CRITICAL SETS OF RANDOM SMOOTH FUNCTIONS ON COMPACT MANIFOLDS

LIVIU I. NICOLAESCU

ABSTRACT. Given a compact, connected Riemann manifold without boundary \((M, g)\) of dimension \(m\) and a large positive constant \(L\) we denote by \(U_L\) the subspace of \(C^\infty(M)\) spanned by eigenfunctions of the Laplacian corresponding to eigenvalues \(\leq L\). We equip \(U_L\) with the standard Gaussian probability measure induced by the \(L^2\)-metric on \(U_L\), and we denote by \(N_L\) the expected number of critical points of a random function in \(U_L\). We prove that \(N_L \sim C_m \dim U_L\) as \(L \to \infty\), where \(C_m\) is an explicit positive constant that depends only on the dimension \(m\) of \(M\) and satisfying the asymptotic estimate \(\log C_m \sim \frac{m}{2} \log m\) as \(m \to \infty\).

CONTENTS

1. Introduction 1
   Notations 4
2. The key integral formula 5
   2.1. A Chern-Lashof type formula 5
   2.2. A Gaussian random field perspective 13
   2.3. Zonal domains of spherical harmonics of large degree 16
3. The proof of Theorem 1.1 19
4. The proof of Theorem 1.2 22
5. The variance of the number of critical points of a random trigonometric polynomial 27
   Appendix A. Gaussian measures and Gaussian random fields 38
   Appendix B. Gaussian random symmetric matrices 42
   Appendix C. Some Gaussian integrals 45
   Appendix D. Some elementary estimates 47
   References 51

1. INTRODUCTION

Suppose that \((M, g)\) is a smooth, compact, connected Riemann manifold of dimension \(m > 1\). We denote by \(|dV_g|\) the volume density on \(M\) induced by \(g\). For any \(u, v \in C^\infty(M)\) we denote by \((u, v)_g\) their \(L^2\) inner product,

\[ (u, v)_g := \int_M u(x)v(x)|dV_g(x)|. \]

The \(L^2\)-norm of a smooth function \(u\) is then

\[ \|u\| := \sqrt{(u, u)_g}. \]


2000 Mathematics Subject Classification. Primary 15B52, 42C10, 53C65, 58K05, 60D05, 60G15, 60G60.

Key words and phrases. Morse functions, critical points, Kac-Price formula, random matrices, gaussian random processes, spectral function.
Let $\Delta_g : C^\infty(M) \to C^\infty(M)$ denote the scalar Laplacian defined by the metric $g$. For $L > 0$ we set

$$ U_L = U_L(M, g) := \bigoplus_{\lambda \in [0, L]} \ker(\lambda - \Delta_g), \ d(L) := \dim U_L. $$

We equip $U_L$ with the Gaussian probability measure.

$$ d\gamma_L(u) := (2\pi)^{-\frac{d(L)}{2}} e^{-\frac{|u|^2}{2}} |du|. $$

For any $u \in U_L$ we denote by $N_L(u)$ the number of critical points of $u$. If $L$ is sufficiently large then $N_L(u)$ is finite with probability 1. We obtain in this fashion a random variable $N_L$, and we denote by $E(N_L)$ its expectation

$$ E(N_L) := \int_{U_L} N_L(u) d\gamma_L(u). $$

In this paper we investigate the behavior of $E(N_L)$ as $L \to \infty$. More precisely we will prove the following result.

**Theorem 1.1.** There exists a positive constant $C = C(m)$ that depends only on the dimension of $M$, such that

$$ E(N_L) \sim C(m) \dim U_L \ as \ L \to \infty. \quad (1.1) $$

The constant $C(m)$ can be expressed in terms of certain statistics on the space $S_m$ the space of symmetric $m \times m$ matrices. We denote $d\gamma_s$ the Gaussian measure\(^1\) on $S_m$ given by

$$ d\gamma_s(X) = \frac{1}{(2\pi)^{\frac{m(m+1)}{4}}} \sqrt{\mu_m} \cdot e^{-\frac{1}{4}\left(\text{tr} X^2 - \frac{1}{m+2} (\text{tr} X)^2\right)} 2^{\frac{1}{2}(m)} \prod_{i \leq j} dx_{ij}, $$

$$ \mu_m = 2^{\frac{m}{2}+1}(m+2)^{m-1}. $$

Then

$$ C(m) = \left(\frac{4\pi}{m+4}\right)^{\frac{m}{2}} \Gamma\left(1 + \frac{m}{2}\right) \int_{S_m} |\det X| d\gamma_s(X). \quad (1.2) $$

We can say something about the behavior of $C(m)$ as $m \to \infty$.

**Theorem 1.2.**

$$ \log C(m) \sim \log I_m \sim \frac{m}{2} \log m \ as \ m \to \infty. \quad (1.3) $$

The proof of (1.1) relies on an integral formula (2.2) that expresses the expected number of critical points of a function in $U_L$ as an integral

$$ E(N_L) = \int_M \rho_L(x) |dV_g(x)|. $$

Using some basic ideas from random field theory we reduce the large $L$ asymptotics of $\rho_L$ to questions concerning the asymptotics of the spectral function $\mathcal{E}_L$ of the Laplacian, i.e., the Schwartz kernel of the orthogonal projection onto $U_L$. Fortunately, these questions were recently settled by X. Bin [5] by refining the wave kernel method of L. Hörmander, [20].

\(^1\)We refer to Appendix B for a detailed description of a 3-parameter family Gaussian measures $d\Gamma_{a,b,c}$ on $S_m$ that includes $d\gamma_s$ as $d\gamma_s = d\Gamma_{3,1,1}$. 
We actually prove a bit more. We show that
\[
\lim_{L \to \infty} L^{-\frac{m}{2}} \rho_L(x) = \frac{C(m) \omega_m}{(2\pi)^m}, \text{ uniformly in } x \in M, \tag{1.4}
\]
where \(\omega_m\) denotes the volume of the unit ball in \(\mathbb{R}^m\). Using the classical Weyl estimates (3.2) we see that (1.4) implies (1.1).

We obtain the asymptotics of \(C(m)\) by relying on a technique used by Y.V. Fyodorov [17] in a related context. This reduces the asymptotics of the integral \(I_m\) to known asymptotics of the 1-point correlation function in random matrix theory, more precisely, Wigner’s semi-circle law.

The equality (1.4) has an interesting interpretation. We can think of \(\rho_L(x)\) as the expected number of critical points of a random function in \(U_L\) inside an infinitesimal region of volume \(|dV_g(x)|\) around the point \(x\). From this point of view we see that (1.4) states that for large \(L\) we expect the critical points of a random function in \(U_L\) to be approximatively uniformly distributed.

