A LAW OF LARGE NUMBERS CONCERNING THE DISTRIBUTION OF CRITICAL POINTS OF RANDOM FOURIER SERIES

QIANGANG "BRANDON" FU AND LIVIU I. NICOLAESCU

ABSTRACT. On the flat torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ with angular coordinates $\vec{\theta}$ we consider the random function $F_R = \mathfrak{a}\big(R^{-1}\sqrt{\Delta}\big)W$, where $R > 0$, Δ is the Laplacian on this flat torus, $\mathfrak{a}$ is an even Schwartz function on $\mathbb{R}$ such that $\mathfrak{a}(0) > 0$ and W is the Gaussian white noise on $\mathbb{T}^m$ viewed as a random generalized function. For any $f \in C(\mathbb{T}^m)$ we set

$$Z_R(f) := \sum_{\nabla F_R(\vec{\theta})=0} f(\vec{\theta})$$

We prove that if the support of f is contained in a geodesic ball of $\mathbb{T}^m$, then the variance of $Z_R(f)$ is asymptotic to $const \times R^m$ as $R \to \infty$. We use this to prove that if $m \geq 2$, then as $N \to \infty$ the random measures $N^{-m}Z_N(-)$ converge a.s. to an explicit multiple of the volume measure on the flat torus.

CONTENTS

1. INTRODUCTION

Denote by $\mathbb{T}^m$ the m-dimensional torus $\mathbb{R}^m/\mathbb{Z}^m$ and by g_1 the flat metric of volume 1. In terms of angular coordinates $\vec{\theta} = (\theta^1, \dots, \vec{\theta}^m)$, $\theta^i \in \mathbb{R} \bmod \mathbb{Z}$,

$$g_1 = (d\theta^1)^2 + \cdots + (d\theta^m)^2.$$

For $R > 0$, meant to be large, we denote by Δ_R the Laplacian of the metric $g_R = R^2 g_1$. Observe that

$$\operatorname{vol}\big[\, M, g_R \,\big] = R^m \operatorname{vol}\big[\, M, g_1 \,\big] = R^m, \;\; \Delta_R = R^{-2}\Delta_1.$$

Last revised December 10, 2024.

A complete orthonormal system of *complex* eigenfunctions of Δ_1 is given by $\big(e_{\vec{\ell}}\big)_{\vec{\ell}\in\mathbb{Z}^m}$

$$e_{\vec{\ell}}\big(\vec{\theta}\,\big)=e^{2\pi\boldsymbol{i}\langle\vec{\ell},\vec{\theta}\rangle},\;\;\langle\vec{\ell},\vec{\theta}\rangle=\sum_{j=1}^m\ell_j\theta^j.$$

To describe a complete orthonormal system of *real* eigenfunctions of Δ_1 we introduce the lexicografic order on $\mathbb{Z}^m$, $\vec{\ell}\succ 0$ iff $\exists i_0:\;\;\ell_{i_0}>0,\;\;\ell_i=0,\;\;\forall i<i_0$. We set $u_0=1$ and, for $\vec{\ell}\succ 0$ we set

$$u_{\vec{k}}\big(\vec{\theta}\,\big)=\sqrt{2}\cos\big(2\pi\langle\vec{k},\vec{\theta}\rangle\big),\;\;v_{\vec{\ell}}\big(\vec{\theta}\,\big)=\sqrt{2}\sin\big(2\pi\langle\vec{\ell},\vec{\theta}\rangle\big),$$
$$u^R_{\vec{k}}=R^{-m/2}u_{\vec{k}},\;\;v^R_{\vec{\ell}}=R^{-m/2}v_{\vec{\ell}}.$$

The collection

$$\big\{\,u^R_{\vec{k}},\;v^R_{\vec{\ell}};\;\vec{k}\succeq 0,\;\;\vec{\ell}\succ 0\,\big\}$$

is a complete $L^2(M,g_R)$-orthonormal system of real eigenfunctions of Δ_R. Moreover

$$\Delta_R u^R_{\vec{k}}=\lambda_{\vec{k}}(R),\;\;\Delta_R v^R_{\vec{\ell}}=\lambda_{\vec{\ell}}(R)v^R_{\vec{\ell}},\;\;\lambda_{\vec{k}}(R)=R^{-2}\big|\,2\pi\vec{k}\,\big|^2=\big(2\pi/R\big)^2\sum_{j=1}^m k_j^2.$$

Fix an even Schwarz function $\mathfrak{a}\in\mathcal{S}(\mathbb{R})$ such that $\mathfrak{a}(0)=1$. We will refer to such an $\mathfrak{a}$ as *amplitude*. Fix independent standard normal random variables

$$\big\{\,A_{\vec{k}},\;B_{\vec{\ell}};\;\vec{k}\succeq 0,\;\;\vec{\ell}\succ 0\,\big\}$$

and consider the random Fourier series

$$\begin{aligned}F^R_{\mathfrak{a}}(\vec{\theta})&=\mathfrak{a}(0)A_0u^R_0\big(\vec{\theta}\,\big)+\sum_{\vec{\ell}\succ 0}\mathfrak{a}\big(\lambda_{\vec{\ell}}(R)^{1/2}\big)\big(A_{\vec{\ell}}u^R_{\vec{\ell}}\big(\vec{\theta}\,\big)+B_{\vec{\ell}}v^R_{\vec{\ell}}\big(\vec{\theta}\,\big)\big)\\&=R^{-m/2}\Big(A_0u_0\big(\vec{\theta}\,\big)+\sum_{\vec{\ell}\succ 0}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)\big(A_{\vec{\ell}}u_{\vec{\ell}}\big(\vec{\theta}\,\big)+B_{\vec{\ell}}v\big(\vec{\theta}\,\big)\big)\Big)\\&=R^{-m/2}\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)Z_{\vec{\ell}}e_{\vec{\ell}}(\vec{\theta}),\end{aligned}\tag{1.1}$$

where

$$Z_{\vec{\ell}}=\begin{cases}A_0, & \vec{\ell}=0,\\ \frac{1}{\sqrt{2}}\big(A_{\vec{\ell}}-\boldsymbol{i}B_{\vec{\ell}}\big), & \vec{\ell}\succ 0,\\ \bar{Z}_{-\vec{\ell}}, & \vec{\ell}\prec 0.\end{cases}$$

Since $\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)$ decays very fast as $|\vec{\ell}|\to\infty$ we deduce from Kolmogorov's two-series theorem that for any $\nu\in\mathbb{N}$ the random series

$$\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)^2\,\|\,e_{\vec{\ell}}\|^2_{C^\nu(\mathbb{T}^m)}$$

converges a.s. and thus the series

$$\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)Z_{\vec{\ell}}e_{\vec{\ell}}$$

converges a.s. in $C^\nu\big(\mathbb{T}^m\big)$. In particular, this shows that the Gaussian function $F^R_{\mathfrak{a}}$ is a.s. smooth. Its covariance kernel is

$$\mathcal{C}^R_{\mathfrak{a}}\big(\vec{\varphi}+\vec{\tau},\vec{\varphi}\big)=C^R_{\mathfrak{a}}(\vec{\tau})=R^{-m}\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)^2e^{2\pi\boldsymbol{i}\langle\vec{\ell},\vec{\tau}\rangle}.$$

Define

$$w_{\mathfrak{a}}:\mathbb{R}^m\to\mathbb{R},\;\; w_{\mathfrak{a}}\big(\,\xi\,\big)=\mathfrak{a}\big(\,|\xi|\,\big)^2,\;\; |\xi|^2:=\sum_{j=1}^m\xi_j^2.$$

Its Fourier transform is

$$\widehat{w}_{\mathfrak{a}}(\boldsymbol{x})=\int_{\mathbb{R}^m}e^{\boldsymbol{i}\langle\xi,\boldsymbol{x}\rangle}w_{\mathfrak{a}}(\xi)d\xi.$$

The Fourier inversion formula shows that

$$\mathfrak{a}\big(\,|\xi|^2\,\big)=\int_{\mathbb{R}^m}e^{\boldsymbol{i}\langle\xi,\boldsymbol{x}\rangle}K_{\mathfrak{a}}(\boldsymbol{x})d\boldsymbol{x},\;\;\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}):=\frac{1}{(2\pi)^m}\widehat{w}_{\mathfrak{a}}(\boldsymbol{x}).$$

Using Poisson's summation formula [15, §7.2] we deduce

$$C^R_{\mathfrak{a}}(\vec{\tau})=\sum_{\vec{k}\in\mathbb{Z}^m}\boldsymbol{K}_{\mathfrak{a}}\big(\,(\vec{k}-\vec{\tau})R\,\big),\;\;\vec{\tau}=\vec{\theta}-\vec{\varphi}. \tag{1.2}$$

We can think of $F^R_{\mathfrak{a}}$ either as a function on $\mathbb{T}^m$, or as a $\mathbb{Z}^m$-periodic function of $\mathbb{R}^m$. If we formally let $R\to\infty$ in the equality

$$R^{m/2}F^R_{\mathfrak{a}}(\vec{\theta})=\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(\,|2\pi\vec{\ell}|/R\,\big)Z_{\vec{\ell}}\,e_{\vec{\ell}}(\vec{\theta})$$

we deduce

$$W_\infty(\vec{\theta})\text{“}=\text{”}\lim_{R\to\infty}R^{m/2}F^R_{\mathfrak{a}}(\vec{\theta})=\sum_{\vec{\ell}\in\mathbb{Z}^m}Z_{\vec{\ell}}\,e_{\vec{\ell}}(\vec{\theta}).$$

The series on the right-hand-side is a.s. divergent but we can still assign a meaning to W_∞ as a random generalized function, i.e., a random linear functional

$$C^\infty(\mathbb{T}^m)\to\mathbb{R},\;\; W_\infty(f)=\sum_{\vec{\ell}\in\mathbb{Z}^m}Z_{\vec{\ell}}\big(\,f,e_{\vec{\ell}}(\vec{\theta})\,\big)_{L^2(\mathbb{T}^m)}.$$

A simple computation shows that for any functions $f_0,f_1\in C^\infty(\mathbb{T}^m)$

$$\operatorname{Cov}\big[\,W_\infty(f_0),W_\infty(f_1)\,\big]=\sum_{\vec{\ell}\in\mathbb{Z}^m}\big(\,f_0,e_{\vec{\ell}}\,\big)_{L^2(\mathbb{T}^m,g_1)}\big(\,f_1,e_{\vec{\ell}}\,\big)_{L^2(\mathbb{T}^m,g_1)}=\big(\,f_0,f_1\,\big)_{L^2(\mathbb{T}^m,g_1)}.$$

The last equality shows that W_∞ is the Gaussian white noise on $\mathbb{T}^m$ driven by the volume measure vol_{g_1}; see [13]. In other words, one could think of the family $\big(\,W_R=R^{m/2}F^R_{\mathfrak{a}}\,\big)_{R>0}$ as a white noise approximation. More precisely, $W^R_{\mathfrak{a}}=\mathfrak{a}\big(\,R^{-1}\sqrt{\Delta}\,\big)W_\infty$. Note that if $\mathfrak{a}(x)=e^{-x^2}$, $t=R^{-2}$, then $\mathfrak{a}\big(\,R^{-1}\sqrt{\Delta}\,\big)=e^{-t\Delta}$, the heat operator.

The main goal of this paper is to investigate the distribution of the critical points of $F^R_{\mathfrak{a}}$ in the white noise limit, $R\to\infty$.

With this in mind, we consider the rescaled function

$$\Phi^R_{\mathfrak{a}}\big(\,\boldsymbol{x}\,\big):=F^R_{\mathfrak{a}}\big(\,\boldsymbol{x}/R\,\big)=R^{-m/2}\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(\,|2\pi\vec{\ell}|/R\,\big)Z_{\vec{\ell}}\,e^{2\pi\boldsymbol{i}\langle\vec{\ell},R^{-1}\boldsymbol{x}\rangle}.$$

We denote by $\mathcal{K}^R_{\mathfrak{a}}$ the covariance kernel of $\Phi^R_{\mathfrak{a}}$. Then $\mathcal{K}^R_{\mathfrak{a}}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y})$, where

$$\boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{z})\overset{(1.2)}{=}\sum_{\vec{k}\in\mathbb{Z}^m}\boldsymbol{K}_{\mathfrak{a}}(R\vec{k}-\boldsymbol{z})=\sum_{\boldsymbol{t}\in(R\mathbb{Z})^m}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{t}-\boldsymbol{z}). \tag{1.3}$$

Since $\boldsymbol{K}_{\mathfrak{a}}$ is a Schwartz function we deduce that

$$\lim_{R\nearrow\infty}\boldsymbol{K}^R_{\mathfrak{a}}=\boldsymbol{K}_{\mathfrak{a}}\;\;\text{ in } C^k\big(\,\mathbb{R}^m\,\big),\;\forall k\in\mathbb{N}. \tag{1.4}$$

The function $\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y})$ is the covariance kernel of an isotropic Gaussian function $\Phi_{\mathfrak{a}}$. The equality (1.4) suggests that $\Phi_{\mathfrak{a}}^R$ approximates $\Phi_{\mathfrak{a}}$ for $R >> 0$.

Suppose that $G : \mathbb{R}^m \to \mathbb{R}$ is a Gaussian C^2-function such that $\nabla G(\boldsymbol{x})$ is a nondegenerate Gaussian vector for any $\boldsymbol{x} \in \mathbb{R}^m$. Then G is a.s. Morse (see Corollary 2.4). Define

$$\mathfrak{C}\big[\,-, G\,\big] = \sum_{\nabla G(\boldsymbol{x})=0} \delta_{\boldsymbol{x}}.$$

This is a locally finite random measure on $\mathbb{R}^m$ in the sense of [8] or [17]. Thus, for every Borel subset $S \subset \mathbb{R}^m$, $\mathfrak{C}[S,G]$ is the number of critical points of G in S. More generally, for any measurable function $\varphi : \mathbb{R}^m \to [0,\infty)$, we set

$$\mathfrak{C}[\varphi, G] := \int_{\mathbb{R}^m} \varphi(\boldsymbol{x})\mathfrak{C}[d\boldsymbol{x}, G] = \sum_{\nabla G(\boldsymbol{x})=0} \varphi(\boldsymbol{x}) \in [0,\infty].$$

Clearly, $\nabla F_{\mathfrak{a}}^R(\boldsymbol{y}) = 0$ iff $\nabla \Phi_{\mathfrak{a}}^R(R\boldsymbol{y}) = 0$ so, for any box $B = [a,b]^m \subset \mathbb{R}^m$, and any $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$, we have

$$\mathfrak{C}\big[\,B, \Phi_{\mathfrak{a}}^R\,\big] = \mathfrak{C}\big[\,RB, F_{\mathfrak{a}}^R\,\big],\;\; \mathfrak{C}\big[\,f, F_{\mathfrak{a}}^R\,\big] = \mathfrak{C}\big[\,f_R, \Phi_{\mathfrak{a}}^R\,\big]$$

where $f_R(\boldsymbol{x}) = f\big(\,R^{-1}\boldsymbol{x}\,\big)$.

In [19, 23] it is shown that the random function $\Phi_{\mathfrak{a}}$ is a.s. Morse and there exists an explicit positive constant $C_m(\mathfrak{a})$ that depends only on $\mathfrak{a}$ and m such that for any box $B \subset \mathbb{R}^m$ and any $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$ we have

$$\mathbb{E}\big[\,\mathfrak{C}[B, \Phi_{\mathfrak{a}}]\,\big] = C_m(\mathfrak{a})\,\mathrm{vol}\,\big[\,B\,\big], \tag{1.5a}$$

$$\mathbb{E}\big[\,\mathfrak{C}[f, \Phi_{\mathfrak{a}}]\,\big] = C_m(\mathfrak{a}) \int_{\mathbb{R}^m} f(\boldsymbol{x})\boldsymbol{\lambda}\big[\,dx\,\big], \tag{1.5b}$$

where $\boldsymbol{\lambda}$ denotes the Lebesgue measure on $\mathbb{R}^m$.

Set $C_1 := [0,1]^m$. In [19] the second author proved that there exists a constant $C_{\mathfrak{a}}'(m) \geq 0$ such that

$$\lim_{R\to\infty} R^{-m}\,\mathrm{Var}\,\big[\,\mathfrak{C}[C_1, F_{\mathfrak{a}}^R]\,\big] = C_m'(\mathfrak{a}). \tag{1.6}$$

The proof of (1.6) in [19] is very laborious and computationally intensive.

The first result of this paper is a functional version of (1.6). We achieve this using a less computational, more robust and more conceptual technique. One consequence of this asymptotic estimate functional strong law of large numbers concerning the random measures $\mathfrak{C}[-, F_{\mathfrak{a}}^N]$, $N \in \mathbb{N}$. Let us provide some more details.

First some notation. Denote by $|-|$ the Euclidean norm on $\mathbb{R}^m$ and by $|-|_\infty$ the sup-norm on $\mathbb{R}^m$. For $\boldsymbol{x}_0 \in \mathbb{R}^m$ and $r > 0$ we set

$$B_r(x_0) := \big\{\,\boldsymbol{x} \in \mathbb{R}^m;\;\; |\boldsymbol{x}| \leq r\,\big\},\;\; B_r^\infty(x_0) := \big\{\,\boldsymbol{x} \in \mathbb{R}^m;\;\; |\boldsymbol{x}|_\infty \leq r\,\big\}.$$

Clearly $B_r(x_0) \subset B_r^\infty(x_0)$.

