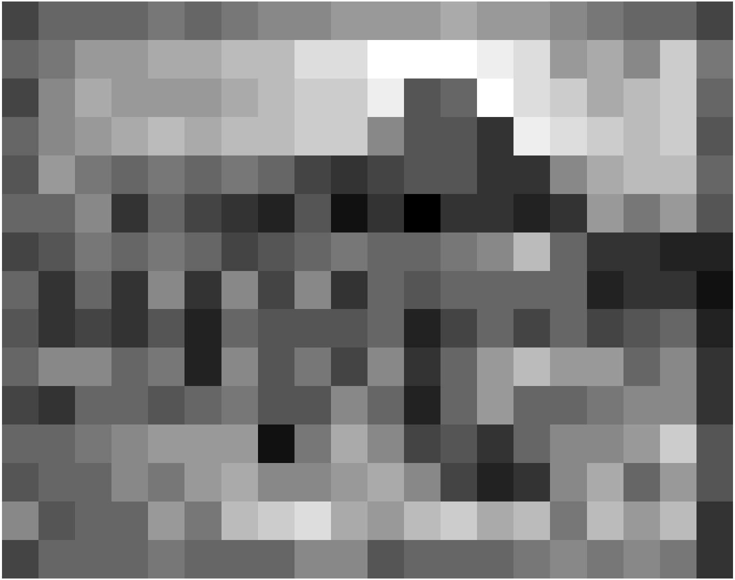


The Geometry of Planar Pixelations and Shape Recognition

Liviu I. Nicolaescu & Brandom Rowekamp

2012



Pixels

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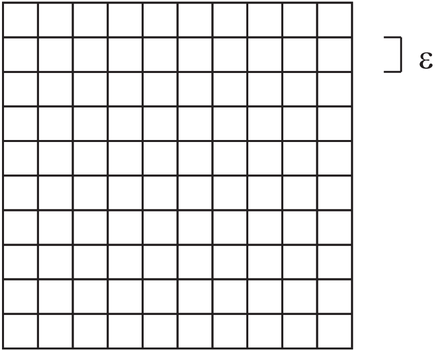
Definition

Let $\varepsilon > 0$. An ε-pixel is a square of the form

$$[\,(m-1)\varepsilon, m\varepsilon\,] \times [\,(n-1)\varepsilon, n\varepsilon\,], \quad m, n \in \mathbb{Z}.$$

The Grid of Pixels



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A pixelation

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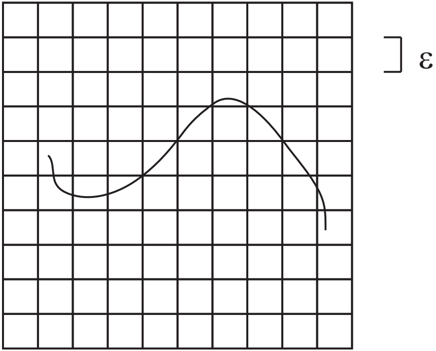

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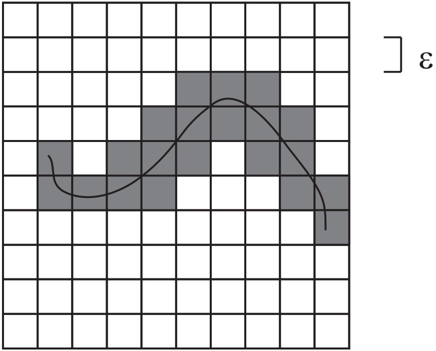

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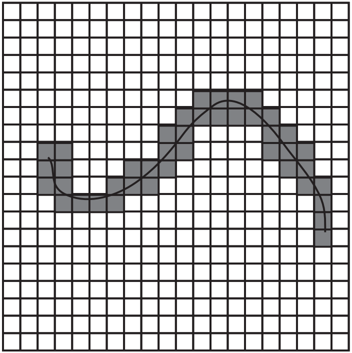

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  - homotopy type, i.e., Betti numbers.

