A LAW OF LARGE NUMBERS CONCERNING THE NUMBER OF CRITICAL POINTS OF ISOTROPIC GAUSSIAN FUNCTIONS

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Abstract. Let Φ be a smooth isotropic random Gaussian function Φ on $\mathbb{R}^m$. Denote by $Z_N(\Phi)$ the number of critical points of Φ inside the cube $[0,N]^m$. We prove that $N^{-m}Z_N(\Phi)$ converges a.s. and L^2 to a universal explicit constant $C_m(\Phi)$.

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1. Introduction

Denote by $\mathrm{Meas}(\mathbb{R}^m)$ the space of finite Borel measures on $\mathbb{R}^m$. Suppose that $\mathfrak{a}:\mathbb{R}\to\mathbb{R}$ is an even Schwartz function such that $\mathfrak{a}(0)=1$. We will refer to such functions as *amplitudes*.

Consider the finite measure $\mu\in\mathrm{Meas}(\mathbb{R}^m)$

$$\mu\big[\,d\xi\,\big]=\mu_{\mathfrak{a}}\big[\,d\xi\,\big]=\frac{1}{(2\pi)^m}w_{\mathfrak{a},m}\big(\,\xi\,\big)\boldsymbol{\lambda}\big[\,d\xi\,\big],\;\; w_{\mathfrak{a},m}\big(\,\xi\,\big)=\mathfrak{a}\big(\,|\xi|\,\big)^2.$$

Its characteristic function is the nonnegative definite function

$$\boldsymbol{K}\big(\,\boldsymbol{x}\,\big)=\boldsymbol{K}_{\mathfrak{a}}\big(\,\boldsymbol{x}\,\big)=\int_{\mathbb{R}^m}e^{\boldsymbol{i}\langle\xi,\boldsymbol{x}\rangle}\mu_{\mathfrak{a}}\big[\,d\xi\,\big]=\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}e^{\boldsymbol{i}\langle\xi,\boldsymbol{x}\rangle}\mathfrak{a}\big(\,|\xi|\,\big)^2\boldsymbol{\lambda}\big[\,d\xi\,\big]. \tag{1.1}$$

Clearly $\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x})$ is an $O(n)$-invariant, real valued Schwartz function. Then $\boldsymbol{K}_{\mathfrak{a}}\big(\,\boldsymbol{x}-\boldsymbol{y}\,\big)$ is the covariance kernel of a real valued, smooth isotropic Gaussian function $\Phi=\Phi_{\mathfrak{a}}$ on $\mathbb{R}^m$ with spectral measure $\mu_{\mathfrak{a}}$.

A box is a subset of $\mathbb{R}^m$ of the form $[a_1,b_1]\times\cdots\times[a_m,b_m]$, $a_i<b_i$, $\forall i$. For any box B we denote by $Z_{\mathfrak{a}}\big(\,B\,\big)$ the number of critical points of $\Phi_{\mathfrak{a}}$ in B. Then a.s. $\Phi_{\mathfrak{a}}$ has no critical points on ∂B and all the critical points of $\Phi_{\mathfrak{a}}$ in B are nondegenerate; see [1, 4].

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Fix numbers $a_1,\dots,a_m>0$ and denote by B the box $[0,a_1]\times\cdots[0,a_m]$. For $\vec{\ell}\in\mathbb{N}$ we denote by $B_{\vec{\ell}}$ the box

$$B_{\vec{\ell}}=\prod_{j=1}^m\big[\,(\ell_j-1)a_j,\ell_ja_j\,\big].$$

For $N\in\mathbb{N}$ we denote by B_N the box

$$B_N=N\cdot B=\prod_{j=1}^m\big[\,0,Na_j\,\big].$$

The main goal of this paper is to prove some laws law of large numbers concerning the random variable $Z_{\mathfrak{a}}(B)$. The first main result is the following.

Theorem 1.1. *Fix an amplitude $\mathfrak{a}\in\mathcal{S}(\mathbb{R})$. Then the following hold.*

(i) *There exists a universal explicit constant $C_m(\mathfrak{a})>0$ such that for any box B we have*

$$\mathbb{E}\big[\,Z_{\mathfrak{a}}(B)\,\big]=C_m(\mathfrak{a})\operatorname{vol}\big[\,B\,\big].$$

(ii) *As $N\to\infty$, the random variable*

$$\frac{1}{N^m}Z_{\mathfrak{a}}\big[\,N\cdot B\,\big]$$

converges a.s. *and L^2 to the (deterministic) constant*

$$\mathbb{E}\big[\,Z_{\mathfrak{a}}(B)\,\big]=C_m(\mathfrak{a})\operatorname{vol}\big[\,B\,\big].$$

□

In the earlier work [17] we proved that the random variable $Z_{\mathfrak{a}}\big[\,N\cdot B\,\big]$ is highly concentrated around its mean. More precisely we showed that as $N\to\infty$ the renormalized random variables

$$N^{-m/2}\Big(\,Z_{\mathfrak{a}}\big(\,B\,\big)-\mathbb{E}\big[\,Z_{\mathfrak{a}}\big(\,B\,\big)\,\big]\,\Big)$$

converge in distribution to a normal random variable with nonzero standard deviation.

Let us offer a different perspective on Theorem 1.1. For $\hbar>0$ we set

$$\mathfrak{a}_\hbar(t):=\mathfrak{a}\big(\hbar t\,\big),\ \ \forall t\in\mathbb{R}.$$

Consider the finite Borel measure $\mu^\hbar_{\mathfrak{a}}\in\operatorname{Meas}(\mathbb{R}^m)$

$$\mu^\hbar_{\mathfrak{a}}\big[\,d\xi\,\big]=\frac{1}{(2\pi)^m}w_{\boldsymbol{a}_\hbar,m}\big(\,\xi\,\big)\boldsymbol{\lambda}\big[\,d\xi\,\big]=\frac{1}{(2\pi)^m}\mathfrak{a}\big(\,|\hbar\xi|\,\big)^2\boldsymbol{\lambda}\big[\,d\xi\,\big].$$

Its characteristic function is the nonnegative definite function

$$\begin{aligned}\boldsymbol{K}^\hbar\big(\,\boldsymbol{x}\,\big)=\boldsymbol{K}^\hbar_{\mathfrak{a}}\big(\,\boldsymbol{x}\,\big)=\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}e^{\boldsymbol{i}\langle\xi,\boldsymbol{x}\rangle}\mathfrak{a}\big(\,|\hbar\xi|\,\big)^2d\xi\\ =\hbar^{-m}\boldsymbol{K}_{\mathfrak{a}}\big(\,\hbar^{-1}\boldsymbol{x}\,\big).\end{aligned}\tag{1.2}$$

We deduce that $\boldsymbol{K}^\hbar_{\mathfrak{a}}\big(\,\boldsymbol{x}-\boldsymbol{y}\,\big)$ is the covariance kernel of the Gaussian function

$$\Phi^\hbar(\boldsymbol{x}):=\hbar^{-m/2}\Phi_{\mathfrak{a}}\big(\,\hbar^{-1}\boldsymbol{x}\,\big).$$

The Fourier inversion formula we deduce

$$\int_{\mathbb{R}^m}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x})d\boldsymbol{x}=\mathfrak{a}(0)^2=1.$$

Since $K_{\mathfrak{a}}(\boldsymbol{x})$ is $O(m)$-invariant and smooth it has the form $\Psi\big(|\boldsymbol{x}|^2\big)$ for some function $\Psi:[0,\infty)\to\mathbb{R}$. According to Schoenberg's characterization theorem [20, Thm.7.13], the function Ψ must be completely monotone. In particular, Ψ is non-increasing, nonnegative and convex, [20, Lemma.7.3]. This implies that the probability measures $\boldsymbol{K}^{\hbar}\big(\,\boldsymbol{x}\,\big)dx$ converge weakly to the Dirac measure δ_0. For example, if $\mathfrak{a}(t)=e^{-t^2/4}$, then

$$\boldsymbol{K}^{\hbar}_{\mathfrak{a}}(\boldsymbol{x})=\frac{1}{(2\pi\hbar^2)^{m/2}}e^{-\frac{|\boldsymbol{x}|^2}{2\hbar^2}}$$

is the probability density the Gaussian measure on $\mathbb{R}^m$ with variance $\hbar^2\mathbb{1}_m$.

To use a terminology favored by physicists, we have $\boldsymbol{K}^{\hbar}\big(\,\boldsymbol{x}\,\big)\to\delta(\boldsymbol{x})$, where $\delta(\boldsymbol{x})$ is Dirac's Delta function. In particular,

$$\boldsymbol{K}^{\hbar}\big(\,\boldsymbol{x}-\boldsymbol{y}\,\big)\to\delta(\boldsymbol{x}-\boldsymbol{y}).$$

In other words, as $\hbar\searrow 0$, the Gaussian random function $\Phi^{\hbar}_{\mathfrak{a}}$ converges in some sense to a Gaussian random "function" Φ^0 whose covariance kernel is $\boldsymbol{K}^0(\boldsymbol{x}-\boldsymbol{y})=\delta(\boldsymbol{x}-\boldsymbol{y})$. This is a Gaussian white noise, [10].

Let us introduce some notation. For any Euclidean space $\boldsymbol{V}$, any function $f:\boldsymbol{V}\to[0,\infty)$ and any Morse function $\Psi\in C^2(\boldsymbol{V})$ we set

$$Z(f,\Psi):=\sum_{\nabla\Psi(\boldsymbol{v})=0}f(\boldsymbol{v}).\tag{1.3}$$

If $S\subset\boldsymbol{V}$ is a subset of $\boldsymbol{V}$ we set

$$Z_S(f,\Psi):=Z(\boldsymbol{I}_Sf,\Psi)=\sum_{\substack{\nabla\Psi(\boldsymbol{v})=0,\\ \boldsymbol{v}\in S}}f(\boldsymbol{v}).\tag{1.4}$$

where $\boldsymbol{I}_S$ denotes the indicator of the set S. Note that

$$Z(\boldsymbol{I}_B,\Phi^{\hbar}_{\mathfrak{a}})=Z_{\mathfrak{a}_{\hbar}}\big[\,B\,\big]=Z_{\mathfrak{a}}\big[\,\hbar^{-1}B\,\big]=Z(\boldsymbol{I}_{\hbar^{-1}B},\Phi_{\mathfrak{a}})$$

Consider the random measure[1]

$$\nu_{\hbar}=\hbar^m\sum_{\nabla\Phi^{\hbar}(\boldsymbol{x})=0}\delta_{\boldsymbol{x}}.$$

Then $\hbar^mZ_{\mathfrak{a}_{\hbar}}\big[\,B\,\big]=\nu_{\hbar}\big[\,B\,\big]$. Now let $\hbar=N^{-1}$. In this case

$$\nu_{N^{-1}}\big[\,B\,\big]=\frac{1}{N^m}Z_{\mathfrak{a}}\big(\,N\cdot B\,\big).$$

Theorem 1.1 shows that in the white noise limit $(N\to\infty)$ the measure $\nu_{N^{-1}}$ converges in some sense to the *deterministic* measure $C_m(\mathfrak{a})\boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ denotes the Lebesgue measure on $\mathbb{R}^m$. In other words, the critical points of $\Phi^{\hbar}$ will a.s. equidistribute as $\hbar\searrow 0$. We can be more precise by investigating the linear statistics.

When $m=1$, M. Ancona and T. Letendre [2, Thm. 1.16] (see also L. Gass [8, Thm.1.6]) proved that any compactly supported continuous function $f\in C^0_{\mathrm{cpt}}\big(\,\mathbb{R}\,\big)$

$$\int_{\mathbb{R}}f(\boldsymbol{x})\nu_{N^{-1}}\big[\,d\boldsymbol{x}\,\big]=\frac{1}{N}Z\big(\,f,\Phi^{\frac{1}{N}}_{\mathfrak{a}}\,\big)\to C_1(\mathfrak{a})\int_{\mathbb{R}}f(\boldsymbol{x})d\boldsymbol{x}\ \text{ a.s.},$$

for any $f\in C^0_{\mathrm{cpt}}(\mathbb{R})$. Our next results shows that ta similar result also holds for $m>1$.

[1] The fact that $\nu_{\hbar}$ is indeed a random measure on kernel follows from [12, Lemma 3.1].