The function $F_{\mathfrak{a}}^R$ is $\mathbb{Z}^m$-periodic and for $r \in (0,1/2)$ the ball $B_r^\infty(0)$ is contained in the interior of a fundamental domain of the $\mathbb{Z}^m$-action since $|\boldsymbol{x}-\boldsymbol{y}|_\infty \leq 2r < 1$ and $|\vec{\ell}|_\infty \geq 1$, $\forall \vec{\ell} \in \mathbb{Z}^m \setminus 0$. This reflects the fact that the injectivity radius of the flat torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ is $\leq \frac{1}{2}$ so $B_r(0)$ can be viewed as a geodesic ball. We can now state the main technical result of this paper.

Theorem 1.1. *Fix an amplitude* $\mathfrak{a}$*, a positive integer* $m \in \mathbb{N}$*, a radius* $r_0 \in (0,1/2)$ *and a nonnegative function* $f : \mathbb{R}^m \to \mathbb{R}$ *with support contained in* $B_{r_0}(0)$*. Then the following hold.*

(i)

$$\lim_{R\to\infty} R^{-m}\mathbb{E}\big[\,\mathfrak{C}[f, F_{\mathfrak{a}}^R]\,\big] = \mathbb{E}\big[\,\mathfrak{C}[f, \Phi_{\mathfrak{a}}]\,\big] = C_m(\mathfrak{a}) \int_{\mathbb{R}^m} f(\boldsymbol{x})\boldsymbol{\lambda}\big[\,dx\,\big]. \tag{1.7}$$

(ii) *There exists a constant* $V_m(\mathfrak{a}) \geq 0$ *that depends only on* m *and* $\mathfrak{a}$ *such that*

$$\lim_{R\to\infty} R^{-m} \operatorname{Var}\big[\,\mathfrak{C}[f, F_{\mathfrak{a}}^R]\,\big] = V_m(\mathfrak{a}) \int_{\mathbb{R}^m} f(\boldsymbol{x})^2 d\boldsymbol{x}. \tag{1.8}$$

If we consider the normalized random measures

$$\bar{\mathfrak{C}}_R := \frac{1}{R^m}\mathfrak{C}[-, F_{\mathfrak{a}}^R], \quad R > 0$$

then we deduce that for any nonnegative $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$, supp $f \in B_{r_0}(0)$, we have

$$\lim_{R\to\infty} \mathbb{E}\big[\,\bar{\mathfrak{C}}_R[f]\,\big] = C_m(\mathfrak{a}) \int_{\mathbb{R}^m} f(\boldsymbol{x})\boldsymbol{\lambda}\big[\,dx\,\big], \tag{1.9}$$

and

$$\operatorname{Var}\big[\,\bar{\mathfrak{C}}_R[f]\,\big] \sim VR^{-m} \ \text{ as } R \to \infty. \tag{1.10}$$

Using finite partitions of unity we deduce from (1.10) that for any nonnegative f with compact support

$$\operatorname{Var}\big[\,\bar{\mathfrak{C}}_R[f]\,\big] = O\big(\,R^{-m}\,\big) \ \text{ as } R \to \infty.$$

If $m \geq 2$, then

$$\sum_{N\in\mathbb{N}} \frac{1}{N^m} < \infty$$

Borel-Cantelli and (1.10) imply that for any nonnegative $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$ we have

$$\lim_{N\to\infty} \bar{\mathfrak{C}}_N[f] = C_m(\mathfrak{a}) \int_{\mathbb{R}^m} f(\boldsymbol{x})\boldsymbol{\lambda}\big[\,dx\,\big] \ \text{ a.s. and in } L^2. \tag{1.11}$$

Thus, in the white noise limit ($R \to \infty$), the critical points of $F_{\mathfrak{a}}^R$ will equidistribute with probability 1. In the case $m = 1$, this law of large numbers is proved in the recent work of L. Gass [11, Thm. 1.6].

To put (1.11) in its proper context we need to recall a few facts about the convergence of random measures. For proofs and details we refer to [8, Chap. 11] and [17, Chap. 4].

We denote by $\operatorname{Prob}(\mathbb{R}^m)$ the space of Borel probability measures on $\mathbb{R}^m$, by $\operatorname{Meas}(\mathbb{R}^m)$ the space of finite probability measures on $\mathbb{R}^m$ and by $\operatorname{Meas}_{\mathrm{loc}}(\mathbb{R}^m)$ the space of locally finite Borel measures on $\mathbb{R}^m$, i.e., Borel measures μ such that $\mu\big[\,S\,\big] < \infty$ for any bounded Borel set $S \subset \mathbb{R}^m$. Any such set S defines an additive map

$$L_S : \operatorname{Meas}_{\mathrm{loc}}(\mathbb{R}^m) \to \mathbb{R}, \ \ \operatorname{Meas}_{\mathrm{loc}}(\mathbb{R}^m) \ni \mu \mapsto L_S(\mu) = \mu\big[\,S\,\big] \in \mathbb{R}.$$

The *vague* topology on the space $\operatorname{Meas}_{\mathrm{loc}}(\mathbb{R}^m)$ is the smallest topology such that all the maps L_S, S bounded Borel, are continuous. The space $\operatorname{Meas}_{\mathrm{loc}}(\mathbb{R}^m)$ equipped with the vague topology is a Polish space, i.e., it is separable and the topology is induced by a complete (Prokhorov-like) metric.

A random locally finite measure on $\mathbb{R}^m$ is a measurable map

$$\mathfrak{M} : \big(\,\Omega, \mathcal{S}, \mathbb{P}\,\big) \to \operatorname{Meas}_{\mathrm{loc}}\big(\,\mathbb{R}^m\,\big),$$

where $\big(\,\Omega, \mathcal{S}, \mathbb{P}\,\big)$ is a probability space. Its distribution is a Borel probability measure $\mathbb{P}_{\mathfrak{M}}$ on $\operatorname{Meas}_{\mathrm{loc}}\big(\,\mathbb{R}^m\,\big)$. It's mean intensity is the measure $\bar{\mathfrak{M}}$ on $\mathbb{R}^m$

$$\bar{\mathfrak{M}}[S] = \mathbb{E}\big[\,\mathfrak{M}[S]\,\big]$$

for any Borel subset $S \subset \mathbb{R}^m$. We say that $\mathfrak{M}$ is locally integrable if its mean intensity is locally finite.

A sequence of random measures $\mathfrak{M}_N : \big(\Omega, \mathcal{S}, \mathbb{P}\big) \to \mathrm{Meas}_{\mathrm{loc}}\big(\mathbb{R}^m\big)$ is said to converge vaguely a.s. to the random measure $\mathfrak{M}$ if

$$\mathbb{P}\Big[\,\big\{\,\omega;\ \mathfrak{M}_N(\omega) \to \mathfrak{M}(\omega)\ \text{ vaguely in }\ \mathrm{Meas}_{\mathrm{loc}}(\mathbb{R}^m)\,\big\}\,\Big] = 1.$$

One can show that the following statements are equivalent

- $\mathfrak{M}_N \to \mathfrak{M}$ vaguely a.s..
- $\mathfrak{M}_N[S] \to \mathfrak{M}[S]$, a.s., for any bounded Borel $S \subset \mathbb{R}^m$.
- $\mathfrak{M}_N[f] \to \mathfrak{M}[f]$, a.s., $\forall f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$.

Similarly we say that $\mathfrak{M}_N \to \mathfrak{M}$ vaguely L^p if the following equivalent conditions hold.

- $\mathfrak{M}_N[S] \to \mathfrak{M}[S]$, L^p, for any bounded Borel $S \subset \mathbb{R}^m$.
- $\mathfrak{M}_N[f] \to \mathfrak{M}[f]$, L^p, $\forall f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$.

We can rephrase the equality (1.11) as a law as large numbers.

Corollary 1.2 (Strong Law of Large Numbers)**.** *Suppose that $m \geq 2$. In white noise limit ($N \to \infty$) the* random *measures $\frac{1}{N^m}\mathfrak{C}\big[\,-, F^N_{\mathfrak{a}}\,\big]$ converge vaguely a.s. and L^2 to the* deterministic *measure $C_m(\mathfrak{a})\boldsymbol{\lambda}$. In particular, for any bounded Borel subset $S \subset \mathbb{R}^m$ we have*

$$\lim_{N\to\infty} \frac{1}{N^m}\mathfrak{C}\big[\,S, F^N_{\mathfrak{a}}\,\big] = C_m(\mathfrak{a})\boldsymbol{\lambda}\big[\,S\,\big]$$

a.s. and L^2. □

In [20] the second author proved that in white noise limit the *random* measures $\mathfrak{C}\big[\,-, F^N_{\mathfrak{a}}\,\big]$ also satisfy a Central Limit Theorem. More precisely, for any $r \in (0,1)$

$$\frac{1}{N^{m/2}}\Big(\,\mathfrak{C}[B^\infty_{r/2}, F^N_{\mathfrak{a}}] - \mathbb{E}\big[\,\mathfrak{C}[B^\infty_{r/2}, F^N_{\mathfrak{a}}]\,\big]\,\Big)$$

converges in distribution to a centered normal random variable with nonzero variance.

The random functions $F^N_{\mathfrak{a}}$ are stationary and so the random measures $\mathfrak{C}\big[\,-, F^N_{\mathfrak{a}}\,\big]$ are stationary as well, i.e., their distributions are invariant with respect to the natural action of $\mathbb{R}^m$ by translations on $\mathrm{Meas}_{\mathrm{loc}}(\mathbb{R}^m)$.

As discussed in [8, Chap.12] or [17, Chap.5], every *stationary* random locally finite measure $\mathfrak{M}$ on $\mathbb{R}^m$ with locally finite mean intensity has an *asymptotic intensity* $\widehat{\mathfrak{M}}$. This is an integrable random variable $\hat{\mathfrak{M}} \in L^1\big(\,\mathrm{Meas}_{\mathrm{loc}}(\mathbb{R}^m), \mathbb{P}_{\mathfrak{M}}\,\big)$ with an ergodic meaning, [8, Sec. 12.2] or [17, Sec. 5.4]. More precisely, for any compact convex subset $C \subset \mathbb{R}^m$ containing the origin in its interior we have

$$\widehat{\mathfrak{M}} = \lim_{N\to\infty} \frac{1}{\mathrm{vol}\,\big[\,NC\,\big]}\mathfrak{M}[NC]\ \text{ a.s. and } L^1.$$

The random measure $\mathfrak{M} = \mathfrak{C}[-, \Phi_{\mathfrak{a}}]$ is stationary and the results of [23] show that the asymptotic intensity of $\mathfrak{C}[-, \Phi_{\mathfrak{a}}]$ is the constant $\widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}} = C_m(\mathfrak{a})$. We set

$$B_R := [-R, R]^m = B^\infty_R(0).$$

For fixed $R \in \mathbb{N}$, the random function $\Phi^R_{\mathfrak{a}}$ is $\big(\,R\mathbb{Z}\,\big)^m$-periodic and we deduce that for any $N \in \mathbb{N}$ we have

$$\mathfrak{C}[NB_R, \Phi^{N_0}_{\mathfrak{a}}] = N^m\mathfrak{C}[NB_R, \Phi^R_{\mathfrak{a}}].$$

Hence

$$\mathfrak{C}[B_R, \Phi^R_{\mathfrak{a}}] = \lim_{N\to\infty} \frac{1}{N^m}\mathfrak{C}[NB_R, \Phi^R_{\mathfrak{a}}] = \widehat{\mathfrak{C}}_{\Phi^R_{\mathfrak{a}}}\,\mathrm{vol}\,\big[\,B_R\,\big],$$

where $\widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}^R}$ denotes the asymptotic intensity of the stationary random measure $\mathfrak{C}[-,\Phi_{\mathfrak{a}}^R]$. Hence

$$\widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}^R}=\frac{1}{R^m}\mathfrak{C}\big[\,B_R,\Phi_{\mathfrak{a}}^R\,\big].$$

Corollary 1.2 shows that

$$\lim_{N_0\to\infty}\widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}^{N_0}}=\widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}}=C_m(\mathfrak{a}),$$

a.s. and L^2.

To prove Theorem 1.1 we relate the critical points of $F_{\mathfrak{a}}^R$ to the critical points of $\Phi_{\mathfrak{a}}^R$. The advantage is that $\Phi_{\mathfrak{a}}^R$ converges in distribution to the smooth isotropic function $\Phi_{\mathfrak{a}}$. To get a handle on the variance we express it as in integral over $(\mathbb{R}^m\times\mathbb{R}^m)\setminus\Delta$ where Δ is the diagonal. To deal with integrability far away from the diagonal we rely on the estimates in Lemma 3.1 that, roughly speaking, states that as $R\to\infty$ the covariance kernel $\boldsymbol{K}_{\mathfrak{a}}$ of $\Phi_{\mathfrak{a}}$ approximates well the covariance kernel $\boldsymbol{K}_{\mathfrak{a}}^R$ of $\Phi_{\mathfrak{a}}^R$ over a *large* ball, of radius $\approx R/2$. To deal with local integrability near the diagonal we rely on the strategy of [2, 6]. The tricky part is proving estimates uniform in R. Appendix B justifies why this is possible.

Let us say a few things about the organization of the paper. In Section 2 we survey a few facts about Gaussian measures and Gaussian random fields and the Kac-Rice formula. Theorem 1.1 is proved in Section 3. We have subdivided this section into several subsections corresponding to the conceptually different steps in proof of Theorem 1.1.

The paper also includes two technical appendices. Appendix A describes several general conditions guaranteeing various forms of ampleness (nondegeneracy) of Gaussian fields. To the best of our knowledge these seem to be new and we believe they will find many other applications. Appendix B contains explicit upper bounds for the variance of the number of zeros of a C^2 Gaussian field F in a given region $\mathcal{R}$ in terms of C^k norms of its covariance kernel of F and the size of $\mathcal{R}$. The finiteness of the variance is ultimately due to Fernique's inequality [10] guaranteeing that $\mathbb{E}\big[\,\|F\|_{C^2}^p\,\big]<\infty$ for all $p\in[1,\infty)$. The fact that these L^p-norms can be controlled by the covariance kernel of F follows from a result of Nazarov and Sodin [18].

2. BASIC GAUSSIAN CONCEPTS AND THE KAC-RICE FORMULA

Let first recall a few things about Gaussian vectors. For details and proofs we refer to [24].

Suppose that $\boldsymbol{X}$ is a finite dimensional real Euclidean space with inner product $(-,-)$. An $\boldsymbol{X}$-valued *Gaussian vector* is a measurable map $X:\big(\Omega,\mathcal{S},\mathbb{P}\big)\to\boldsymbol{X}$ such that, for any $\xi\in\boldsymbol{X}$, the random variable $\big(\xi,X\big)$ is Gaussian. Above and in the sequel $\big(\Omega,\mathcal{S},\mathbb{P}\big)$ denotes a probability space. The Gaussian vector X is called *centered* if $\big(\xi,X\big)$ has mean zero, $\forall\xi\in\boldsymbol{X}$.

If X is a centered Gaussian random vector valued in the finite dimensional Euclidean space $\boldsymbol{X}$, then its distribution is uniquely determined by its *variance*. This is the symmetric nonnegative operator $\operatorname{Var}\big[\,X\,\big]:\boldsymbol{X}\to\boldsymbol{X}$ uniquely determined by the equality

$$\mathbb{E}\big[\,e^{\boldsymbol{i}(\xi,X)}\,\big]=e^{-\frac{1}{2}(\operatorname{Var}[X]\xi,\xi)},\ \ \forall\xi\in\boldsymbol{X}.$$

The Gaussian vector X is called nondegenerate if $\operatorname{Var}\big[\,X\,\big]$ is nonsingular.

Suppose that X_1,X_2 are centered Gaussian vectors valued in the Euclidean spaces $\boldsymbol{X}_1$ and respectively $\boldsymbol{X}_2$. If X_1,X_2 are *jointly Gaussian,* i.e., $X_1\oplus X_2$ is also Gaussian, then the variance of $X_1\oplus X_2$ has the block decomposition

$$\operatorname{Var}\big[\,X_i\oplus X_2\,\big]=\left[\begin{array}{cc}\operatorname{Var}\big[\,X_1\,\big] & \operatorname{Cov}\big[\,X_1,X_2\,\big]\\ \operatorname{Cov}\big[\,X_2,X_1 & \operatorname{Var}\big[\,X_2\,\big]\end{array}\right]$$

where the *covariance* $\operatorname{Cov}\big[\,X_1, X_2\,\big]$ is a linear map $\boldsymbol{X}_2 \to \boldsymbol{X}_1$ and

$$\operatorname{Cov}\big[\,X_1, X_2\,\big] = \operatorname{Cov}\big[\,X_1, X_2\,\big]^*.$$

If X_1, X_2 are jointly Gaussian and X_1 is nondegenerate, then $\mathbb{E}\big[\,X_2 \,\|\, X_1\,\big]$, the conditional expectation of X_2 given X_1, is an explicit linear function of X_1. Moreover, for any continuous function $f : \boldsymbol{X}_2 \to \mathbb{R}$ with at most polynomial growth at ∞ we have the *regression formula* (see [4, Prop. 12] or [11, Sec. 2.3.2])

$$\mathbb{E}\big[\,f(X_2)\big|\,X_1 = 0\,\big] = \mathbb{E}\big[\,f(Z)\,\big]$$

where $Z = X_2 - \mathbb{E}\big[\,X_2 \,\|\, X_1\,\big]$ is a centered Gaussian vector with variance

$$\Delta_{X_2,X_1} = \operatorname{Var}\big[\,X_2\,\big] - \operatorname{Cov}\big[\,X_2, X_1\big]\operatorname{Var}[X_1]^{-1}\operatorname{Cov}\big[\,X_1, X_2\,\big]. \tag{2.1}$$

Suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are finite dimensional real Euclidean spaces and $\mathscr{V} \subset \boldsymbol{V}$ is an open set. For a map $F \in C^k(\mathscr{V}, \boldsymbol{U})$ we denote by $F^{(k)}(v)$ the k-order differential of F at v. It is an element of the vector space $\mathbf{Sym}_k\big(\,\boldsymbol{V}, \boldsymbol{U}\,\big)$ consisting of symmetric k-linear maps

$$\underbrace{\boldsymbol{V} \times \cdots \times \boldsymbol{V}}_{k} \to \boldsymbol{U}.$$

The *k-th jet* of F at $\boldsymbol{v} \in \mathscr{V}$ is the vector

$$J_k F(\boldsymbol{v}) := F(\boldsymbol{v}) \oplus F'(\boldsymbol{v}) \oplus \cdots \oplus F^{(v)}(\boldsymbol{v}).$$

The *Jacobian of F at $v \in \mathscr{V}$* is

$$J_F(v) = \sqrt{\det\big(\,F'(v(F'(v)^*\,\big)}.$$

When $\boldsymbol{U} = \boldsymbol{V}$ we have

$$J_F(v) = \big|\,\det F'(v)\,\big|$$

A $\boldsymbol{U}$-valued *random field* on $\mathscr{V}$ is a family of random variables

$$F(v) : \big(\,\Omega, \mathcal{S}, \mathbb{P}\,\big) \to \boldsymbol{U},\;\; \Omega \ni \omega \mapsto F_\omega(\boldsymbol{v}) \in \boldsymbol{U},\;\; \boldsymbol{v} \in \mathscr{V}.$$

We will work with measurable random fields , i.e., random fields F such that the the map

$$\Omega \times \mathscr{V} \to \boldsymbol{U},\;\; (\omega, \boldsymbol{v}) \mapsto F_\omega(\boldsymbol{v})$$

is measurable with respect to the product sigma-algebra on $\Omega \times \mathscr{V}$. The maps $F_\omega : \mathscr{V} \to \boldsymbol{U}$ are called the *sample maps* of the random field F. The random field is called C^ℓ if all its sample maps belong to $C^\ell(\mathscr{V}, \boldsymbol{U})$.

