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Rigidity of generalized laplacians and some geometric applications

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Summary. Every generalized laplacian L defined on a manifold M determines a sheaf of "L-harmonic" sections namely the sheaf of local solutions of $Lu = 0$. We study the converse problem: to what extent this sheaf determines the operator. Our main result states that the sheaf of L-harmonic sections determines the operator up to a conformal factor. Moreover, when the operator is a covariant laplacian and the dimension of M is greater than 2, the sheaf determines L up to a multiplicative constant. An interesting consequence is the following: if two Riemann metrics on a smooth manifold of dimension greater than 2 have the same sheaves of harmonic functions then they are homothetic.

0. Introduction

Let (M, g) be an N-dimensional Riemann manifold and consider the associated Laplace–Beltrami operator

$$\Delta_g = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j), \qquad \text{with } \partial_i = \partial / \partial x_i. \tag{1}$$

Here $(g^{ij}(x))$ is the inverse matrix of $(g_{ij}(x))$ while $|g| = \det(g_{ij}(x))$. We now call a function *g-harmonic* if it satisfies the equation

$$\Delta_g u = 0. \tag{2}$$

The g harmonic functions form a sheaf $\mathscr{H}_g$ on M. This paper deals with the following

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PROBLEM. *Does the sheaf of g-harmonic functions determine the metric g?*

We shall prove that the sheaf $\mathscr{H}_g$ determines the Laplace–Beltrami operator up to a multiplicative constant and hence determines the metric up to a multiplicative constant.

We approach this problem by asking what is the relationship between two Laplace–Beltrami operators on a smooth manifold M that have the same sheaf of harmonic functions. This suggests a more general question. What is the relationship between two generalized laplacians (cf. [BGV]) that have the same sheaves of local solutions? The main result of this paper shows that the two operators are identical up to a multiplicative term (possible nonconstant). We call this property of elliptic operators *rigidity*. In the case of covariant laplacians their special algebraic form enables one to prove that the multiplicative term mentioned above is constant. We called this property of covariant laplacians *strong rigidity*.

The main idea of the proof is to approximate on small open sets an elliptic operator with variable coefficients by one with constant coefficients (as Schauder did when he proved holder estimates for second order elliptic equations; see [GT]). The proof then consists of two parts. First we prove the rigidity of the operators with constant coefficients and we estimate "the extent of this rigidity". Then we show that this rigidity is transmitted to variable coefficient operators through the above approximation.

The paper is divided into four sections. Section 1 contains the statements of the main results. Section 2 is devoted to technical details while Section 3 contains the proof of the main results. We conclude with a speculative Section 4 where we list a number of problems which seemed interesting to the author.

1. The main results

We begin by recalling the notion of generalized laplacian (see [BGV]). Let $\mathscr{E} \to M$ be a (rank p) real vector bundle over the oriented manifold M. *Fix once and for all a metric h on $\mathscr{E}$.*

DEFINITION 1.1. A second order differential operator $L: C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E})$ is called a *generalized laplacian* if there exists a Riemann metric g on M such that

$$\sigma(L)(x, \xi) = |\xi|_g^2 I_{\mathscr{E}_x}$$

where $\sigma(L)$ is the principal symbol of L.

A generalized laplacian is determined by three pieces of geometrical data:

(i) a metric g on M which determines the second-order pieces;
(ii) a connective $\nabla^{\mathscr{E}}$ on the vector bundle $\mathscr{E}$ which determines the first-order piece;
(iii) a section F of the bundle $End(\mathscr{E})$ which determines the zeroth order piece.

Using local coordinates $(x^1, \ldots, x^N)$ $(N = \dim M)$ in some open set $U \subset M$ and a local trivialization of $\mathscr{E}|_U \cong U \times \mathbf{R}^p$, L can be written as

$$\begin{aligned} Lu &= -\sum_{i,j=1}^{N} g^{ij}\partial_{ij}^2 u + \sum_{i=1}^{n} F_i \partial_i u + Hu \\ &= Gu + Fu + Hu \end{aligned} \tag{1}$$

where the metric $g = (g_{ij})$, F_i, H: $U \to End(\mathbf{R}^p)$ and u: $U \to \mathbf{R}^p$ is a local section of $\mathscr{E}|_U$. We will denote the space of generalized laplacians in $\mathscr{E}$ by $\mathscr{L}(\mathscr{E})$. Geometrically $\mathscr{L}(\mathscr{E})$ is a convex open cone. Moreover

$$\forall \phi \in C^\infty(M) \ (\phi > 0) \quad \text{and} \quad L \in \mathscr{L}(\mathscr{E}): \phi L \in \mathscr{L}(\mathscr{E}). \tag{2}$$

The operators L and ϕL above are called *conformally equivalent*. We denote the conformal equivalence class of L by $[L]_c$. Two laplacians L_1, L_2 will be called *homothetically equivalent* if there exists a positive constant λ so that $L_1 = \lambda L_2$. The homothety equivalence class of L will be denoted by $[L]_h$.