Theorem 1.2. *Fix $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$, $m \geq 2$ and set $Y_N(f) = \frac{1}{N^m} Z(f, \Phi_{\mathfrak{a}}^{\frac{1}{N}})$. Then*

$$\operatorname{Var}\big[\, Y_N(f) \,\big] = O\big(N^{-m} \big), \quad \text{as } N \to \infty \tag{1.5}$$

In particular, as $N \to \infty$,

$$\int_{\mathbb{R}^m} f(\boldsymbol{x}) \nu_{N^{-1}}\big[\, d\boldsymbol{x} \,\big] = Y_N(f) \to C_m(\mathfrak{a}) \int_{\mathbb{R}^m} f(\boldsymbol{x}) \boldsymbol{\lambda}\big[\, d\boldsymbol{x} \,\big] \ \text{ in } L^2 \text{ and a.s..}$$

□

The strategy we employ in the proof of Theorem 1.1 can be easily explained. Set $\mathbb{I}_N := [1, N] \cap \mathbb{N}$. Bulinskaya's Lemma [1, Lemma 11.2.10] implies that

$$\frac{1}{N^m} Z_{\mathfrak{a}}\big(N \cdot B \big) = \frac{1}{N^m} \sum_{\vec{\ell} \in \mathbb{I}_N^m} Z_{\mathfrak{a}}\big(B_{\vec{\ell}} \big) \text{ a.s..}$$

Since the Gaussian function $\Phi_{\mathfrak{a}}$ is stationary, the family of random variables $\big(Z_{\mathfrak{a}}\big(B_{\vec{\ell}} \big) \big)_{\vec{\ell} \in \mathbb{Z}^m}$ is stationary.

Using the multiparametric ergodic theorem [13, Ch.6.Thm. 3.5], [18, Thm.5] we conclude that $\frac{1}{N^m} Z_{\mathfrak{a}}\big[\, N \cdot B \,\big]$ converge a.s. and L^1 to some random variable Z_∞. To show that Z_∞ is a.s. constant we prove that

$$\operatorname{Var}\big[\, N^{-m} Z_{\mathfrak{a}}\big(N \cdot B \big) \,\big] \to 0 \ \text{ as } N \to \infty.$$

We achieve this by observing that

$$\operatorname{Var}\big[\, N^{-m} Z_{\mathfrak{a}}\big(N \cdot B \big) \,\big] = N^{-2m} \sum_{\vec{k}, \vec{\ell} \in \mathbb{I}_N^m} \underbrace{\operatorname{Cov}\big[\, Z_{\mathfrak{a}}\big(B_{\vec{k}} \big), Z_{\mathfrak{a}}\big(B_{\vec{\ell}} \big) \,\big]}_{=: C(\vec{k}, \vec{\ell})}.$$

The above sum consists of N^{2m} terms, but the terms that are far from the diagonal $\vec{k} = \vec{\ell}$ contribute very little. More precisely, using the Kac-Rice formula we will prove that for any $p > 0$ there exists $C = C(p) > 0$ such that

$$\big|\, C(\vec{k}, \vec{\ell}) \,\big| \leq \frac{C(p)}{\big(1 + |\vec{k} - \vec{\ell}|_1 \big)^p}, \ \ \forall (\vec{k}, \vec{\ell}) \in \mathbb{N}^m \times \mathbb{N}^m,$$

where

$$\big|\, \boldsymbol{x} \,\big|_1 = \sum_{i=1}^m \big|\, x_i \,\big|, \ \ \forall \boldsymbol{x} = (x_1, \dots, x_m) \in \mathbb{R}^m.$$

According to Lemma 3.1, this implies that

$$\sum_{\vec{k}, \vec{\ell} \in \mathcal{R}_N} C(\vec{k}, \vec{\ell}) = O\big(N^{2m-1} \big) \ \text{ as } N \to \infty.$$

In fact, as shown in Lemma 3.3, if $p > m$, then

$$\sum_{\vec{k}, \vec{\ell} \in \mathcal{R}_N} C(\vec{k}, \vec{\ell}) = O\big(N^m \big) \ \text{ as } N \to \infty.$$

In particular, this guarantees the a.s. convergence in the case $m > 1$, without having to invoke the ergodic theory.

There is more to this ergodic angle. The Gaussian function $\Phi_{\mathfrak{a}}$ defines a Gaussian measure $\boldsymbol{\Gamma}$ on $C^2(\mathbb{R}^m)$. The additive group $\mathbb{R}^m$ acts on $C^2(\mathbb{R}^m)$ by translations $T_{\boldsymbol{x}}, \boldsymbol{x} \in \mathbb{R}^m$

$$T_{\boldsymbol{x}} F(\boldsymbol{y}) = F(\boldsymbol{y} - \boldsymbol{x}).$$

Since the Gaussian function $\Phi_{\mathfrak{a}}$ is stationary, the translations $T_{\boldsymbol{x}}$ are $\boldsymbol{\Gamma}$-preserving. Since the spectral measure of $\Phi_{\mathfrak{a}}$ is absolutely continuous with respect to the Lebesgue measure, the above action of $\mathbb{R}^m$ is ergodic; see [5] or [16, App. C].

In the case $m=1$ of Theorem 1.1. was proved in 1960 by V. Volkovski [19] using the ergodicity of the action of $\mathbb{R}$ on $C^2(\mathbb{R})$. We refer to [6, Sec. 11.5] for details.

The proof of Theorem 1.2 uses a similar strategy but we can now longer invoke ergodic theory. We rely instead on Lemma 3.3.

Theorem 1.1 is valid for a wider family of homogeneous Gaussian functions on $\mathbb{R}^m$ with fewer constraints on their spectral functions. We chose to work in this more restrictive context for two reasons. First, relaxing the assumptions on the Gaussian function would have added a few more technical complications that would have blurred the main arguments.

The second reason is that the class of Gaussian functions discussed in this paper is exactly the class of function that arises when studying critical points of certain random Fourier series of eigenvalues on Riemann manifolds.

More precisely given a Riemann manifold (M,g) with Laplacian Δ we denote by $\mathcal{K}^g_{\mathfrak{a}}$ the Schwartz kernel of the smoothing operator $\mathfrak{a}(\Delta_g)^2$. This is the covariance kernel of a smooth random function on M that can be represented as a random eigen-series. Upon rescaling the metric $g\to g_\hbar=\hbar^{-2}g$ we have $\Delta_{g_\hbar}=\hbar^2\Delta_g$ and, as $\hbar\searrow 0$, the kernel $\mathcal{K}^{g_\hbar}_{\mathfrak{a}}$ converges in a precise sense to the kernel $\boldsymbol{K}_{\mathfrak{a}}$ in (1.1).

The paper is organized as follows. In Section 2 we describe some basic properties of $\Phi_{\mathfrak{a}}$. Section 3 describes an abstract weak law of large numbers for families of mean zero random variables $(X_{\vec{\ell}})_{\vec{\ell}\in\mathbb{Z}^m}$ such that $X_{\vec{\ell}}$, $X_{\vec{k}}$ are weakly correlated if $\vec{k}$ and $\vec{\ell}$ are far apart. It is based on the technical Lemma 3.3 that seems to be new and we believe will find other uses. Theorem 1.1 is proved in Section 4 and Theorem 1.2 is proved in Section 5.

2. Some properties of the function $\Phi_{\mathfrak{a}}$

We collect in this section several useful properties of the random function $\Phi_{\mathfrak{a}}$ used throughout the paper. For $k\in\mathbb{N}$ we denote by $D^k\Phi_{\mathfrak{a}}$ the k-th order differential of $\Phi_{\mathfrak{a}}$

Proposition 2.1. *Let $N\in\mathbb{N}$. Then the following hold.*

(i) *The function $\Phi_{\mathfrak{a}}$ is N-ample, i.e., for any are distinct points $\boldsymbol{x}_1,\dots,\boldsymbol{x}_N\in\mathbb{R}^m$, the Gausian vector*

$$\big(\,\Phi_{\mathfrak{a}}(\boldsymbol{x}_1),\dots,\Phi_{\mathfrak{a}}(\boldsymbol{x}_N)\,\big).$$

is nondegenerate.

(ii) *The function $\Phi_{\mathfrak{a}}$ is J_N-ample, i.e., the Gaussian vector*

$$\Phi_{\mathfrak{a}}(\boldsymbol{x})\oplus D\Phi_{\mathfrak{a}}(\boldsymbol{x})\oplus\cdots\oplus D^N\Phi_{\mathfrak{a}}(\boldsymbol{x})$$

is nondegenerate for any $\boldsymbol{x}\in\mathbb{R}^m$.

Proof. (i) Since $w_{\mathfrak{a},m}\big(\,|\,\xi\,|\,\big)=\mathfrak{a}\big(\,|\xi|\,\big)^2$ is positive on an open neighborhood of $0\in\mathbb{R}^m$ we deduce from [20, Thm. 6.8] that if $\boldsymbol{x}_1,\dots,\boldsymbol{x}_N\in\mathbb{R}^m$ are distinct points, then the symmetric $N\times N$ matrix

$$\big(\,K(\boldsymbol{x}_i-\boldsymbol{x}_j)\,\big)_{1\leq i,j\leq N}$$

is *positive* definite. This matrix is the variance matrix of the Gaussian vector

$$\big(\,\Phi_{\mathfrak{a}}(\boldsymbol{x}_1),\dots,\Phi_{\mathfrak{a}}(\boldsymbol{x}_N)\,\big).$$

(ii) Observe that for any multi-indices $\alpha\in\big(\mathbb{Z}_{\geq 0}\big)^m$, we have

$$\mathbb{E}\big(\,\partial^\alpha\Phi_{\mathfrak{a}}(\boldsymbol{x})\partial^\beta\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big)=\partial^\alpha_x\partial^\beta_y\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y})\,\big|_{\boldsymbol{x}=\boldsymbol{y}}$$

$$= \int_{\mathbb{R}^m} \xi^\alpha \xi^\beta \mu_{\mathfrak{a}}\big[\,d\xi\,\big],\;\; \xi^\alpha := \xi_1^{\alpha_1}\cdots\xi_m^{\alpha_m}$$

This shows that for any $N \in \mathbb{N}$ and any $\boldsymbol{x} \in \mathbb{R}^n$ the variance the Gaussian vector $\big(\partial^\alpha \Phi_{\mathfrak{a}}(\boldsymbol{x})\big)_{|\alpha|\leq N}$ is the Gramian matrix of the functions $\big(\xi^\alpha\big)_{|\alpha|\leq N}$ with respect to the inner product in $L^2\big(\mathbb{R}^m, \mu_{\mathfrak{a}}\big)$. Since $\mathfrak{a}(0) = 1$ we deduce that the functions ξ^α are linearly independent in $L^2\big(\mathbb{R}^m, \mu_{\mathfrak{a}}\big)$ so the determinant of their Gramian matrix is nonzero. Hence the Gaussian vector

$$\Phi_{\mathfrak{a}}(\boldsymbol{x}) \oplus D\Phi_{\mathfrak{a}}(\boldsymbol{x}) \oplus \cdots \oplus D^N\Phi_{\mathfrak{a}}(\boldsymbol{x})$$

is nondegenerate, for any $k \in \mathbb{N}$ and any $\boldsymbol{x} \in \mathbb{R}^m$. □

We deduce from the above proposition and the results of Ancona-Letendre [3] or Gass-Stecconi [9] that for any box B the random variable $Z_{\mathfrak{a}}\big[\,B\,\big]$ has finite moments of any order. To compute the expectation of $Z_{\mathfrak{a}}(B)$ we rely on the Kac-Rice formula [1, 4].

Theorem 2.2. *Let $\boldsymbol{V}$ be a finite dimensional Euclidean space, $\mathcal{V} \subset \boldsymbol{V}$ an open set and $F : \mathcal{V} \to \mathbb{R}$ a Gaussian random function that is* a.s. C^2 *and such that the Gaussian vector $\nabla F(\boldsymbol{v})$ is nondegenerate for any $\boldsymbol{v} \in \mathcal{V}$. We denote by $p_{\nabla F(v)}$ is probability* density. *For a subset $S \subset \mathcal{V}$ we denote by $Z(S, F)$ the number of critical points of F in S. Then F is* a.s. *Morse, for any box $B \subset \mathcal{V}$, the function F* a.s. *has no critical points on ∂B, and for any continuous function $\varphi : B \to \mathbb{R}$*

$$\mathbb{E}\big[\,Z_B(\varphi, F)\,\big] = \int_B \mathbb{E}\big[\,|\det \operatorname{Hess} F(\boldsymbol{v})|\;\big|\;\nabla F(\boldsymbol{v}) = 0\,\big] p_{\nabla F(v)}(0)\varphi(\boldsymbol{v})d\boldsymbol{v}, \tag{2.1}$$

where $Z_B(\varphi, F)\big]$ is defined as in (1.4). □

We want to apply the above result to the function $\Phi_{\mathfrak{a}}$. Proposition 2.1 shows that the Gaussian vector $\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})$ is nondegenerate for any $\boldsymbol{x} \in \mathbb{R}^m$.