The random field is called *Gaussian* if, for any $n \in \mathbb{N}$, and any $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathscr{V}$, the random vector

$$\big(\,F(\boldsymbol{v}_1), \dots, F(\boldsymbol{v}_n)\,\big) \in \boldsymbol{U}^n$$

is Gaussian. In the sequel we will work exclusively with centered Gaussian fields, i.e., $F(\boldsymbol{v})$ is centered, $\forall \boldsymbol{v} \in \mathscr{V}$.

If $F : \Omega \times \mathscr{V} \to \boldsymbol{U}$, the the covariance kernel of F is the map

$$\mathcal{K}_F : \mathscr{V} \times \mathscr{V} \to \operatorname{End}\big(\,\boldsymbol{U}\,\big),\; \mathcal{K}_F(\boldsymbol{v}_1, \boldsymbol{v}_0) = \operatorname{Cov}\big[\,F(\boldsymbol{v}_1), F\boldsymbol{v}_0)\,\big]$$

Work going back to Kolmogorov shows that if the covariance kernel of F is sufficiently regular, then so is F. More precisely, we will need the following more precise result, [18, Appendices A.9-A.11].

Theorem 2.1. *Fix $\ell \in \mathbb{N}_0$ and $\alpha \in (0,1)$. Suppose that $\mathcal{V}$ is an open subset of $\mathbb{R}^m$ and $X : \Omega \times \mathcal{V} \to \mathbb{R}$ is a centered Gaussian function with covariance kernel $\mathcal{K}_X$. Assume that $\mathcal{K} \in C^{2\ell+2}\big(\mathcal{V} \times \mathcal{V}\big)$. Then X admits a $C^{\ell,\alpha}$-modification. Moreover, for every box $B \subset \mathcal{V}$ and every $p \geq 1$ there exists a constant $C_p = C(B, \mathcal{V}, \ell, \alpha)$ such that*

$$\mathbb{E}\big[\, \|X\|^p_{C^{\ell,\alpha}(B)}\,\big] \leq C \big\|\, \mathcal{K}_X\, \big\|^{p+1}_{C^{2\ell+2}(B \times B)}, \tag{2.2}$$

where $C^{k,\alpha}$ denotes the spaces of functions that are k times differentiable and the k-th differential is Hölder continuous with exponent α. □

Suppose that $F : \Omega \times \mathscr{V} \to \boldsymbol{U}$ is a C^ℓ Gaussian field. F is said to be *ample* if for any $\boldsymbol{v} \in \mathscr{V}$ the Gaussian vector $F(\boldsymbol{v})$ is nondegenerate. More generally, if n is a positive integer, then we say that F is n-*ample* if, for any pairwise distinct points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathscr{V}$, the Gaussian vector $F(\boldsymbol{v}_1) \oplus \cdots \oplus F(\boldsymbol{v}_n)$ is nondegenerate. Let $0 \leq k \leq \ell$. We say that F is J_k-*ample* if, for any $\boldsymbol{v} \in \mathscr{V}$, the k-th jet $J_k F(\boldsymbol{v})$ is a nondegenerate Gaussian vector.

We have the following result. For a proof we refer to [1, Lemma 11.2.10] or [3, Sec. 4].

Lemma 2.2 (Bulinskaya)**.** *Suppose now that $\dim \boldsymbol{U} = \dim \boldsymbol{V} = m$ and $F : \Omega \times \mathcal{V} \to \boldsymbol{V}$ is an ample C^1, Gaussian random field. Then, if $K \subset \mathscr{V}$ is a compact set of Hausdorff dimension $\leq m-1$, then $F^{-1}(0) \cap K = \emptyset$* a.s.. *Moreover*

$$\mathbb{P}\big[\, \{F(v) = 0,\ \ J_F(v) = 0\}\,\big] = 0$$

□

The Kac-Rice formula plays a central role in this paper. We state below one version of this formula. For a proof we refer to [3] or [4, Thm.6.2].

Theorem 2.3 (Local Kac-Rice formula)**.** *Suppose that $F : \Omega \times \mathcal{V} \to \boldsymbol{U}$ is an ample C^1, Gaussian random field, $m = \dim \boldsymbol{V} = \dim \boldsymbol{U}$. Denote by $p_{F(v)}$ the probability density of the nondegenerate Gaussian vector $F(v)$. For any box $B \subset \mathscr{V}$ and any nonnegative continuous function $w \in C(B)$ we set*

$$\mathcal{Z}_B(w, F) = \sum_{\substack{F(v)=0,\\ v \in B}} w(v) \in [0, \infty].$$

In particular $\mathcal{Z}(B, F) := \mathcal{Z}_B(1, F)$ is the number of zeros of F in B. Then $\mathcal{Z}_B(\varphi, F)$, is measurable, a.s. *finite and*

$$\mathbb{E}\big[\, \mathcal{Z}_B(w, F)\,\big] = \int_B w(v)\rho_{KR}(v)dv \tag{2.3}$$

where ρ_{KR} is the Kac-Rice density

$$\rho_{KR}(v) := \mathbb{E}\big[\, J_F(v) \big|\, F(v) = 0\,\big] p_{F(v)}(0). \tag{2.4}$$

□

Corollary 2.4. *Let $\mathcal{V} \subset \mathbb{R}^m$ an open set. Suppose that $\Phi : \mathcal{V} \to \mathbb{R}$ a Gaussian random function that is* a.s. *C^2 and such that the Gaussian vector $\nabla \Phi(\boldsymbol{v})$ is nondegenerate for any $\boldsymbol{v} \in \mathcal{V}$. We denote by $p_{\nabla\Phi(v)}$ is probability* density. *The following hold.*

(i) *The random function Φ is* a.s. *Morse*
(ii) *We set*

$$\mathfrak{C}[-, \Phi] := \sum_{\nabla F(\boldsymbol{v})=0} \delta_v \tag{2.5}$$

Then $\mathfrak{C}[-,\Phi]$ is a random locally finite measure on $\mathscr{V}$ in the sense of [8] *or* [17]. *For any nonnegative measurable function $\varphi:\mathscr{V}\to[0,\infty)$ we set*

$$\mathfrak{C}[\varphi,\phi]=\int\varphi(v)\mathfrak{C}[dv,\Phi]=\sum_{\nabla\Phi(v)=0}\varphi(v).$$

(iii) *For any box $B\subset\mathscr{V}$, the function Φ* a.s. *has no critical points on ∂B and*

$$\mathbb{E}\big[\,\mathfrak{C}^{\Phi}[\boldsymbol{I}_B\varphi]\,\big]=\int_B\mathbb{E}\big[\,|\det\mathrm{Hess}_{\Phi}(\boldsymbol{v})|\;\big|\,\nabla\Phi(\boldsymbol{v})=0\,\big]p_{\nabla\Phi(v)}(0)\varphi(\boldsymbol{v})d\boldsymbol{v}. \tag{2.6}$$

The quantity

$$\rho_{\Phi}:=\mathbb{E}\big[\,|\det\mathrm{Hess}_{\Phi}(\boldsymbol{v})|\;\big|\,\nabla\Phi(\boldsymbol{v})=0\,\big]p_{\nabla F(v)}(0) \tag{2.7}$$

is called the Kac-Rice (or KR) density of Φ. □

We conclude this section with a technical result that will be used several times in the proof of Theorem 1.1.

Suppose that $\boldsymbol{V}$ is an m-dimensional real Euclidean space with inner product $(-,-)$. Denote by $S_1(\boldsymbol{V})$ the unit sphere in $\boldsymbol{V}$, by $\mathbf{Sym}(\boldsymbol{V})$ the space of symmetric operators $\boldsymbol{V}\to\boldsymbol{V}$, and by $\mathbf{Sym}_{\geq 0}(\boldsymbol{V})\subset\mathbf{Sym}(\boldsymbol{V})$ the cone of nonnegative ones. For $A\in\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ we denote by Γ_A the centered Gaussian measure on $\boldsymbol{V}$ with variance A.

The space $\mathbf{Sym}(\boldsymbol{V})$ is equipped with an inner product

$$\big(\,A,B\,\big)_{\mathrm{op}}=\mathrm{tr}(AB),\;\;\forall A,B\in\mathbf{Sym}(\boldsymbol{V}).$$

We denote by $\|-\|_{\mathrm{op}}$ the associated norm.

We have a natural map $\mathbf{Sym}_{\geq 0}(\boldsymbol{V})\to\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$, $A\mapsto A^{1/2}$. We will have the following result, [14, Prop.2.1].

Proposition 2.5. *For any $\mu>0$ and $\forall A,B\in\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$, such that $A^{1/2}+B^{1/2}\geq\mu\mathbb{1}$ we have*

$$\mu\big\|\,A^{1/2}-B^{1/2}\,\big\|_{\mathrm{op}}\leq\big\|\,A-B\,\big\|_{\mathrm{op}}^{1/2}. \tag{2.8}$$

□

Lemma 2.6. *Fix $A_0\in\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ such that $A_0^{1/2}\geq\mu_0\mathbb{1}$, $\mu_0>0$. Suppose that $f:\mathbb{V}\to\mathbb{R}$ is a locally Lipschitz function that is homogeneous of degree $k\geq 1$. For $A\in\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ we set*

$$\mathcal{I}_A(f):=\int_{\mathbf{V}}f(\boldsymbol{v})\mathbf{\Gamma_A}\big[\,\mathbf{d}\boldsymbol{v}\,\big].$$

Then for and $R\geq\|A_0\|_{\mathrm{op}}$ there exists a constant $C=C(f,R,\mu_0)>0$ with the following property: for any $A\in\mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ such that $\|,A\|_{\mathrm{op}}\leq R$

$$\big|\,\mathcal{I}_{A_0}(f)-\mathcal{I}_A(f)\,\big|\leq C\|A-A_0\|^{1/2}\leq C(k,R)\|A-A_0\|_{\mathrm{op}}^{1/2}. \tag{2.9}$$

In other words, $A\mapsto\mathcal{I}_A(f)$ is locally Hölder continuous with exponent $1/2$ in the open set $\mathbf{Sym}_{>0}\big(\boldsymbol{V}\big)$.

Proof. The function f is Lipschitz on the ball

$$B_R(\boldsymbol{V}):=\big\{\,\boldsymbol{v}\in\boldsymbol{V};\|\boldsymbol{v}\|\leq R\,\big\},$$

so there exists $L=L(R)>0$ such that

$$\big[\,f(\boldsymbol{u})-f(\boldsymbol{v})\,\big|\leq L\|\boldsymbol{u}-\boldsymbol{v}\|,\;\;\forall\boldsymbol{u},\boldsymbol{v}\in B_R(\boldsymbol{V}). \tag{2.10}$$

Note that

$$\mathcal{I}_A(f) = \int_{\boldsymbol{V}} f\big(A^{1/2}\boldsymbol{v}\big)\Gamma_{\mathbb{1}}\big[\,d\boldsymbol{v}\,\big],$$

so

$$\big|\,\mathcal{I}_{A_0}(f) - \mathcal{I}_A(f)\,\big| \le \int_{\boldsymbol{V}} \big|\,f\big(A^{1/2}\boldsymbol{v}\big) - f\big(A_0^{1/2}\boldsymbol{v}\big)\,\big|\,\Gamma_{\mathbb{1}}\big[\,d\boldsymbol{v}\,\big]$$

$$= \underbrace{\frac{1}{(2\pi)^{m/2}}\left(\int_0^\infty r^{n+k-1}e^{-r^2/2}dr\right)}_{C_{m,k}}\int_{S_1(\boldsymbol{V})}\big|\,f\big(A^{1/2}\boldsymbol{v}\big) - f\big(A_0^{1/2}\boldsymbol{v}\big)\,\big|\,\mathrm{vol}_{S_1(\boldsymbol{V})}\big[\,d\boldsymbol{v}\,\big]$$

$$\stackrel{(2.10)}{\le} C_{m,k}L(R)\int_{S_1(\boldsymbol{V})}\|A^{1/2} - A_0^{1/2}\|_{\mathrm{op}}\,\mathrm{vol}_{S^1(\boldsymbol{V})}\big[\,d\boldsymbol{v}\,\big] \stackrel{(2.8)}{\le} C(k,R,\mu_0)\|A - A_0\|_{\mathrm{op}}^{1/2}.$$

□

3. Proof of Theorem 1.1

We split the proof of Theorem 1.1 into several conceptually distinct parts.

3.1. **The key estimate.** The following technical result will play a key role.

Lemma 3.1. *Fix $r_0 \in (0,1)$ and a box $B = B^\infty_{r_0/2}(0) = [-r_0/2, r_0/2]^m$. Then the following hold.*

(i) *For any $\ell \in \mathbb{N}_0$ and any $p > m$ there exists $C = C(p,m,\ell,\mathfrak{a}) > 0$, independent of R, such that, $\forall R > 2$*

$$\big\|\,\boldsymbol{K}_{\mathfrak{a}}^R - \boldsymbol{K}_{\mathfrak{a}}\,\big\|_{C^\ell(RB)} \le CR^{-p}$$

(ii) *For any $\ell \in \mathbb{N}_0$ and any $p > m$ there exists $C = C(p,m,\ell,\mathfrak{a}) > 0$, independent of R, such that, $\forall R > 2$, $\forall \boldsymbol{x},\boldsymbol{y} \in RB$*

$$\big|\,D^\ell \boldsymbol{K}_{\mathfrak{a}}^R(\boldsymbol{x}-\boldsymbol{y})\,\big| \le \frac{C}{\big(1 + |\boldsymbol{x}-\boldsymbol{y}|_\infty\big)^p}.$$

Proof. (i) Denote by $\mathcal{T}_{R\vec{k}}\boldsymbol{K}_{\mathfrak{a}}$ the translate

$$\mathcal{T}_{R\vec{k}}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}) := \boldsymbol{K}\big(\boldsymbol{x} - R\vec{k}\big).$$

We have

$$\boldsymbol{K}_{\mathfrak{a}}^R(\boldsymbol{x}) - \boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}) = \sum_{\vec{k}\in\mathbb{Z}^m\setminus 0} \mathcal{T}_{R\vec{k}}\boldsymbol{K}_{\mathfrak{a}}\big(\boldsymbol{x}\big).$$

Now observe that $\forall R > 0$, $\forall \boldsymbol{x} \in RB$, and any $\vec{k} \in \mathbb{Z}^m \setminus 0$ we have

$$\big|\,\boldsymbol{x} - R\vec{k}\,\big|_\infty \ge N\big|\,\vec{k}\,\big|_\infty - \big|\,\boldsymbol{x}\,\big|_\infty \ge R\big(\,\big|\,\vec{k}\,\big|_\infty - r_0/2\big).$$

Since $\boldsymbol{K}_{\mathfrak{a}}$ and all its derivatives are Schwartz functions we deduce that for any $p > m$, and any $\vec{k} \in \mathbb{Z}^m \setminus 0$

$$\big\|\,\mathcal{T}_{R\vec{k}}\boldsymbol{K}_{\mathfrak{a}}\,\big\|_{C^\ell(NB)} \le C(p,m,\ell,\mathfrak{a})R^{-p}\big(\,\big|\,\vec{k}\,\big|_\infty - r_0/2\,\big)^{-p}.$$

The last expression is well defined since $r < 1 \le \big|\,\vec{k}\,\big|_\infty$ for any $\vec{k} \in \mathbb{Z}^m \setminus 0$. Hence

$$\big\|\,\boldsymbol{K}_{\mathfrak{a}}^R - \boldsymbol{K}_{\mathfrak{a}}\,\big\|_{C^\ell(NB)} \le C(p,m,\ell,\mathfrak{a})R^{-p}\sum_{\vec{k}\in\mathbb{Z}^m\setminus 0}\big(\,\big|\,\vec{k}\,\big|_\infty - r_0/2\,\big)^{-p}$$

The above series is convergent since $p > m$.