Inside the space $\mathscr{L}(\mathscr{E})$ sits the distinguished family of geometric laplacians (see [BGV]). Namely given a metric g on M and a connection $\nabla^{\mathscr{E}}$ on $\mathscr{E}$ *compatible with the metric h* we can construct the *geometric* (or covariant) *laplacian*

$$L(\nabla^{\mathscr{E}}, g) = -Tr_g(\nabla^{T^*M \otimes \mathscr{E}} \nabla^{\mathscr{E}})$$

where we denote by $Tr_g(S) \in C^\infty(M, \mathscr{E})$ the contraction of an element $S \in C^\infty(M, TM \otimes TM \otimes \mathscr{E})$ with the metric $g \in C^\infty(M, T^*M \otimes T^*M)$. We denote the space of geometric laplacians in $\mathscr{E}$ by $\mathscr{L}_{geom}(\mathscr{E})$.

To any $L \in \mathscr{L}(\mathscr{E})$ we can associate the (pre-)sheaf of local solutions $\mathscr{H}_L$ defined by

$$\forall U \text{ open} \subset M \qquad \mathscr{H}_L(U) = \{u \in C^\infty(U, \mathscr{E}) / Lu = 0\}.$$

The generalized laplacians satisfy the unique continuation property ([Ar], [J]). This shows that the above sheaves have a built-in "rigidity". Namely, if U is a connected open subset of M and V is open $V \subset U$ then the restriction map

$$r_V^U : \mathscr{H}_L(U) \to \mathscr{H}_L(V)$$

is injective (roughly speaking they are far from being flabby).

DEFINITION 1.2. If $\mathscr{S}_1$ and $\mathscr{S}_2$ are two sheaves over a topological space X we say that $\mathscr{S}_2$ contains $\mathscr{S}_1$ (and write $\mathscr{S}_1 \subset \mathscr{S}_2$) if for any open subset U of X we have

$$\mathscr{S}_1(U) \subset \mathscr{S}_2(U).$$

We can now state the main results of this paper.

THEOREM 1.1. *Let $L_1, L_2 \in L(\mathscr{E})$. The following are equivalent:*
(i) $[L_1]_c = [L_2]_c$
(ii) $\mathscr{H}_{L_1} \subset \mathscr{H}_{L_2}$.

We call the above property of generalized laplacians *rigidity.*

THEOREM 1.2. *Assume $N \geq 3$. Let $L_1, L_2 \in \mathscr{L}_{geom}(\mathscr{E})$. The following are equivalent:*
(i) $[L_1]_h = [L_2]_h$
(ii) $\mathscr{H}_{L_1} \subset \mathscr{H}_{L_2}$.

We call the above property of geometric laplacians *strong rigidity.* As a consequence of Theorem 1.2 we have

COROLLARY 1.1. *If two Riemann metrics on a connected smooth manifold of dimension greater than 2 have the same sheaves of harmonic functions then there exists a positive constant μ such that*

$$g = \mu h.$$

Proof. By Theorem 1.2 we have

$$\Delta_g = \lambda^2 \Delta_h \qquad \text{for some } \lambda > 0.$$

Since the principal part of Δ_g is g^{ij} and the principal part of Δ_h is h^{ij} we get the desired result. □

REMARK 1.1. The above result is not true in dimension 2. Indeed consider the euclidian laplacian in $\mathbf{R}^2$, $\Delta = \partial_x^2 + \partial_y^2$. Let $g = (f^2(x, y)\delta_{ij})$ be a conformal metric. Using formula (1) in the introduction, we see that $\Delta_g = -(1/f)\Delta$. Thus $\mathscr{H}_\Delta = \mathscr{H}_{\Delta_g}$. However, clearly the two operators are not homothetically equivalent.

2. Some technical results

We have gathered in this section the main estimates needed in the proof of Theorem 1.1. The preferred functional framework will be that of Sobolev spaces $L^{k,p}$ (functions k-times "differentiable" with derivatives in L^p; see [GT], [M]). Theorem 1.1, 2 have a local nature so we loose no generality if we assume that $M = \mathbf{R}^N$ and $\mathscr{E} \cong M \times \mathbf{R}^p$. We will denote by B_r the open ball of radius r of $\mathbf{R}^N$ (in the euclidian metric) centered at the origin. All the Sobolev or sup norms we will use are defined in the euclidian context. Δ will denote the euclidian laplacian and $\lambda_1(r)$ will denote the first eigenvalue of Δ on B_r with homogeneous Dirichlet conditions. $\lambda_1(r) = \lambda_1(1)/r^2$. Set

$$Du = \sum \partial_i u \, dx^i \quad \text{and} \quad D^2 u = \sum \partial_{ij}^2 u \, dx^i \otimes dx^j$$