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- Things go downhill from here.

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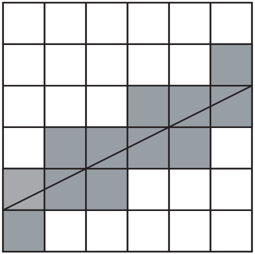

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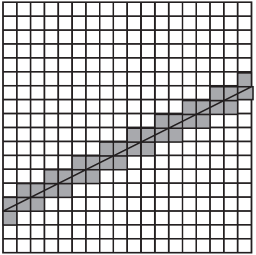

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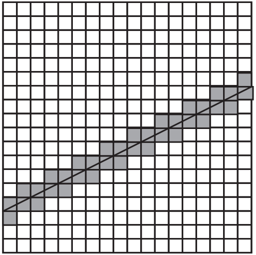

The total curvature of the boundary of the pixelation goes to ∞ as $\varepsilon \searrow 0$.

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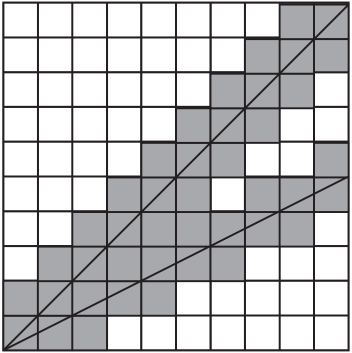

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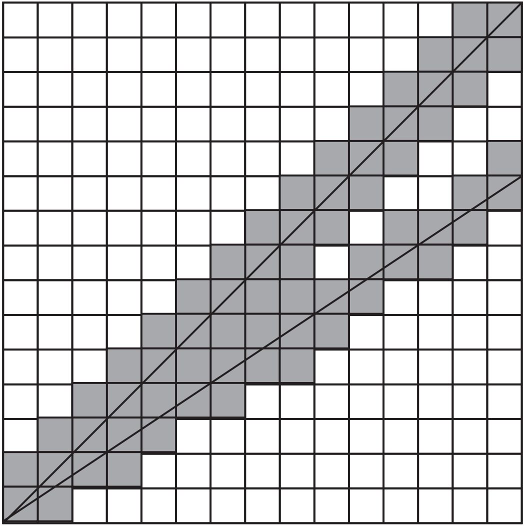

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- More cycles can be found in pixelations of two lines by altering slopes.
- An angle between a line of slope 1 and a line of slope $\frac{n}{n+1}$ will have n cycles appear in each ε-pixelation.

Elementary regions

Definition

A subset $S \subset \mathbb{R}^2$ is said to be elementary if it can be defined as

$$S = S(\beta, \tau) := \{ (x, y) : x \in [a, b], \beta(x) \leq x \leq \tau(x) \},$$

where $\beta, \tau : [a, b] \to \mathbb{R}$ are continuous semialgebraic functions such that $\beta(x) \leq \tau(x)$, $\forall x \in [a, b]$. The function β is called the *bottom* of S while τ is called the *top* of S.

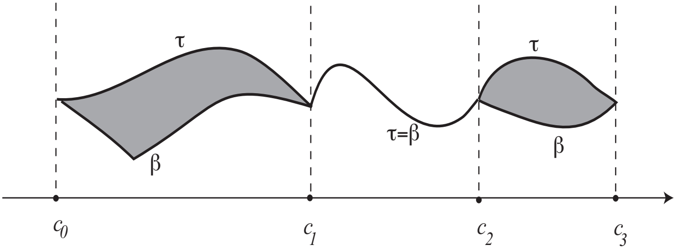

Figure: *An elementary set.*

Columns, stacks and elementary sets

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- The connected components of a column are called stacks.
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A pixelation of an elementary set