We are inclined to believe that as \(L \to \infty\) the ratio
\[
q_L = \frac{\text{var}(N_L)}{E(N_L)}
\]
has a finite limit \(q(M, g)\). Such a result would show that \(N_L\) is highly concentrated near its mean value as \(L \to \infty\). In Theorem 5.1 we prove that this is the case when \(M = S^1\) and moreover, \(q(S^1) \approx 0.4518\ldots\)

The proof of Theorem 1.1 reveals two additional interesting phenomena. To describe the first one let us first observe that for large \(L\) we have a natural smooth map
\[
M \ni p \xrightarrow{\Phi} \Phi_p \in U_L
\]
uniquely determined by the requirement
\[
(\Phi_p, u)_{L^2(M, g)} = u(p), \quad \forall u \in U_L.
\]
For large \(L\) this map is an embedding, and we denote by \(\sigma_L\) the pullback to \(M\) via \(\Phi\) of the \(L^2\)-metric on \(U_L\). The equality (3.5) in the proof of Theorem 1.1 shows that the rescaled metric \(L^{-\frac{m+2}{4}} \sigma_L\) converges in the \(C^0\) topology to \(K_m g\), where \(g\) is the original metric on \(M\) and \(K_m\) is a certain, explicit constant that depends only on \(m\).

To describe the second rather surprising consequence of the proof of Theorem 1.1 let us denote by \(\text{Hess}_p(u, g)\) the Hessian at \(p\) of the a smooth function \(u\) defined in terms of the Levi-Civita connection \(\nabla^g\), see (2.19). Fix an orthonormal basis of \(T_p M\).

Then the correspondence \(U_L \ni u \mapsto \text{Hess}_p(u, g) \in S_m\) defines a Gaussian random symmetric matrix. The equality (3.10) in the proof of Theorem 1.1 shows that the family of Gaussian symmetric matrices
\[
L^{-\frac{m+4}{2}} \text{Hess}_p(u, g)
\]
converges to a random symmetric matrix \(A_\infty\) whose entries \(a_{ij}\) are centered Gaussian variables satisfying the correlation equalities
\[
E(a_{ii}^2) = 3c_m^2, \quad \forall i
\]
\[
E(a_{ii} a_{jj}) = E(a_{ii}^2) = c_m^2, \quad \forall i < j
\]
\[
E(a_{ij} a_{k\ell}) = 0, \quad \forall i < j, \quad k \leq \ell, \quad (i, j) \neq (k, \ell), \tag{1.5}
\]
where \(c_m\) is a certain, explicit universal constant that depends only on \(m\). Observe that the random matrix \(A_\infty\) is independent of the point \(p\), the choice of metric \(g\) and even on the manifold \(M\)!

More surprisingly, at least to this author, the universal gaussian ensemble described by the random matrix \(A_\infty\) is not the GOE (Gaussian Orthogonal Ensemble) typically investigated in random matrix
theory. Indeed, in the above ensemble the entries $a_{ii}$ and $a_{jj}$, $i \neq j$ are dependent random variables. The entries $a_{ii}$ and $a_{jj}$ do depend on the choice of orthonormal frame of $T_p M$, but the correlation equalities (1.5) are independent of this choice. In Appendix B we describe a 2-parameter family

$$\{ \Gamma_{a,b,c}; \ a, b, c \in \mathbb{R}, \ a = b + 2c \}$$

of $O(m)$-invariant Gaussian ensembles of symmetric $m \times m$ matrices. The usual GOE corresponds to the family $\Gamma_{2c,0,c}$, while the above ensemble corresponds to the family $\Gamma_{3c,c,3c}$.

We want to comment on a setup a bit more general than the one in Theorem 1.1. Suppose that we are given an increasing family $(V_{\nu})_{\nu \geq 0}$ of finite dimensional subspaces of $C^\infty(M)$ with the following properties.

- $\Delta V_{\nu} \subset V_{\nu}, \forall \nu \geq 0$, where $\Delta$ is the Laplace operator of the metric $g$.
- The union of the subspaces $V_{\nu}$ is dense in $C^\infty(M)$ with respect to the usual locally convex topology.

We will refer to such a family as an exhaustion of $C^\infty(M)$. The $L^2(M)$-metric defines natural Euclidean structures and Gaussian probability measures on each of the spaces $V_{\nu}$. We denote by $N(V_{\nu})$ the expected number of critical points of a random function in $V_{\nu}$ and we can ask that an asymptotic estimate of the type (1.1) holds in this more general context. More specifically, we can ask if

$$\log N(V_{\nu}) \sim \log \dim V_{\nu} \ \text{as} \ \nu \to \infty,$$

for any exhaustion of $C^\infty(M)$. In [29] we show that if $M = S^2 \times S^k$, $k \geq 2$, then for any $r \in (0, 1]$ we can find an exhaustion $(V^r_{\nu})$ of $C^\infty(M)$ such that

$$\log N(V^r_{\nu}) \sim r \log \dim V^r_{\nu} \ \text{as} \ \nu \to \infty.$$

The present paper is structured as follows. Section 2 contains the formulation and the proof of the key integral formula (2.2) alluded to above, including several reformulations in the language of random processes. In this section we also present a simple application of this formula to the number of critical points of random spherical harmonics of large degree on $S^2$. This sheds additional light on a recent result of Nazarov and Sodin [26] on the number of nodal domains of random spherical harmonics. More precisely, the inequality (2.40) shows that the expected number $\delta_n$ of zonal domain on $S^2$ of a random harmonic polynomial of large degree $n$ satisfies the upper bound $\delta_n < 0.29 n^2$.

Section 3 contains the proof of the asymptotic estimate (1.1), Section 4 contains the proof of the estimate (1.2), and in Section 5 we investigate the variance of the number of critical points of a random trigonometric polynomial of large degree $\nu$.

For the reader’s convenience we have included in Appendix A a brief survey of the main facts about Gaussian measures and Gaussian processes used throughout the paper, while Appendix B contains a detailed description of a 3-parameter family of Gaussian measures on the space $S_m$ of real, symmetric $m \times m$ matrices. These measures play a central role in the proof of (1.1) and we could not find an appropriate reference for the mostly elementary facts discussed in this appendix.

Appendix C contains the computations of several Gaussian integrals involving random $2 \times 2$ matrices, while Appendix D contains the proofs of some technical results used in Section 5.

**Notations**

(i) For any random variable $\xi$ we denote by $E(\xi)$ and respectively $var(\xi)$ its expectation and respectively its variance.

(ii) $i := \sqrt{-1}$.

(iii) For any Euclidean space $V$, we denote by $S(V)$ the unit sphere in $V$ centered at the origin and by $B(V)$ the unit ball in $V$ centered at the origin.
(iv) We will denote by $\sigma_n$ the “area” of the round $n$-dimensional sphere $S^n$ of radius 1, and by $\omega_n$ the “volume” of the unit ball in $\mathbb{R}^n$. These quantities are uniquely determined by the equalities (see [27, Ex. 9.1.11])

$$\sigma_{n-1} = n\omega_n = 2 \pi^{\frac{n}{2}} \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}$$

where $\Gamma$ is Euler’s Gamma function.