(ii) Note that $\forall \boldsymbol{x},\boldsymbol{y} \in RB$ we have $\big|\,\boldsymbol{x} - \boldsymbol{y}\,\big|_\infty \le Rr_0$. Set $\boldsymbol{z} := \boldsymbol{x} - \boldsymbol{y}$. We discuss only the case $\ell = 0$. The general case can be reduced to this case by taking partial derivatives.

Using (i) we deduce that

$$C=\sup_R\sup_{|\boldsymbol{z}|_\infty<r_0}\big|\boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{z})\big|<\infty$$

and thus, $\forall R\geq 2$, $\forall|\boldsymbol{z}|_\infty<r_0$,

$$\big|\boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{z})\big|<\frac{C\big(1+r_0\big)^p}{\big(1+|\boldsymbol{z}|_\infty\big)^p}.$$

Assume now that $|\boldsymbol{z}|_\infty\geq r_0$. We have

$$\boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{z})=\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{z})+\sum_{\vec{k}\in\mathbb{Z}^m\setminus 0}\mathcal{T}_{R\vec{k}}\boldsymbol{K}_{\mathfrak{a}}\big(\boldsymbol{z}\big),$$

and thus,

$$\big|\boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{z})\big|\leq\big|\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{z})\big|+\sum_{\vec{k}\in\mathbb{Z}^m\setminus 0}\big|\mathcal{T}_{R\vec{k}}\boldsymbol{K}_{\mathfrak{a}}\big(\boldsymbol{z}\big)\big|.$$

Since $\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x})$ is Schwartz we deduce that there exists a constant $C=C(p,\mathfrak{a})$ such that

$$\frac{C_p}{\big(1+|\boldsymbol{z}|_\infty\big)^p}+C_p\sum_{\vec{k}\in\mathbb{Z}^m\setminus 0}\frac{1}{\big(1+|\boldsymbol{z}-R\vec{k}|_\infty\big)^p}.$$

We have $|\boldsymbol{z}|_\infty\leq Rr_0$ and

$$\big|\boldsymbol{z}-Z\vec{k}\big|_\infty\geq|\boldsymbol{z}|_\infty\left(\frac{R|\vec{k}|_\infty}{|\boldsymbol{z}|_\infty}-1\right)\geq|\boldsymbol{z}|_\infty\left(\frac{1}{r_0}|\vec{k}|_\infty-1\right).$$

Thus

$$\sum_{\vec{k}\in\mathbb{Z}^m\setminus 0}\frac{1}{\big(1+|\boldsymbol{z}-R\vec{k}|_\infty\big)^p}\leq|\boldsymbol{z}|_\infty^{-p}\underbrace{\sum_{\vec{k}\in\mathbb{Z}^m\setminus 0}\left(\frac{1}{r_0}|\vec{k}|_\infty-1\right)^{-p}}_{<\infty}.$$

□

3.2. **An integral formula.** Set $B=B^\infty_{r_0/2}(0)$, $f_R(\boldsymbol{x})=f(\boldsymbol{x}/R)$

$$Z^R[f]=\mathfrak{C}[f,F^R_{\mathfrak{a}}]=\mathfrak{C}[f_R,\Phi^R_{\mathfrak{a}}],\;\;Z[f]=\mathfrak{C}[f,\Phi_{\mathfrak{a}}].$$

Denote by $\rho^R_{\mathfrak{a}}$ the Kac-Rice density of $\Phi^R_{\mathfrak{a}}$ and by $\rho_{\mathfrak{a}}$ the Kac-Rice density of $\Phi_{\mathfrak{a}}$; see (2.7). Since both $\Phi^R_{\mathfrak{a}}$ and $\Phi_{\mathfrak{a}}$ are stationary random functions we deduce that both $\rho^R_{\mathfrak{a}}$ and $\rho_{\mathfrak{a}}$ are constant functions. We set and $C_m(\mathfrak{a}):=\rho_{\mathfrak{a}}(0)$. For an explicit description of $C_m(\mathfrak{a})$ we refer to [23] .

The covariance functions $\boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{z})$ and $\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{z})$ are even so the odd order derivatives of these functions vanish at 0. This implies that the Gaussian vectors $\mathrm{Hess}_{\Phi^R_{\mathfrak{a}}}(0)$ and $\nabla\Phi^R_{\mathfrak{a}}(0)$ are independent. A similar phenomenon is true for $\Phi_{\mathfrak{a}}$. Thus, the conditional expectations in (2.7) are usual expectations. Using Lemma 2.6 and Lemma 3.1(i) we deduce that for any $\boldsymbol{x}\in\mathbb{R}^m$

$$\sup_{\boldsymbol{x}\in RB}\big|\rho^R_{\mathfrak{a}}(\boldsymbol{x})-\rho_{\mathfrak{a}}(\boldsymbol{x})\big|=\big|\rho^R_{\mathfrak{a}}(0)-\rho_{\mathfrak{a}}(0)\big|=O\big(R^{-\infty}\big),\tag{3.1}$$

where $O(N^{-\infty})$ is short-hand for $O(N^{-p})$, $\forall p>0$. We deduce that

$$R^{-m}\big(\mathbb{E}\big[Z^R[f]\big]-\mathbb{E}\big[Z[f]\big]\big)=R^{-m}\int_{RB}f_R(\boldsymbol{x})\big(\rho^R_{\mathfrak{a}}(0)-\rho_{\mathfrak{a}}(0)\big)d\boldsymbol{x}$$

$$= \int_B f(\boldsymbol{y})\big(\rho_{\mathfrak{a}}^R(0) - \rho_{\mathfrak{a}}(0)\big)d\boldsymbol{y} = O\big(R^{-\infty}\big).$$

On the other hand

$$\mathbb{E}\big[Z[f]\big] = C_m(\mathfrak{a})\int_{\mathbb{R}^m} f(\boldsymbol{x})d\boldsymbol{x}.$$

We need to introduce some notation. Set

- $\Phi_{\mathfrak{a}}^{\infty} = \Phi_{\mathfrak{a}}$.
- For any $R \in (0,\infty]$ we define
$$\widehat{\Phi}^R,\ \widehat{\Phi} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R},$$
$$\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y}) = \Phi_{\mathfrak{a}}^R(\boldsymbol{x}) + \Phi_{\mathfrak{a}}^R(\boldsymbol{y}),\ \ \widehat{\Phi}(\boldsymbol{x},\boldsymbol{y}) = \Phi_{\mathfrak{a}}(\boldsymbol{x}) + \Phi_{\mathfrak{a}}(\boldsymbol{y}),$$
$$\hat{\mathfrak{C}}^R := \mathfrak{C}[-,\widehat{\Phi}_{\mathfrak{a}}^R],\ \ \widehat{H}_R(\boldsymbol{x},\boldsymbol{y}) := \operatorname{Hess}_{\widehat{\Phi}^R}(\boldsymbol{x},\boldsymbol{y}),\ \ H_R(\boldsymbol{x}) := \operatorname{Hess}_{\Phi_{\mathfrak{a}}^R}(\boldsymbol{x}).$$
- Choose an independent copy $\Psi_{\mathfrak{a}}^R$ of $\Phi_{\mathfrak{a}}^R$ and for $R \in (0,\infty]$ set
$$\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y}) := \Phi_{\mathfrak{a}}^R(\boldsymbol{x}) + \Psi_{\mathfrak{a}}^R(\boldsymbol{y}),\ \ \widetilde{H}_R(\boldsymbol{x},\boldsymbol{y}) := \operatorname{Hess}_{\widetilde{\Phi}^R}(\boldsymbol{x},\boldsymbol{y}),$$
$\tilde{\mathfrak{C}}^R = \mathfrak{C}[-,\widetilde{\Phi}^N]$.
- For $R \in (0,\infty)$ define
$$f_R^{\boxtimes 2} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R},\ \ f_R^{\boxtimes}(\boldsymbol{x},\boldsymbol{y}) = f_R(\boldsymbol{x})f_R(\boldsymbol{y})$$
and set $\|f\| := \|f\|_{C^0(\mathbb{R}^m)}$.
- Set
$$\mathfrak{X} = \mathbb{R}^m \times \mathbb{R}^m \setminus \Delta = \big\{ (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^m \times \mathbb{R}^m;\ \ \boldsymbol{x} \neq \boldsymbol{y} \big\}.$$

Observe that the random function on $\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})$ is *stationary* with respect to the action of $\mathbb{R}^{2m}$ on itself by translations.

We have

$$\hat{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}] = \sum_{\substack{\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{x}) = \nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{y}) = 0,\\ \boldsymbol{x}\neq\boldsymbol{y}}} f_R(\boldsymbol{x})f_R(\boldsymbol{y}) = Z^R[f]^2 - Z^R[f^2].$$

Bulinskaya's lemma implies that

$$\mathbb{P}\big[\,\exists \boldsymbol{x}:\ \ \nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}) = \nabla\Psi_{\mathfrak{a}}(\boldsymbol{x}) = 0\,\big] = 0$$

and we deduce

$$\tilde{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}] = \sum_{\substack{\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{x}) = \nabla\Psi_{\mathfrak{a}}^R(\boldsymbol{y}) = 0,\\ \boldsymbol{x}\neq\boldsymbol{y}}} f_R(\boldsymbol{x})f_R(\boldsymbol{y})$$

$$= \sum_{\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{x}) = \nabla\Psi_{\mathfrak{a}}^R(\boldsymbol{y}) = 0} f_R(\boldsymbol{x})f_R(\boldsymbol{y}) = \mathfrak{C}[f,\Phi_{\mathfrak{a}}^R]\,\big]\mathfrak{C}[f,\Psi_{\mathfrak{a}}^R],\ \ \text{a.s.}.$$

Hence

$$\mathbb{E}\big[\,\mathfrak{C}[f,\Phi_{\mathfrak{a}}^R]\mathfrak{C}[f,\Psi_{\mathfrak{a}}^R]\,\big] = \mathbb{E}\big[\,\mathfrak{C}[f,\Phi_{\mathfrak{a}}^R]\,\big]\ \cdot\ \mathbb{E}\big[\,\mathfrak{C}[f,\Psi_{\mathfrak{a}}^R]\,\big] = \mathbb{E}\big[\,\mathfrak{C}[f,\Phi_{\mathfrak{a}}^R]\,\big]^2$$

so that

$$\mathbb{E}\big[\,\hat{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\,\big] - \mathbb{E}\big[\,\tilde{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\,\big] = \underbrace{\mathbb{E}\big[\,Z^R[f]^2\,\big] - \mathbb{E}\big[\,Z^R[f]\,\big]^2}_{=\operatorname{Var}\left[\,Z^R[f]\,\right]} - \mathbb{E}\big[\,Z^R[f^2]\,\big]$$

We have seen that

$$\lim_{R\to\infty} R^{-m}\mathbb{E}\big[\,Z^R[f^2]\,\big] = C_m(\mathfrak{a})\int_{\mathbb{R}^m} f^2(\boldsymbol{x})d\boldsymbol{x}$$

so we have to show that

$$I(R) := \mathbb{E}\big[\,\hat{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\,\big] - \mathbb{E}\big[\,\tilde{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\,\big] \sim Z_m(\mathfrak{a}) R^m \int_{\mathbb{R}^m} f^2(\boldsymbol{x}) d\boldsymbol{x} \;\text{ as } R \to \infty \tag{3.2}$$

for some constant $Z_m(\mathfrak{a}) \in \mathbb{R}$ that depends only on m and $\mathfrak{a}$.

According to Corollary A.12, there exists $R_0 > 0$ such that for $R \geq R_0$, the gradient $\nabla\Phi_{\mathfrak{a}}^R$ is 2-ample and Φ_R is J_1-ample so, for $R \geq R_0$ the gradient $\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})$ is nondegenerate for any $\boldsymbol{x} \neq \boldsymbol{y}$ and the random vector $\big(\,\Phi_{\mathfrak{a}}^R(\boldsymbol{x}), \nabla\Phi_{\mathfrak{a}}^R\,\big)$ is nondegenerate for any $\boldsymbol{x} \in \mathbb{R}^n$. As shown in [23] this is true also for $R = \infty$, where we recall that $\Phi_{\mathfrak{a}}^\infty = \Phi_{\mathfrak{a}}$.

We can apply the Kac-Rice formula and we deduce that for any $R > R_0$ we have

$$\begin{gathered}\mathbb{E}\big[\,\hat{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\,\big] \\ = \int_{\mathbb{R}^m \times \mathbb{R}^m \setminus \Delta} \underbrace{\mathbb{E}\big[\,|\det \widehat{H}_R(\boldsymbol{x},\boldsymbol{y})|\,\big|\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y}) = 0\,\big] p_{\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})}(0)}_{=\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y})} f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{\lambda}\big[\,d\boldsymbol{x}d\boldsymbol{y}\,\big].\end{gathered} \tag{3.3}$$

The gradient $\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})$ is nondegenerate for any $\boldsymbol{x},\boldsymbol{y}$ and invoking Kac-Rice again we obtain

$$\begin{gathered}\mathbb{E}\big[\,\tilde{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\,\big] \\ = \int_{\mathbb{R}^m \times \mathbb{R}^m \setminus \Delta} \underbrace{\mathbb{E}\big[\,|\det \widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})|\,\big|\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y}) = 0\,\big] p_{\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})}(0)}_{=\widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})} f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{\lambda}\big[\,d\boldsymbol{x}d\boldsymbol{y}\,\big].\end{gathered} \tag{3.4}$$

The function $\widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})$ is independent of $\boldsymbol{x},\boldsymbol{y}$ since the random function $\widetilde{\Phi}^R$ is stationary. Thus

$$\begin{aligned} I(R) &= \int_{\mathfrak{X}} \big(\,\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})\,\big) f_R(\boldsymbol{x}) f_R(\boldsymbol{y}) \boldsymbol{\lambda}\big[\,d\boldsymbol{x}d\boldsymbol{y}\,\big] \\ &= \int_{\substack{|\boldsymbol{x}|,\,|\boldsymbol{y}| \leq Rr_0/2,\\ \boldsymbol{x}\neq\boldsymbol{y}}} \big(\,\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})\,\big) f_R(\boldsymbol{x}) f_R(\boldsymbol{y}) \boldsymbol{\lambda}\big[\,d\boldsymbol{x}d\boldsymbol{y}\,\big]. \end{aligned} \tag{3.5}$$

Let us observe that for any $\boldsymbol{x} \neq \boldsymbol{y}$ we have

$$\lim_{R\to\infty} \big(\,\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})\,\big) = \big(\,\widehat{\rho}_\infty(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}_\infty(\boldsymbol{x},\boldsymbol{y})\,\big).$$

Moreover

$$\lim_{R\to\infty} f_R(\boldsymbol{x}) = f(0)$$

uniformly on compacts.

3.3. **Off-diagonal behavior.** Note that

$$\operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big] = \begin{bmatrix} \operatorname{Var}\big[\,H_R(\boldsymbol{x})\,\big] & 0 \\ 0 & \operatorname{Var}\big[\,H_R(\boldsymbol{y})\,\big] \end{bmatrix}.$$

For every $\boldsymbol{z} \in \mathbb{R}^m$ we set

$$T_R(\boldsymbol{z}) := \sum_{|\alpha|\leq 4} \big|\,\partial^\alpha \boldsymbol{K}_{\mathfrak{a}}^R(\boldsymbol{z})\,\big|.$$

Lemma 3.1(ii) shows that for every $p > 0$ there exists $C_p = C_p(\mathfrak{a}, m, r) > 0$ such that, $\forall R$, $\forall \big|\,\boldsymbol{z}\,\big|_\infty < Nr$

$$\forall N,\;\; \forall \big|\,\boldsymbol{z}\,\big|_\infty < Rr_0,\;\; T_R(z) \leq C_p\big(\,1 + \big|\,\boldsymbol{z}\,\big|_\infty\,\big)^{-p}. \tag{3.6}$$

We want to emphasize that C_p *is independent of* R.