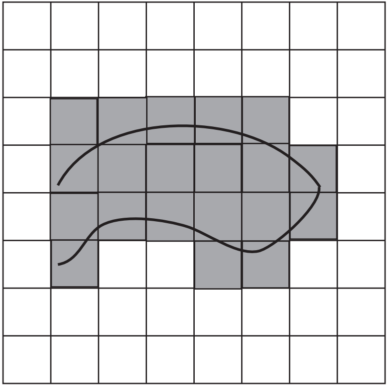

Figure: *The columns of the above pixelation consist of single stacks.*

Approximating the graph of a function

- Let f be a a continuous semialgebraic function and denote by $P_\varepsilon(f)$ the ε-pixelation of its graph.

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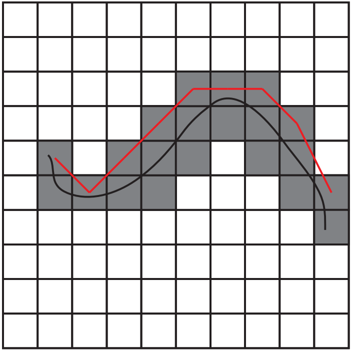

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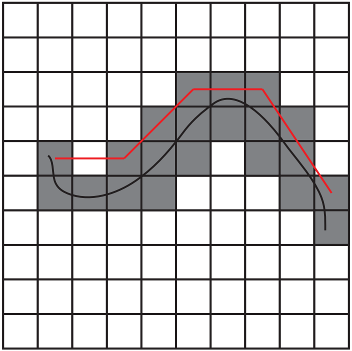

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- Skipping more columns means more possible slopes, but fewer line segments.
- We want to skip a lot of columns, but want many line segments.

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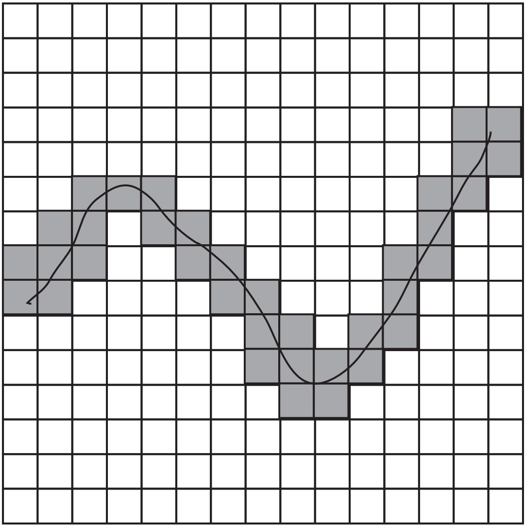

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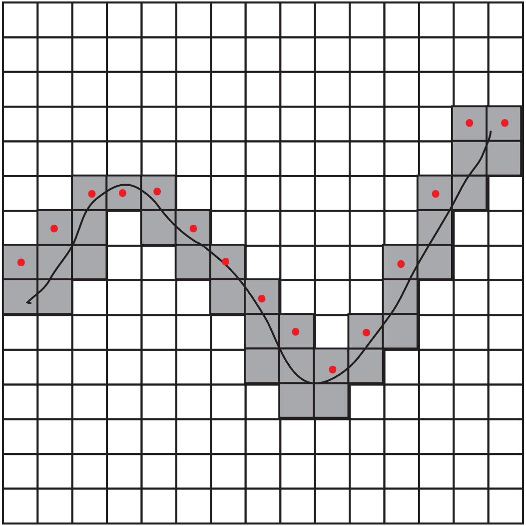