(v) If $V_0$ and $V_1$ are two Euclidean spaces of dimensions $n_0, n_1 < \infty$ and $A : V_0 \to V_1$ is a linear map, then the Jacobian of $A$ is the nonnegative scalar $J(A)$ defined as the norm of the linear map

$$\Lambda^k A : \Lambda^k V_0 \to \Lambda^k V_1, \quad k := \min(n_0, n_1).$$

More concretely, if $n_0 \leq n_1$, and $\{e_1, \ldots, e_{n_0}\}$ is an orthonormal basis of $V_0$, then

$$J(A) = \left(\det G(A)\right)^{1/2},$$

where $G(A)$ is the $n_0 \times n_0$ Gramm matrix with entries

$$G_{ij} = (Ae_i, Ae_j)_{V_1}.$$ If $n_1 \geq n_0$ then

$$J(A) = J(A^\dagger) = \left(\det G(A^\dagger)\right)^{1/2},$$

where $A^\dagger$ denotes the adjoint (transpose) of $A$. Equivalently, if $d\text{Vol}_i \in \Lambda^{n_1} V_i^*$ denotes the metric volume form on $V_1$, and $d\text{Vol}_A$ denotes the metric volume form on $\ker A$, then $J(A)$ is the positive number such that

$$d\text{Vol}_0 = \pm d\text{Vol}_A \wedge A^* d\text{Vol}_1.$$  

2. THE KEY INTEGRAL FORMULA

2.1. A Chern-Lashof type formula. As we mentioned in the introduction key component in the proof of Theorem 1.1 is an integral formula that describes the expected number of critical points as an integral over the background manifold $M$. The goal of this section is to state and prove this formula.

The literature on random fields contains many formulæ of this type, and their proofs follow the strategy pioneered by M. Kac and S. Rice, [22, 33]. In particular, we would like to mention the results of M. Douglas, B. Shiffman, S. Zelditch, [12, 13] where they investigate the number of critical points of a random holomorphic section of a positive holomorphic line bundle over a Kähler manifold.

The general integral formula [12, Thm. 4.4] covers the situation of interest to us, but we prefer to give an independent and detailed description of our concrete problem using a language that is better suited for the applications we have in mind. Moreover, we present an alternate proof that does not follow the usual probabilistic pattern pioneered by Kac and Rice, but instead relies on a basic trick in integral geometry. As a matter of fact, our main integral formula (2.2) contains as special cases the integral formulæ of Chern-Lashof, [9] and Milnor, [24].

Suppose that $M$ is a smooth, compact, connected manifold without boundary. Set $m := \dim M$.

**Definition 2.1.** (a) For any nonnegative integer $k$, any point $p \in M$ and any $f \in C^\infty(M)$ we will denote by $j_k(f, p)$ the $k$-th jet of $f$ at $p$.

(b) Suppose that $U \subset C^\infty(M)$ is a linear subspace. If $k$ is nonnegative integer then we say that $U$ is $k$-ample if for any $p \in M$ and any $f \in C^\infty(M)$ there exists $u \in U$ such that

$$j_k(u, p) = j_k(f, p).$$
Fix a finite dimensional vector space \( U \subset C^\infty(M) \). Set \( N := \dim U \). We have an evaluation map
\[
ev = ev^U : M \rightarrow U' := \text{Hom}(U, \mathbb{R}), \quad p \mapsto ev_p,
\]
where for any \( p \in M \) the linear map \( ev_p : U \rightarrow \mathbb{R} \) is given by
\[
ev_p(u) = u(p), \quad \forall u \in U.
\]
In the remainder of this section we will assume that \( U \) is 1-ample. This implies that the evaluation map \( ev^U \) is an immersion. Moreover, as explained in [28, §1.2], the 1-ampleness condition also implies that almost all functions \( u \in U \) are Morse functions and thus have finite critical sets. For any \( u \in U \) we denote by \( N(u) \) the number of critical points of \( u \).

We fix an inner product \( h = (-,-)_h \) on \( U \). We denote by |−| the resulting Euclidean norm and by |du| the associated volume density on \( U \). We will refer to the pair \( (U,h) \) as the sample space. Using the metric \( h \) we can regard the evaluation map a smooth map \( ev : M \rightarrow U \). We define the expected number of critical points of a function in \( U \) to be the quantity
\[
N(U,h) := \frac{1}{\sigma_{n-1}} \int_{S(U)} N(u) |dA_h(u)| = \int_U N(u) e^{-\frac{|u|^2_h}{2}} |dV_h(u)|,
\]
where \( \sigma_{n-1} \) denotes the ”area” of the unit sphere in \( \mathbb{R}^n \), \( |dA_h| \) denotes the ”area” density on \( S_h(U) \), and \( |dV_h(u)| \) denotes the volume density on \( U \) determined by the metric \( h \). A priori, the expected number of critical points could be infinite, but in any case, it is independent of any choice of metric on \( M \). The space \( U \) equipped with the Gaussian probability measure \( d\gamma_h \) is a probability space. We denote by \( N_U \) the random variable \( U \ni u \mapsto N(u) \in \mathbb{Z} \) so that
\[
N(U,h) = E(N_U, d\gamma_h),
\]
where \( E(−, d\gamma_h) \) denotes the expectation computed with respect to the probability measure \( d\gamma_h \).

The main goal of this section is to express \( N(U,h) \) as the integral of an explicit density on \( M \). To describe this formula it is convenient to fix a metric \( g \) on \( M \). We will express \( N(U,h) \) as an integral
\[
\int_M \rho_g(p) |dV_g(p)|.
\]
The function \( \rho_g \) does depend on \( g \) but the density \( \rho_g(p) |dV_g(p)| \) is independent of \( g \). The concrete description of \( \rho_g(p) \) relies on several fundamental objects naturally associated to the triplet \( (U,h,g) \).

For any \( p \in M \) we set
\[
U_p^0 := \{ u \in U ; \quad du(p) = 0 \}.
\]
The 1-amplessness assumption on \( U \) implies that for any \( p \in M \) the subspace \( U_p^0 \) has codimension \( m \) in \( U \) so that
\[
\dim U_p^0 = N - m.
\]
Denote by \( dA_{S(U_p^0)} \) the area density along \( S(U_p^0) \).