For any multi-indices $\alpha, \beta \in \big(\mathbb{Z}_{\geq 0}\big)^m$ we have

$$\mathbb{E}\big[\,\partial^\alpha\Phi_{\mathfrak{a}}(\boldsymbol{x})\partial^\beta\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]_{\boldsymbol{x}=\boldsymbol{y}} = \partial^\alpha_{\boldsymbol{x}}\partial^\beta_{\boldsymbol{y}}\mathcal{K}^{\mathfrak{a}}(\boldsymbol{x},\boldsymbol{y})\big|_{\,\boldsymbol{x}=\boldsymbol{y}} = \frac{(-1)^{|\beta||}\boldsymbol{i}^{|\alpha|+|\beta|}}{(2\pi)^m}\int_{\mathbb{R}^m} \xi^{\alpha+\beta}\mathfrak{a}(|\xi|)^2 d\xi. \tag{2.2}$$

For any multi-index $\alpha \in \big(\mathbb{Z}_{\geq 0}\big)^m$ we set

$$M^{\mathfrak{a}}_\gamma := \int_{\mathbb{R}^m} \xi^\alpha \mu_{\mathfrak{a}}\big[\,d\xi\,\big] = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} \xi^\alpha \mathfrak{a}(|\xi|)^2 d\xi.$$

We say that the multi-index $\alpha = (\alpha_1, \dots, \alpha_m)$ is *even* if α_j is even for any $j = 1, \dots, m$. The multi-index α is called *odd* if it is not even. The radial symmetry of $\mathfrak{a}\big(\,|\xi|\,\big)$ implies that

$$M^{\mathfrak{a}}_\alpha = 0 \;\text{ if } \alpha \text{ is odd.} \tag{2.3}$$

Using spherical coordinates on $\mathbb{R}^m$ we deduce that for any α we have

$$M^{\mathfrak{a}}_\alpha = \frac{1}{(2\pi)^m}\left(\int_0^\infty r^{m-1+|\alpha|}\mathfrak{a}(r)^2 dr\right) \times \underbrace{\int_{S^{m-1}} \xi^\alpha \operatorname{vol}_{S^{m-1}}\big[\,d\xi\,\big]}_{=:\boldsymbol{m}_\alpha}. \tag{2.4}$$

Note that $\boldsymbol{m}_\alpha$ is independent of $\mathfrak{a}$. If we let $\mathfrak{a}_0 := (2\pi)^{m/2}e^{-\frac{t^2}{4}}$, then

$$M^{\mathfrak{a}_0}_\alpha = \int_{\mathbb{R}^m} \xi^\gamma e^{-|\xi|^2/2}d\xi = (2\pi)^{m/2}\prod_{j=1}^m \int_{\mathbb{R}} \xi^{\alpha_j}\boldsymbol{\gamma}_1\big[\,d\xi\,\big]$$

where $\boldsymbol{\gamma}_1$ denotes the Gaussian measure on $\mathbb{R}$ with mean zero and variance 1. If α is even, $\alpha = 2\kappa$, then

$$M_{2\kappa}^{\mathfrak{a}_0} = (2\pi)^{m/2} \prod_{j=1}^{m} (2\kappa_j - 1)!!.$$

On the other hand, using (2.4) we deduce

$$M_{2\kappa}^{\mathfrak{a}_0} = \boldsymbol{m}_{2\kappa} \int_0^\infty r^{m+2|\kappa|-1} e^{-r^2/2} dr = \sqrt{\frac{\pi}{2}} \boldsymbol{m}_{2\kappa} \int_{\mathbb{R}} |x|^{m+2|\kappa|-1} \boldsymbol{\gamma}_1 \big[\, dx \,\big]$$
$$= 2^{|\kappa|+m/2-1} \boldsymbol{m}_{2\kappa} \Gamma\big(\, |\kappa| + m/2 \,\big).$$

Hence

$$\boldsymbol{m}_{2\kappa} = 2^{|\kappa|+m/2-1} \frac{\prod_{j=1}^{m} (2\kappa_j - 1)!!}{\Gamma\big(\, |\kappa| + m/2 \,\big)} = \frac{2 \prod_{j=1} \Gamma(\kappa_j + 1/2)}{\Gamma\big(\, |\kappa| + m/2 \,\big)}. \tag{2.5}$$

For every $k \in \mathbb{Z}_{\geq 0}$ we set

$$I_k(\mathfrak{a}) := \int_0^\infty r^k \mathfrak{a}(r)^2 dr.$$

We deduce

$$(2\pi)^m M_{2\kappa}^{\mathfrak{a}} = I_{m-1+2|\kappa|}(\mathfrak{a}) \frac{2 \prod_{j=1} \Gamma(\kappa_j + 1/2)}{\Gamma\big(\, |\kappa| + m/2 \,\big)}. \tag{2.6}$$

We set

$$s_m := \int_{\mathbb{R}^m} \mu_{\mathfrak{a}} \big[\, d\xi \,\big], \;\; d_m = \int_{\mathbb{R}^m} \xi_1^2 \mu_{\mathfrak{a}} \big[\, d\xi \,\big], \;\; h_m := \int_{\mathbb{R}^m} \xi_1^2 \xi_2^2 \mu_{\mathfrak{a}} \big[\, d\xi \,\big]. \tag{2.7}$$

Then

$$\int_{\mathbb{R}^m} \mathfrak{a}(|\xi|)^2 d\xi = \frac{2\pi^{m/2}}{\Gamma(m/2)} I_{m-1}(\mathfrak{a}) = (2\pi)^m s_m, \tag{2.8}$$

$$\int_{\mathbb{R}^m} \xi_j^2 \mathfrak{a}(|\xi|)^2 d\xi = \frac{2\pi^{m/2}}{\Gamma(m/2+1)} I_{m+1}(\mathfrak{a}) = (2\pi)^m d_m, \;\; \forall j, \tag{2.9}$$

$$\int_{\mathbb{R}^m} \xi_j^2 \xi_k^2 \mathfrak{a}(|\xi|)^2 d\xi = \frac{(2\pi)^{m/2}}{\Gamma(m/2+2)} I_{m+3}(\mathfrak{a}) = (2\pi)^m h_m, \;\; \forall j \neq k, \tag{2.10}$$

$$\int_{\mathbb{R}^m} \xi_j^4 \mathfrak{a}(|\xi|)^2 d\xi = \frac{6\pi^{m/2}}{\Gamma(m/2+1)} I_{m+3}(\mathfrak{a}) = 3(2\pi)^m h_m, \;\; \forall j. \tag{2.11}$$

Using (2.2) and (2.3) we deduce that for any $\boldsymbol{x} \in \mathbb{R}^m$ *the Gaussian vectors* $\nabla\Phi(\boldsymbol{x})$ *and* $\mathrm{Hess}_\Phi(\boldsymbol{x})$ *are independent*. Hence

$$\mathbb{E}\big[\, \det \mathrm{Hess}_{\Phi_{\mathfrak{a}}}(\boldsymbol{x})| \,\big|\, \nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}) = 0 \,\big] = \mathbb{E}\big[\, \det \mathrm{Hess}_{\Phi_{\mathfrak{a}}}(\boldsymbol{x})| \,\big].$$

Using (2.2)and (2.9) we deduce that the variance matrix of $\nabla\Phi(\boldsymbol{x})$ is

$$\mathrm{Var}\,\big[\, \nabla\Phi(\boldsymbol{x}) \,\big] = d_m \mathbb{1}_m, \;\; \forall \boldsymbol{x} \in \mathbb{R}^m, \tag{2.12}$$

where $\mathbb{1}_m$ denotes the identity $m \times m$ matrix. Hence

$$p_{\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})}(0) = (2\pi d_m)^{-m/2}.$$

The space $\mathbf{Sym}(\mathbb{R}^m)$ of real symmetric $m \times m$ matrices is equipped with an inner product $(A, B) = \mathrm{tr}(A)$. Moreover the linear functions $\ell_{ij}, \omega_{ij} : \mathbf{Sym}(\mathbb{R}^m) \to \mathbb{R}$, $1 \leq i \leq j \leq m$,

$$\ell_{ij}(A) = a_{ij}, \;\; \omega_{ij}(A) = \begin{cases} a_{ii}, & i = j, \\ \sqrt{2} a_{ij}, & i < j \end{cases} \tag{2.13}$$

define coordinates on $\mathbf{Sym}(\mathbb{R}^m)$ that are orthonormal with respect to the above inner product. We set

$$L_{ij}(\boldsymbol{x}) := \ell_{ij}\big(\operatorname{Hess}_\Phi(\boldsymbol{x})\big),\;\; \Omega_{ij}(\boldsymbol{x}) := \omega_{ij}\big(\operatorname{Hess}_\Phi(\boldsymbol{x})\big). \tag{2.14}$$

Then

$$\mathbb{E}\big[\,\partial^2_{x_ix_j}\Phi_{\mathfrak{a}}(x)\partial^2_{x_kx_\ell}\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\xi_i\xi_j\xi_k\xi_\ell a\big(|\xi|^2\big)d\xi,\;\; i\leq j,\;\; k\leq \ell.$$

Note that if $i<j$, then the above integral is nonzero iff $(i,j)=(k,\ell)$ in which case

$$\mathbb{E}\big[\,L_{ij}(\boldsymbol{x})L_{ij}(\boldsymbol{x})\,\big] = \mathbb{E}\Big[\,\big(\,\partial^2_{x_ix_j}\Phi_{\mathfrak{a}}(x)\,\big)^2\,\Big]$$

$$= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\xi_i^2\xi_j^2\mathfrak{a}\big(|\xi|^2\big)d\xi \stackrel{(2.10)}{=} h_m.$$

If $i=j$, then the above integral is nonzero iff $k=\ell$, in which case we deduce from (2.10) and (2.11)

$$\mathbb{E}\big[\,\partial^2_{x_i}\Phi_{\mathfrak{a}}(x)\partial^2_{x_k}\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] = \begin{cases} h_m & i\neq k,\\ 3h_m, & i=k.\end{cases}$$

The above equalities can be rewritten in the more compact form

$$\mathbb{E}\big[\,L_{ij}(\boldsymbol{x})L_{k\ell}(\boldsymbol{x})\,\big] = h_m\big(\,\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}\,\big),\;\; \forall i\leq j,\; k\leq\ell. \tag{2.15}$$

These equalities show that the off-diagonal entries of Hess_Φ are i.i.d., and also independent of the diagonal entries. The diagonal entries have identical distributions but are dependent. The parameter h_m describes the various variances and covariances.