$L_0^{1,2}(B_r)$ is the completion of $C_0^\infty(B_r, \mathbf{R}^p)$ in the $L^{1,2}$ norm. The norm in $L_0^{1,2}(B_r)$ is

$$\|u\|_{L_0^{1,2}(B_r)}^2 = \int_{B_r} |Du|^2.$$

We will frequently use the Poincaré inequality in the form (see [M])

$$\|u\|_{L^2(B_r)}^2 \leq r^2/\lambda_1 \|Du\|_{L^2(B_r)}^2, \qquad \forall u \in L_0^{1,2}(B_r, \mathbf{R}^p). \tag{1}$$

Instead of the usual $C^{2,\alpha}(B_r)$-norm $(0 < \alpha < 1)$ we will use the conformal invariant one

$$|||u|||_r = r^{2+\alpha}\|D^2 u\|_{C^\alpha(B_r)} + r^2\|D^2 u\|_{C^0(B_r)} + r\|Du\|_{C^0(B_r)} + \|u\|_{C^0(B_r)}.$$

All differential operators we will deal with have smooth coefficients. To any metric $g = (g_{ij})$ on $\mathbf{R}^N$ one can associate a second order operator

$$G: C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E})$$

by

$$G = \sum g^{ij}\partial_{ij}^2.$$

The standard estimates and existence results for scalar second order elliptic operators continue to hold for vectorial operators G as above. In particular we will use the following

PROPOSITION 2.1. *For any $f \in L^2(\overline{B_r}, \mathscr{E})$ the problem*

$$\begin{cases} Gu = f & \text{in } B_r \\ u = 0 & \text{on } \partial B_r \end{cases} \tag{2}$$

has a unique weak solution $u = T_r(f) \in L^{2,2} \cap L_0^{1,2}(B_r)$ which satisfies

$$\|D(T_r f)\|_{L^2(B_r)}^2 \leq 1/\mu_1(r)\|f\|_{L^2(B_r)}^2 \tag{3}$$

$$\|T_r f\|_{L^2(B_r)} \leq Cr^2\|f\|_{L^2(B_r)} \tag{4}$$

where $0 < \mu_1(r) \leq Cr^{-2}$ (for r small) is the first eigenvalue of (G + Dirichlet conditions) on B_r. If moreover $f \in C^\alpha(\overline{B_r}, \mathscr{E})$ then

$$|||T_r f|||_r \leq Cr^2\|f\|_{C^\alpha(\overline{B_r})} \tag{5}$$

Here as throughout the paper C will denote various constants independent of r and f.

For proof of this result we refer to [GT]. Using standard perturbation techniques (as in [GT] or [M]) we can extend the above result to more general classes of elliptic systems (with scalar principal symbol)

LEMMA 2.1 (Key Estimates). *Let $L \in \mathscr{L}(\mathscr{E})$, $L = G + F + H$ (see (1.1)). Then there exists $\varrho = \varrho(L) > 0$ such that $\forall r < \varrho$ and $\forall f \in C^\alpha(\overline{B_r})$ $(0 < \alpha < 1)$ the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } B_r \\ u = 0 & \text{on } \partial B_r \end{cases} \tag{6}$$

has a unique solution $u = u(f)$ *satisfying*

$$\|u\|_{L^2(B_r)} \leq Cr^2 \|f\|_{L^2(B_r)} \tag{7}$$

and

$$|||u|||_r \leq Cr^2 \|f\|_{C^\alpha(B_r)}. \tag{8}$$

The proof is left to the reader.

REMARK 2.1. From Lemma 2.1 one can deduce in a standard manner that, for $0 < r < \varrho(L)$, $f \in C^\alpha(B_r)$, $g \in C^{2,\alpha}(\partial B_r)$, the nonhomogeneous Dirichlet problem

$$\begin{cases} Lu = f & \text{in } B_r \\ u = g & \text{on } \partial B_r \end{cases}$$

has a unique solution $u \in C^{2,\alpha}(\overline{B_r})$ *(see* [*GT*]).

As a consequence we have the following

LEMMA 2.2. *Let* $L_1, L_2 \in \mathscr{L}(\mathscr{E})$ *such that* $\mathscr{H}_{L_1} \subset \mathscr{H}_{L_2}$. *Then for any* $0 < r <$ $\min(\varrho(L_1), \varrho(L_2))$

$$\mathscr{H}_{L_1}(\overline{B_r}) = \mathscr{H}_{L_2}(\overline{B_r}).$$

Here $\mathscr{H}_{L_1}(\overline{B_r}) = \mathscr{H}_{L_1}(B_r) \cap C^{2,\alpha}(\overline{B_r})$.