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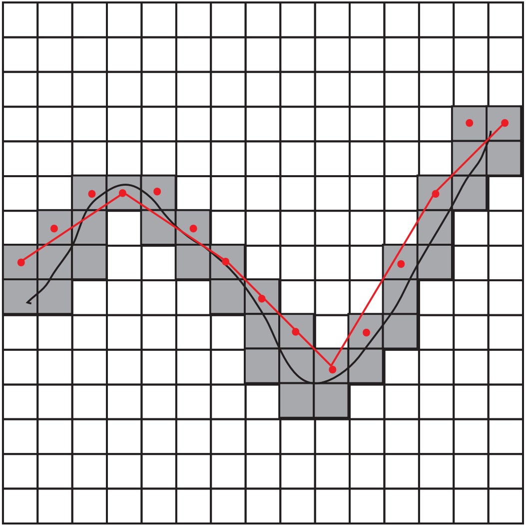

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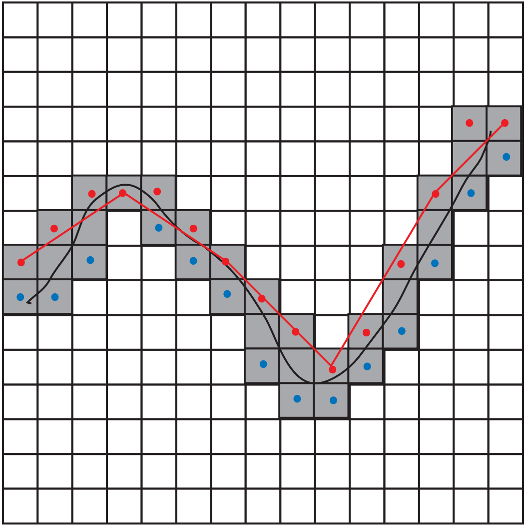

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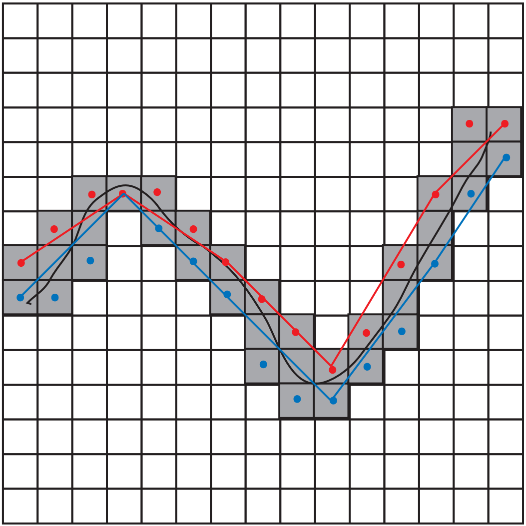

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Theorem (N.& Rowekamp)

Let f be a continuous semialgebraic function with graph Γ_f, and $\sigma : \mathbb{R}_{>0} \to \mathbb{Z}_{>0}$ be a spread function, i.e.,

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then the total curvature of $\mathcal{L}_{\varepsilon}(f)$ *converges to the total curvature of* Γ_f*.*

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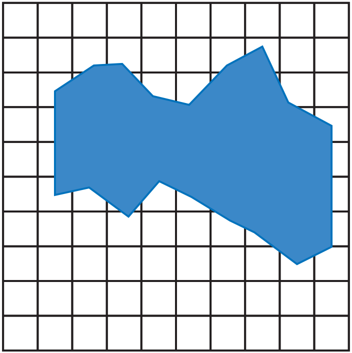
- We denote by $\mathfrak{L}_\varepsilon(S)$ the region between $\mathfrak{L}_\varepsilon^-(S)$ and $\mathfrak{L}_\varepsilon^+(S)$.