The Gaussian measure on $\mathbf{Sym}(\mathbb{R}^m)$ determined by these covariance equalities is invariant with respect to the action of $O(m)$ by conjugation on $\mathbf{Sym}(\mathbb{R}^m)$. For $v>0$ denote by $\mathcal{S}^v_m$ the space $\mathbf{Sym}(\mathbb{R}^m)$ equipped with the $O(m)$-invariant Gaussian measure on $\mathbf{Sym}(\mathbb{R}^m)$ determined by the covariances

$$\mathbb{E}\big[\,L_{ij}(\boldsymbol{x})L_{k\ell}(\boldsymbol{x})\,\big] = v\big(\,\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}\,\big),\;\; \forall i\leq j,\; k\leq\ell.$$

Hence

$$\mathbb{E}\big[\,\det\operatorname{Hess}_{\Phi_{\mathfrak{a}}}(\boldsymbol{x})|\,\big] = \mathbb{E}_{\mathcal{S}^{h_m}_m}\big[\,|\det H|\,\big].$$

For any box $B\subset\mathbb{R}^m$ we denote by $Z_{\mathfrak{a}}(B)=Z_{\mathfrak{a}}(B)$ the number of critical points of $\Phi_{\mathfrak{a}}$ in B. We deduce from the Kac-Rice formula (2.1) that

$$\mathbb{E}\big[\,Z_{\mathfrak{a}}(B)\,\big] = \int_B \mathbb{E}_{\mathcal{S}^{h_m}_m}\big[\,|\det H|\,\big]p_{\nabla\Phi(\boldsymbol{x})}(0)\boldsymbol{\lambda}\big[\,d\boldsymbol{x}\,\big]$$

$$\stackrel{(2.12)}{=} (2\pi d_m)^{-m/2}\mathbb{E}_{\mathcal{S}^{h_m}_m}\big[\,|\det H|\,\big]\operatorname{vol}\big[\,B\,\big]$$

$(X=(2h_m)^{-1/2}H)$

$$= \underbrace{\left(\frac{h_m}{\pi d_m}\right)^{m/2}\mathbb{E}_{\mathcal{S}^{1/2}_m}\big[\,|\det X|\,\big]}_{=:C_m(\mathfrak{a})}\operatorname{vol}\big[\,B\,\big].$$

Hence

$$\mathbb{E}\big[\,Z_{\mathfrak{a},m}(B)\,\big] = C_m(\mathfrak{a})\operatorname{vol}\big[\,B\,\big]. \tag{2.16}$$

Remark 2.3. One can prove that as $m \to \infty$

$$C_m(\mathfrak{a}) \sim 2^{\frac{5}{2}}\Gamma\left(\frac{m+3}{2}\right)\left(\frac{h_m}{\pi d_m}\right)^{m/2}\left(\frac{1}{m+1}\right)^{1/2}.$$

Using (2.9) and (2.10) we deduce

$$\frac{h_m}{d_m} = \frac{\Gamma(1+m/2)}{\Gamma(2+m/2)} \times \frac{I_{m+3}(\mathfrak{a})}{I_{m+1}(\mathfrak{a})} = \frac{2I_{m+3}(\mathfrak{a})}{(m+2)I_{m+1}(\mathfrak{a})}.$$

Hence

$$\begin{aligned} C_m(\mathfrak{a}) &\sim 2^{5/2}\left(\frac{h_m(\mathfrak{a})}{d_m(\mathfrak{a})}\right)^{m/2}\Gamma\left(\frac{m+3}{2}\right)m^{-1/2} \\ &\sim 2^{\frac{m+5}{2}}\left(\frac{I_{m+3}(\mathfrak{a})}{(m+2)I_{m+1}(\mathfrak{a})}\right)^{m/2}\Gamma\left(\frac{m+3}{2}\right)m^{-1/2} \ \text{ as } m \to \infty. \end{aligned} \tag{2.17}$$

The constant $C_m(\mathfrak{a})$ tends to grow very fast as $m \to \infty$, but its large m behavior depends on the tail of the amplitude $\mathfrak{a}$. Roughly speaking, the slower the decay at ∞ of $\mathfrak{a}$ the faster the growth of $C_m(\mathfrak{a})$. □

3. An abstract law of large numbers

Fix $m \in \mathbb{N}$. For $N \in \mathbb{N}$, let $\mathbb{I}_N := [1, N] \cap \mathbb{N}$. Suppose that we have an even continuous function $\rho : \mathbb{R}^m \to (0, \infty)$ that decays sufficiently fast to 0 as $|x| \to \infty$. Then

$$\int_{NB\times NB}\rho(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y} = N^{2m}\int_{B\times B}\rho_N\big(\boldsymbol{u}-\boldsymbol{v}\big)d\boldsymbol{u}d\boldsymbol{v}$$

where $\rho_N(\boldsymbol{x}) = \rho(N\boldsymbol{x})$. Observing that $\rho_N(\boldsymbol{x}) \to 0$ almost everywhere on B we deduce from the dominated convergence theorem that

$$\int_{B\times B}\rho_N\big(\boldsymbol{u}-\boldsymbol{v}\big)d\boldsymbol{u}d\boldsymbol{v} \to 0$$

as $N \to \infty$. Hence

$$\int_{NB\times NB}\rho(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y} = o\big(N^{2m}\big) \ \text{ as } N \to \infty$$

In fact we can be more precise. If we use Fubini theorem and integrate ρ along the m-planes orthogonal to the ‘diagonal $\Delta_N = \{\boldsymbol{x} = \boldsymbol{y}\} \subset NB \times nB$ we deduce that

$$\int_{NB\times NB}\rho(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y} \le C\,\mathrm{vol}_m\big[\,\Delta_N\,\big]\int_{|\boldsymbol{x}|<N}\rho(\boldsymbol{x})d\boldsymbol{x} \le CN^m\int_{|\boldsymbol{x}|<N}\rho(\boldsymbol{x})d\boldsymbol{x}.$$

If we specialize further, $\rho(\boldsymbol{x}) = \frac{1}{1+|\boldsymbol{x}|^p}$, $p > 0$, $p \neq m$, then

$$\int_{|\boldsymbol{x}|<N}\rho(\boldsymbol{x})d\boldsymbol{x} = O\big(N^{\max(m-p,0)}\big)$$

so that

$$\int_{NB\times NB}\rho(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y} = O\big(N^{m+\max(m-p,0)}\big).$$

The sum

$$\sum_{(\vec{k},\vec{\ell})\in\mathbb{I}_N^m\times\mathbb{I}_N^m}\rho(\vec{k}-\vec{\ell})$$

is a very rough Riemann sum approximation of the above integral when $B=[0,1]^N$. The next results show that if $\rho(\boldsymbol{x})=\frac{1}{1+|\boldsymbol{x}|^p}$, then this Riemann sum is also $o\big(N^{2m}\big)$ as $N\to\infty$.

Lemma 3.1. *Fix $m\in\mathbb{N}$. For any $N\in\mathbb{N}$ we set $\mathcal{R}_{N,m}:=\mathbb{I}_N^m\times\mathbb{I}_N^m$.*

(i) *If $m>1$, then there exists a constant $K=K(\alpha,p,m)>0$ such that*

$$\sum_{(\vec{k},\vec{\ell})\in\mathcal{R}_{N_m}}\frac{1}{\big(1+\alpha|\vec{\ell}-\vec{k}|_1\big)^p}\leq KN^{2m-\kappa(p)},\;\;\kappa(p)=\min(p,1).$$

(ii) *If $m=1$, then there exists a constant $K=K(\alpha,p)>0$ such that*

$$\sum_{k,\ell\in\mathbb{I}_n}\frac{1}{(1+\alpha|k-\ell|)^p}\leq K\begin{cases}N^{2m-\kappa(p)} & ,p\neq 1,\\ N\log N & p=1.\end{cases}$$

Proof. (i) $m>1$. For any $N\in\mathbb{N}$ define

$$D_{N,m}:=\big\{\,(\vec{k},\vec{\ell})\in\mathcal{R}_{N_m};\;\;\exists j=1,\dots,m,\;\;k_j=\ell_j\,\big\},\;\;\mathcal{R}^*_{N,m}:=\mathcal{R}_{N_m}\setminus D_{N,m}$$

Note that

$$D_N=\bigcup_{i=1}^m D_N^i,\;\;D_N^i=\big\{\,(\vec{k},\vec{\ell})\in\mathcal{R}_N;\;\;k_i=\ell_i\,\big\}.$$

For $1\leq i_1<\cdots<i_r\leq m$ we have

$$\#\big(\,D_N^{i_1}\cap\cdots\cap D_N^{i_p}\,\big)=N^{2m-2r+1}$$

Using the Inclusion-Exclusion Principle we deduce that

$$\#D_N=\sum_{r=1}^m(-1)^{p-1}\binom{m}{p}N^{2m-2r+1}\leq 2^mN^{2m-1}.$$

We have

$$\sum_{(\vec{k},\vec{\ell})\in\mathcal{R}_N}\frac{1}{\big(1+\alpha|\vec{\ell}-\vec{k}|_1\big)^p}=\underbrace{\sum_{(\vec{k},\vec{\ell})\in D_N}\frac{1}{\big(1+\alpha|\vec{\ell}-\vec{k}|_1\big)^p}}_{=:Y_N}+\underbrace{\sum_{(\vec{k},\vec{\ell})\in\mathcal{R}^*_N}\frac{1}{\big(1+\alpha|\vec{\ell}-\vec{k}|_1\big)^p}}_{=:Z_N}.$$

Note that

$$Y_N\leq\#D_N\leq 2^mN^{2m-1}.$$

To estimate Z_N we need to first analyze the structure of the region $\mathcal{R}^*_N$. Denote by R_i the reflection

$$R_i:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m\times\mathbb{R}^m,\;\;\begin{bmatrix}x_1 & y_1\\ \vdots & \vdots\\ \textcircled{x_i} & \boxed{y_i}\\ \vdots & \vdots\\ x_m & y_m\end{bmatrix}\mapsto\begin{bmatrix}x_1 & y_1\\ \vdots & \vdots\\ \boxed{y_i} & \textcircled{x_i}\\ \vdots & \vdots\\ x_m & y_m\end{bmatrix}.$$

Denote by G_m the direct product of cyclic groups

$$G_m=\big(\mathbb{Z}/2\mathbb{Z}\big)^m=\big\{\,\vec{\epsilon}=(\epsilon_1,\dots,\epsilon_m);\;\;\epsilon_k=0,1\,\big\}.$$

The group G_m acts freely on $\mathcal{R}^*_N$

$$\vec{\epsilon}\cdot\big(\vec{k},\vec{\ell}\big)=R^{\vec{\epsilon}}\big(\vec{k},\vec{\ell}\big),\;\;R^{\vec{\epsilon}}=R_1^{\epsilon_1}\cdots R_m^{\epsilon_m}.$$

We denote by $\mathcal{C}_N^+$ the positive chamber of $\mathcal{R}_N^*$,

$$\mathcal{C}_n^+ := \left\{ \left((\vec{k},\vec{\ell}) \in \mathcal{R}_N^*;\ \ell_j > k_j,\ \ \forall 1 \leq j \leq m \right\},$$

and we observe that

$$\mathcal{C}_n^+ = \mathcal{T}_N^m,\ \ \mathcal{T}_N := \left\{ (k,\ell) \in \mathbb{I}_N \times \mathbb{I}_N;\ \ \ell > k \right\}.$$

We have

$$\mathcal{R}_N^* = \bigcup_{\vec{\epsilon}\in G_m} R^{\vec{\epsilon}}\mathcal{C}_n^+$$

The function ρ is G_m-invariant so

$$\sum_{(\vec{k},\vec{\ell})\in\mathcal{R}_N^*} \frac{1}{\left(1+\alpha|\vec{\ell}-\vec{k}|_1\right)^p} = \sum_{\vec{\epsilon}\in G_m}\sum_{(\vec{k},\vec{\ell})\in R^{\vec{\epsilon}}\mathcal{C}_N^+} \frac{1}{\left(1+\alpha|\vec{\ell}-\vec{k}|_1\right)^p}$$

$$= 2^m \sum_{(\vec{k},\vec{\ell})\in\mathcal{C}_N^+} \frac{1}{\left(1+\alpha|\vec{\ell}-\vec{k}|_1\right)^p} < \frac{2^m}{\alpha^p}\sum_{(\vec{k},\vec{\ell})\in\mathcal{C}_N^+}\frac{1}{|\vec{\ell}-\vec{k}|_1^p}$$

(use the AM-GM inequality)

$$\leq \frac{2^m}{(m\alpha)^p}\sum_{(\vec{k},\vec{\ell})\in\mathcal{C}_N^+}\Big(\prod_{j=1}^m(\ell_j-k_j)\Big)^{-p/m}.$$

$(\mathcal{C}_n^+ = \mathcal{T}_N^m)$

$$= \frac{2^m}{(m\alpha)^p}\prod_{j=1}^m\sum_{(k_j,\ell_j)\in\mathcal{T}_N}\left(\ell_j-k_j\right)^{-p/m} = \frac{2^m}{(m\alpha)^p}\Big(\sum_{(k,\ell)\in\mathcal{T}_N}\left(\ell-k\right)^{-p/m}\Big)^m.$$

Now observe that

$$\sum_{(k,\ell)\in\mathcal{T}_N}\left(\ell-k\right)^{-p/m} = \sum_{k=1}^{N-1}\sum_{j=1}^{N-k} j^{-p/m}.$$

To proceed further we need to use the following result.