The differential of the evaluation map at \( p \) is a linear map \( A_p : T_pM \rightarrow U \). We will refer to \( A_p \) as the adjunction map and we will denote by \( J_g(p) = J_g(p,U) \) its Jacobian, i.e., the norm of the induced map \( \Lambda^m : \Lambda^m T_pM \rightarrow \Lambda^m U \). Equivalently, if \( (e_1, \ldots, e_m) \) is \( g \)-orthonormal basis of \( T_pM \), then
\[
J_g(p)^2 = \det \left[ ( A_pe_i, A_pe_j )_h \right]_{1 \leq i,j \leq m}.
\]
Since \( ev^U \) is an immersion we have \( J_g(p) \neq 0, \forall x \in M \).
For any \( p \in M \) and any \( u \in U_p^0 \) the Hessian of \( u \) at \( p \) is a well defined symmetric bilinear form on \( T_pM \) that can be identified via the metric \( g \) with a symmetric endomorphism \( \text{Hess}_p(u, g) \) of \( T_pM \). We denote this symmetric endomorphism by \( \text{Hess}_p(u, g) \).

**Theorem 2.2.** If \((U, h)\) is a 1-ample sample space on \( M \), then

\[
N(U, h) = \frac{1}{\sigma_{N-1}} \int_M \frac{1}{J_g(p)} \left( \int_{S(U_p^0)} |\det \text{Hess}_p(v, g)||dA_{S(U_p^0)}(v)| \right) |dV_p(p)|
\]

\[
= (2\pi)^{-\frac{m}{2}} \int_M \frac{1}{J_g(p)} \left( \int_{U_p^0} |\det \text{Hess}_p(u, g)| \frac{e^{-\frac{|u|^2}{2}}}{(2\pi)^{\frac{N-m}{2}}} |dV_h(u)| \right) |dV_p(p)|. \tag{2.2}
\]

**Proof.** Denote by \( U_M \) the trivial vector bundle over \( M \) with fiber \( U \),

\[
U_M := (U \times M \to M).
\]

This bundle is equipped with a canonical trivial connection \( D \). More precisely, if we regard a section of \( U \) of \( U_M \) as a smooth map \( u : M \to U \), then for any vector field \( X \) on \( M \) we define \( D_X u \) as the smooth function \( M \to U \) obtained by derivating \( u \) along \( X \).

For any \( p \in M \) we denote by \( K_p \) the orthogonal complement of \( U_p^0 \) in \( U \). Let us observe that the subspace \( K_p \) coincides with the range of the adjunction map \( A_p \). Indeed, if \((\Psi_n)_{1 \leq n \leq N} \) is an orthonormal basis of \((U, h)\), then

\[
ev_p = \sum_n \Psi_n(p) \Psi_n \in U.
\]

and for any vector field \( X \) on \( M \) we have

\[
A_pX_p = \sum_n (X \Psi_n)_p \Psi_n.
\]

Hence \( u = \sum_n u_n \Psi_n \in U, u_n \in \mathbb{R} \), belongs to \( K_p \) if and only if for any vector field \( X \) on \( M \) we have

\[
0 = \sum_n u_n (X \Psi_n)_p \quad \iff \quad u \in U_p^0.
\]

This proves that the collection \((K_p)\) defines a subbundle \( K \) of \( U_M \) and the adjunction map induces an isomorphism of vector bundle \( A : TM \to K \). We deduce that the collection of spaces \((U_p^0)_{p \in M}\) also forms a vector subbundle \( U^0 \) of the trivial bundle \( U_M \) and we have an orthogonal direct sum decomposition

\[
U_M = U^0 \oplus K.
\]

For any section \( u \) of \( U_M \) we denote by \( u^0 \) its \( U^0 \)-component.

The **shape operator** of the subbundle \( K \) is the bundle morphism \( \Xi : TM \otimes K \to U^0 \) defined by the equality

\[
\Xi(X, u) := (D_X u)^0, \quad \forall X \in C^\infty(TM), \ u \in C^\infty(K).
\]

For every \( p \in M \), we denote by \( \Xi_p : T_p M \otimes K_p \to U_p^0 \) the induced linear map \( \Xi_p : T_p M \otimes K_p \to U_p^0 \). If we denote by \( \text{Gr}_m(U) \) the Grassmannian of \( m \)-dimensional subspaces of \( U \), then we have a Gauss map

\[
M \ni p \overset{\Xi}{\to} \mathcal{S}(p) := K_p \in \text{Gr}_m(U).
\]

The shape operator \( \Xi_p \) can be viewed as a linear map

\[
\Xi_p : T_p M \to \text{Hom}(K_p, U_p^0) = T_{K_p} \text{Gr}_m(U),
\]
and, as such, it can be identified with the differential of $\mathcal{J}$ at $p$, [27, §9.1.2]. Any $v \in U^0_p$ determines a bilinear map
$$\Xi_p \cdot v : T_pM \otimes K_p \to \mathbb{R}, \quad \Xi_p \cdot v(e, u) := (\Xi_p(e, u), v)_h,$$
By choosing orthonormal bases $(e_i)$ in $T_pM$ and $(u_j)$ of $K_p$ we can identify this bilinear form with an $m \times m$-matrix. This matrix depends on the choices of bases, but the absolute value of its determinant is independent of these bases. It is thus an invariant of the pair $(\Xi_p, v)$ that we will denote by $|\det\Xi_p \cdot v|$.

**Lemma 2.3.**

$$\mathcal{N}(U, h) = \frac{1}{\sigma_{N-1}} \int_M \left( \int_{S(U^0_p)} |\det\Xi_p \cdot v| |dA_{S(U^0_p)}(v)| \right) |dV_g(p)|, \quad (2.3)$$

**Proof.** Consider the incidence set
$$\mathcal{I} := \{ (p, v) \in M \times S(U); \quad dv(p) = 0 \} = \{ (x, v) \in M \times S(U); \quad v \in S(U^0_p) \}.$$ We have natural (left/right) smooth projections
$$M \xrightarrow{\lambda} \mathcal{I} \xrightarrow{\rho} S(U).$$
The left projection $\lambda : \mathcal{I} \to M$ describes $\mathcal{I}$ as the unit sphere bundle associated to the metric vector bundle $U^0$. In particular, this shows that $\mathcal{I}$ is a compact, smooth manifold of dimension $(N - 1)$. For generic $v \in S(U)$ the fiber $\rho^{-1}(v)$ is finite and can be identified with the set of critical points of $\nu : M \to \mathbb{R}$. We deduce
$$\mathcal{N}(U, h) = \frac{1}{\text{area}(S(U))} \int_{S(U)} \#\rho^{-1}(v) |dA_{h}(u)|.$$

Denote by $g_\mathcal{I}$ the metric on $\mathcal{I}$ induced by the metric on $M \times S(U)$ and by $|dV_\mathcal{I}|$ the induced volume density. The coarea formula, [8, §13.4], implies that
$$\int_{S(U)} \#\rho^{-1}(v)|dA_{h}(v)| = \int_{\mathcal{I}} J_\rho(p, v)|dV_\mathcal{I}(p, v)|,$$
where the nonnegative function $J_\rho$ is the Jacobian of $\rho$ defined by the equality
$$\rho^*|dA_h| = J_\rho \cdot |dV_\mathcal{I}|.$$ To compute the integral in the right-hand side of (2.5) we need a more explicit description of the geometry of $\mathcal{I}$.