Example of Approximation



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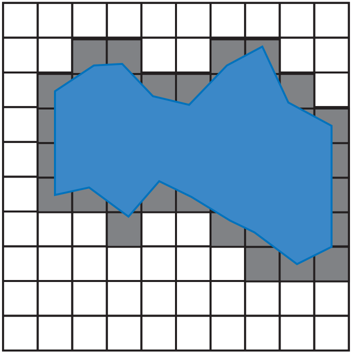

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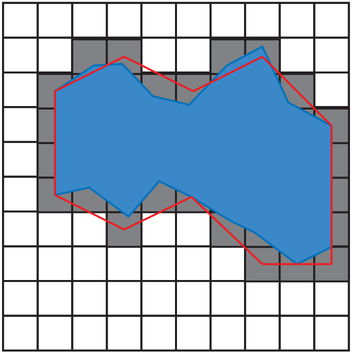

Morse theory to the rescue

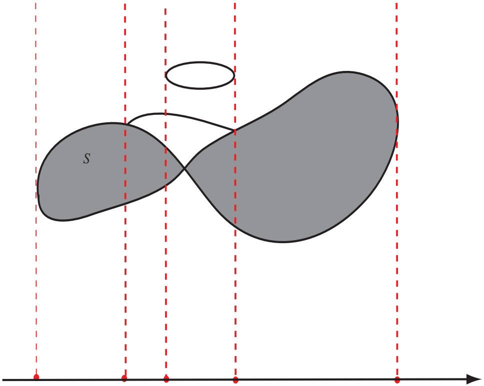

Figure: *A planar semialgebraic set.*

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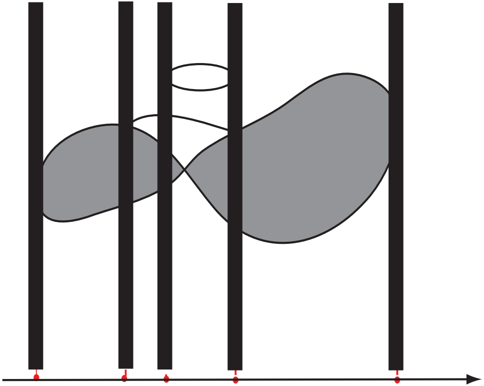

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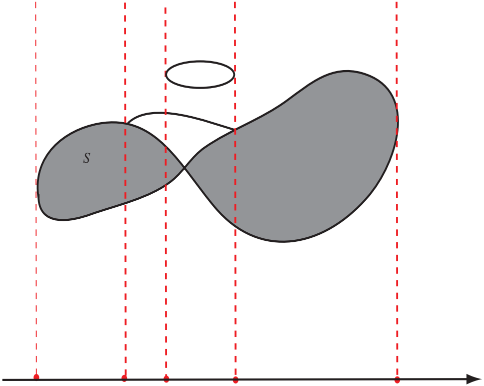

Figure: *A planar semialgebraic set. The •'s mark the jump set $\mathfrak{J}_S$.*

Locating the jump set

Theorem (N. & Rowekamp)

Suppose that $S \subset \mathbb{R}^2$ is a compact semialgebraic set satisfying the following condition.

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Then there exist $\hbar > 0$, $\nu_0 > 0$ and $\kappa_0 \in (0, 1] \cap \mathbb{Q}$, depending only on S such that for $\varepsilon < \hbar$ we have

$$\mathrm{dist}(\mathcal{J}_S, \mathcal{J}_{P_\varepsilon(S)}) < \nu_0 \varepsilon^{\kappa_0}.$$

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- The condition **(G)** is generic.
- If S itself is PL, then we can choose $\kappa_0 = 1$. In general κ_0 depends on the order of contact of the various branches of S at singular points.
- A spread function $\sigma(\varepsilon)$ is said to be compatible with S if it satisfies

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{1}{2}} \sigma(\varepsilon) = \lim_{\varepsilon \searrow 0} \frac{\varepsilon\sigma(\varepsilon)}{\nu_0 \varepsilon^{\kappa_0}} = \infty.$$

For example if $\alpha := \max(\frac{1}{2}, 1 - \kappa_0)$ then

$$\sigma(\varepsilon) = \left\lfloor \varepsilon^{-\frac{1+\alpha}{2}} \right\rfloor$$

will do the trick.