Sublemma 3.2. *Let $r\in\mathbb{R}$. Then for any $M\in\mathbb{N}$*

$$S_r(M) := \sum_{j=1}^M j^r \leq u_r(M) := \begin{cases} \frac{1}{r+1}M^{r+1}, & r\geq 0,\\ \frac{1}{r+1}M^{r+1}+1, & r\in(-1,0),\\ \log M+1 & r=-1\\ 2, & r<-1.\end{cases}$$

Proof. Using approximations by Riemann sums for the integral

$$I_r(M) := \int_1^M x^r dx$$

we deduce

$$S_r(M) \leq \begin{cases} I_r(M), & r>0,\\ I_r(M)+1, & r<0\end{cases} = \begin{cases} \frac{1}{r+1}\left(M^{r+1}-1\right), & r\geq 0,\\ \frac{1}{r+1}\left(M^{r+1}-1\right)+1, & r\in(-1,0),\\ 1+\log M, & r=-1,\\ \frac{1}{|r+1|}\left(1-M^{r+1}\right)+1, & r<-1.\end{cases}$$

$$\leq \begin{cases} \frac{1}{r+1}M^{r+1}, & r \geq 0, \\ \frac{1}{r+1}M^{r+1}, & r \in (-1,0), \\ 1+\log M, & r=-1, \\ 2, & r<-1. \end{cases}$$

□

Suppose that $p \neq m$. Using Sublemma 3.2 we deduce that

$$\sum_{j=1}^{N-k} j^{-p/m} \leq u_{-p/m}(N-k) = \begin{cases} \frac{1}{(1-p/m)}(N-k)^{1-p/m}+1, & p/m<1, \\ 2 & 1<p/m. \end{cases}$$

Next, using the sublemma again we deduce

$$\sum_{k=1}^{N} u_r(N-k) \leq \begin{cases} \frac{1}{(1-p/m)(2-p/m)}N^{2-p/m}+N, & p/m<1, \\ 2N & 1<p/m. \end{cases}$$

Hence

$$\sum_{(k,\ell)\in \mathfrak{T}_N} \bigl(\ell-k\bigr)^{-p/m} \leq \begin{cases} \frac{1}{(1-p/m)(2-p/m)}N^{2-p/m}+N, & p/m<1, \\ 2N & 1<p/m. \end{cases}$$

and thus

$$Z_N = \sum_{(\vec{k},\vec{\ell})\in \mathcal{R}_N^*} \frac{1}{(1+|\vec{\ell}-\vec{k}|_1)^p} \leq \frac{2^m}{(m\alpha)^p} \leq C(m,\alpha,p) \begin{cases} N^{2m-p}, & p<m, \\ N^m & p>m. \end{cases} \tag{3.1}$$

If $p=m$, then Sublemma 3.2 implies that

$$\sum_{j=1}^{N-k} j^{-1} \leq u_{-1}(N-k) = 1+\log(N-k),$$

and

$$\sum_{k=1}^{N} \bigl(1+\log(N-k)\bigr) = N+\log N!.$$

The conclusion follows from Stirling's formula which implies that

$$\log N! = O\bigl(N\log N\bigr).$$

(ii) Suppose that $p=1$. Then

$$\sum_{k,\ell=\mathbb{I}_N} \frac{1}{(1+\alpha|k=\ell|)^p} = N+2\sum_{1} \leq k<\ell\leq n\frac{1}{(1+\alpha|k-\ell|)^p}$$

$$< N+\frac{2}{\alpha^p}\sum_{1\leq k<\ell} \frac{1}{(\ell-k)^p} = N+\frac{2}{\alpha^p}\sum_{j=1}^{N-1}\frac{N-j}{j^p}$$

$$= N+\frac{2}{\alpha^p}\sum_{k=1}^{N-1}\sum_{j=1}^{k}\frac{1}{j^p} \leq N+\frac{2}{\alpha^p}\sum_{k=1}^{N-1}u_{-p}(j).$$

The conclusion now follows exactly as in (i). □

Lemma 3.3. *Fix $m \in \mathbb{N}$. For any $N \in \mathbb{N}$ we set $\mathcal{R}_{N,m} := \mathbb{I}_N^m \times \mathbb{I}_N^m$. For any $\alpha > 0$ and any $p > m$ there exists a constant $K = K(\alpha, p, m) > 0$ such that*

$$\sum_{(\vec{k},\vec{\ell})\in\mathcal{R}_{N_m}} \frac{1}{\big(1+\alpha|\vec{\ell}-\vec{k}|_1\big)^p} \leq KN^m.$$

Proof. We argue by induction. The case $m = 1$ is covered in Lemma 3.1 (ii). Define

$$\rho_m : \mathbb{N}^m \times \mathbb{N}^m \to (0,\infty),\ \ \rho_m\big(\vec{k},\vec{\ell}\big) = \frac{1}{\big(1+\alpha|\vec{k}-\vec{\ell}|_1\big)^p}.$$

For any region $\mathcal{R} \subset \mathbb{N}^m \times \mathbb{N}^m$ we set

$$S(\mathcal{R},\rho_m) = \sum_{(\vec{k},\vec{\ell})\in\mathcal{R}} \rho_m\big(\vec{k},\vec{\ell}\big).$$

For any $N \in \mathbb{N}$ define

$$D_{N,m} := \big\{\, (\vec{k},\vec{\ell}) \in \mathcal{R}_{N,m};\ \ \exists j = 1,\dots,m,\ \ k_j = \ell_j \,\big\},\ \ \mathcal{R}^*_{N,m} := \mathcal{R}_{N_m} \setminus D_{N_m}.$$

We have

$$S\big(\mathcal{R}_{N,m},\rho_m\big) = S\big(D_{N,m},\rho_m\big) + S\big(\mathcal{R}^*_{N,m},\rho_m\big).$$

The inequality (3.1) implies that

$$S\big(\mathcal{R}^*_{N,m},\rho_m\big) \leq KN^m.$$

As before we have

$$D_N = \bigcup_{i=1}^m D^i_{N,m},\ \ D^i_{N,m} = \big\{\,(\vec{k},\vec{\ell}) \in \mathcal{R}_{N_m};\ \ k_i = \ell_i\,\big\}.$$

Using Inclusion-Exclusion Principle we deduce

$$S\big(D_{N,m},\rho_m\big) = \sum_{p=1}^m (-1)^{p-1} \sum_{1\leq i_1<\cdots<i_p\leq m} S\big(D^{i_1}_{N,m}\cap\cdots\cap D^{i_p}_{N,m},\rho_m\big)$$

$$= \sum_{p=1}^m (-1)^{p-1}\binom{m}{p} S\big(D^1_{N,m}\cap\cdots\cap D^p_{N,m},\rho_m\big)$$

$$= \sum_{p=1}^m (-1)^{p-1}\binom{m}{p} S\big(\mathcal{R}_{N,m-p},\rho_{m-p}\big) \leq \sum_{p=1}^m \binom{m}{p} S\big(\mathcal{R}_{N,m-p},\rho_{m-p}\big)$$

(use the induction assumption)

$$\leq K\sum_{p=1}^m \binom{m}{p} N^{m-p} \leq K(N+1)^m \leq 2^m KN^m.$$

□

Corollary 3.4. *Consider a family of random variable $\big(X_{\vec{\ell}}\big)_{\vec{\ell}\in\mathbb{N}^M}$ defined on the same probability space $(\Omega,\mathcal{S},\mathbb{P})$ such that there exist constants $C,\alpha,p>0$ such that*

$$\big|\ \mathrm{Cov}\big[\,X_{\vec{k}},X_{\vec{\ell}}\big]\,\big| \leq \frac{C}{\big(1+\alpha|\vec{k}-\vec{\ell}\,|_1\big)^p},\ \ \forall\vec{k},\vec{\ell}\in\mathbb{N}^m.$$

Then, as $N\to\infty$

$$A_N(X) := \frac{1}{N^m}\sum_{\vec{k}\in\mathbb{I}_N^m}\big(X_{\vec{k}} - \mathbb{E}\big[X_{\vec{k}}\big]\big)\to 0$$

in L^2*. In particular* $A_N(X)$ *converges in probability to zero as* $N\to\infty$*. Moreover, if* $p>m$*, then* A_N *converges* a.s. *to* 0*.*

Proof. We have

$$\mathbb{E}\big[A_N(X)^2\big] = \frac{1}{N^{2m}}\sum_{(\vec{k},\vec{\ell})\in\mathbb{I}_N^m\times\mathbb{I}_N}\operatorname{Cov}\big[X_{\vec{k}},X_{\vec{\ell}}\big].$$

Lemma 3.1 implies that $\mathbb{E}\big[A_N(X)^2\big]\to 0$. Moreover if $p>m$, then $\mathbb{E}\big[A_N(X)^2\big] = O(N^{-m})$ as $N\to\infty$. Hence, by Chebysev's inequality there exists a constant $C>0$ such that, $\forall N$,

$$\mathbb{P}\big[A_N(x)\geq\varepsilon\big]\leq\frac{C}{\varepsilon^2N^m}.$$

In particular, for $m>1$ we have

$$\sum_N\mathbb{P}\big[A_N(x)\geq\varepsilon\big]<\infty$$

so $A_N\to 0$ a.s.. For $m=1$, the result follows from [14, Thm.10]. □

This follows from the Strong Law of Large Numbers [15, Thm. 4]. For $r\in\mathbb{N}$ we denote by C_r the lattice cube $\mathbb{I}_{2^r}^m$. Set $N_r := 2^{r+1}$.

Remark 3.5. (a) We have a.s. convergence in general without assuming $p>m>1$. This follows from the Strong Law of Large Numbers [15, Thm. 4]. For $r\in\mathbb{N}$ we denote by C_r the lattice cube $\mathbb{I}_{2^r}^m$. Set $N_r := 2^{r+1}$.

Then

$$u_r := \sum_{\vec{k},\vec{\ell}\in C_{r+1}\setminus C_r}\big|\,\mathbb{E}\big[X_{\vec{k}}X_{\vec{\ell}}\big]\,\big|\leq\sum_{\vec{k},\vec{\ell}\in C_{r+1}}\big|\,\mathbb{E}\big[X_{\vec{k}}X_{\vec{\ell}}\big]\,\big|\leq KN_r^{2m-1},$$

where $K>0$ is a universal constant. We deduce that

$$\sum_{r\geq 0}\frac{(r+1)^2}{N_r^{2m}}u_r\leq K\sum_{r\geq 0}\frac{(r+1)^2}{N_r}<\infty.$$

According to [15, Thm. 4], this implies that $A_N(X)\to 0$ a.s..

(b) The estimate $\mathbb{E}\big[A_N^2\big] = O\big(N^{-m}\big)$ used in the above proof is optimal. For example, if the random variables $X_{\vec{k}}$ are independent and their variances are equal to 1, then $\mathbb{E}\big[A_N^2\big] = N^{-m}$. □

4. Proof of Theorem 1.1

We have

$$Z_{\mathfrak{a}}(N\cdot B) = \sum_{\vec{\ell}\in\mathbb{I}_N^m}Z_{\mathfrak{a}}(B_{\vec{\ell}})$$

since , according to Bulinskaya's lemma, the probability that $\Phi_{\mathfrak{a}}$ has a critical point on the boundary of some B_ℓ is zero.

The Gaussian function $\Phi_{\mathfrak{a}}$ is a stationary is stationary, so the collection $\big(Z_{\mathfrak{a}}(B_{\vec{\ell}})\big)_{\vec{\ell}\in\mathbb{N}^m}$ is stationary as well, i.e., for any $n\in\mathbb{N}$ and any $\vec{\ell},\vec{\ell}_1,\dots,\vec{\ell}_n\in\mathbb{N}^n$ the random vectors

$$\big(Z_{\mathfrak{a}}(B_{\vec{\ell}_1}),\dots,Z_{\mathfrak{a}}(B_{\vec{\ell}_n})\big) \text{ and } \big(Z_{\mathfrak{a}}(B_{\vec{\ell}_1+\vec{\ell}}),\dots,Z_{\mathfrak{a}}(B_{\vec{\ell}_n+\vec{\ell}})\big)$$

have the same distribution. We deduce from the ergodic theorem [13, Chap.6, Thm.2.5] that the averages

$$\frac{1}{\mathrm{vol}(NB)} Z_{\mathfrak{a}}(N\cdot B) = \frac{1}{N^m\,\mathrm{vol}(B)}\sum_{\vec{\ell}\in\mathbb{I}_N} Z_{\mathfrak{a}}(B_{\vec{\ell}})$$

converge a.s. and in L^1 to some random variable Z_∞. We need to prove that the limit Z_∞ is the constant

$$\frac{1}{\mathrm{vol}(B)}\mathbb{E}\big[\, Z_{\mathfrak{a}}(B)\,\big].$$

To achieve this we set

$$X_{\vec{\ell}} := Z_{\mathfrak{a}}(B_{\vec{\ell}}) - \mathbb{E}\big[\, Z_{\mathfrak{a}}(B_{\vec{\ell}})\,\big].$$

Then

$$\sum_{\vec{\ell}\in\mathbb{I}_N^m} X_{\vec{\ell}} = Z_{\mathfrak{a}}\big(NB\,\big) - \mathbb{E}\big[\, Z_{\mathfrak{a}}\big(NB\,\big)\,\big].$$