Fix a local orthonormal frame $(e_1, \ldots, e_m)$ of $TM$ defined in a neighborhood $\mathcal{O}$ in $M$ of a given point $p_0 \in M$. We denote by $(e^1, \ldots, e^n)$ the dual co-frame of $T^*M$. Set
$$f_i(p) := A_p e_i(p) \in U, \quad i = 1, \ldots, m, \quad p \in \mathcal{O}.$$ More explicitly, $f_i(u)$ is defined by the equality
$$\left( f_i(p), v \right)_h = \partial_{e_i} u(p), \quad \forall u \in U.$$ (2.6)

Fix a neighborhood $\mathcal{U} \subset \lambda^{-1}(0)$ in $M \times S(U)$ of the point $(p_0, v_0)$, and a local orthonormal frame $u_1(p, v), \ldots, u_{N-1}(p, v)$ over $\mathcal{U}$ of the bundle $\rho^*TS(U) \to M \times S(U)$ such that the following hold.

- The vectors $u_1(p, v), \ldots, u_m(p, v)$ are independent of the variable $v$ and form an orthonormal basis of $K_p^\perp$. (E.g., we can obtain such vectors from the vectors $f_1(p), \ldots, f_m(p)$ via the Gramm-Schmidt process.)
- For $(p, v) \in \mathcal{U}$, the space $T_p E_p$ is spanned by the vectors $u_{m+1}(p, v), \ldots, u_{N-1}(p, v)$.
The collection \( u_1(p), \ldots, u_m(p) \) is a collection of smooth sections of \( U_M \) over \( \emptyset \). For any \( p \in \emptyset \) and any \( e \in T_pM \), we obtain the vectors (functions).

\[
D_e u_1(p), \ldots, D_p u_m(x) \in U,
\]

where we recall that \( D \) denotes the trivial connection on \( U_M \). Observe that

\[
J \cap U = \{(p,v) \in U; \quad U_i(p,v) = 0, \quad \forall i = 1, \ldots, m\},
\]

where \( U_i \) is the function \( U_i : \emptyset \times U \rightarrow \mathbb{R} \) given by

\[
U_i(p,v) := (u_i(p),v)_h.
\]

Thus, the tangent space of \( J \) at \((p,v)\) consists of tangent vectors \( \hat{p} \oplus \dot{v} \in T_xM \oplus T_vS(V) \) such that

\[
dU_i(\hat{p}, \dot{v}) = 0, \quad \forall i = 1, \ldots, m.
\]

We let \( \omega_U \) denote the \( m \)-form

\[
\omega_U := dU_1 \wedge \cdots \wedge dU_m \in \Omega^m(U),
\]

and we denote by \( \|\omega_U\| \) its norm with respect to the product metric on \( M \times S(U) \). Denote by \( |\hat{dV}| \) the volume density on \( M \times S(U) \) induced by the product metric. The equality (2.7) implies that

\[
|\hat{dV}| = \frac{1}{\|\omega_U\|} |\omega_U \wedge dV_E|.
\]

Hence

\[
J_\rho |\hat{dV}| = \frac{1}{\|\omega_U\|} |\omega_U \wedge \rho^*dA|.
\]

We deduce

\[
J_\rho(p_0,v_0) = J_\rho(p_0,v_0) |\hat{dV}|(e_1, \ldots, e_m, u_1, \ldots, u_{N-1})
\]

\[
= \frac{1}{\|\omega_U\|} \big| \omega_U \wedge \rho^*dS|(e_1, \ldots, e_m, u_1, \ldots, u_{N-1}) = \frac{1}{\|\omega_U\|} \omega_U(e_1, \ldots, e_m)|_{(p_0,v_0)}.
\]

Hence,

\[
\int_{S(U)} \#\rho^{-1}(w)|dA_h(v)| = \int_{J} \frac{\Delta_U}{\|\omega_U\|} |dV_3(p,v)|.
\]

**Sublemma 2.4.** We have the equality

\[
J_\lambda = \frac{1}{\|\omega_U\|},
\]

where \( J_\lambda \) denotes the Jacobian of the projection \( \lambda : J \rightarrow M \).

**Proof.** Along \( U \) we have

\[
|\hat{dV}| = \frac{1}{\|\omega_U\|} \big| \omega_U \wedge dV_3 \big|,
\]

while the definition of the Jacobian implies that

\[
|dV_3| = \frac{1}{J_\lambda} |dV_g \wedge dA_{S(U^0_p)}|.
\]

Therefore, it suffices to show that along \( U \) we have

\[
|\hat{dV}| = \big| \omega_U \wedge dV_3 \wedge dA_{S(U^0_p)} \big|,
\]

i.e.,

\[
\big| \omega_U \wedge dV_3 \wedge dA_{S(U^0_p)}(e_1, \ldots, e_m, u_1, \ldots, u_{N-1}) \big| = 1.
\]
Since \(dU_i(u_k) = 0, \forall k \geq m + 1\) we deduce that
\[
|\omega_U \wedge dV_g \wedge dA_{S(U_p^0)}(e_1, \ldots, e_m, u_1, \ldots, u_{N-1})| = |\omega_U(u_1, \ldots, u_m)|.
\]
Thus, it suffices to show that
\[
|\omega_U(u_1, \ldots, u_m)| = 1.
\]
This follows from the elementary identities
\[
dU_i(u_j) = (u_i, u_j)_h = \delta_{ij}, \quad \forall 1 \leq i, j \leq m,
\]
where \(\delta_{ij}\) is the Kronecker symbol.

Using (2.9) in (2.8) and the coarea formula we deduce
\[
\int_{S(U)} \# \rho^{-1}(w)|dA_h(v)| = \int_M \left( \int_{S(U_p^0)} \Delta_U(p, v)|dA_{S(U_p^0)}(v)| \right) |dV_g(p)|. \tag{2.10}
\]
Observe that at a point \((p, v) \in \lambda^{-1}(0) \subset J\) we have
\[
dU_i(e_j) = (D_v u_i(p), v)_h.
\]
We can rewrite this in terms of the shape operator \(\Xi_p : T_p M \otimes K_p \rightarrow U_p^0\). More precisely,
\[
dU_i(e_j) = (\Xi_p(e_j, u_i), v)_h.
\]
Hence,
\[
\Delta_U(x, v) = |\det(\Xi_p(e_j, u_i), v)|_h.
\]
We conclude that
\[
\int_{S_h(U)} \# \rho^{-1}(w)|dA_h(v)| = \int_M \left( \int_{S(U_p^0)} |\det \Xi_p \cdot v||dA_{S(U_p^0)}(v)| \right) |dV_M(p)|.
\]
This proves (2.3)

To proceed further observe that the left-hand side of (2.3) is plainly independent of the metric \(g\) on \(M\). This raises the hope that if we judiciously choose the metric on \(M\) we can obtain a more manageable expression for \(\mu(M, V)\). One choice presents itself. Namely, we choose the metric \(\sigma\), the pullback to \(M\) of the metric on \(V\) via the immersion \(ev : M \rightarrow U\). More concretely, for any \(p \in M\) and any \(X, Y \in T_p M\), we have
\[
\sigma_p(X, Y) = (A_p X, A_p Y)_h.
\]
With this choice of metric, the equality (2.3) is precisely the main theorem of Chern and Lashof, [9].