The construction of $\mathcal{L}_\varepsilon$

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(b) For $\varepsilon > 0$ we denote by $\mathcal{N}_\varepsilon$ the union of the open vertical strips

$$V_{\varepsilon,x_0} := \{|x - x_0| < \sigma(\varepsilon)\}, \;\; x_0 \in \mathcal{J}_{P_\varepsilon(S)}$$

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(c) Each of the components of $P_\varepsilon(S) \setminus \mathcal{N}_\varepsilon$ is the pixelation of an elementary region. Using the $\sigma(\varepsilon)$-interpolation method outlined earlier, replace each of these components C with the PL-set $\mathcal{L}_\varepsilon(C)$.

Approximating Noise

(d) Replace each component of $\mathcal{N}_\varepsilon \cap P_\varepsilon(S)$ with the smallest rectangle that contains it.

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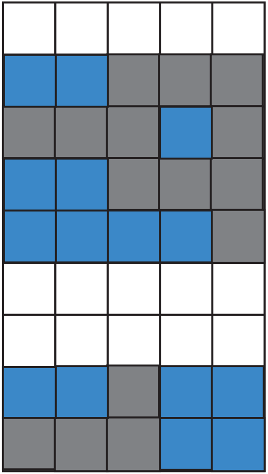
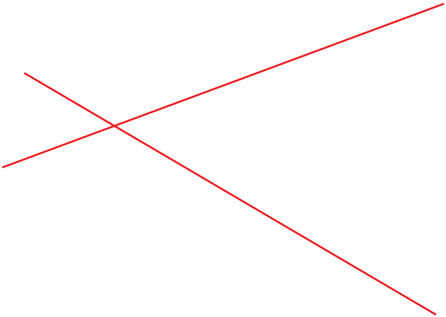


The *PL*-set $\mathcal{L}_\varepsilon(S)$

Define $\mathcal{L}_\varepsilon(S)$ as the union of the *PL*-sets constructed at (c) and (d) above.

Example



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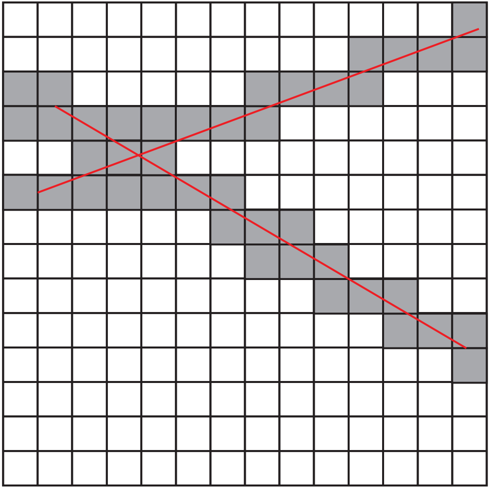

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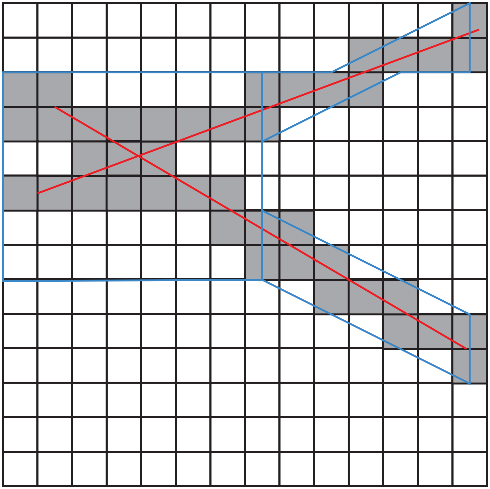