We have to show that

$$\lim_{N\to\infty}\frac{1}{N^m}\Bigg\|\sum_{\vec{\ell}\in\mathbb{I}_N^m} X_{\vec{\ell}}\Bigg\|_{L^2}\to 0.$$

We will deduce this from Corollary 3.4. We set

$$C(\vec{k},\vec{\ell}) := \mathrm{Cov}\,\big[\, Z_{\mathfrak{a}}(B_{\vec{k}}), Z_{\mathfrak{a}}(B_{\vec{\ell}})\,\big] = \mathbb{E}\big[\, X_{\vec{k}}X_{\vec{\ell}}\,\big],\ \ \forall \vec{k},\vec{\ell}\in\mathbb{I}_N^m.$$

Notice that since $\Phi_{\mathfrak{a}}$ is a stationary Gaussian function we have

$$\mathbb{E}\big[\, Z_{\mathfrak{a}}(B_\ell)^p\,\big] = \mathbb{E}\big[\, Z_{\mathfrak{a}}(B)^p\,\big] < \infty,\ \ \forall p\in[1,\infty),\ \ \vec{\ell}\in\mathbb{N}^m$$

and we deduce that

$$\exists K_1 = K_1(\mathfrak{a},m) > 0 : \big|\, C(\vec{k},\vec{\ell})\,\big| < K_1,\ \ \forall \vec{k},\vec{\ell}\in\mathbb{N}^m. \tag{4.1}$$

To proceed further we define

$$\widehat{\Phi}:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R},\ \ \widehat{\Phi}(\boldsymbol{x},\boldsymbol{y}) = \Phi_{\mathfrak{a}}(\boldsymbol{x}) + \Phi_{\mathfrak{a}}(\boldsymbol{y})$$

and we set

$$\widehat{H}(\boldsymbol{x},\boldsymbol{y}) := \mathrm{Hess}_{\widehat{\Phi}}(\boldsymbol{x},\boldsymbol{y}),\ \ H(\boldsymbol{x}) := \mathrm{Hess}_{\Phi_{\mathfrak{a}}}(\boldsymbol{x}).$$

Denote for any Borel subset $\widehat{B}\in\mathbb{R}^m\times\mathbb{R}^m$ by $\widehat{Z}\big(\,\widehat{B}\,\big)$ the number of critical points of $\widehat{\Phi}$ in $\widehat{B}$. Note that if $B_{\vec{k}}\cap B_{\vec{\ell}} = \emptyset$, then

$$\widehat{Z}(B_{\vec{k}}\times B_{\vec{\ell}}) = Z_{\mathfrak{a}}(B_{\vec{k}})Z_{\mathfrak{a}}(B_{\vec{\ell}})$$

so

$$\mathbb{E}\big[\, Z_{\mathfrak{a}}(B_{\vec{k}})Z_{\mathfrak{a}}(B_{\vec{\ell}})\,\big] = \mathbb{E}\big[\,\widehat{Z}(B_{\vec{k}}\times B_{\vec{\ell}})\,\big].$$

We want to compute $\mathbb{E}\big[\,\widehat{Z}(B_{\vec{k}}\times B_{\vec{\ell}})\,\big]$ using the Kac-Rice formula. We first need to verify that $\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})$ is nondegenerate for any $\boldsymbol{x}\neq\boldsymbol{y}$.

Lemma 4.1. *For any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^m$, $\boldsymbol{x}\neq\boldsymbol{y}$, the Gaussian vector $\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})$ is nondegenerate.*

Proof. We have

$$\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]=\left[\begin{array}{cc}\operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\end{array}\right].$$

As shown in(2.12), for any $\boldsymbol{x}\in\mathbb{R}^n$ we have

$$\operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big]=d_m\mathbb{1}_m,\;\; d_m=\int_{\mathbb{R}^n}\xi_1^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big].$$

We have

$$\operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]=\big(\,\partial_{x_j}\partial_{y_k}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y})\,\big)_{1\leq j,k\leq m}$$

and

$$\partial_{x_j}\partial_{y_k}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y})=\int_{\mathbb{R}^m}e^{-\boldsymbol{i}\langle\xi,\boldsymbol{x}-\boldsymbol{y}\rangle}\xi_j\xi_k\mu_{\mathfrak{a}}\big[\,d\xi\,\big].\tag{4.2}$$

Since $\Phi_{\mathfrak{a}}$ is stationary it suffice to consider only the case $\boldsymbol{x}=0$. On the other hand, $\Phi_{\mathfrak{a}}$ is $O(m)$-invariant so, up to a rotation we can assume that $\boldsymbol{x}-\boldsymbol{y}=-t\boldsymbol{e}_1$, $t\neq 0$, where $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_m\}$ is the canonical basis of $\mathbb{R}^m$. Hence

$$\partial_{x_j}\partial_{y_k}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y})=\int_{\mathbb{R}^m}e^{\boldsymbol{i}t\xi_1}\xi_j\xi_k\mu_{\mathfrak{a}}\big[\,d\xi\,\big].$$

Let us observe that if $j\neq k$, then either $j\neq 1$, or $k\neq 1$. Suppose $j\neq 1$. The function $e^{\boldsymbol{i}t\xi_1}\xi_j\xi_k$ is odd with respect to the reflection $\xi_j\mapsto-\xi_j$ so

$$\partial_{x_j}\partial_{y_k}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x},\boldsymbol{y})=\int_{\mathbb{R}^m}e^{\boldsymbol{i}t\xi_1}\xi_j\xi_k\mu_{\mathfrak{a}}\big[\,d\xi\,\big]=0,\;\;\forall j\neq k.$$

If $j=k$, then

$$d_m(j):=\partial_{x_j}\partial_{y_j}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x},\boldsymbol{y})=\int_{\mathbb{R}^m}e^{\boldsymbol{i}t\xi_1}\xi_j^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big]=\int_{\mathbb{R}^m}\cos(t\xi_1)\xi_j^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big]$$

and we deduce[2]

$$|v_m(j)|\leq\int_{\mathbb{R}^m}\big|\cos(t\xi_1)\big|\xi_j^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big]<\int_{\mathbb{R}^m}\xi_j^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big]=d_m.$$

After a reordering

$$\begin{gathered}\big(\,\partial_{x_1}\Phi_{\mathfrak{a}}(\boldsymbol{x}),\dots,\partial_{x_m}\Phi_{\mathfrak{a}}(\boldsymbol{x}),\partial_{y_1}\Phi_{\mathfrak{a}}(\boldsymbol{y}),\dots\partial_{y_m}\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big)\\ \downarrow\\ \big(\,\partial_{x_1}\Phi_{\mathfrak{a}}(\boldsymbol{x}),\partial_{y_1}\Phi_{\mathfrak{a}}(\boldsymbol{y}),\dots,\partial_{x_m}\Phi_{\mathfrak{a}}(\boldsymbol{x}),\partial_{y_m}\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big)\end{gathered}$$

we see that

$$\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]=\bigoplus_{j=1}^m\underbrace{\left[\begin{array}{cc}d_m & d_m(j)\\ d_m(j) & d_m\end{array}\right]}_{=:V_j}.$$

Note that, for each j, the symmetric matrix V_j is positive definite since

$$\det V_j=d_m^2-d_m(j)^2>0.$$

□

[2]At this point we use the fact that $\mathfrak{a}\big(\,|\xi|\,\big)>0$ for $|\xi|$ sufficiently small.

Suppose that $B_{\vec{k}}\cap B_{\vec{\ell}}=\emptyset$. We deduce from Lemma 4.1 that $\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})$, is nondegenerate for any $(\boldsymbol{x},\boldsymbol{y})\in B_{\vec{k}},\times B_{\vec{\ell}}$. We can the apply the Kac-Rice formula to deduce that

$$\begin{aligned}\mathbb{E}\big[\,Z_{\mathfrak{a}}(B_{\vec{k}})Z_{\mathfrak{a}}(B_{\vec{\ell}})\,\big] &= \mathbb{E}\big[\,\widehat{Z}(B_{\vec{k}}\times B_{\vec{\ell}})\,\big]\\ &= \int_{B_{\vec{k}}\times B_{\vec{\ell}}}\underbrace{\mathbb{E}\big[\,|\det\widehat{H}(\boldsymbol{x},\boldsymbol{y})|\,\big|\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})=0\,\big]p_{\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})}(0)}_{=\widehat{\rho}(\boldsymbol{x},\boldsymbol{y})}\boldsymbol{\lambda}\big[\,d\boldsymbol{x}d\boldsymbol{y}\,\big],\\ &\quad\text{if } B_{\vec{k}}\cap B_{\vec{\ell}}=\emptyset \qquad .\end{aligned}\tag{4.3}$$

For $\boldsymbol{x}\in\mathbb{R}^m$ we denote by $\big|\,\boldsymbol{x}\,\big|_\infty$ the sup-norm of $\boldsymbol{x}$

$$\big|\,\boldsymbol{x}\,\big|_\infty := \max_{1\leq i\leq m}|x_i|.$$

Note that

$$\big|\,\boldsymbol{x}\,\big|_1\leq m\big|\,\boldsymbol{x}\,\big|_\infty.$$

and $B_{\vec{k}}\cap B_{\vec{\ell}}=\emptyset$ if $\big|\,\vec{k}-\vec{\ell}\,\big|_\infty>1$. Hence,

$$\big|\,\vec{k}-\vec{\ell}\,\big|_1>m\Rightarrow B_{\vec{k}}\cap B_{\vec{\ell}}=\emptyset.$$

Hence

$$\begin{aligned}\mathbb{E}\big[\,Z_{\mathfrak{a}}(B_vk)Z_{\mathfrak{a}}(B_{\vec{\ell}})\,\big] &= \int_{B_{\vec{k}}\times B_{\vec{\ell}}}\underbrace{\mathbb{E}\big[\,|\det\widehat{H}(\boldsymbol{x},\boldsymbol{y})|\,\big|\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})=0\,\big]p_{\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})}(0)}_{=\widehat{\rho}(\boldsymbol{x},\boldsymbol{y})}\boldsymbol{\lambda}\big[\,d\boldsymbol{x}d\boldsymbol{y}\,\big],\\ &\quad\text{if } \big|\,\vec{k}-\vec{\ell}\,\big|_1>m.\end{aligned}\tag{4.4}$$

Let us now express $\mathbb{E}\big[\,Z_{\mathfrak{a}}(B_{\vec{k}})\,\big]\mathbb{E}\big[\,Z_{\mathfrak{a}}(B_{\vec{\ell}})\,\big]$ as an integral over $B_{\vec{k}}\times B_{\vec{\ell}}$. Choose an independent copy $\Psi_{\mathfrak{a}}$ of $\Phi_{\mathfrak{a}}$. We set

$$\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y}):=\Phi_{\mathfrak{a}}(\boldsymbol{x})+\Psi_{\mathfrak{a}}(\boldsymbol{y}),\ \ \widetilde{H}(\boldsymbol{x},\boldsymbol{y}):=\mathrm{Hess}_{\widetilde{\Phi}}(\boldsymbol{x},\boldsymbol{y}).$$

Then

$$\begin{aligned}\mathbb{E}\big[\,Z_{\mathfrak{a}}(B_{\vec{k}})\,\big]\mathbb{E}\big[\,Z_{\mathfrak{a}}(B_{\vec{\ell}})\,\big] &= \mathbb{E}\big[\,Z_{\widetilde{\Phi}}(B_{\vec{k}}\times B_{\vec{\ell}})\,\big]\\ = \int_{B_{\vec{k}}\times B_{\vec{\ell}}}&\underbrace{\mathbb{E}\big[\,\big|\,\det\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big|\,\big|\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})=0\,\big]p_{\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})}(0)}_{=\widetilde{\rho}(\boldsymbol{x},\boldsymbol{y})}\boldsymbol{\lambda}\big[\,d\boldsymbol{x}d\boldsymbol{y}\,\big],\end{aligned}\tag{4.5}$$

if $\big|\,\vec{k}-\vec{\ell}\,\big|_1>m$. Thus

$$\mathrm{Cov}\,\big[\,Z_{\mathfrak{a}}(B_{\vec{k}}),Z_{\mathfrak{a}}(B_{\vec{\ell}})\big]=\int_{B_{\vec{k}}\times B_{\vec{\ell}}}\big(\,\widehat{\rho}(\boldsymbol{x},\boldsymbol{y})-\widetilde{\rho}(\boldsymbol{x},\boldsymbol{y})\,\big)\boldsymbol{\lambda}\big[\,\boldsymbol{x}d\boldsymbol{y}.\,\big]\tag{4.6}$$

To proceed further, we need a few technical results that seem to be part of the mathematical folklore, but whose proofs are difficult to locate in the existing literature.