Observe next that

$$\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]=\begin{bmatrix}\operatorname{Var}\big[\,\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\,\big] & 0\\ 0 & \operatorname{Var}\big[\,\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\,\big]\end{bmatrix},$$

is independent of $\boldsymbol{x}$ and $\boldsymbol{y}$.

$$\operatorname{Var}\big[\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]=\begin{bmatrix}\operatorname{Var}\big[\,\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Var}\big[\,\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\,\big]\end{bmatrix}$$
$$=\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]+\underbrace{\begin{bmatrix}0 & \operatorname{Cov}\big[\,\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\,\big] & 0\end{bmatrix}}_{=:\mathcal{E}^R_{\nabla}(\boldsymbol{x},\boldsymbol{y})}.$$

Hence

$$\big\|\,\operatorname{Var}\big[\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]-\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]\,\big\|_{\mathrm{op}}=\|\mathcal{E}^R_{\nabla}(\boldsymbol{x},\boldsymbol{y})\|_{\mathrm{op}}=O\big(\,T_R(\boldsymbol{x}-\boldsymbol{y})\,\big),\tag{3.7}$$

where $\|-\|_{\mathrm{op}}$ denotes the operator norm. Above and in the sequel, the *constant implied by the Landau symbol O is independent of R as long as $\boldsymbol{x},\boldsymbol{y}\in RB$*. In particular

$$\begin{aligned}\operatorname{Var}\big[\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}&=\Big(\,\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]+\mathcal{E}^R_{\nabla}(\boldsymbol{x},\boldsymbol{y})\,\Big)^{-1}\\ &=\Big(\,\mathbb{1}+\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\mathcal{E}^R_{\nabla}(\boldsymbol{x},\boldsymbol{y})\,\Big)^{-1}\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}.\end{aligned}\tag{3.8}$$

As shown in [23], there exists an explicit positive constant d_m such that

$$\operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big]=d_m\mathbb{1}_m,\ \ \forall\boldsymbol{x}.$$

Then $\operatorname{Var}\big[\,\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\,\big]=\operatorname{Var}\big[\,\nabla\Phi^R_{\mathfrak{a}}(0)\,\big]$, $\forall\boldsymbol{x}\in\mathbb{R}^m$ and

$$\operatorname{Var}\big[\,\nabla\Phi^R_{\mathfrak{a}}(0)\,\big]=d_m\mathbb{1}_m+O\big(\,R^{-\infty}\,\big).$$

The variance $\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]$ is independent of $\boldsymbol{x}$ and $\boldsymbol{y}$ and

$$\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]=\operatorname{Var}\big[\,\nabla\Phi^R_{\mathfrak{a}}(0)\,\big]\oplus\operatorname{Var}\big[\,\nabla\Phi^R_{\mathfrak{a}}(0)\,\big]=d_m\mathbb{1}_{2m}+O\big(\,R^{-\infty}\,\big).\tag{3.9}$$

From (3.8) and (3.9) we conclude that there exists $C_0>0$, independent of $R>R_0$, such that

$$\|\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\mathcal{E}^R_{\nabla}(\boldsymbol{x},\boldsymbol{y})\|_{\mathrm{op}}<\frac{1}{2},\ \ \forall\boldsymbol{x},\boldsymbol{y}\in RB,\ \ |\boldsymbol{x}-\boldsymbol{y}|_{\infty}>C_0,$$

and thus

$$\begin{aligned}&\big\|\,\operatorname{Var}\big[\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}-\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\,\big\|_{\mathrm{op}}\\ &=O\big(\,T_R(\boldsymbol{x}-\boldsymbol{y})\,\big),\ \ \forall\boldsymbol{x},\boldsymbol{y}\in RB,\ \ |\boldsymbol{x}-\boldsymbol{y}|_{\infty}>C_0.\end{aligned}\tag{3.10}$$

Note that since $\Phi^R_{\mathfrak{a}}$ is stationary, $\operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big]$ is *independent* of $\boldsymbol{x}$ and $\boldsymbol{y}$.

$$\operatorname{Var}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big]=\begin{bmatrix}\operatorname{Var}\big[\,H_R(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,H_R(\boldsymbol{x}),H_R(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,H_R(\boldsymbol{y}),H_R(\boldsymbol{x})\,\big] & \operatorname{Var}\big[\,H_R(\boldsymbol{y})\,\big]\end{bmatrix}$$
$$=\operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big]+\underbrace{\begin{bmatrix}0 & \operatorname{Cov}\big[\,H_R(\boldsymbol{x}),H_R(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,H_R(\boldsymbol{y}),H_R(\boldsymbol{x})\,\big] & 0\end{bmatrix}}_{=:\mathcal{E}^R_{H}(\boldsymbol{x},\boldsymbol{y})}.$$

We deduce

$$\big\| \operatorname{Var}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big] - \operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big]\,\big\|_{\mathrm{op}} = \|\mathcal{E}_H^R(\boldsymbol{x},\boldsymbol{y})\|_{\mathrm{op}} = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big). \tag{3.11}$$

We denote by $\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat$ the Gaussian random matrix

$$\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat = \widetilde{H}_R(\boldsymbol{x},\boldsymbol{y}) - \mathbb{E}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\,\|\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big].$$

We define $\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat$ similarly

$$\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat = \widehat{H}_R(\boldsymbol{x},\boldsymbol{y}) - \mathbb{E}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\,\|\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big].$$

The distributions of $\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat$ and $\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat$ are determined by the Gaussian regression formula (2.1). We have

$$\operatorname{Cov}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big] = \begin{bmatrix} \operatorname{Cov}\big[\,H_R(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,H_R(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{y})\,\big] \\ \operatorname{Cov}\big[\,H_R(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}^N(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,H_R(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{y})\,\big] \end{bmatrix}$$

$$= \begin{bmatrix} \operatorname{Cov}\big[\,H_R(0),\nabla\Phi_{\mathfrak{a}}^R(0)\,\big] & \operatorname{Cov}\big[\,H_R(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{y})\,\big] \\ \operatorname{Cov}\big[\,H_R(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,H_R(0),\nabla\Phi_{\mathfrak{a}}^R(0)\,\big] \end{bmatrix}.$$

The covariance $\operatorname{Cov}\big[\,H_R(0),\nabla\Phi_{\mathfrak{a}}^R(0)\,\big]$ involves only third order partial derivatives of $\boldsymbol{K}_{\mathfrak{a}}^N$ at 0, and these are all trivial since $\boldsymbol{K}_{\mathfrak{a}}^R$ is an even function. Hence

$$\operatorname{Cov}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big] = \begin{bmatrix} 0 & \operatorname{Cov}\big[\,H_R(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{y})\,\big] \\ \operatorname{Cov}\big[\,H_R(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{x})\,\big] & 0 \end{bmatrix}.$$

Similarly

$$\operatorname{Cov}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y}),\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big] = \begin{bmatrix} \operatorname{Cov}\big[\,H_R(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{x})\,\big] & 0 \\ 0 & \operatorname{Cov}\big[\,H_R(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{y})\,\big] \end{bmatrix} = 0.$$

Lemma 3.1(ii) implies that

$$\big\| \operatorname{Cov}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y}),\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]\,\big\|_{\mathrm{op}} = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big),$$
$$\big\| \operatorname{Cov}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]\,\big\|_{\mathrm{op}} = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big).$$

Since $\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^N(\boldsymbol{x},\boldsymbol{y})\,\big]$ and we deduce from the regression formula (2.1) that

$$\operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big] = \operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\,\big] + O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big),$$

$$\operatorname{Var}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big] = \operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\,\big] + O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big).$$

The regression formula (2.1) shows that

$$\operatorname{Var}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\,\big] = \operatorname{Var}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big]$$
$$- \operatorname{Cov}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]\operatorname{Var}\big[\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\operatorname{Cov}\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y}),\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big].$$

$$= \operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\,\big] + O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big)$$
$$- \operatorname{Cov}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]\operatorname{Var}\big[\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\operatorname{Cov}\big[\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y}),\widehat{H}^R(\boldsymbol{x},\boldsymbol{y})\,\big].$$

Since $\operatorname{Cov}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]=O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big)$ we deduce from (3.9) and (3.10) that there exists $C_1>0$, independent of $R>R_0$, such that

$$\begin{gathered}\operatorname{Cov}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]\operatorname{Var}\big[\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\operatorname{Cov}\big[\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y}),\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big]\\ =O\big(T_R(\boldsymbol{x},\boldsymbol{y})\big),\;\;\forall\boldsymbol{x},\boldsymbol{y}\in RB,\;\;|\boldsymbol{x}-\boldsymbol{y}|_\infty>C_1,\end{gathered}$$

and thus

$$\begin{gathered}\big\|\operatorname{Var}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]-\operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]\big\|_{\mathrm{op}}\\ =O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big),\;\;\forall\boldsymbol{x},\boldsymbol{y}\in RB,\;\;|\boldsymbol{x}-\boldsymbol{y}|_\infty>C_2=\max(C_0,C_1).\end{gathered}$$

Since $\operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\,\big]=\operatorname{Var}\big[\,H_R(0)\,\big]\oplus\operatorname{Var}\big[\,H_R(0)\,\big]$ we deduce from Lemma 3.1(i) that there exists $\mu_0>0$ such that

$$\operatorname{Var}\big[\,\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]\geq\mu_0\mathbb{1},\;\;\forall R\geq R_0.$$

Note also that (3.7) implies that there exists $C_3>0$, independent of $R>R_0$, such that

$$\sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in RB\\ |\boldsymbol{x}-\boldsymbol{y}|_\infty>C_3}}\big\|\operatorname{Var}\big[\,\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]\big\|_{\mathrm{op}}=O(1).$$

Lemma 2.6 implies that

$$\Big|\,\mathbb{E}\big[\,|\det\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat|\,\big]-\mathbb{E}\big[\,|\det\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat|\,\big]\,\Big|=O\big(T_R(\boldsymbol{x}-\boldsymbol{y})^{1/2}\big).\tag{3.12}$$

Using (3.10) we deduce that there exists $C_4>0$, independent of $R>R_0$, such that

$$\begin{gathered}\Big|\,p_{\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})}(0)-p_{\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})}(0)\,\Big|\\ =\frac{1}{(2\pi)^{m/2}}\Big|\,\det\operatorname{Var}\big[\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}-\det\operatorname{Var}\big[\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\,\Big|\\ =O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big),\;\;\forall\boldsymbol{x},\boldsymbol{y}\in RB,\;\;|\boldsymbol{x}-\boldsymbol{y}|_\infty>C_4.\end{gathered}\tag{3.13}$$

We can now estimate the right-hand-side of (3.5). For any $\boldsymbol{x},\boldsymbol{y}\in RB$

$$O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big)\stackrel{(3.6)}{=}O\big(\,|\,\boldsymbol{x}-\boldsymbol{y}\,|_\infty^{-p/2}\big),\;\;\forall p>0.$$

Using (3.10), (3.11), (3.12) and (3.13) that we conclude that there exists $C_5>1$, independent of $R>R_0$ such that, for any $p>m$,

$$\forall\boldsymbol{x},\boldsymbol{y}\in RB,\;\;|\boldsymbol{x}-\boldsymbol{y}|_\infty>C_5,\;\;\big|\,\underbrace{\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y})-\widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})}_{=\Delta_R(\boldsymbol{x},\boldsymbol{y})}\,\big|=O\big(\,|\,\boldsymbol{x}-\boldsymbol{y}\,|_\infty^{-p/2}\big).\tag{3.14}$$

Since the random function $\Phi^R_{\mathfrak{a}}$ is stationary, we deduce that for any $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^m$ such that $\boldsymbol{x}\neq\boldsymbol{y}$ we have

$$\Delta_R(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{y}+\boldsymbol{z})=\Delta_R(\boldsymbol{x},\boldsymbol{y})$$

so $\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y})$, $\widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})$ and $\Delta_R(\boldsymbol{x},\boldsymbol{y})$ depend only on $\boldsymbol{y}-\boldsymbol{x}$.

3.4. **Conclusion.** Assume now that $\boldsymbol{x},\boldsymbol{y}\in RB$ and $|\boldsymbol{x}-\boldsymbol{y}|_\infty\leq C_5$. Denote by $\widehat{\mathfrak{X}}$ the radial-blowup of $\mathbb{R}^m\times\mathbb{R}^m$ along the diagonal. It is diffeomorphic to the product $\mathbb{R}^m\times S^{m-1}\times[0,\infty)$.

Choose new orthogonal coordinates (ξ,η) given by

$$\xi=\boldsymbol{x}+\boldsymbol{y},\;\;\eta=\boldsymbol{x}-\boldsymbol{y}\Longleftrightarrow\boldsymbol{x}=\frac{1}{2}(\xi+\eta),\;\;\boldsymbol{y}=\frac{1}{2}(\xi-\eta)$$

then

$$|x-y|=|\eta|,\;\;d\boldsymbol{x}d\boldsymbol{y}=2^{-2m}d\xi d\eta.$$

Note that if $\boldsymbol{x},\boldsymbol{y}\in\operatorname{supp}f_R$, then $|\boldsymbol{x}|,|\boldsymbol{y}|\leq Rr_0/2$ and thus

$$\boldsymbol{x},\boldsymbol{y}\in\operatorname{supp}f_R\;\Rightarrow\;|\xi|,\;|\eta|<\frac{1}{2}\big(\,|\xi+\eta|+|\xi-\eta|\,\big)=|\boldsymbol{x}|+|\boldsymbol{y}|\leq Rr_0. \tag{3.15}$$

The natural projection $\pi:\widehat{\mathfrak{X}}\to\mathbb{R}^m\times\mathbb{R}^m$ can given the explicit description

$$\mathbb{R}^m\times S^{m-1}\times[0,\infty)\ni(\xi,\boldsymbol{\nu},r)\mapsto(\xi,\eta)=(\xi,r\boldsymbol{\nu})\in\mathbb{R}^m\times\mathbb{R}^m.$$

Set for $R\in(R_0,\infty]$ we set

$$w_R(\boldsymbol{x},\boldsymbol{y})=|\boldsymbol{x}-\boldsymbol{y}|^{m-2}\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y}).$$

Lemma 3.1(i) implies that for any $C>0$

$$\sup_{R\in(R_0,\infty]}\sup\|K^R_{\mathfrak{a}}\|_{C^6(RB)}<\infty.$$

We deduce from Proposition B.2 and Lemma B.1 that

$$\sup_{R\in(R_0,\infty]}\sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in RB\\0<|\boldsymbol{x}-\boldsymbol{y}|\leq C_5}}\big|\,w_R(\boldsymbol{x},\boldsymbol{y})\,\big|<\infty. \tag{3.16}$$

It is easy to see that $\widetilde{\rho}_R\circ\pi$ admits a continuous extension to the blow-up. Using (3.14) and (3.16) we deduce that for any $p>0$ there exists a constant $K_p>0$, independent of R, such that

$$|x-y|^{m-1}\big|\,\Delta_R(\boldsymbol{x},\boldsymbol{y})\,\big|\leq K_p\big(\,1+|x-y|\,\big)^{-p+m-1},\;\;\forall\boldsymbol{x},\boldsymbol{y}\in RB \tag{3.17}$$

Set

$$\delta_R(\xi,\eta)=\Delta_R\big(\,\pi(\xi,\eta)\,\big)$$

Since $\Delta_R(\boldsymbol{x},\boldsymbol{y})$ depends only on $\boldsymbol{y}-\boldsymbol{x}$ we deduce that $\delta_R(\xi,\eta)$ is independent of ξ. We have

$$I(R)=\int_{\mathfrak{X}}\Delta_R(\boldsymbol{x},\boldsymbol{y})f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}=\int_{\substack{|\boldsymbol{x}|,|\boldsymbol{y}|\leq Rr_0/2\\\boldsymbol{x}\neq\boldsymbol{y}}}\Delta_R(\boldsymbol{x},\boldsymbol{y})f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$

$$\stackrel{(3.15)}{=}\frac{1}{2^{2m}}\int_{\substack{|\xi|<Rr_0,\\|\boldsymbol{\nu}|=1,\,r\in(0,Rr_0)}}r^{m-1}\delta_R\big(\,\xi,r\boldsymbol{\nu}\,\big)f_R\Big(\,\frac{\xi+r\boldsymbol{\nu}}{2}\,\Big)f_R\Big(\,\frac{\xi-r\boldsymbol{\nu}}{2}\,\Big)dr\operatorname{vol}_{S^{m-1}}[d\boldsymbol{\nu}]d\xi$$

$(\xi=2R\zeta,\ \delta_R(\xi,r\nu)=\delta_R(0,r\nu))$

$$=R^m\int_{|\zeta|\leq}\underbrace{\left(2^{-m}\int_{\substack{|\boldsymbol{\nu}|=1\\r\in(0,Rr_0)}}r^{m-1}\delta_R\big(\,0,r\boldsymbol{\nu}\,\big)f\Big(\,\zeta+\frac{r\boldsymbol{\nu}}{2R}\,\Big)f\Big(\zeta-\frac{r\boldsymbol{\nu}}{2R}\,\Big)dr\operatorname{vol}_{S^{m-1}}[d\boldsymbol{\nu}]\right)d\zeta}_{=:J(R)}.$$

Note that

$$\delta_R\big(\,0,r\boldsymbol{\nu}\,\big)=\widehat{\rho}_R(r\boldsymbol{\nu}/2,-r\boldsymbol{\nu}/2)-\widetilde{\rho}_R(r\boldsymbol{\nu}/2,-r\boldsymbol{\nu}/2)$$

and for $r>0$, $|\boldsymbol{\nu}|=1$ fixed

$$\lim_{R\to\infty}\delta_R\big(\,0,r\boldsymbol{\nu}\,\big)=\delta_\infty(0,r\nu)=\widehat{\rho}_\infty(r\boldsymbol{\nu}/2,-r\boldsymbol{\nu}/2)-\widetilde{\rho}_\infty(r\boldsymbol{\nu}/2,-r\boldsymbol{\nu}/2).$$

We deduce from (3.15) and (3.16) that for any $p>0$ there exists $K_p>0$ such that for any $R>R_0$, $|\zeta|<r_0/2$, $|\boldsymbol{\nu}|=1$ and $r\leq Rr_0$ we have

$$\Big|\, r^{m-1}\delta_R\big(\,0,r\boldsymbol{\nu}\,\big)f\Big(\,\zeta+\frac{r\boldsymbol{\nu}}{2R}\Big)f\Big(\,\zeta-\frac{r\boldsymbol{\nu}}{2R}\Big)\,\Big|\leq K_p\|f\|^2\big(\,1+r\,\big)^{-p+m-1}.$$

For $p>m$ we have

$$\int_{|\zeta|\leq r_0/2}\left(\int_{(0,\infty)\times S^{m-1}}\big(\,1+r\,\big)^{-p+m-1}dr\,\mathrm{vol}_{S^{m-1}}\big[\,d\boldsymbol{\nu}\big)\right)d\zeta<\infty.$$

The dominated convergence theorem implies that $J(R)$ has a finite limit as $R\to\infty$. More precisely

$$\lim_{R\to\infty}J(R)=\int_{|\zeta|\leq r_0/2}\underbrace{2^{-m}\left(\int_{\substack{|\boldsymbol{\nu}|=1\\ r>0}}r^{m-1}\delta_\infty\big(\,0,r\boldsymbol{\nu}\,\big)dr\,\mathrm{vol}_{S^{m-1}}[d\boldsymbol{\nu}]\right)}_{=:Z_m(\mathfrak{a})}f(\zeta)^2d\zeta.$$

This concludes the proof of Theorem 1.1.