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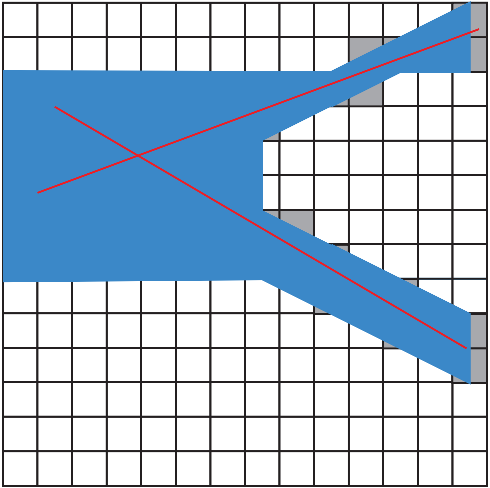

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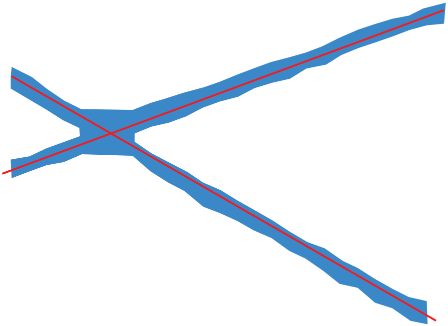

The normal cycle of a planar semialgebraic set

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- It is a current of integration along a union of oriented, semialgebraic Legendrian arcs in $S(T\mathbb{R}^2)$.
- A simple description of $\boldsymbol{N}^S$ when S is a *PL*-set was given by Cheeger-Muller-Schrader-Wintgen.
- The correspondence $S \mapsto \boldsymbol{N}^S$ is injective! Its inverse was explicitly described by Kashiwara-Schapira using the concept of local Euler obstruction. They also gave the first general construction of $\boldsymbol{N}^S$ using micro-local theory of sheaves.

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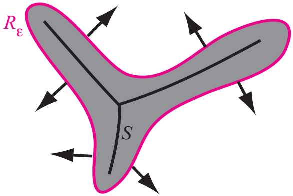

Figure: *An ε-tube around a semialgebraic set S.*

The construction of the normal cycle

Consider the Gauss map $\boldsymbol{n}_\varepsilon : \partial R_\varepsilon \to S^1$ that associates to a point on the boundary its unit outer normal. Its graph

$$\Gamma_{f,\varepsilon} := \{(x, \boldsymbol{n}_\varepsilon(x));\ \ x \in \partial R_\varepsilon\} \subset S(T\mathbb{R}^2)$$

is an oriented Legendrian cycle defining a current of integration $[\Gamma_{f,\varepsilon}]$.

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Theorem (J. Fu)

As $\varepsilon \searrow 0$ the current $[\Gamma_{f,\varepsilon}]$ converges in the sense of currents to a current $[\Gamma_f]$. This limit is independent of the choice of function f and it is, by definition, the normal cycle of S.

The main approximation result

Theorem (N.& Rowekamp)

Suppose that $S \subset \mathbb{R}^2$ is a compact semialgebraic set. For $\varepsilon > 0$ denote by $\mathfrak{L}_\varepsilon(S)$ the PL-set obtained from the pixelation $P_\varepsilon(S)$ by the algorithm outlined earlier. Then the normal cycle $\boldsymbol{N}^{\mathfrak{L}_\varepsilon(S)}$ converges to the normal cycle $\boldsymbol{N}^S$ as $\varepsilon \searrow 0$.

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Suppose S and $(S_n)_{n\geq 0}$ are compact semialgebraic sets in the plane satisfying the following conditions.

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(c) *For almost any closed half-plane $H \subset \mathbb{R}^2$ we have*

$$\lim_{n\to\infty} \chi\big(H \cap S_n \big) = \chi(S),$$

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Then $\boldsymbol{N}^{S_n} \to \boldsymbol{N}^S$ as $n \to \infty$.

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- In our case the mass condition (b) is guaranteed by the fact that the perimeters and total curvatures of $\mathcal{L}_\varepsilon(S)$ are $O(1)$ as $\varepsilon \searrow 0$.

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- The tricky part is proving the condition (c) on convergence of Euler characteristics.


That's all Folks!