Proof. Let $v \in \mathscr{H}_{L_2}(\overline{B_r})$. Choose $u \in C^{2,\alpha}(\overline{B_r})$ solving

$$\begin{cases} L_1 u = L_1 & \text{in } B_r \\ u = 0 & \text{on } \partial B_r. \end{cases}$$

Then $w = v - u$ satisfies

$$\begin{cases} L_i w = 0 & \text{in } B_r \\ w = v & \text{on } \partial B_r \end{cases}$$

so that $w \in \mathscr{H}_{L_1}(\overline{B_r}) \subset \mathscr{H}_{L_2}(\overline{B_r})$. Since on the boundary $w = v \in \mathscr{H}_{\underline{L_1}}(\overline{B_r})$ the uniqueness part in Lemma 2.1 applied to L_2 implies $w = v$ i.e. $v \in \mathscr{H}_{L_1}(B_r)$. Lemma 2.2 is proved. □

Our next result establishes the strong rigidity of constant-coefficient laplacians. The vectorial case is slightly more complicated than the scalar case since in the scalar case we have a simple way of constructing enough solutions namely the exponential solutions. In the scalar case the problem is essentially algebraic. The rigidity is a consequence of Hilbert Nullstellensatz. In the vectorial case we will produce enough solutions via power series but unfortunately there is no elegant theorem such as Hilbert Nullstellensatz.

LEMMA 2.3. *Let* $L, \bar{L} \in \mathscr{L}(\mathscr{E})$ *be two operators with constant coefficients. Then the following are equivalent*

(i) $[L]_h = [\bar{L}]_h$

(ii) $\mathscr{H}_L = \mathscr{H}_{\bar{L}}$

Proof. (i) $\Rightarrow$ (ii) is trivial. We prove (ii) $\Rightarrow$ (i). Let

$$L = G + F + H \quad \text{and} \quad \bar{L} = \bar{G} + \bar{F} + \bar{H}.$$

We first construct a local solution of $Lu = 0$ depending on a single variable $x = x^j$ (for a fixed $j = 1, \ldots, N$). We search for u as a power series

$$u = \sum_{k=0}^{\infty} A_k x^k \qquad (A_k \in \mathbf{R}^p). \tag{9}$$

Since $Lu = 0$, we deduce

$$k(k-1)g^{jj}A_k + (k-1)F^jA_{k-1} + HA_{k-2} = 0 \qquad k \geq 2$$

with A_0, A_1 free parameters. Thus

$$A_k = -\frac{1}{g^{jj}k(k-1)}((k-1)F^jA_{k-1} + HA_{k-2}). \tag{10}$$

We obtain from (10)

$$|A_k| \leq \frac{C}{k}(|A_{k-1}| + |A_{k-2}|) \qquad k \geq 2 \tag{11}$$

for some $C > 0$ independent of k. A simple induction on (11) shows that

$$|A_k| \leq \max(|A_0|, |A_1|)C^k, \qquad k \geq 2.$$

In particular this shows the series (9) is convergent for $|x| < 1/C$ so u is a genuine analytic solution of $Lu = 0$. Since $\mathscr{H}_L = \mathscr{H}_{\bar{L}}$ we deduce $\bar{L}u = 0$. From (10) with $k = 2$ we get

$$\frac{g^{jj}}{2}(2F^j A_1 + HA_0) = \frac{\bar{g}^{jj}}{2}(2\bar{F}^j A_1 + \bar{H}A_0) \qquad \forall A_0, A_1 \in \mathbf{R}^p.$$

Therefore

$$1/g^{jj}F^j = 1/\bar{g}^{jj}\bar{F}^j, \qquad 1/g^{jj}H = 1/\bar{g}^{jj}\bar{H}, \qquad \forall j = 1, \ldots, N. \tag{12}$$

We distinguish three situations

(a) $H = F^j = 0$, $\forall j = 1, \ldots, N$.

Then (12) implies that the lower order terms of $\bar{L}$ vanish. L and $\bar{L}$ become essentially scalar operators and by looking at the exponential solutions we obtain (i).

(b) $H = 0$.

We may assume (after a linear change in coordinates) that $g^{ij} = \delta^{ij}$. Thus we get from (12) that

$$F^j = \bar{F}^j, \qquad H = \bar{H}, \qquad \bar{g}^{11} = \cdots = \bar{g}^{NN} = \lambda^2.$$

The above equalities continue to hold in any coordinate system in which $g^{ij} = \delta^{ij}$. If we now make the orthogonal changes in variables

$$y^1 = x^1 \cos\theta + x^2 \sin\theta$$

$$y^2 = -x^1 \sin\theta + x^2 \cos\theta$$

$$y^k = x^k \qquad k \geq 2$$

we see that $g^{11} = \delta^{11}$ does not change while $\bar{g}^{11}$ becomes

$$\bar{g}^{11} \mapsto g^{11}(\theta) = (\lambda^2 \cos^2\theta + \lambda^2 \sin^2\theta + 2\bar{g}^{12} \sin\theta\cos\theta) = (\lambda^2 + 2\bar{g}^{12}\sin\theta\cos\theta).$$

Since $\bar{g}^{11}(\theta) = \bar{g}^{11}(0) = \lambda^2\ \forall\theta$ we deduce $\bar{g}^{12} = 0$. Similarly one shows $\bar{g}^{ij} = 0\ \forall i \neq j$ so we get (ii).

(c) $H = 0$ and $F^j \neq 0$ for some $j = 1, \ldots, N$.