Appendix A. Some abstract ampleness criteria

We begin we an abstract result that will be our main tool for detecting ample Gaussian fields.

Proposition A.1. *Let $\boldsymbol{X}$ be a separable Banach space with norm $\|-\|$. Let $(x_n)_{n\geq 0}$ be a sequence in $\boldsymbol{X}$ and $(c_n)_{n\geq 0}$ a sequence of positive real numbers such that*

$$\sum_{n\geq 1}c_n\|x_n\|<\infty.$$

Denote by $\boldsymbol{Y}$ the closure of the span of $(x_n)_{n\geq 1}$. Let $(A_n)_{n\geq 1}$ be a sequence of independent standard normal random variables defined on the probability space $(\Omega,\mathcal{S},\mathbb{P})$. Then the following hold.

(i) *There exists a negligible subset $\mathcal{N}\in\mathcal{S}$ such that the series*

$$\sum_{n\geq 1}A_n(\omega)c_nx_n$$

converges in $\boldsymbol{X}$ to an element in $\boldsymbol{Y}$ for any $\omega\in\Omega\setminus\mathcal{N}$.

(ii) *The map $S:\Omega\to\boldsymbol{Y}$ defined by*

$$S(\omega)=\begin{cases}\sum_{n\geq 1}A_n(\omega)c_nx_n, & \omega\in\Omega\setminus\mathcal{N},\\ 0, & \omega\in\mathcal{N}\end{cases}$$

is Borel measurable and the push-forward $\Gamma_S:=S_\#\mathbb{P}$ is a nondegenerate Gaussian measure on $\boldsymbol{Y}$.

(iii) *For any nonempty open subset $\mathcal{O}\subset\boldsymbol{Y}$, $\mathbb{P}\big[\,S\in\mathcal{O}\,\big]>0$.*

Proof. (i) We will show that the random scalar series

$$\sum_n|A_n|c_n\|x_n\|$$

is a.s. convergent. According to Kolmogorov's two-series theorem this happens if the positive random variables $X_n=|A_n|\cdot c_n\|x_n\|$ satisfy

$$\sum_{n\geq 1}\mathbb{E}\big[\,X_n\,\big]<\infty\ \text{ and }\ \sum_{n\geq 1}\mathbb{E}\big[\,X_n^2\,\big]<\infty.$$

Now observe that

$$\mathbb{E}\big[\,|A_n|\,\big] = 2\frac{1}{\sqrt{2\pi}}\int_0^\infty xe^{-x^2/2}dx = \sqrt{\frac{2}{\pi}},$$

$$\sum_{n\geq 1}\mathbb{E}\big[\,X_n\,\big] = \sqrt{\frac{2}{\pi}}\sum_{n\geq 1}c_n\|x_n\| < \infty$$

and

$$\sum_{n\geq 1}\mathbb{E}\big[\,X_n^2\,\big] = \sum_{n\geq 1}c_n^2\|x_n\|^2 < \infty.$$

(ii) Define $S_n : \Omega \to \boldsymbol{Y}$

$$S_n(\omega) = \begin{cases}\sum_{k=1}^n A_k(\omega)c_kx_k, & \omega \in \Omega\setminus\mathcal{N},\\ 0, & \omega\in\mathcal{N}.\end{cases}$$

The maps S_n are measurable since the addition operation on a separable Banach space is a measurable map. The map S is measurable since for any $\xi \in \boldsymbol{Y}^*$ the function $\langle \xi, S\rangle$ is measurable as limit of the measurable functions $\langle \xi, S_n\rangle$.

To see that Γ_S is a Gaussian map let $\xi\in\boldsymbol{Y}^*$. Then

$$\langle \xi, S(\omega)\rangle = \lim_{n\to\infty}\langle \xi, S_n(\omega)\rangle.$$

The random variables

$$\langle \xi, S_n\rangle = \sum_{k=1}^n A_nc_n\langle \xi, x_n\rangle$$

are Gaussian as sum of independent Gaussians. Since the limit of Gaussian random variables is also Gaussian we deduce that $\langle \xi, S\rangle$ is Gaussian with variance

$$v[\xi] = \sum_{n\geq 1}c_n^2\big|\,\langle \xi, x_n\rangle\,\big|^2.$$

Since (x_n) spans a dense subspace of $\boldsymbol{Y}$, we deduce that for any $\xi\in\boldsymbol{Y}^*\setminus 0$ there exists n such that $\langle\xi, x_n\rangle \neq 0$. This proves that Γ_S is nondegenerate. Part (iii) now follows from the Support Theorem [7, Thm. 3.6.1] or [24, Cor. 4.2.1]. □

Proposition A.2. *Suppose that $\boldsymbol{U}$ is a Banach space with norm $\|-\|$, $\boldsymbol{T}$ is a compact metric space, $N\in\mathbb{N}$ and*

$$G : \boldsymbol{U}^N\times\boldsymbol{T}\to[0,\infty),\;\;(u_1,\dots,u_N,t)\mapsto G(u_1,\dots,u_N,t)\in[0,\infty)$$

is a continuous function. We define

$$G_* : \boldsymbol{U}^N\to[0,\infty),\;\;G_*(u_1,\dots,u_N) := \min_{t\in\boldsymbol{T}}G(u_1,\dots,u_N,t).$$

Suppose that there exist $v_1,\dots,v_N\in\boldsymbol{U}$ such that $G_*(v_1,\dots,v_N) = r_0 > 0$. Then, for any $r\in(0,r_0)$, there exists $\varepsilon=\varepsilon(r)>0$ such that

$$\forall u_1,\dots,u_N\in\boldsymbol{U},\;\;\forall i=1,\dots N,\;\;\|u_i-v_i\|<\varepsilon\Rightarrow G_*(u_1,\dots,u_N)>r.$$

In particular if

$$U_1\subset U_2\subset\cdots$$

is an increasing sequence of finite dimensional subspaces of $\boldsymbol{U}$ whose union is a dense subspace of $\boldsymbol{U}$, then there exists $\nu\in\mathbb{N}$ and

$$u_{1,\nu},\dots,u_{N,\nu}\in U_\nu$$

such that $G_*\big(\,u_{1,\nu},\dots,u_{N,\nu}\,\big)>0$.

Proof. We argue by contradiction suppose there exists $r_1 \in (0, r_0)$ and sequences in $\boldsymbol{U}$

$$\big(\, u_{i,\nu}\,\big)_{\nu\in\mathbb{N}},\ \ i = 1, \dots, N,$$

such that

$$\lim_{\nu\to\infty} \|u_{i,\nu} - v_i\| = 0,\ \ \forall i = 1, \dots, N,$$

and

$$G_*(u_{1,\nu}, \dots, u_{N,\nu}) \leq r_1,\ \ \forall \nu.$$

Next choose $t_\nu \in \boldsymbol{T}$ such that

$$G\big(\, u_{1,\nu}, \dots, u_{N,\nu}, t_\nu\,\big) = G_*\big(\, u_{1,\nu}, \dots, u_{N,\nu}\,\big).$$

Upon extracting a subsequence we can assume that t_ν converges in $\boldsymbol{T}$ to some point t_∞. Then

$$r_1 \geq \liminf_{\nu\to\infty} G_*\big(\, u_{1,\nu}, \dots, u_{N,\nu}\,\big) = \liminf_{\nu\to\infty} G\big(\, u_{1,\nu}, \dots, u_{N,\nu}, t_\nu\,\big)$$

$$= G\big(\, v_1, \dots, v_N, t_\infty\big) \geq r_0 > r_1.$$

□

With $\boldsymbol{T}$ a compact metric space as above. Let $E \to \boldsymbol{T}$ be a rank r topological real vector bundle over $\boldsymbol{T}$ equipped with a continuous metric h. We will refer to the pair (E, h) as a metric vector bundle. For $t \in \boldsymbol{T}$ we denote by $|-|_t$ the norm on the fiber E_t induced by h. The space $C^0(E)$ of continuous sections E is a Banach space with respect to the norm

$$\|u\| := \sup_{t\in\boldsymbol{T}} \big|\, u(t)\,\big|_t,\ \ u \in C(E).$$

Definition A.3. An *ample Banach space of sections* of E is a Banach space $\boldsymbol{U} \subset C^0(E)$ continuously embedded in $C^0(E)$ such that

$$\forall t \in \boldsymbol{T},\ \ \operatorname{span}\big\{\, u(t),\ \ u \in \boldsymbol{U}\,\big\} = E_t.$$

Let $k \in \mathbb{N}$. We say that the Banach space $\boldsymbol{U}$ is *k-ample* if for any *distinct* points $t_1, \dots, t_k \in \boldsymbol{T}$ the map

$$\boldsymbol{U} \ni u \mapsto u(t_1) \oplus \cdots \oplus u(t_k) \in E_{t_1} \oplus \cdots \oplus E_{t_k}$$

is onto. □

Example A.4. The space $C^0(E)$ is a k-ample Banach space of continuous sections of $E \to \boldsymbol{T}$ for any $k \in \mathbb{N}$. If $\boldsymbol{T}$ is a compact smooth manifold and $E \to \boldsymbol{T}$ is a smooth vector bundle, then each of the spaces $C^\ell(E)$, $\ell \in \mathbb{N}$, is a k-ample Banach space of sections of E for any $k \in \mathbb{N}$. □

Corollary A.5. *Let $E \to \boldsymbol{T}$ be a real metric vector bundle over the compact metric space $\boldsymbol{T}$. Suppose that $\boldsymbol{U} \subset C^0(E)$ is an ample Banach space of sections of E and*

$$U_1 \subset U_2 \cdots$$

is an increasing sequence of finite dimensional subspaces of $\boldsymbol{U}$ such that

$$U_\infty = \bigcup_{\nu\in\mathbb{N}} U_\nu$$

is dense in $\boldsymbol{U}$. Then there exists $\nu \in \mathbb{N}$, for any $t \in \boldsymbol{T}$, the evaluation map

$$\mathbf{Ev}_t : U_\nu \to E_t \ \ \textit{is onto}.$$

Proof. Using the compactness of $\boldsymbol{T}$ and the openess of the surjectivity condition we can find $v_1, \dots, v_N \in \boldsymbol{U}$ such that

$$\forall t \in \boldsymbol{T}, \;\; \operatorname{span}\big\{\, v_1(t), \dots, v_N(t) \,\big\} = E_t.$$

For every $u_1, \dots, u_N \in U$ and $t \in \boldsymbol{T}$ define

$$S_{u_1,\dots,u_N,t} : \mathbb{R}^N \to E_t, \;\; S_{u_1,\dots,u_N,t}(\boldsymbol{x}) = \sum_{k=1}^{N} x_k u_k(t)$$

and

$$G(u_1, \dots, u_N, t) = \det\big(\, S_{u_1,\dots,u_N,t} S^*_{u_1,\dots,u_N,t} \,\big) \geq 0.$$

Note that

$$\operatorname{span}\big\{\, u_1(t), \dots, u_N(t) \,\big\} = E_t \Longleftrightarrow G(u_1, \dots, u_N, t) > 0.$$

The resulting map $G : \boldsymbol{U}^N \times \boldsymbol{T} \to [0, \infty)$ is continuous and

$$G(v_1, \dots, v_N, t) > 0, \;\; \forall t \in \boldsymbol{T}$$

Hence

$$G_*(v_1, \dots, v_N) = \inf_{t \in \boldsymbol{T}} G(v_1, \dots, v_N, t) > 0.$$

Using Proposition A.2, we deduce that there exists $\nu \in \mathbb{N}$ and $u_{1,\nu}, \dots, u_{N,\nu} \in U_\nu$ such that

$$G_*(u_{1,\nu}, \dots, u_{N,\nu}) > 0.$$

Hence

$$\mathbf{Ev}_t : \operatorname{span}\big\{\, u_1, \dots, u_N \,\big\} \subset \boldsymbol{U} \to E_t \text{ is onto}, \;\; \forall t \in \boldsymbol{T}.$$

A fortiori, this implies that

$$\mathbf{Ev}_t : U_\nu \to E_t \text{ is onto}, \;\; \forall t \in \boldsymbol{T}.$$

□

Corollary A.6. *Let $E \to \boldsymbol{T}$ be a real metric vector bundle over the compact metric space $\boldsymbol{T}$. Suppose that $\boldsymbol{U} \subset C^0(E)$ is a 2-ample Banach space of sections and*

$$U_1 \subset U_2 \cdots$$

is an increasing sequence of finite dimensional subspaces of $\boldsymbol{U}$ such that

$$U_\infty = \bigcup_{\nu \in \mathbb{N}} U_\nu$$

is dense in $\boldsymbol{U}$. Then, for any open neighborhood $\mathcal{O}$ of the diagonal $\Delta \subset \mathbb{T} \times \mathbb{T}$, there exists $\nu \in \mathbb{N}$ such that for any $(t_1, t_2) \in \boldsymbol{T}^2 \setminus \mathcal{O}$, the evaluation map

$$\mathbf{Ev}_{t_1,t_2} : U_\nu \to E_{t_1} \oplus E_{t_2} \;\; \textit{is onto}.$$

Proof. For $\underline{t} \in \boldsymbol{T}^2$ and $u \in \boldsymbol{U}$ we set

$$u(\underline{t}) := u(t_1) \oplus u(t_2), \;\; E_{\underline{t}} = E_{t_1} \oplus E_{t_2}, \;\; \mathbf{Ev}_{\underline{t}}(u) = u(\underline{t}).$$

Using the compactness of $\boldsymbol{T}^2 \setminus \mathcal{O}$ and the openness of the surjectivity condition we can find $v_1, \dots, v_N \in \boldsymbol{U}$ such that

$$\forall \underline{t} \in \boldsymbol{T}^2 \setminus \mathcal{O}, \;\; \operatorname{span}\big\{\, v_1(\underline{t}), \dots, v_N(\underline{t}) \,\big\} = E_{\underline{t}}.$$

For every $u_1,\dots,u_N\in\boldsymbol{U}$ and $\underline{t}\in\boldsymbol{T}^2$ define

$$S_{u_1,\dots,u_N,\underline{t}}:\mathbb{R}^N\to E_{\underline{t}},\;\; S_{u_1,\dots,u_N,\underline{t}}(\boldsymbol{x})=\sum_{k=1}^N x_k u_k(\underline{t})$$

and

$$G(u_1,\dots,u_N,\underline{t})=\det\big(\,S_{u_1,\dots,u_N,\underline{t}}S^*_{u_1,\dots,u_N,\underline{t}}\,\big)\geq 0.$$

Note that

$$\operatorname{span}\big\{\,u_1(\underline{t}),\dots,u_N(\underline{t})\,\big\}=E_{\underline{t}}\Longleftrightarrow G(u_1,\dots,u_N,\underline{t})>0.$$

Thus

$$G(u_1,\dots,u_N,\underline{t})>0\iff \mathbf{Ev}_t:\operatorname{span}\big\{\,u_1,\dots,u_N\,\big\}\subset\boldsymbol{U}\to E_{\underline{t}}\text{ is onto.}$$

The resulting map $G:\boldsymbol{U}^N\times\big(\boldsymbol{T}^2\setminus\mathcal{O}\big)\to[0,\infty)$ is continuous and

$$G_*(v_1,\dots,v_N)=\inf_{\underline{t}\in\boldsymbol{T}^2\setminus\mathcal{O}}G(v_1,\dots,v_N,\underline{t})>0.$$

Using Proposition A.2, we deduce that there exists $\nu\in\mathbb{N}$ and $u_{1,\nu},\dots,u_{N,\nu}\in U_\nu$ such that

$$G_*(u_{1,\nu},\dots,u_{N,\nu})>0.$$

Hence

$$\mathbf{Ev}_{\underline{t}}:\operatorname{span}\big\{\,u_1,\dots,u_N\,\big\}\subset\boldsymbol{U}\to E_{\underline{t}}\text{ is onto},\;\;\forall\underline{t}\in\boldsymbol{T}^2\setminus\mathcal{O}.$$

A fortiori, this implies that

$$\mathbf{Ev}_{\underline{t}}:U_\nu\to E_{\underline{t}}\text{ is onto},\;\;\forall\underline{t}\in\boldsymbol{T}^2\setminus\mathcal{O}.$$

□

Proposition A.7. *Suppose that $E\to\boldsymbol{T}$ is a topological metric vector bundle over the compact metric space $\boldsymbol{T}$. Let $\boldsymbol{X}\subset C^0(E)$ be an ample Banach space of sections of E embedded continuously in $C^0(T)$.*

Suppose that $(u_n)_{n\in\mathbb{N}}$ is a sequence of sections in $\boldsymbol{X}$ such that $\operatorname{span}\big\{u_n,\;\; n\in\mathbb{N}\big\}$ *is dense in $\boldsymbol{X}$ and exists $\alpha>0$ such that*

$$\|u_n\|_{\boldsymbol{U}}=O(n^\alpha)\;\;\textit{as } n\to\infty. \tag{A.1}$$

Fix a sequence of positive real numbers $(\lambda_n)_{n\geq 0}$ such that

$$\liminf_{n\to\infty}\frac{\lambda_n}{n^\beta}>0, \tag{A.2}$$

for some $\beta>0$. Let $\mathfrak{a}\in\mathcal{S}(\mathbb{R})$ be an even Schwartz function such that $\mathfrak{a}(0)=1$. Fix a sequence of i.i.d. *standard normal random variables $(X_n)_{n\geq 0}$. Then the following hold.*

(i) *For any $R>0$ the random series*

$$\sum_{n\in\mathbb{N}}\mathfrak{a}\big(\,\lambda_n/R\,\big)X_n u_n \tag{A.3}$$

converges a.s. *in $\boldsymbol{X}$. Denote by Φ^R the resulting continuous Gaussian section of E.*

(ii) *There exists R_0 such that $\forall R>R_0$ the Gaussian section Φ^R is ample.*

Proof. (i) Since $\mathfrak{a}$ is a Schwartz function we deduce from (A.1) and (A.2) that

$$\sum_{n\to\infty} \big|\,\mathfrak{a}\big(\lambda_n/R\big)\,\big|\|u_n\|_{\boldsymbol{X}} < \infty,\;\;\forall \hbar > 0$$

The convergence of the random series (A.3) in $C^0(E)$ follows from Proposition A.1.
(ii) For $\hbar > 0$ we set

$$\mathcal{N}_R := \big\{ n \in \mathbb{N};\;\; \mathfrak{a}\big(\lambda_n/R\big) \neq 0\,\big\}$$

and denote by $\boldsymbol{Y}^R$ the closure in $\boldsymbol{X}$ of

$$\operatorname{span}\big\{\,u_n;\; n\in\mathcal{N}_R\,\big\}.$$

According to Proposition A.1 the above random series defines a *nondegenerate* Gaussian $\boldsymbol{\Gamma}^\hbar$ measure on the Banach space $\boldsymbol{Y}^\hbar$.