Digression 4.2. Suppose that $\boldsymbol{V}$ is an m-dimensional real Euclidean space with inner product $(-,-)$. Denote by $S_1(\boldsymbol{V})$ the unit sphere in $\boldsymbol{V}$ and by $\mathbf{Sym}(\boldsymbol{V})$ the space of symmetric operators $\boldsymbol{V}\to\boldsymbol{V}$ and by $\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ the cone of nonnegative ones. For $A\in\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ we denote by Γ_A the centered Gaussian measure with variance A.

The space $\mathbf{Sym}(\boldsymbol{V})$ is equipped with an inner product

$$\big(\,A,B\,\big)_{\mathrm{op}}=\mathrm{tr}(AB),\ \ \forall A,B\in\mathbf{Sym}(\boldsymbol{V}).$$

We denote by $\|-\|_{\mathrm{op}}$ the associated norm.

We have a natural map $\mathbf{Sym}_{\geq 0}(\boldsymbol{V})\to\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$, $A\mapsto A^{1/2}$. We will need the following result, [11, Prop.2.1]. For any $\mu>0$ and $\forall A,B\in\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$, such that $A^{1/2}+B^{1/2}\geq\mu\mathbb{1}$ we have

$$\mu\big\|\,A^{1/2}-B^{1/2}\,\big\|_{\mathrm{op}}\leq\big\|\,A-B\,\big\|_{\mathrm{op}}^{1/2}. \tag{4.7}$$

Lemma 4.3. *Fix $A_0\in\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ such that $A_0^{1/2}\geq\mu_0\mathbb{1}$, $\mu_0>0$. Suppose that $f:\mathbb{V}\to\mathbb{R}$ is a locally Lipschitz function that is homogeneous of degree $k\geq 1$. For $A\in\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ we set*

$$\mathcal{I}_A(f):=\int_{\boldsymbol{V}}f(\boldsymbol{v})\boldsymbol{\Gamma}_{\mathbf{A}}\big[\,\mathbf{d}\boldsymbol{v}\,\big].$$

Then for and $R\geq\|A_0\|_{\mathrm{op}}$ there exists a constant $C=C(f,R,\mu_0)>0$ with the following property: for any $A\in\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ such that $\|,A\|_{\mathrm{op}}\leq R$

$$\big|\,\mathcal{I}_{A_0}(f)-\mathcal{I}_A(f)\,\big|\leq C\|A-A_0\|^{1/2}\leq C(k,R)\|A-A_0\|_{\mathrm{op}}^{1/2}. \tag{4.8}$$

In other words, $A\mapsto\mathcal{I}_A(f)$ is locally Hölder continuous with exponent $1/2$ in the open set $\mathbf{Sym}_{>0}\big(\boldsymbol{V}\big)$.

Proof. The function f is Lipschitz on the ball

$$B_R(\boldsymbol{V}):=\big\{\,\boldsymbol{v}\in\boldsymbol{V};\,\|\boldsymbol{v}\|\leq R\,\big\},$$

so there exists $L=L(R)>0$ such that

$$\big[\,f(\boldsymbol{u})-f(\boldsymbol{v})\,\big|\leq L\|\boldsymbol{u}-\boldsymbol{v}\|,\ \ \forall\boldsymbol{u},\boldsymbol{v}\in B_R(\boldsymbol{V}). \tag{4.9}$$

Note that

$$\mathcal{I}_A(f)=\int_{\boldsymbol{V}}f\big(\,A^{1/2}\boldsymbol{v}\,\big)\Gamma_{\mathbb{1}}\big[\,d\boldsymbol{v}\,\big],$$

so

$$\big|\,\mathcal{I}_{A_0}(f)-\mathcal{I}_A(f)\,\big|\leq\int_{\boldsymbol{V}}\big|\,f\big(\,A^{1/2}\boldsymbol{v}\,\big)-f\big(\,A_0^{1/2}\boldsymbol{v}\,\big)\,\big|\,\Gamma_{\mathbb{1}}\big[\,d\boldsymbol{v}\,\big]$$

$$=\underbrace{\frac{1}{(2\pi)^{m/2}}\left(\int_0^\infty r^{n+k-1}e^{-r^2/2}dr\right)}_{C_{m,k}}\int_{S_1(\boldsymbol{V})}\big|\,f\big(\,A^{1/2}\boldsymbol{v}\,\big)-f\big(\,A_0^{1/2}\boldsymbol{v}\,\big)\,\big|\,\mathrm{vol}_{S_1(\boldsymbol{V})}\big[\,d\boldsymbol{v}\,\big]$$

$$\stackrel{(4.9)}{\leq}C_{m,k}L(R)\int_{S_1(\boldsymbol{V})}\|A^{1/2}-A_0^{1/2}\|_{\mathrm{op}}\,\mathrm{vol}_{S^1(\boldsymbol{V})}\big[\,d\boldsymbol{v}\,\big]\stackrel{(4.7)}{\leq}C(k,R,\mu_0)\|A-A_0\|_{\mathrm{op}}^{1/2}.$$

□

This ends the digression. □

For every $\boldsymbol{z}\in\mathbb{R}^m$ we set

$$T(z):=\sum_{|\alpha|\leq 4}\big|\,\partial^\alpha\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{z})\,\big|.$$

Since $\boldsymbol{K}_{\mathfrak{a}}$ is a Schwartz function we deduce that

$$T(\boldsymbol{z})=O\big(\,|\boldsymbol{z}|_1^{-\infty}\,\big)\ \text{ as }|\boldsymbol{z}|_1\to\infty.$$

This means that

$$\forall p>0,\ \ T(\boldsymbol{z})=O\big(\,|\boldsymbol{z}|_1^{-p}\,\big)\ \text{ as }|\boldsymbol{z}|_1\to\infty.$$

Observe that

$$\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]=\begin{bmatrix}\operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & 0\\ 0 & \operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\end{bmatrix}=d_m\mathbb{1}_{2m},$$

and

$$\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]=\begin{bmatrix}\operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\end{bmatrix}$$

$$=\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]+\underbrace{\begin{bmatrix}0 & \operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & 0\end{bmatrix}}_{=:R_{\nabla}(\boldsymbol{x},\boldsymbol{y})}.$$

In particular

$$\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}=\Big(\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]+R_{\nabla}(\boldsymbol{x},\boldsymbol{y})\Big)^{-1}$$
$$=\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\Big(\mathbb{1}+\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}R_{\nabla}(\boldsymbol{x},\boldsymbol{y})\Big)^{-1}.$$

Hence

$$\Big\|\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]-\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]\Big\|_{\mathrm{op}}=\|R_{\nabla}(\boldsymbol{x},\boldsymbol{y})\|_{\mathrm{op}}=O\big(T(\boldsymbol{x}-\boldsymbol{y})\big). \tag{4.10}$$

Since $\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]$ is independent of $\boldsymbol{x}$ and $\boldsymbol{y}$

$$\Big\|\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}-\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\Big\|_{\mathrm{op}}=O\big(T(\boldsymbol{x}-\boldsymbol{y})\big). \tag{4.11}$$

Note that

$$\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big]=\begin{bmatrix}\operatorname{Var}\big[\,H(\boldsymbol{x})\,\big] & 0\\ 0 & \operatorname{Var}\big[\,H(\boldsymbol{y})\,\big]\end{bmatrix}.$$

Since $\Phi_{\mathfrak{a}}$ is stationary, $\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big]$ is *independent* of $\boldsymbol{x}$ and $\boldsymbol{y}$. We have

$$\operatorname{Var}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big]=\begin{bmatrix}\operatorname{Var}\big[\,H(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,H(\boldsymbol{x}),H(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,H(\boldsymbol{y}),H(\boldsymbol{x})\,\big] & \operatorname{Var}\big[\,H(\boldsymbol{y})\,\big]\end{bmatrix}$$

$$=\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big]+\underbrace{\begin{bmatrix}0 & \operatorname{Cov}\big[\,H(\boldsymbol{x}),H(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,H(\boldsymbol{y}),H(\boldsymbol{x})\,\big] & 0\end{bmatrix}}_{=:R_H(\boldsymbol{x},\boldsymbol{y})}.$$

We deduce

$$\Big\|\operatorname{Var}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big]-\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big]\Big\|_{\mathrm{op}}=\|R_H(\boldsymbol{x},\boldsymbol{y})\|_{\mathrm{op}}=O\big(T(\boldsymbol{x}-\boldsymbol{y})\big). \tag{4.12}$$

We denote by $\widetilde{H}(\boldsymbol{x},y)^{\flat}$ the Gaussian random matrix obtained from $\widetilde{H}(\boldsymbol{x},\boldsymbol{y})$ by conditioning on $\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})=0$. Similarly, we denote by $\widehat{H}(\boldsymbol{x},y)^{\flat}$ the Gaussian random matrix obtained from $\widehat{H}(\boldsymbol{x},\boldsymbol{y})$ by conditioning on $\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})=0$. The distributions of $\widetilde{H}(\boldsymbol{x},y)^{\flat}$ and $\widehat{H}(\boldsymbol{x},y)^{\flat}$ are determined by the Gaussian regression formula.

Since $\widetilde{H}(\boldsymbol{x},y)$ and $\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})$ are independent we deduce

$$\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})^{\flat}\,\big]=\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big].$$

Using the regression formula we deduce

$$\begin{gathered}\operatorname{Var}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]=\operatorname{Var}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big]\\ -\operatorname{Cov}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\operatorname{Cov}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y}),\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big]|\\ =\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]+R_H(\boldsymbol{x},\boldsymbol{y})\\ -\operatorname{Cov}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\operatorname{Cov}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y}),\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big].\end{gathered}$$

We have

$$\operatorname{Cov}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]=\left[\begin{array}{cc}\operatorname{Cov}\big[\,H(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,H(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,H(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,H(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\end{array}\right]$$

$$=\left[\begin{array}{cc}\operatorname{Cov}\big[\,0 & \operatorname{Cov}\big[\,H(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,H(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & 0\end{array}\right].$$

This implies

$$\operatorname{Cov}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]=O\big(\,T(\boldsymbol{x}-\boldsymbol{y})\,\big).$$

Since $\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]$ is independent of $\boldsymbol{x}$ and $\boldsymbol{y}$ we deduce from (4.11) that

$$\operatorname{Cov}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\operatorname{Cov}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y}),\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big]=O\big(\,T(\boldsymbol{x}-\boldsymbol{y})\,\big),$$

Hence

$$\sup_{\boldsymbol{x}\neq\boldsymbol{y}}\|\operatorname{Var}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]\|_{\mathrm{op}}<\infty,\tag{4.13}$$

Since $\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big]$ is independent of $\boldsymbol{x},\boldsymbol{y}$ we deduce that there exists $\mu_0>0$ such that

$$\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]\geq\mu_0\mathbb{1},\ \ \forall\boldsymbol{x}\neq\boldsymbol{y}.$$

We deduce from (4.13) and (4.8) that

$$\Big|\,\mathbb{E}\big[\,|\det\widehat{H}(\boldsymbol{x},\boldsymbol{y})^\flat|\,\big]-\mathbb{E}\big[\,|\det\widetilde{H}(\boldsymbol{x},\boldsymbol{y})^\flat|\,\big]\,\Big|=O\big(\,T(\boldsymbol{x}-\boldsymbol{y})^{1/2}\,\big).\tag{4.14}$$

Using (4.10) we deduce

$$\begin{gathered}\Big|\,p_{\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})}(0)-p_{\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})}(0)\,\Big|\\ =(2\pi)^{-m/2}\Big|\,\det\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}-\det\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\,\Big|=O\big(\,T(\boldsymbol{x}-\boldsymbol{y})\,\big).\end{gathered}\tag{4.15}$$

We can now estimate the right-hand-side of (4.6). Note that if $B_{\vec{k}}\cap B_{\vec{\ell}}=\emptyset$ we have

$$\sup_{(\boldsymbol{x},\boldsymbol{y})\in B_{\vec{k}}\times B_{\vec{\ell}}}O\big(\,T(\vec{k}-\vec{\ell})\,\big)=O\big(\,|\vec{k}-\vec{\ell}|_1^{-\infty}\,\big).$$

We deduce from(4.10), (4.11), (4.14) and (4.15) that

$$\sup_{(\boldsymbol{x},\boldsymbol{y})\in B_{\vec{k}}\times B_{\vec{\ell}}}\big|\,\widehat{\rho}(\boldsymbol{x},\boldsymbol{y})-\widetilde{\rho}(\boldsymbol{x},\boldsymbol{y})\,\big|=O\big(\,\big|\,\vec{k}-\vec{\ell}\,\big|_1^{-\infty}\,\big),\ \ \big|\,\vec{k}-\vec{\ell}\,\big|_1>m.\tag{4.16}$$

Hence

$$C(\vec{k},\vec{\ell})=O\big(\,\big|\,\vec{k}-\vec{\ell}\,\big|_1^{-\infty}\,\big)\ \text{ as }\big|\,\vec{k}-\vec{\ell}\,\big|_1\to\infty.$$

The above estimate coupled with (4.1) implies that, for any $p>0$, there exists a constant $C=C(p)>0$ such that

$$\big|\,C(\vec{k},\vec{\ell})\,\big|\leq \frac{C(p)}{\big(\,1+|\vec{k}-\vec{\ell}|_1\,\big)^p},\;\;\forall(\vec{k},\vec{\ell})\in\mathbb{N}^m\times\mathbb{N}^m.$$

Theorem 1.1 now follows from the above estimate and Corollary 3.4.