Assume for simplicity that the index j above is $j = 1$. As in case (b) assume that $g^{ij} = \delta^{ij}$. Equations (12) imply

$$\bar{F}^1 = \bar{g}^{11}F^1, \qquad \bar{F}^2 = \bar{g}^{22}F^2. \tag{13}$$

Consider the change of variables

$$y^1 = ax^1 + bx^2$$
$$y^2 = cx^1 + dx^2$$
$$y^k = x^k \qquad k \geq 2$$

$(ad - bc \neq 0)$. Then the coefficients of L and $\bar{L}$ change as indicated below

$$F^1 \mapsto aF_1 + bF_2, \qquad \bar{F}^1 \mapsto a\bar{F}^1 + b\bar{F}^2$$
$$g^{11} \mapsto a^2 + b^2, \qquad \bar{g}^{11} \mapsto \bar{g}^{11}a^2 + 2\bar{g}^{12}ab + \bar{g}^{22}b^2.$$

Equations (12) continue to hold for the new coefficients as well. Thus

$$(\bar{g}^{11}a^2 + 2\bar{g}^{12}ab + \bar{g}^{22}b^2)(aF_1 + bF_2) = (a^2 + b^2)(a\bar{F}^1 + b\bar{F}^2).$$

Using (13), we get

$$(\bar{g}^{11}a^2 + 2\bar{g}^{12}ab + \bar{g}^{22}b^2)(aF_1 + bF_2) = (a^2 + b^2)(\bar{g}^{11}aF_1 + \bar{g}^{22}bF_2).$$

Identifying the coefficients of a^2b and ab^2 in the above polynomials, we deduce

$$\begin{cases} 2\bar{g}^{12}F^1 + (\bar{g}^{11} - \bar{g}^{22})F^2 = 0 & (a^2b) \\ (\bar{g}^{22} - \bar{g}^{11})F^1 + 2\bar{g}^{12}F^2 = 0 & (ab^2). \end{cases} \tag{14}$$

Since the homogeneous system has a nontrivial solution $(F^1 \neq 0)$, we deduce

$$4(\bar{g}^{12})^2 + (\bar{g}^{11} - \bar{g}^{22})^2 = 0.$$

In particular, we deduce

$$\bar{g}^{11} = \bar{g}^{22}, \qquad \bar{g}^{12} = 0.$$

Applying this technique to the other pairs $(1, j)$ of indices we deduce

$$\bar{g}^{11} = \cdots = \bar{g}^{NN}.$$

The above equality holds in any orthonormal basis of $\mathbf{R}^N$. The trace

$$\sum \bar{g}^{jj}$$

is an orthonormal invariant so the common value of $\bar{g}^{11}, \ldots, \bar{g}^{NN}$ is independent of the orthonormal basis. Now we conclude with the same argument we used in case (b). □

As a consequence we have

COROLLARY 2.1. *Let $L_1, L_2 \in \mathscr{L}(\mathscr{E})$ be two operators with constant coefficients such that $[L_1]_h \neq [L_2]_h$. Then there exists $\bar{\varrho} = \bar{\varrho}(L_1, L_2)$ such that for any $0 < r < \bar{\varrho} < \min(\varrho(L_1), \varrho(L_2))$ there exist $u_j \in C^{2,\alpha}(\overline{B_r})$ $j = 1, 2$ satisfying*

$$\begin{cases} L_1 u_1 = L_2 u_2 = 0 & \text{in } B_r \\ u_1 = u_2 & \text{on } \partial B_r \end{cases}$$

and $u_1 \neq u_2$.

Proof. From Lemma 2.3 we deduce that there exists $\bar{\varrho} = \bar{\varrho}(L_1, L_2) > 0$ such that

$$\mathscr{H}_{L_1}(\overline{B_r}) \neq \mathscr{H}_{L_2}(\overline{B_r}) \qquad \forall r < \bar{\varrho}.$$

Pick $u_1 \in \mathscr{H}_{L_1}(\overline{B_r}) \backslash \mathscr{H}_{L_2}(\overline{B_r})$ (or the other way around). Using Remark 2.1 we can find $u_2 \in \mathscr{H}_{L_2}(B_r)$ satisfying

$$\begin{cases} L_2 u_2 = 0 & \text{in } B_r \\ u_2 = u_1 & \text{on } \partial B_r \end{cases}$$

u_1, u_2 have the desired properties. Corollary 2.1 is proved. □

LEMMA 2.4. *Let $L_1, L_2 \in \mathscr{L}(\mathscr{E})$ be two operators with constant coefficients such that $[L_1]_h \neq [L_2]_h$. Then there exists $k = k(L_1, L_2) > 0$ such that for any $0 < r < \bar{\varrho}(L_1, L_2)$ there exist $u_r^j \in \mathscr{H}_{L_j}(\overline{B_r})$ $(j = 1, 2)$ such that $u_r^1 = u_r^2$ on ∂B_r and*