Set

$$U_\nu := \operatorname{span}\big\{\,u_1,\dots,u_\nu\,\big\}.$$

Since $\mathfrak{a}(0)=1$, we deduce that

$$\exists r_0 > 0,\;\; \forall |t|\leq r_0,\;\; \big|\,\mathfrak{a}(t)\,\big| \geq 1/2.$$

Hence, for any $\nu\in\mathbb{N}$ there exists $R = R(\nu) > 0$ so that

$$\forall R \geq R(\nu),\;\; \max_{1\leq k\leq \nu} \lambda_k/R < r_0,$$

i.e.,

$$U_\nu \subset \boldsymbol{Y}^R,\;\; \forall R\geq R(\nu).$$

Corollary A.5 implies that there exists $\nu_0\in\mathbb{N}$ such that

$$\forall t\in\boldsymbol{T},\;\; \mathbf{Ev}_t : U_{\nu_0} \to E_t \;\text{ is onto.}$$

Set $\hbar_0 = \hbar(\nu_0)$ such that $U_{\nu_0}\subset \boldsymbol{Y}^R, \forall R\leq R_0$.

We will show that for any $t\in\boldsymbol{T}$ and any $R\geq R_0$, the Gaussian vector $\Phi^R(t)$ is nondegenerate, i.e., for any open set $\mathcal{O}\subset E_t$, $\mathbb{P}\big[\,\Phi^R(t)\in\mathcal{O}\,\big] > 0$. Equivalently, this means

$$\boldsymbol{\Gamma}^R\big[\,\mathbf{Ev}_t^{-1}(\mathcal{O})\,\big] > 0.$$

Since $\boldsymbol{\Gamma}^R$ is a nondegenerate Gaussian measure on $\boldsymbol{Y}^R$, it suffice to show that the open subset $\mathbf{Ev}_t^{-1}(\mathcal{O})\subset\boldsymbol{Y}^R$ is nonempty. This is indeed the case since $\mathbf{Ev}_t^{-1}(\mathcal{O})\supset\mathbf{Ev}_t^{-1}(\mathcal{O})\cap U_{\nu_0}\neq\emptyset$. □

Suppose that M is a smooth compact manifold. Denote by $C^k(E)$ the vector space of sections of E that are k-times continuously differentiable. We need to define on $C^k(E)$ a structure of separable Banach space and to do so we need to make some choices.

- Fix a smooth Riemannian metric g on M.
- Fix a smooth h metric on E. We denote by $(-,-)_{E_x}$ the induced inner product on E_x.
- Fix a connection (covariant derivative) ∇^h on E that is compatible with the metric h.

We will refer to such choices as *standard choices*. There are several geometric objects canonically induced by these choices; see [22, Sec. 3.3].

First, the metric g determines a a Borel measure vol_g on M, classically referred to as the *volume element* or the *volume density*. Next, the metric determines the *Levi-Civita connection* ∇^g on TM. The metric g also determines metrics on all the tensor bundles $TM^{\otimes p}\otimes(T^*M)^{\otimes q}$ and the connection ∇^g determines connections on these bundles compatible with the metrics induced by g. To ease the notational burden we will denote by ∇^g each of these connections.

Similarly, the metric h induces metrics in all the bundles $E^{\otimes p}\otimes(E^*)^{\otimes q}$ and the connection ∇^h determines connections on these bundles compatible with the induced metrics. We will denote by $\big|-\big|_x$ the Euclidean norms in any of the spaces $(T^*_xM)^{\otimes q}\otimes E^{\otimes p}$. We define the *jet bundle*

$$J_k(E):=\bigoplus_{j-0}^{k}T^*M^{\otimes j}\otimes E. \tag{A.4}$$

The connections ∇^g and ∇^h induce a connection $\nabla=\nabla^{g,h}$ on the bundle $(T^*M)^{\otimes k}\otimes E$

$$\nabla: C^1\big(\,(T^*M)^{\otimes k}\otimes E\,\big)\to C^0\big(\,(T^*M)^{\otimes k+1}\otimes E\,\big).$$

We denote by ∇^q the composition

$$C^q(E)\stackrel{\nabla}{\to}C^{m-1}\big(T^*M\otimes E\,\big)\stackrel{\nabla}{\to}\cdots\stackrel{\nabla}{\to}C^1\big(\,(T^*M)^{\otimes q-1}\otimes E\,\big)\stackrel{\nabla}{\to}C^0\big(\,(T^*M)^{\otimes q}\otimes E\,\big).$$

For every section $\psi\in C^k(E)$ we set

$$J_k(\psi)=J_k(\psi,\nabla)=\bigoplus_{k=0}^{k}\nabla^k\psi$$

$$\|u\|_{C^k}=\sum_{j=0}^{q}\|\nabla^j\psi\|,$$

where

$$\|\nabla^ju\|=\sup_{x\in M}\big|\,\nabla^ju(x)\,\big|_x.$$

Note that $J_k(\psi)$ is a continuous section of $J_k(E)$.The resulting normed spaces is a separable Banach space. The norm $\|-\|_{C^k}$ depends on the standard choices, but different standard choices yield equivalent norms. Fix one such norm and denote by $C^k(E)$ the resulting separable Banach space.

Corollary A.8. *Suppose that $E\to M$ is a smooth real vector bundle over the compact smooth manifold M. Fix a smooth Riemann metric g on M, a smooth metric h on E and a smooth connection on E compatible with h. Let $k\in\mathbb{N}$ and suppose that $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of C^k sections of E that span a dense subset of $C^k(E)$. Suppose that*

$$\|\phi_n\|_{C^k(E)}=O(n^\alpha)\ \text{ as } n\to\infty, \tag{A.5}$$

for some $\alpha>0$. Fix a sequence of positive numbers $(\lambda_n)_{n\in\mathbb{N}}$ satisfying (A.2). Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. *standard normal random variables and suppose that $\mathfrak{a}\in\mathcal{S}(\mathbb{R})$ is an even Schwartz function such that $\mathfrak{a}(0)=1$. Then the following hold.*

(i) *For any $R>0$ the random series*

$$\sum_{n\in\mathbb{N}}\mathfrak{a}\big(\,\lambda_n/R\,\big)X_n\phi_n \tag{A.6}$$

converges a.s. *in $C^k(E)$. Denote by Φ^R the resulting C^k Gaussian section of E.*

(ii) *There exists $R_0>$ such that $\forall R>R_0$ the Gaussian section Φ^R is J_k-ample, i.e., for any $\boldsymbol{x}\in M$ the Gaussian vector*

$$J_k\Phi^R(\boldsymbol{x})=\bigoplus_{j=0}^{k}\nabla^j\Phi^R(\boldsymbol{x})$$

is nondegenerate.

Proof. (i) This follows from Proposition A.7.
(ii) Consider the jet bundle $J_k(E)\to M$; see (A.4) We have a continuous linear

$$C^k(E)\to C^0\big(\,J^k(E)\,\big),\;\;\phi\mapsto J_k(\phi).$$

Denote by $\boldsymbol{U}$ the image of this map. It is a closed[1] subspace of $C^0\big(\,J^k(E)\,\big)$. Then the random series

$$\sum_{n\in\mathbb{N}}\mathfrak{a}\big(\,\lambda_n/R\,\big)X_nJ_k(\phi_n)$$

converges a.s. uniformly to $J_k(\Phi^R)$. Now observe that $\boldsymbol{U}$ is an ample Banach space of sections of $J^k(E)$. Indeed, using smooth partitions of unity we can find $\psi_1,\dots,\psi_N\in C^k(E)$ such that, for any $x\in M$,

$$\operatorname{span}\big\{\,J_k(\psi_1(x),\dots,J_k(\psi_N)(x)\,\big\}=J_k(E)_x.$$

Proposition A.7 now implies that $J_k(\Phi^R)$ is an ample Gaussian section of $J_k(E)$. □

Remark A.9. In applications (ϕ_n) are eigenfunctions of the Laplacian Δ on a Riemann manifold (M,g) and $\Delta\phi_n=\lambda_n^2\phi_n$. The covariance kernel of Φ^R is then the Schwartz kernel of the smoothing operator $\mathfrak{a}\big(\,\hbar\sqrt{\Delta}\,\big)^2$, $\hbar=R^{-1}$. If $\mathfrak{a}(x)=e^{-x^2/2}$ $\hbar=t^{1/2}$, then $\mathfrak{a}\big(\,\hbar\sqrt{\Delta}\,\big)^2=e^{-t\Delta}$, the heat operator on M. □

Corollary A.10. *Fix an even Schwartz function $\mathfrak{a}\in\mathcal{S}(\mathbb{R})$ and consider the random Fourier series $F^R_{\mathfrak{a}}$ defined in (1.1). We regard it as a random smooth function on the torus $\mathbb{T}^m$. Then for any $k\in\mathbb{R}$ there exists $R=R_k>0$ such that, for any $R>R_k$ the function $F^R_{\mathfrak{a}}$ is J_k-ample. In particular it is* a.s. *Morse for $R>R_1$.* □

Lemma A.11. *Suppose that $E\to M$ is a smooth real vector bundle over the compact smooth manifold M. Fix a smooth Riemann metric g on M, a smooth metric h on E and a smooth connection on E compatible with h. Let $k\in\mathbb{N}$ and suppose that $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of C^k sections of E that span a dense subset of $C^k(E)$. Set*

$$U_\nu:=\operatorname{span}\big\{\,\phi_1,\dots,\phi_\nu\,\big\}$$

Then there exists $\nu_0>0$ such that $\forall\nu\geq\nu_0$ the following hold.

(i) *For any $t\in M$ and any $\nu\geq\nu_0$ the map*

$$U_\nu\ni u\mapsto J_1(u)_t\in J_1(E_t)$$

is onto. Above, $J_1(u)_t$ is the 1-jet of u at t, $J_1(u)_t=u(t)\oplus\nabla u(t)\in E_t\oplus T^*_tM\otimes E_t$.

(ii) *For any $\underline{t}\in M^2\setminus\Delta$ the map*

$$U_\nu\ni u\mapsto u(\underline{t})\in E_{\underline{t}}$$

is onto.

Proof. The space $C^k(E)$ is J_1-ample and arguing as in the proof of Corollary A.5 we deduce that there exists $\nu_1\in\mathbb{N}$ such that for any $\nu\geq\nu_1$ and $t\in M$ the map

$$U_\nu\ni u\mapsto J_1(u)_t\in J_1(E)_t$$

is ample.

[1]Here we are using the classical fact that if a sequence of C^1-function (u_n) has the property that both (u_n) and their differentials (du_n) converge uniformly to u and respectively v, then u is C^1 and $du=v$.

The argument at the beginning of [12, Sec. 3.3] based on Kergin interpolation shows that there exists an open neighborhood $\mathcal{O}$ of the diagonal $\Delta \in M^2$ such that $\forall \nu \geq \nu_1$ and any $\underline{t} \in \mathcal{O} \setminus \Delta$ the map

$$U_\nu \ni u \mapsto u(\underline{t}) \in E_{\underline{t}}$$

is onto.

Corollary A.6 implies that there exists $\nu_0 > 0$ such that $\forall \nu \geq \nu_2$ and any $\underline{t} \in M^2 \setminus \mathcal{O}$ the map

$$U_\nu \ni u \mapsto u(\underline{t}) \in E_{\underline{t}}$$

is onto. Then $\nu_0 = \max(\nu_1, \nu_2)$ has all the claimed properties.

□

Corollary A.12. *Fix an even Schwartz function $\mathfrak{a} \in \mathcal{S}(\mathbb{R})$ and consider the random Fourier series $F^R_{\mathfrak{a}}$ defined in (1.1). We regard it as a random smooth function on the torus $\mathbb{T}^m$. Then there exists $R = R_{2,2} > 0$ such that, for any $R > R_{2,2}$ the function $F^R_{\mathfrak{a}}$ is J_2-ample and $\nabla F^R_{\mathfrak{a}}$ is 2-ample.* □

APPENDIX B. VARIANCE ESTIMATES

It has been known for some time that under certain conditions the number of zeros in a box of a Gaussian field F has finite variance, [2, 6, 9, 12]. In this paper we need an upper estimate for this variance in terms of the covariance kernel of F. In this appendix we use the ideas in the above references to obtain such estimates and a bit more.

Suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are finite dimensional real Euclidean spaces of the same dimension m and $\mathscr{V} \subset \boldsymbol{V}$ is an open set. Let $F : \mathscr{V} \to \boldsymbol{U}$ be a Gaussian random field whose covariance kernel $\mathcal{K}_F$ is C^6. In particular, this implies that F is a.s. C^2.

Theorem 2.1 implies the following result.

Lemma B.1. *For any box $B \subset \mathscr{V}$ the random variable*

$$C_F(B) := \sup_{v \in B} \Big(\big| F(v) \big| + \big\| F'(v) \big\| + \big\| F^{(2)}(v) \big\| \Big)$$

is p-integrable for any $p \in [1, \infty)$. More precisely there exists a constant $M = M_p(B) > 0$ such that

$$\mathbb{E}\big[C_F(B)^p \big] \leq M \|\mathcal{K}_F\|^{p+1}_{C^6(B \times B)}.$$

□

For any box $B \subset \mathscr{V}$ we denote by Z_B the number of zeros of F in B, i.e., $Z_B = Z[B, F]$. Let $\mathscr{V}^2_* := \mathscr{V}^2 \setminus \Delta$, where Δ is the diagonal

$$\Delta := \big\{ (v_0, v_1) \in \mathscr{V}^2; \;\; v_0 = v_1 \big\}.$$

Define B^2_* in a similar fashion. Consider the random field

$$\widehat{F} =: \mathscr{V}^2_* \to \boldsymbol{U} \oplus \boldsymbol{U}, \;\; \hat{F}(v_0, v_1) = F(v_0) \oplus F(v_1).$$

Note that

$$Z[\widehat{F}, B^2_*] = Z_B\big(Z_B - 1 \big).$$

Suppose that $F|_B$ is 2-ample, i.e., for any $\underline{v} = (v_0, v_1) \in B^2_*$ the Gaussian vector $F(v_0) \oplus F(v_1)$ is nondegenerate. We deduce from the local Kac-Rice formula that (2.3), $E\big[Z_B \big] < \infty$, and

$$\mathbb{E}\big[Z_B\big(Z_B - 1 \big) \big] = \int_{B^2_*} \rho^{(2)}_G(v_0, v_1) dv_0 dv_1,$$

where $\rho_F^{(2)}$ is the Kac-Rice density

$$\rho_F^{(2)}(v_0, v_1) := \mathbb{E}\big[\, |\det F'(v_0)\det F'(v_1)| \;\big|\; F(v_0) = F(v_1) = 0\,\big] p_{\widehat{F}(v_0,v_1)}(0). \tag{B.1}$$

Note that

$$p_{\widehat{F}(v_0,v_1)}(0) = \frac{1}{\sqrt{\det\big(\,2\pi\,\mathrm{Var}[F(v_0)\oplus F(v_1)]\,\big)}},$$

so $p_{\widehat{F}(v_0,v_1)}(0)$ explodes as (v_0, v_1) approaches the diagonal since $F(v)\oplus F(v)$ is degenerate for any $v \in \mathscr{V}$. Thus the function $\rho_F^{(2)}(v_0, v_1)$ might have a non integrable singularity along the diagonal so $E\big[\, Z_B^2\,\big]$ could be infinite.

We want to show that this is not the case and a bit more. We will use the gauge-change trick outlined in the introduction.

Denote by $\widehat{B^2}$ the radial blow-up of B^2 along the diagonal and by $\pi : \widehat{B^2} \to B^2$ the canonical projection.

Proposition B.2. *Suppose that $F\big|_B$ is C^2, 2-ample and J_1-ample, i.e., for any $\boldsymbol{v} \in B$ the Gaussian vector $\big(\,F(v), F'(v)\,\big)$ is nondegenerate. Define*

$$w_F : B_*^2 \to \mathbb{R},\;\; w_F(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}|^{m-2}\rho_F^{(2)}(\boldsymbol{x},\boldsymbol{y}).$$

Then there exists a constant $K > 0$, independent of B, such that

$$\sup_{\boldsymbol{p}\in B_*^2} |w_F(\boldsymbol{p})\big| \leq K C_F(B). \tag{B.2}$$

Proof. Our approach is a modification of the arguments in [6, Sec. 4.2]. For any $v_0, v_1 \in B$, $v_0 \neq v_1$, the Gaussian vector $\hat{F}(v_0, v_1) = F(v_0) \oplus F(v_1)$ is nondegenerate. We denote by $p_{F(v_0),F(v_1)}$ the probability density of $\hat{F}(v_0, v_1)$.