Fix \(p \in M\) and a \(\sigma\)-orthonormal frame \((e_i)_{1 \leq i \leq m}\) of \(TM\) defined in a neighborhood \(\emptyset\) of \(p\). Then the collection \(u_j = Ae_j, 1 \leq j\), is a local orthonormal frame of \(K|_\emptyset\). The shape operator has the simple description
\[
\Xi_p(e_i, u_j) = (D_v Ae_j)^0.
\]
Fix an orthonormal basis \((\Psi_n)_{1 \leq n \leq N}\) of \(U\) so that every \(v \in U\) has a decomposition
\[
v = \sum_\alpha v_n \Psi_n, \quad v_n \in \mathbb{R}.
\]
Then
\[
A_p e_j(p) = \sum_n (\partial_{e_j} \Psi_n)_p \Psi_n, \quad D_v A^1 e_j(p) = \sum_n (\partial_{e_n e_j} \Psi_n)_p \Psi_n.
\]
and
\[(\langle D_{e_i}A e_j \rangle_p, v \rangle_h = \sum_{\alpha} v_n(\partial^2_{e_i,e_j} \Psi_{\alpha})_p = \partial^2_{e_i,e_j} v(p).\]

If \(v \in U^0_p\), then the Hessian of \(v\) at \(p\) is a well-defined, symmetric bilinear form \(\text{Hess}_p(v)\) on \(T_pM\) that can be identified via the metric \(\sigma\) with a symmetric linear operator
\[\text{Hess}_p(v, \sigma) : T_pM \to T_pM.\]

If we fix a \(\sigma\)-orthonormal frame \((e_i)\) of \(T_pM\), then the operator \(\text{Hess}_p(v, \sigma)\) is described by the symmetric \(m \times m\) matrix with entries \(\partial^2_{e_i,e_j} v(x)\). We deduce that
\[|\det \Xi_p \cdot v| = |\det \text{Hess}_p(v, \sigma)|, \quad \forall v \in S(U^0_p).\]

In particular, we deduce that
\[N(U, h) = \frac{1}{\sigma_{n-1}} \int_M \left( \int_{S(U^0_p)} |\det \text{Hess}_p(v, \sigma)||dA_S(v_p^0)(v)| \right) |dV_p(p)|. \quad \text{(2.11)}\]

Finally, we want to express (2.11) entirely in terms of the adjunction map \(A\). For any \(p \in M\) and any \(v \in U_p\), we define the density
\[\rho_{p,v} : \Lambda^m T_x M \to \mathbb{R},\]
\[\rho_{p,v}(X_1 \wedge \cdots \wedge X_m) = |\det(\partial^2_{X_i,X_j} v(p))_{1 \leq i,j \leq m}| \cdot |\det(\langle A_p X_i, A_p X_j \rangle_h)_{1 \leq i,j \leq m}|^{-1/2}\]
\[= |\det(\text{Hess}_p(v)(X_i, X_j))_{1 \leq i,j \leq m}| \cdot |\det(\sigma(X_i, X_j))_{1 \leq i,j \leq m}|^{-1/2}.\]

Observe that for any \(\sigma\)-orthonormal frame of \(T_x M\) we have
\[\rho_{p,v}(e_1 \wedge \cdots \wedge e_m) = |\det \text{Hess}_p(v, \sigma)|.\]

If we integrate \(\rho_{p,v}\) over \(v \in S(U^0_p)\), we obtain a density
\[|d\mu_U(p)| : \Lambda^m T_p M \to \mathbb{R},\]
\[|d\mu_U(p)|(X_1 \wedge \cdots \wedge X_m) = \int_{S(U^0_p)} \rho_{p,v}(X_1 \wedge \cdots \wedge X_m) |dA_S(v_p^0)(v)|, \quad \forall X_1, \ldots, X_m \in T_p M.\]
Clearly \(|d\mu_U(p)|\) varies smoothly with \(p\), and thus it defines a density \(|d\mu_U(\cdot)|\) on \(M\). We want to emphasize that this density depends on the metric on \(U\) but it is independent of any metric on \(M\). We will refer to it as the density of \(U\). By construction
\[N(U, h) = \frac{1}{\sigma_{n-1}} \int_M |d\mu_U(p)|.\]

If we now return to our original metric \(g\) on \(M\), then we can express \(|d\mu_U(\cdot)|\) as a product
\[|d\mu_U(p)| = \rho_g(p) \cdot |dV_g(p)|,\]
where \(\rho_g = \rho_g_U : M \to \mathbb{R}\) is a smooth nonnegative function.

To find a more useful description of \(\rho_g\), we choose local coordinates \((x^1, \ldots, x^m)\) near \(p\) such that \((\partial_{x^i})\) is a \(g\)-orthonormal basis of \(T_p M\). Then
\[\rho_{p,v}(\partial_{x_1} \wedge \cdots \wedge \partial_{x_m}) = |\det(\partial^2_{x_i,x_j} v(p))_{1 \leq i,j \leq m}| \cdot |\det(\langle A \partial_{x_i}, A \partial_{x_j} \rangle_h)_{1 \leq i,j \leq m}|^{-1/2}.\]

Observe that the matrix \((\partial^2_{x_i,x_j} v(p))_{1 \leq i,j \leq m}\) describes the Hessian operator
\[\text{Hess}_p(v, g) : T_p M \to T_p M\]
induced by the Hessian of \(v\) at \(p\) and the metric \(g\).
The scalar \( \left( \det A \partial_x A \partial_x \right)_{1 \leq i, j \leq m} \right)^{1/2} \) is precisely the Jacobian \( J_g(p) \) of the adjunction map \( A_p : T_pM \to U \) defined in terms of the metric \( g \) on \( T_pM \) and the metric \( h \) on \( U \). We set

\[
\Delta_x(V, g) := \int_{S(U_p)} |\det \text{Hess}_x(v, g)| |dA_{S(U_p)}(v)|.
\]