$$\|u_r^1 - u_r^2\|_{L^2(B_r)}^2 = kr^{N+4} + O(r^{N+6}) \tag{15}$$

$$|||u|||_r \leq Cr^2. \tag{16}$$

Proof. We assume for simplicity that $\bar{\varrho}(L_1, L_2) > 1$ (the general case can be reduced to this situation via a rescaling). By Corollary 2.1 we can find $v^j \in \mathscr{H}_{L_j}(\overline{B_1})$ $(j = 1, 2; v^1 \neq v^2)$ such that $v^1 = v^2$ on ∂B_1. Set

$$v_r^j(x) = r^2 v^j(x/r), \qquad j = 1, 2.$$

Then

$$\|v_r^1 - v_r^2\|_{L^2(B_r)}^2 = kr^{N+4} \tag{17}$$

and

$$|||v_r^j|||_r \leq Cr^2 \tag{18}$$

where

$$k = \int_{B_r} |v^1 - v^2|^2 > 0.$$

Now define $w_r^j \in \mathscr{H}_{L_j}(\overline{B_r})$ by

$$\begin{cases} L_j w_r^j \equiv G_j u_r^j + F_j u_r^j + H_j u_r^j = L v_r^j & \text{in } B_r \\ \qquad\qquad\qquad\qquad w_r^j = 0 & \text{on } \partial B_r. \end{cases} \tag{19}$$

$L_j v_r^j$ satisfies

$$\|L_j v_r^j\|_{L^2(B_r)}^2 \leq Cr^{N+2} \tag{20}$$

and

$$\|L_j v_r^j\|_{C^\alpha(B_r)} \leq C. \tag{21}$$

Using (20), (21) and the key estimates in (19) we deduce

$$\|w_r^j\|_{L^2(B_r)}^2 \le Cr^4\|L_j v_r^j\|_{L^2(B_r)}^2 = O(r^{N+6)})$$
$$|||w_r^j|||_r \le Cr_2.$$

In particular, the $u_r^j + w_r^j$ satisfy all the conditions of Lemma 2.4. □

The last result of this section is a perturbation result. Consider $L \in \mathscr{L}(\mathscr{E})$,

$$L = -\sum_{i,j} g^{ij}\partial_{ij}^2 + \sum_i F^i\partial_i + H.$$

Consider the "tangent" operator

$$L^0 = -g^{ij}(0)\partial_{ij}^2 + F^i(0)\partial_j + C(0).$$

With these notations we have

LEMMA 2.5. *Let $(u_r)_{r>0}$ be a family of solutions of*

$$L^0u = 0 \quad \text{in } B_r$$

such that

$$|||u_r|||_r \le Cr^2. \tag{22}$$

If v_r is the solution of the boundary value problem

$$\begin{cases} Lv_r = 0 & \text{in } B_r \\ v_r = u_r & \text{on } \partial B_r \end{cases}$$

then

$$\int_{B_r} |v_r - u_r|^2 \le Cr^{N+6}. \tag{23}$$

Proof. The operator $L - L^0$ has coefficients

$$\bar{G}(x) = G(x) - G(0), \qquad \bar{F}(x) = F(x) - F(0), \qquad \bar{H}(x) = H(x) - H(0).$$

Obviously

$$\bar{G}(x), \bar{F}(x), \bar{H}(x) = O(|x|). \tag{24}$$

We have $(w_r = u_r - v_r)$

$$L(w_r) = \bar{G}(\partial_{ij}^2 u_r) + \bar{F}(\partial_i u_r) + \bar{H}u_r. \tag{25}$$

By (22) the C^α-norm of the right hand side of (25) is bounded (uniformly in r). Using the key estimates we deduce

$$|||w_r|||_r \le Cr^2. \tag{26}$$

Multiply (25) by w_r and integrate by parts in the left hand side, noting that $w_r = 0$ on ∂B_r. We get (μ is the smallest eigenvalue of the matrix $(g^{ij}(0))$ that

$$\mu \int_{B_r} |Dw_r|^2 \le \int_{B_r} (|\bar{G}||D^2 u_r| + |\bar{F}||Du_r| + |\bar{H}||u_r|)|w_r| + \int_{B_r} |\bar{F}||Dw_r||w_r| + \int_{B_r} \bar{H}|w_r|^2. \tag{27}$$

By (22) and (24) the first term in the right hand side of (27) is bounded by

$$C \int_{B_r} |w_r||x|\, dx.$$

Using (26), Hölder and Poincaré inequalities we can bound the second term in the right hand side of (27) by

$$Cr \sup_{B_r} |\bar{F}| \int_{B_r} |Dw_r|^2\, dx.$$

Using (26) we can bound the third term by

$$Cr^2 \int_{B_r} |w_r||x|\, dx.$$

(27) becomes

$$\mu \int_{B_r} |Dw_r|^2 \leq C\left(\int_{B_r} |w_r|\,|x|\,dx + r \int_{B_r} |Dw_r|^2 \right),$$

i.e., for r small

$$\int_{B_r} |Dw_r|^2 \leq C \int_{B_r} |w_r|\,|x|\,dx. \tag{28}$$

Setting $p = 2N/(2N-2)$ the Sobolev embedding theorem implies $w_r \in L^p$ and

$$\|w_r\|_{L^p} \leq C\|Dw_r\|_{L^2} \tag{29}$$

with C independent of r. Apply Hölder ($q = 2N/(N+2)$) in (28)

$$\begin{aligned} \|Dw_r\|_{L^2}^2 &\leq C\left(\int |x|^q\,dx\right)^{1/q} \|w_r\|_{L^p} \\ &\leq Cr^{N/q+1}\|w_r\|_{L^p}. \end{aligned} \tag{30}$$