5. Proof of Theorem 1.2

Note that $\boldsymbol{x}$ is a critical point of $\Phi_{\mathfrak{a}}$ iff $N^{-1}\boldsymbol{x}$ is a critical point of $\Phi^{\frac{1}{N}}$. If we set

$$f_N(\boldsymbol{x}):=f\big(\,N^{-1}\boldsymbol{x}\,\big).$$

Then

$$Z(f,\Phi_{\mathfrak{a}}^{\frac{1}{N}})=Z(f_N,\Phi_{\mathfrak{a}})$$

since $\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})=0$ iff $\nabla\Phi_{\mathfrak{a}}(Nx)=0$ and $f(x)=f_N(Nx)$. Hence

$$Y_N(f)=\frac{1}{N^m}Z(f_N,\Phi_{\mathfrak{a}})$$

Using the weighted local Kac-Rice formula [4, Thm.6.4] we deduce

$$\mathbb{E}\big[\,Z(f_N,\Phi_{\mathfrak{a}})\,\big]=\int_{\mathbb{R}^n}f_N(\boldsymbol{x})\underbrace{\mathbb{E}\big[\,|\det\operatorname{Hess}_{\Phi_{\mathfrak{a}}}|\,\big|\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})=0\,\big]p_{\nabla\Phi_{\mathfrak{a}}}(0)}_{=\rho_{\mathfrak{a}}(\boldsymbol{x})}\,d\boldsymbol{x}.$$

In the proof of (2.16) we showed that the density $\rho_{\mathfrak{a}}(\boldsymbol{x})$ is constant, $\rho_{\mathfrak{a}}(\boldsymbol{x})=\rho_{\mathfrak{a}}(0)=C_m(\mathfrak{a})$. Hence

$$\mathbb{E}\big[\,Z(f_N,\Phi_{\mathfrak{a}})\,\big]=C_m(\mathfrak{a})\int_{\mathbb{R}^m}f_N(\boldsymbol{x})d\boldsymbol{x}\stackrel{\boldsymbol{x}\to N\boldsymbol{y}}{=}N^mC_m(\mathfrak{a})\int_{\mathbb{R}^m}f(\boldsymbol{y})d\boldsymbol{y}.$$

We have to prove that

$$\frac{1}{N^m}\big(\,Z(f_N,\Phi_{\mathfrak{a}})-\mathbb{E}\big[\,Z(f,\Phi_{\mathfrak{a}})\,\big]\,\big)\to 0.$$

Using the decomposition $f=f_+-f_-$ we see that it suffices to consider only the case $f\geq 0$. By replacing $f(\boldsymbol{x})$ with a translate $\boldsymbol{x}\mapsto f(\boldsymbol{x}-\boldsymbol{y})$, we can assume that $\operatorname{supp}f\subset[0,R]^m$, for some $R>0$ sufficiently large. Set $B=[0,R]^m$. Then $\operatorname{supp}f_N\subset N\cdot B$.

We use the same strategy as in the proof of Theorem 1.1. For $\vec{\ell}\in\mathbb{I}_N^M$ we set

$$X_{\vec{\ell}}^N(f)=Z_{B_{\vec{\ell}}}(f_N,\Phi_{\mathfrak{a}})=\sum_{\substack{\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})=0,\\ \boldsymbol{x}\in B_{\vec{\ell}}}}f_N(\boldsymbol{x}).$$

Since $\operatorname{supp}f_N\subset NB$ we deduce Then

$$Y_N(f)=\underbrace{\frac{1}{N^m}\sum_{\vec{\ell}\in\mathbb{I}_N^m}X_{\vec{\ell}}^N(f)}_{=:S_N(f)}\;\;\text{a.s.},$$

and

$$\sum_{\vec{\ell}\in\mathbb{I}_N^m}\mathbb{E}\big[\,X_{\vec{\ell}}^N(f)\,\big]=\mathbb{E}\big[\,Z(f_N,\Phi_{\mathfrak{a}})\,\big]$$

so tha

$$\operatorname{Var}\big[\,Y_N(f)\,\big]=\frac{1}{N^{2m}}\operatorname{Var}\big[\,S_N(f)\,\big]=\frac{1}{N^{2m}}\sum_{\vec{k},\vec{\ell}\in\mathbb{I}_N^m}\operatorname{Cov}\big[\,X^N_{\vec{k}}(f),X^N_{\vec{\ell}}(f)\,\big].$$

Hence we have to prove that

$$\operatorname{Var}\big[\,S_N(f)\,\big]=O\big(\,N^m\,\big)\ \text{ as } N\to\infty.$$

Lemma 5.1. *For any $m\in\mathbb{N}$ and any $p>m$ there exists a constant $C_p=C_p(\mathfrak{a},m,f)$,* independent of N *such that, for any $N>0$ and any $\vec{k},\vec{\ell}\in\mathbb{I}_N^m$, we have*

$$\big|\operatorname{Cov}\big[\,X^N_{\vec{k}}(f),X^N_{\vec{\ell}}(f)\,\big]\,\big|\leq C\big(\,1+\big|\,\vec{k}-\vec{\ell}\,\big|_\infty\,\big)^{-p}. \tag{5.1}$$

Proof. For any Morse function $\Psi:\mathbb{R}^N\to\mathbb{R}$, any Borel subset $S\subset\mathbb{R}^n$ and any nonnegative continuous function $g\in C^0_{\mathrm{cpt}}(\mathbb{R}^m)\to 0$ we set

$$Z_S(g)=Z_S\big(\,g,\Psi\,\big)=\sum_{\substack{\nabla\Psi(\boldsymbol{x})=0,\\ \boldsymbol{x}\in S}}g(\boldsymbol{x})$$

Then $X^N_{\vec{k}}(f)=Z_{B_{\vec{k}}}\big(\,f_N,\Phi_{\mathfrak{a}}\,\big)$. Define

$$f_N^{\boxtimes 2}:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R},\ \ f_N^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y})=f_N(\boldsymbol{x})f_N(\boldsymbol{y}).$$

As in the proof of Theorem 1.1 we choose an independent copy $\Psi_{\mathfrak{a}}$ of $\Phi_{\mathfrak{a}}$ and we define

$$\widehat{\Phi},\ \widetilde{\Phi}:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R},\ \ \widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})=\Phi_{\mathfrak{a}}(\boldsymbol{x})+\Phi_{\mathfrak{a}}(y),\ \widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})=\Phi_{\mathfrak{a}}(\boldsymbol{x})+\Psi_{\mathfrak{a}}(y).$$

Note that for any cube $C=[a,b]^m\subset NB$ we have

$$Z_C(f_N,\Phi_{\mathfrak{a}})^2=Z_{C^2_*}\big(\,f_N^{\boxtimes 2},\widehat{\Phi}\,\big)+Z_C\big(\,f_N^2,\Phi_{\mathfrak{a}}\,\big),$$

where

$$C^2_*=\big\{(\boldsymbol{x},\boldsymbol{y})\in C^2;\ \ \boldsymbol{x}\neq\boldsymbol{y}\,\big\}.$$

Note that

$$\mathbb{E}\big[\,Z_C\big(\,f_N^2\,\big)\,\big]=C_m(\mathfrak{a})\int_C f_N^2(\boldsymbol{x})d\boldsymbol{x}\leq C_m(\mathfrak{a})\|f\|_{C^0}\operatorname{vol}\big[\,C\,\big].$$

Using the weighted local Kac-Rice formula we deduce

$$\mathbb{E}\big[\,Z_{C^2_*}\big(\,f_N^{\boxtimes 2},\widehat{\Phi}\,\big)\,\big]=\int_{C^2_*}\widehat{\rho}(\boldsymbol{x},\boldsymbol{y})f(x)f(y)d\boldsymbol{y}),$$

where $\widehat{\rho}$ is defined in (4.4). On the other hand, as shown in [7, App.1] or [9, Sec.1.2] there exsts $K>0$ depending only on $\mathfrak{a}$ and m such that

$$\forall\boldsymbol{x}\neq\boldsymbol{y},\ \ \big|\,\widehat{\rho}(\boldsymbol{x},\boldsymbol{y})\,\big|<K\max\big(\,1,\big|\,\boldsymbol{x}-\boldsymbol{y}\,\big|^{2-m}\,\big).$$

$$\big|\,\mathbb{E}\big[\,Z_{C^2_*}\big(\,f_N^{\boxtimes 2},\widehat{\Phi}\,\big)\,\big]\,\big|\leq K\|f\|^2_{C^0}\int_{C^2}\big|\,\boldsymbol{x}-\boldsymbol{y}\,\big|^{2-m}d\boldsymbol{x}d\boldsymbol{y}\leq K\|f\|^2_{C^0}\operatorname{vol}\big[\,C\,\big]^2,$$

where K denotes a positive constant that depends only on $\mathfrak{a}$ and m. We deduce that

$$\forall\vec{k}\in\mathbb{I}_N^m:\ \ \operatorname{Var}\big[\,X^N_{\vec{\ell}}(f)\,\big]=\operatorname{Var}\big[\,Z_{B_{\vec{\ell}}}(f_N)\,\big]\leq K$$

If $B_{\vec{k}}\cap B_{\vec{\ell}}=\emptyset$, $\vec{k},\vec{\ell}\in\mathbb{I}_N^m$ and $p>m$, then with $\widetilde{\rho}$ defined in (4.5) we have

$$\operatorname{Cov}\big[\,X^N_{\vec{k}}(f),X^N_{\vec{\ell}}(f)\,\big]=\operatorname{Cov}\big[\,Z_{B_{\vec{k}}}(f_N),Z_{B_{\vec{\ell}}}(f_N)\,\big]$$

$$= \int_{B_{\vec{k}}\times B_{\vec{\ell}}} \big(\,\widehat{\rho}(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}(\boldsymbol{x},\boldsymbol{y})\,\big) f_N(\boldsymbol{x}) f_N(\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y}$$

$$\leq \sup_{(\boldsymbol{x},\boldsymbol{y})\in B_{\vec{k}}\times B_{\vec{\ell}}} \big|\,\widehat{\rho}(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}(\boldsymbol{x},\boldsymbol{y})\,\big| \cdot \|f\|^2_{C^0} \operatorname{vol}\big[\,B_{\vec{k}}\times B_{\vec{\ell}}\,\big]$$

$$\stackrel{(4.16)}{\leq} K_p\big(\,1+\big|\,\vec{k}-\vec{\ell}\,\big|_\infty\,\big)^{-p}\|f\|^2_{C^0} \operatorname{vol}\big[\,B\,\big]^2,$$

where K_p is a positive constant that depends only on $\mathfrak{a}$, m and p and it is independent of N. □

The estimate (1.5) now follows from Lemmas 3.3 and 5.1 . □

Remark 5.2. Set $\overline{Y}_N(f) = Y_N(f) - \mathbb{E}\big[\,Y_N(f)\,\big]$. We proved that

$$\mathbb{E}\big[\,\overline{Y}_N(f)^2\,\big| = O(N^{-m}), \text{ as } N\to\infty.$$

This implies a.s. convergence for $m>1$ since $\sum_{N>0} N^{-m} < \infty$ for $m>1$. This argument fails in dimension $m=1$. In [2, 8] the authors prove that when $m=1$

$$\mathbb{E}\big[\,\overline{Y}_N(f)^4\,\big| = O(N^{-2}), \text{ as } N\to\infty.$$

This estimate implies a.s.-convergence. □

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