Since

\[
|dV_g(p)|(\partial x_1 \wedge \cdots \wedge \partial x_m) = 1,
\]

we deduce

\[
\rho_g(V) = \Delta_p(V, g) : J_g(p)^{-1}. \tag{2.12}
\]

This proves the first equality in (2.2). The second equality follows from the first by invoking (A.6) and the explicit formula \((\sigma)\) for \( \sigma_{N-1} \). \( \square \)

**Remark 2.5 (A Gauss-Bonnet type formula).** With a little care, the above arguments lead to a Gauss-Bonnet type theorem. More precisely, if we assume that \( M \) is oriented, then, under appropriate orientation conventions, the Morse inequalities imply that the degree of the map \( \rho : J \to S_h(U) \) is equal to the Euler characteristic of \( M \). If instead of working with densities, we work with forms, then we conclude that

\[
\chi(M) = (2\pi)^{-1} \int_M \frac{1}{\rho_g(p)} \left( \int_{U_p} \det \text{Hess}_g(u, g) \frac{e^{-|u|^2_{g}}}{(2\pi)^{n-m}} |dV_g(u)| \right) |dV_g(p)|. \tag{2.13}
\]

When \( M \) is a submanifold of the Euclidean space \( U \), and \( g = \sigma \), then the above argument yields the Gauss-Bonnet theorem for submanifolds of a Euclidean space. \( \square \)

**Remark 2.6.** The equality (2.2) can be used in some instances to compute the variance of the number of critical points of a random function in \( U \). More precisely, to a sample space \((U, h) \subset C^\infty(M)\) we associate a sample space \((U^\Delta, h_\Delta) \subset C^\infty(M \times M)\), the image of \( V \) via the diagonal injection

\[
\mathcal{D} : C^\infty(M) \to C^\infty(M \times M), \quad u \mapsto u^\Delta,
\]

\[
u^\Delta(x, y) = u(x) + u(y), \quad \forall u \in U, \; x, y \in M.
\]

The metric \( h_\Delta \) is defined by requiring that the map \( \mathcal{D} : (U, h) \to (U^\Delta, h_\Delta) \) is an isometry.

There is a slight problem. The sample space \( U^\Delta \) is never 1-ample, no matter how large we choose \( U \). More precisely, the ampltensess is always violated along the diagonal \( \Delta_M \subset M \times M \). We denote by \( U^\Delta_{M_2^3} \) the space of restrictions to \( M_2^3 : M \times M \setminus \Delta_M \) of the functions in \( U^\Delta \). Note that for any \( u \in U \) we have

\[
N_{U^\Delta_{M_2^3}}(u) = N(u^\Delta |_{M_2^3}) = N(u)^2 - N(u).
\]

If \( U \) is sufficiently large, then the sample space \( V^\Delta_{U^\Delta} \) is 1-ample and we deduce that

\[
E(N_{U^\Delta_{M_2^3}} - N_U, d\gamma_h) = E(N_{U^\Delta_{M_2^3}}, d\gamma_{h_\Delta}). \tag{M2}
\]

The quantity in the left-hand side of \((M2)\) is the so called second combinatorial momentum of the random variable \( N_U \). Using (2.2) we can express the right-hand side of \((M2)\) as an integral of a density over \( M_2^3 \). Since \( U^\Delta \) fails to be 1-ample along the diagonal so that the corresponding Jacobian term vanishes along the diagonal. We deduce that

\[
E(N_{U^\Delta_{M_2^3}} - N_U, d\gamma_h)
\]
2.2. A Gaussian random field perspective. The formula (2.2) looks hopeless for two immediately visible reasons.

- The Jacobian $J_g(p)$ seems difficult to compute.
- The integral $I_p$ in (2.2) may be difficult to compute since the domain of integration $U_p^0$ may be impossible to pin down.

We will deal with these difficulties simultaneously by relying on some probabilistic principles inspired from [1]. For the reader’s convenience we have gathered in Appendix A the basic probabilistic notions and facts need in the sequel.

Consider again the metric $\sigma = \sigma_{U}$, the pullback of the metric $h$ on $U$ via the evaluation map. We will refer to it as the stochastic metric associated to the sample space $(U, h)$. It is convenient to have a local description of the stochastic metric.

Fix an orthonormal basis $\psi_1, \ldots, \psi_N$ of $U$. The evaluation map $\text{ev}^U : M \to U$ is then given by

$$M \ni x \mapsto \sum_n \psi_n(x) \cdot \psi_n \in U.$$  

If $p \in M$ and $\mathbf{u}$ is an open coordinate neighborhood of $p$ with coordinates $x = (x^1, \ldots, x^m)$ then

$$\sigma_p(\partial_{x^i}, \partial_{x^j}) = \sum_n \frac{\partial \psi_n}{\partial x^i}(p) \frac{\partial \psi_n}{\partial x^j}(p), \quad \forall 1 \leq i, j \leq m. \quad (2.14)$$

Note that if the collection $(\partial_{x^i})_{1 \leq i \leq m}$ forms an $g$-orthonormal frame of $T_PM$ then

$$J_g(p)^2 = \det \left[ \sigma_p(\partial_{x^i}, \partial_{x^j}) \right]_{1 \leq i, j \leq m}. \quad (2.15)$$

To the sample space $(U, h)$ we associate in a tautological fashion a Gaussian random field on $M$ as follows. The measure $d\gamma_h$ in (2.1) is a probability measure and thus $(U, d\gamma_h)$ is naturally a probability space. We have a natural map

$$\xi : M \times U \to R, \quad M \times U \ni (p, \mathbf{u}) \mapsto \xi_p(\mathbf{u}) := \mathbf{u}(p)$$

The collection of random variables $\{\xi_p\}_{p \in M}$ is a Gaussian random field on $M$.

Using the orthonormal basis $(\psi_k)_k$ of $U$ we obtain a linear isometry

$$\mathbb{R}^N \ni \mathbf{t} = (t_1, \ldots, t_n) \mapsto \mathbf{u}_t = \sum_k t_k \psi_k \in U,$$

with inverse $\mathbf{u} \mapsto t_k(\mathbf{u}) = h(\mathbf{u}, \psi_k)$. For any $p \in M$ and any $\mathbf{t} \in \mathbb{R}^N$ we have

$$\xi_p(\mathbf{u}_t) = \sum_k t_k \psi_k(p).$$
The covariance kernel of this field is the function \( \mathcal{E} = \mathcal{E}_U : M \times M \to \mathbb{R} \) given by
\[
\mathcal{E}(p, q) = E(\xi_p, \xi_q) = \sum_{j,k=1}^{\infty} \left( \int_{\mathbb{R}^N} t_j t_k d\gamma_N(t) \right) \psi_j(p) \psi_k(q),
\]
(2.16)
where \( d\gamma_N \) is the canonical Gaussian measure on \( \mathbb{R}^N \).