We now use (29) in (30) to get

$$\|Dw_r\|_{L^2} \leq Cr^{N/q+1}. \tag{31}$$

At this point we apply Poincaré inequality (1) to get

$$\|w_r\|_{L^2}^2 \leq Cr^{(2N/q)+4} = Cr^{N+6}.$$

Lemma 2.5 is proved. □

REMARK 2.2. Using a terminology analogous to that introduced in [Ar] we can restate Lemma 2.4 by saying that two elliptic operators of second order with constant coefficients, which have different kernels, have contact of order at most 2 in the 2-mean (i.e. infinitesimally they are measurably far apart). Similarly Lemma 2.5 can be restated by saying that a second order elliptic operator and the "tangent" operator have contact of order at least 3 in the 2-mean. Thus constant-coefficients operators are quite rigid. The rigidity of variable-coefficients operators is infinitesimally inherited from the "tangent" operators.

3. Proof of the main results

In the previous section we defined a sort of distance between two laplacians using their sheaves of local solutions. Lemma 2.5 will be used to transfer the problem to the constant-coefficient case. Lemma 2.5 shows that the error we get after this transfer is inessential for our purposes.

Proof of Theorm 1.1. (i) $\Rightarrow$ (ii) is trivial. We prove the converse. Let $L_j \in \mathscr{L}(\mathscr{E})$ $(j = 1, 2)$ such that $\mathscr{H}_{L_1} \subset \mathscr{H}_{L_2}$. (Again assume $M = \mathbf{R}^N$ etc. as in Section 2.) It suffices to show that their "tangent" operators L_j^0 at $x = 0$ are homothetically equivalent. Assume the contrary. Using Lemma 2.4 we can find $u_r^j \in \mathscr{H}_{L_j}(\overline{B}_r)$ $j = 1, 2$ such that $u_r^1 = u_r^2$, on ∂B_r

$$\|u_r^1\| - u_r^2\|_{L^2(B_r)}^2 = kr^{N+4} + O(r^{N+6}), \qquad k > 0 \tag{1}$$

$$|||u_r^j|||_r \leq Cr^2. \tag{2}$$

Let v_r^j be the solution of the boundary value problem

$$\begin{cases} Lv_r^j = 0 & \text{in } B_r \\ v_r^j = u_r^j & \text{on } \partial B_r. \end{cases}$$

Since

$$v_r^1 = u_r^1 = u_r^2 = v_r^2, \qquad \text{on } \partial B_r,$$

we deduce from Lemma 2.2 that

$$v_r^1 = v_r^2, \qquad \text{in } B_r. \tag{3}$$

By Lemma 2.5,

$$\|u_r^j - v_r^j\|_{L^2(B_r)}^2 = O(r^{N+6}). \tag{4}$$

(4) and (3) contradict (1). Theorem 1.1 is proved. $\square$

Proof of the Theorem 1.2. Suppose $L, \bar{L} \in \mathscr{L}_{geom}(\mathscr{E})$ have $\mathscr{H}_L \subset \mathscr{H}_{\bar{L}}$. Because these are geometric laplacians, L is given by (1.3) for some metric g and connection

∇ and $\bar{L}$ is similarly specified by a metric $\bar{g}$ and connection $\bar{\nabla}$. By Theorem 1.1 we have

$$\bar{L} = \phi L \tag{5}$$

for some smooth positive function ϕ. Comparing the principal symbols, we see that $\bar{g} = \phi^{-1} g$. Now note that for any $f \in C^\infty(M)$ and $\psi \in \Gamma(\mathscr{E})$ we have ([BGV])

$$[L, f]\psi = \Delta_g f \cdot \psi - 2df(\nabla\psi).$$

Combining this with the corresponding formula for the $[\bar{L}, f] = [\phi L, f]$ gives

$$\Delta_{\bar{g}} f \cdot \psi - 2df(\bar{\nabla}\psi) = \phi(\Delta_g f \cdot \psi - 2df(\nabla\psi)). \tag{6}$$

The laplacians of g and $\bar{g}$ are related by

$$\Delta_{\bar{g}} f = -\frac{\phi^{N/2}}{\sqrt{|g|}} \partial_i (\phi^{1 - N/2} g^{ij} \sqrt{|g|} \partial_j) f = \phi \Delta_g f + \left(\frac{N-2}{2}\right) df(\mathrm{grad}_g\, \phi).$$