We set

$$r(\underline{v}) := \|v_1 - v_0\|,\;\; \Xi(\underline{v}) := \frac{1}{r(\underline{v})}\big(\,F(v_1) - F(v_0)\,\big).$$

Note that

$$\hat{F}(\underline{v}) = 0 \Longleftrightarrow F(v_0) = \Xi(\underline{v}) = 0.$$

Denote by $A(\underline{v})$ the linear map $\boldsymbol{U}^2 \to \boldsymbol{U}^2$ given by

$$A(\underline{v})\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} u_0 \\ u_0 + r(\underline{v})u_1 \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{\boldsymbol{U}} & 0 \\ \mathbb{1}_{\boldsymbol{U}} & r(\underline{v})\mathbb{1}_{\boldsymbol{U}} \end{bmatrix}\cdot\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}. \tag{B.3}$$

Thus

$$\begin{bmatrix} F(v_0) \\ F(v_1) \end{bmatrix} = A(\underline{v})\begin{bmatrix} F(v_0) \\ \Xi(\underline{v}) \end{bmatrix}.$$

Then the Gaussian regression formula implies that

$$\begin{gathered}\mathbb{E}\big[\,|\det F'(v_0)\det F'(v_1)| \;\big|\; F(v_0) = F(v_1) = 0\,\big] \\ = \mathbb{E}\big[\,|\det F'(v_0)\det F'(v_1)| \;\big|\; F(v_0) = \Xi(\underline{v}) = 0\,\big].\end{gathered}$$

Note that

$$p_{F(v_0),F(v_1)} = \frac{1}{\sqrt{\det\big(\,2\pi\,\mathrm{Var}[F(v_0)\oplus F(v_1)]\,\big)}}$$

$$= \frac{1}{|\det A|\sqrt{\det\big(\,2\pi\,\mathrm{Var}[F(v_0)\oplus\Xi(\underline{v}))]\big)}} = r(\underline{v})^{-m}p_{F(v_0)\oplus\Xi(\underline{v})}(0).$$

We deduce that for any $\underline{u} \in B_*^2$ we have

$$\rho_F^{(2)}(\underline{v}) := r(\underline{v})^{-m}\mathbb{E}\big[\,|\det F'(v_0)\det F'(v_1)|\;\big|\;F(v_0) = \Xi(\underline{v}) = 0\,\big]p_{F(v_0)\oplus\Xi(\underline{v})}(0). \tag{B.4}$$

Lemma B.3. *There exists a constant $C = C(B) > 0$ such that for $i = 0, 1$, and any $\underline{v} \in B_*^2$*

$$\big|\,\mathbb{E}\big[\,|\det F(v_i)|^2\big|\,F(v_0) = \Xi(\underline{v}) = 0\,\big]\,\big| \leq Cr(\underline{v})^2.$$

Proof. It suffices to consider only the case $i = 0$ since

$$F(v_0) = \Xi(\underline{v}) = 0 \Longleftrightarrow F(v_1) = \Xi(\underline{v}) = 0.$$

We argue by contradiction. Suppose that there exists a sequence $\underline{v}^n = (v_0^n, v_1^n)$ in B_*^2 converging to $\underline{v} \in B^2$ such that

$$\frac{1}{r(\underline{v}^n)^2}\big|\,\mathbb{E}\big[\,|\det F'(v_0^n)|^2\big|\,F(v_0^n) = \Xi(\underline{v}^n) = 0\,\big]\,\big| \to \infty. \tag{B.5}$$

Note that $\mathbb{E}\big[\,|\det F'(v_0)|^2\big|\,F(v_0) = \Xi(\underline{v}) = 0\,\big]$ depends continuously on $\underline{v} \in B_*^2$. The assumption (B.5) shows that the limit point $\underline{v}$ must live on the diagonal $\underline{v} = (v, v)$. Upon extracting a subsequence we can assume that

$$\nu_n := \frac{1}{r(\underline{v}^n)}\big(\,v_n^1 - v_0^n\,\big)$$

converges to a unit vector $\boldsymbol{e}$. Extend $\boldsymbol{e}$ to an orthonormal basis $(\boldsymbol{e}_k)$ of $\boldsymbol{U}$ such that $\boldsymbol{e} = \boldsymbol{e}_1$ Note that

$$\lim_{n\to\infty}\Xi(\underline{v}^n) = \partial_{\boldsymbol{e}}F(v).$$

Moreover, the Gaussian random vector $F(v) \oplus \partial_{\boldsymbol{e}}F(v)$ is nondegenerate since F was assumed J_1-ample. The regression formula shows that for any continuous and homogeneous function $f = f\big(\,F'(v_0), F'(v_1)\,\big)$ we have

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\big[\,f\big(\,F'(v_0^n), F'(v_1^n)\,\big)\,\big|\,F(v_0^n) = \Xi(\underline{v}^n) = 0\,\big]\\ = \mathbb{E}\big[\,f(F'(v), F'(v)\,)\,\big|\,F(v) = \partial_{\boldsymbol{e}}F(v) = 0\,\big].\end{aligned}$$

Since $F(v)$ is a.s. C^2 we deduce from the first order Taylor approximation with integral remainder that

$$\big|\,F(v_1^n) - F(v_0^n) - r(\underline{v}^n)\partial_{\nu_n}F(v_0^n)\,\big| \leq K(B)r(\underline{v})^2 \tag{B.6}$$

where, according to Lemma B.1, the quantity $K(B)$ is a nonnegative random variable that is p-integrable $\forall p \in [1, \infty)$. According to the regression formula the quantity

$$\mathbb{E}\big[\,\big|\,F(v_1^n) - F(v_0^n) - r(\underline{v}^n)\partial_{\nu_n}F(v_0^n)\,\big|^p\;\big|\,F(v_0^n) = \Xi(\underline{v}^n) = 0\,\big]$$

can be expressed as integral with respect to a Gaussian measure $\boldsymbol{\Gamma}_n$ whose variance is given by the regression fomula (2.1). Since the limiting condition $F(v) = \partial_{\boldsymbol{e}}F(v) = 0$ is also nondegenerate we deduce that the variances $\operatorname{Var}\big[\,\boldsymbol{\Gamma}_n\,\big]$ stay bounded as $n \to \infty$. The variances of the unconditioned vectors $F(v_1^n) - F(v_0^n) - r(\underline{v}^n)\partial_{\nu_n}F(v_0^n)$ also stay bounded as $n \to \infty$ Thus, $\forall p \in [1, \infty)$ there exists a positive constant $C_p(B)$ such that $\forall n$ we have

$$\begin{aligned}&\mathbb{E}\big[\,\big|\,r(\underline{v}^n)\partial_{\nu_n}F(v_0^n)\,\big|^p\;\big|\,F(v_0^n) = \Xi(\underline{v}^n) = 0\,\big]\\ &= \mathbb{E}\big[\,\big|\,F(v_1^n) - F(v_0^n) - r(\underline{v}^n)\partial_{\nu_n}F(v_0^n)\,\big|^p\;\big|\,F(v_0^n) = \Xi(\underline{v}^n) = 0\,\big]\\ &= \mathbb{E}_{\boldsymbol{\Gamma}_n}\big[\,\big|\,F(v_1^n) - F(v_0^n) - r(\underline{v}^n)\partial_{\nu_n}F(v_0^n)\,\big|^p\,\big] \overset{(B.6)}{\leq} C_p(B)\mathbb{E}\big[\,K(B)^p\,\big]r(\underline{v})^{2p}.\end{aligned}$$

Hence

$$\mathbb{E}\big[\,\big|\,\partial_{\nu_n}F(v_0^n)\,\big|^p\;\big|\,F(v_0^n) = \Xi(\underline{v}^n) = 0\,\big] \leq C_p(B)r(\underline{v})^{2p}. \tag{B.7}$$

Extend (ν_n) to an orthonormal basis $(\boldsymbol{e}_k^n)$ of $\boldsymbol{V}$ such that $\nu_n = \boldsymbol{e}_1^n$ and

$$\lim_{n\to\infty} \boldsymbol{e}_k^n = \boldsymbol{e}_k.$$

Then, using Hadamard's inequality [16, Cor. 7.8.2] we deduce

$$\begin{aligned}\big|\det F'(v_0^n)\big| &= \big|\det\big(\partial_{\boldsymbol{e}_1^n}F(v_0^n), \partial_{\boldsymbol{e}_2^n}F(v_0^n), \dots, \partial_{\boldsymbol{e}_m^n}F(v_0^n)\big)\big| \\ &\le \big|\partial_{\boldsymbol{e}_1^n}F(v_0^n)\big|\prod_{k=2}^m \big|\partial_{\boldsymbol{e}_k^n}F(v_0^k)\big|.\end{aligned}$$

Hence

$$\mathbb{E}\Big[\big|\det F'(v_0^n)\big|^2 \,\big|F(v_0^n) = \Xi(\underline{v}^n) = 0\;\Big] \le \prod_{k=1}^m \mathbb{E}\Big[\big|\partial_{\boldsymbol{e}_k^n}F(v_0^n)\big|^{2m}\,\big|\,\big|F(v_0^n) = \Xi(\underline{v}^n) = 0\Big]\Big]^{\frac{1}{m}}.$$

For $k = 2, \dots, m$ we have

$$\mathbb{E}\Big[\big|\partial_{\boldsymbol{e}_k^n}F(v_0^n)\big|^{2m}\,\big|\,\big|F(v_0^n) = \Xi(\underline{v}^n) = 0\Big]^{\frac{1}{m}} = O(1), \;\; \text{as } n\to\infty,$$

since

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\Big[\big|\partial_{\boldsymbol{e}_k^n}F(v_0^n)\big|^{2m}\,\big|\,\big|F(v_0^n) = \Xi(\underline{v}^n) = 0\Big] \\ &= \lim_{n\to\infty}\mathbb{E}\Big[\big|\partial_{\boldsymbol{e}_k^n}F(v_0^n)\big|^{2m}\,\big|\,\big|F(v_0^n) = \Xi(\underline{v}^n) = 0\Big] \\ &= \mathbb{E}\Big[\big|\partial_{\boldsymbol{e}_k}F(v)\big|^{2m}\,\big|\,\big|F(v) = \partial_{\boldsymbol{e}}F(v) = 0\Big] < \infty.\end{aligned}$$

The last equality follows from the Gaussian regression formula which is valid also in the limit since the limiting condition is also nondegenerate.

On the other hand (B.7), shows that

$$\mathbb{E}\Big[\big|\partial_{\boldsymbol{e}_1^n}F(v_0^n)\big|^{2m}\,\big|F(v_1^n) = F(v_0^n) = 0\Big]^{\frac{1}{m}} = O\big(r(\underline{v}^n)^2\big), \;\; \text{as } n\to\infty.$$

$$\mathbb{E}\Big[\big|\det F'(v_0^n)\big|^2\,\big|\,F(v_1^n) = F(v_0^n) = 0\Big] = O\big(r(\underline{v}^n)^2\big), \;\; \text{as } n\to\infty.$$

This contradicts (B.5) and thus completes the proof of Lemma B.3. □

Lemma B.3 implies

$$\begin{aligned}&\mathbb{E}\Big[\,|\det F'(v_0)\det F'(v_1)|\,\big|\,F(v_0) = F(v_1) = 0\Big] \\ &\le \mathbb{E}\Big[\big|\det F'(v_0)\big|^2\,\big|\,F(v_0) = F(v_1) = 0\Big]\Big|^{1/2}\mathbb{E}\Big[\big|\det F'(v_1)\big|^2\,\big|\,F(v_0) = F(v_1) = 0\Big]\Big|^{1/2} \\ &= O\big(r(\underline{v})^{-2}\big).\end{aligned}$$

Hence

$$\rho_F^{(2)}(\underline{v}) = O\big(r(\underline{v})^{2-m}\big) \;\text{ on } B_*^2. \tag{B.8}$$

This shows that $\rho_F^{(2)} \in L^1\big(B_*^2\big)$ and completes the proof of Proposition B.2. □

We can extract from the above proof a more precise result. For any box B in a Euclidean space $\boldsymbol{V}$ we set

$$\mathfrak{q}(B) := \int_{B_*^2} r(\underline{v})^{2-m}dv_0dv_1.$$

Note that $\mathfrak{q}(B)$ is a translation invariant and for any $t > 0$, $\mathfrak{q}(tB) = t^{m+2}\mathfrak{q}(B)$. In particular, if B is is the cube $B_r = [0, r]^m$, then

$$\mathfrak{q}(B_r) = \mathfrak{q}(B_1)r^{m+2} = \mathfrak{q}(B_1)\operatorname{vol}\big[\,B_r\,\big]^{\frac{m+2}{m}}.$$

Corollary B.4. *Let $\mathcal{V}$ be an open subset of $\boldsymbol{V}$. There exists a function*

$$\mathfrak{F} : (0,\infty) \to (0,\infty)$$

with the following property: for any $m_0 > 0$, any box $B \subset \mathcal{W}$ and any Gaussian field $F : \Omega\times\mathcal{V} \to \boldsymbol{U}$ such that

- *the covariance kernel $\mathcal{K}_F$ is C^6,*
- *the restriction of F to B is 2-ample,*
- *and $\| \mathcal{K} \|_{C^6(B\times B)} < m_0$*

we have

$$\| \rho^{(2)}_{KR} \|_{L^1(B\times B)} < \mathfrak{F}(m_0)\mathfrak{q}(B).$$

□

Remark B.5. One can show that if F is a.s. C^3, then the function w_F in Proposition B.2 admits an extension to a continuous function on the radial blow-up of B^2 along the diagonal. □

REFERENCES

[1] R. Adler, R. J. E. Taylor: *Random Fields and Geometry*, Springer Monographs in Mathematics, Springer Verlag, 2007.

[2] M. Ancona, T. Letendre: *Multijet bundles and applications to the finiteness of moments for zeros of Gaussian fields*, arXiv.2307.10659

[3] D. Armentao, J.-M. Azaïs, J.-R. León: *On the general Kac-Rice formula for the measure of a level set*, arXiv: 2304.0742v1.

[4] J.-M. Azaïs, M. Wschebor: *Level Sets and Extrema of Random Processes*, John Wiley & Sons, 2009.

[5] P. Baxendale: *Gaussian measures on function spaces*, Amer. J. Math., **98**(1976), 891-952.

[6] D. Beliaev, M. McAuley, S. Muirhead: *A central limit theorem for the number of excursion set components of Gaussian fields*, arXiv.2205.0985

[7] V. I. Bogachev: *Gaussian Measures*, Amer. Math. Soc., 1998.

[8] D. J. Daley, D. Vere-Jones, *An Introduction to the Theory of Point Processes. Vol II: General Theory and Structure*, 2nd Edition, Springer Verlag, 2008.

[9] A. Estrade, J. R. León: *A central limit theorem for the Euler characteristic of a Gaussian excursion set*, Ann. of Probab., **44**(2016), 3849-3878. https://hal.archives-ouvertes.fr/hal-00943054v3

[10] X. Fernique: *Integrabilité des vecteurs gaussiens*, C. R. Acad. Paris, S'erie A, **270**(1970), 1698-1699.

[11] L. Gass: *Cumulants asymptotics of the zeros counting measure of real Gaussian processes*, Electron. J. Probab., **28**(2023), 1-45, arXiv: 2112.08247

[12] L. Gass, M. Stecconi: *The number of critical points of a Gaussian field: finiteness of moments*, arxiv.org: 2305.17586v2

[13] I. M. Gelfand, N. Ya. Vilenkin: *Generalized Functions. Volume 4. Applications of Harmonic Analysis*, Academic Press, 1964.

[14] J. L. van Hemmen, J. Ando: *An inequality for trace ideals*, Comm. Math. Phys., **76**(1980), 143-148.

[15] L. Hörmander: *The Analysis of Linear Partial Differential Operators I.*, Springer Verlag, 2003.

[16] R. A. Horn, C. R. Johnson: *Matrix Analysis*, Cambridge University Press, 1990 Reprint.

[17] O. Kallenberg: *Random measures, Theory and Applications*, Springer Verlag 2017.

[18] F. Nazarov, M. Sodin: *Asymptotic laws for the spatial distribution and the number of components of zero sets of Gaussian random functions*, J. Math. Phys., Analysis, Geom., **12**(2016), 205-278. https://core.ac.uk/download/pdf/159626247.pdf

[19] L. I. Nicolaescu: *Critical points of multidimensional random Fourier series: variance estimates*, J. of Math. Phys., **57**(2016), 083304, arXiv: 1310.5571

[20] L. I. Nicolaescu: *Critical points of multidimensional random Fourier series: central limits*, Bernoulli, **24**(2018), 1128-1170, arXiv:1511.04965v2.

[21] *A stochastic Gauss-Bonnet-Chern formula*, Probab. Theory Relat. Fields, **165**(2016), 235-265.

[22] L. I. Nicolaescu: *Lectures on the Geometry of Manifolds*, 3rd Edition, World Scientific, 2021.

[23] L. I. Nicolaescu: *A law of large numbers concerning the number of critical points of isotropic Gaussian functions*, arXiv: 2408.14383.

[24] D. W. Stroock: *Gaussian Measures in Finite and Infinite Dimensions*, Springer Verlag, 2023.

BRANDON FU: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556-4618.
Email address: qfu3@nd.edu

LIVIU NICOLAESCU: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556-4618.
Email address: lnicolae@nd.edu
URL: http://www.nd.edu/~lnicolae/