Hence (6) simplifies to

$$\left(\frac{N-2}{4}\right) df(\mathrm{grad}_g\, \phi) \cdot \psi = df((\bar{\nabla} - \nabla)\psi).$$

Now fix a point $p \in M$, a vector $X \in T_p M$ (with $\mathrm{grad}_g f)_p = X$. Then as endomorphisms of $\mathscr{E}_p$

$$\left(\frac{N-2}{4}\right)(X \cdot \phi) Id = \bar{\nabla}_X - \nabla_X.$$

But $\bar{\nabla}_X - \nabla_X$ is a skewsymmetric endomorphism on $\mathscr{E}_p$ since ∇ and $\bar{\nabla}$ are compatible with the metric on $\mathscr{E}$. Taking the trace shows that $X \cdot \phi = 0$ provided $N \geq 3$. Since p and X are arbitrary we conclude that ϕ is constant. Theorem 1.2 is proved. □

4. Final comments

The rigidity phenomenon described in the previous sections raises (at least in author's mind) more questions than it answers. First the proof. What made everything work was the possibility of measuring the "distance" between two sheaves of local solutions of laplacians. We essentially utilized local existence and uniqueness for the Dirichlet problem. Another crucial aspect is the "blow-up analysis" in Lemma 2.5 (which seems to be known but we could not find any reference). It essentially measures the infinitesimal "distance" between an operator and its "germ" at a fixed point. For elliptic operators these germs give rather good local approximations. (This idea is not new. It goes back to the 1930s when Schauder proved his celebrated holder estimates; see [GT].) We are inclined to believe that a similar approach can work for other classes of elliptic operators (with the Dirichlet condition replaced by a suitable elliptic boundary condition). In particular we formulate

PROBLEM 4.1. *The generalized Dirac operators (cf. [BGV]) are rigid.*

The Dolbeaut operator $\bar{\partial}$ is a Dirac operator on Kahler manifolds. Having a $\bar{\partial}$ (satisfying $\bar{\partial}^2 = 0$) is equivalent to having a complex structure and the rigidity means that the sheaf of holomorphic functions on a complex manifold determines its complex structure. The similarity with complex geometry suggests other problems.

PROBLEM 4.2. *Study the cohomology of the sheaves $\mathscr{H}_L$. In particular find sufficient conditions (à la Cartan–Serre) for the finite dimensionality of these groups. Is the compactness of M such a condition?*

Even in the simplest case when L is a Laplace–Beltrami operator the above problem is not studied. One can ask

PROBLEM 4.3. *If L is a Laplace–Beltrami on a compact manifold are the cohomology groups of $\mathscr{H}_L$ diffeomorphism invariants of the manifold or do they capture some of the global geometry of the manifold? Compare these groups with the deRham cohomology.*

The following example suggests that the cohomology of $\mathscr{H}_L$ is actually the deRham cohomology.

EXAMPLE 4.1. Let $M \cong S^1$ with the standard euclidian metric. Denote by θ the angular coordinate. The laplacian is simply $L = -\partial^2/\partial\theta^2$. The harmonic functions are the functions affine (in θ). Denote by $\underline{\mathbf{R}}$ the sheaf of locally constant real functions. We may think of $\underline{\mathbf{R}}$ as a subsheaf of $\mathscr{H}_L$. Since the tangent bundle of S^1 is trivial we may think of 1-forms as functions on the circle. In particular if u is a harmonic function on an open set $U \subset S^1$ we see that the exterior derivative du is a locally constant function. One can then check easily that the sequence of sheaves

$$0 \to \underline{\mathbf{R}} \hookrightarrow \mathscr{H}_L \xrightarrow{d} \underline{\mathbf{R}} \to 0$$

is exact. Playing with the associated long exact sequence in cohomology one deduces that

$$H^1(S^1; \mathscr{H}_L) \cong H^1(S^1; \underline{\mathbf{R}}) \cong \mathbf{R}.$$

Obviously $H^0(S^1; \mathscr{H}_L) \cong \mathbf{R}$.

On a more analytical level one may ask why restrict to elliptic operators. If we allow complex coefficients, we encounter wild examples of linear differential operators (Lewy's example is a possible candidate for nonrigidity; see [H]).

Let $\mathscr{F}$ be a family of differential operators of fixed order M acting between sections of two fixed vector bundles $\mathscr{E}_1$, $\mathscr{E}_2$ on a manifold M. The family $\mathscr{F}$ will be called rigid if any two operators L_1 and L_2 in $\mathscr{F}$ which have the same sheaves of local solutions are equivalent i.e. there exists an automorphism ϕ of $\mathscr{E}_2$ so that $L_1 = \phi L_2$.

PROBLEM 4.4. (i) *Find (nontrivial) necessary conditions for a family to be rigid.*
(ii) *Find (nontrivial) sufficient conditions for a family to be rigid.*

We see that this paper offers one solution to (ii).

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