If \( p \in M \) and \( U \) is an open coordinate neighborhood of \( p \) with coordinates \( x = (x^1, \ldots, x^m) \) such that \( x(p) = 0 \), then we can rewrite (2.14) in terms of the covariance kernel alone
\[
\sigma_p(\partial_{x^i}, \partial_{x^j}) = \frac{\partial^2 E(x,y)}{\partial x^i \partial y^j}|_{x=y=0}.
\]
(2.17)

Note that any vector field \( X \) determines a new Gaussian random field on \( M \), the derivative of \( u \) along \( X \). We obtain Gaussian random variables
\[
u \mapsto (Xu)_p, \quad u \mapsto (Yu)_p,
\]
and we have
\[
\sigma_p(X,Y) = E( (Xu)_p, (Yu)_p ).
\]
(2.18)

The last equality justifies the attribute stochastic attached to the metric \( \sigma \).

We denote by \( \nabla \) the Levi-Civita connection of the metric \( g \). The Hessian of a smooth function \( f : \rightarrow \mathbb{R} \) with respect to the metric \( g \) is the symmetric \((0,2)\)-tensor \( \nabla^2 f \) on \( M \) defined by the equality
\[
\nabla^2 f(X,Y) := XY f - (\nabla_X Y) f, \quad \forall X,Y \in \text{Vect}(M).
\]
(2.19)

If \( p \) is a critical point of \( f \) then \( \nabla^2_p f \) is the usual Hessian of \( f \) at \( p \). More generally, if \((x^1, \ldots, x^m)\) are \( g \)-normal coordinates at \( p \) then
\[
\nabla^2_p \left( \partial_{x^i}, \partial_{x^j} \right) = \partial_{x^i x^j} f(p), \quad \forall 1 \leq i, j \leq m.
\]

For any \( p \in M \) and any \( f \in C^\infty(M) \) we use the metric \( g_p \) to identify the bilinear form \( \nabla^2_p f \) on \( T_p M \) with an element of \( S(T_p M) \), the vector space of symmetric endomorphisms of the Euclidean space \( (T_p M, g_p) \). For any \( p \in M \) we have two random Gaussian vectors
\[
U \ni u \mapsto \nabla^2_p u \in S(T_p M), \quad U \ni u \mapsto du(p) \in T_x^* M.
\]

Note that the expectation of both random vectors are trivial while (2.17) shows that the covariance form of \( du(p) \) is the metric \( \sigma_p \).

To proceed further we need to make an additional assumption on the sample space \( U \). Namely, in the remainder of this section we will assume that it is \( 2 \)-ample. In this case the map
\[
U \ni u \mapsto \nabla^2_p u \in S(T_p M)
\]
is surjective so the Gaussian random vector \( \nabla^2_p u \) is nondegenerate. A simple application of the co-area formula shows that the integral \( I_p \) in (2.2) can be expressed as a conditional expectation
\[
I_p = E \left( \left| \det \nabla^2_p u \right| \bigg| du(p) = 0 \right).
\]

The covariance form of the pair of random variables \( \nabla^2_p u \) and \( du(p) \) is the bilinear map
\[
\Omega : S(T_p M)^\vee \times T_p M \to \mathbb{R},
\]
\[
\Omega(\xi, \eta) = E \left( \langle \xi, \nabla^2_p u \rangle \cdot \langle du, \eta \rangle \right), \quad \forall \xi \in S_m^\vee, \ \eta \in T_p M.
\]
Using the natural inner products on $S(T_P M)$ and $T_P M$ defined by $g_P$ we can regard the covariance form as a linear operator

$$\Omega_P : T_P M \to S(T_P M).$$

Similarly, we can identify the covariance forms of $\nabla_P^2 u$ and $d u$ with symmetric positive definite operators

$$S_{\nabla_P^2 u} : S(T_P M) \to S(T_P M)$$

and respectively

$$S_{d u(p)} : T_P M \to T_P M.$$

Using the regression formula (A.4) we deduce that

$$E\left( | \det \nabla_P^2 u | \right) = E( | \det Y_p |), \quad (2.20)$$

where $Y_p : U \to S(T_P M)$ is a Gaussian random vector with mean value zero and covariance operator

$$\Xi_p = \Xi_{\nabla_P} := S_{\nabla_P^2 u} - \Omega S_{d u(p)}^{-1} \Omega^T : S(T_P M) \to S(T_P M). \quad (2.21)$$

Since $U$ is 2-ample the operator $\Xi_P$ is invertible and we have

$$E( | \det Y_p |) = (2\pi)^{-\dim S(T_P M)/2} (\det \Xi_P)^{-\frac{1}{2}} \int_{S(T_P M)} | \det Y | e^{-\frac{(\Xi_P^{-1}) Y \cdot Y}{2}} dV_Y(Y). \quad (2.22)$$

Observing that

$$J_g(p) = (\det S_{d u(p)})^{\frac{1}{2}}, \quad (2.23)$$

we deduce that when $U$ is 2-ample we have

$$\mathcal{N}(U, h) = \frac{1}{(2\pi)^{m/2}} \int_M (\det S_{d u(p)})^{-\frac{1}{2}} E( | \det Y_p |) | dV_g(p) |, \quad (2.24)$$

where $Y_p$ is a Gaussian random symmetric endomorphism of $T_P M$ with expectation 0 and covariance operator $\Xi_P$ described by (2.21).

To compute the above integral we choose normal coordinates $(x^1, \ldots, x^n)$ near $p$ and thus we can orthogonally identify $T_P M$ with $\mathbb{R}^m$. We can view the random variable $\nabla_P^2 u$ as a random variable

$$H_P : U \to S_m := S(\mathbb{R}^n), \quad U \ni u \mapsto H_P(u) \in S_m, \quad H_P(u) = \partial^2_{x \times} u(p),$$

and the random variable $d u(p)$ as a random variable

$$D_P : U \to \mathbb{R}^m, \quad u \mapsto D_P u \in \mathbb{R}^m, \quad D_P u = \partial_{x_1} u(p).$$

The covariance operator $S_{d u(p)}$ of the random variable $D_P$ is given by the symmetric $m \times m$ matrix with entries

$$\sigma_p(\partial_{x^1}, \partial_{x^j}) = \frac{\partial^2 E(x, y)}{\partial x^i \partial y^j} \big|_{x=y=0}. \quad (2.25)$$

To compute the covariance form $\Sigma_{HP}$ of the random matrix $H_P$ we observe first that we have a canonical basis $(\xi_{ij})_{1 \leq i \leq j \leq m}$ of $S_m$ so that $\xi_{ij}$ associates to a symmetric matrix $A$ the entry $a_{ij}$ located in the position $(i, j)$. Then

$$\Sigma_{HP}(\xi_{ij}, \xi_{kl}) = E( H_{ij}^P(u), H_{kl}^P(u) ) = E\left( \partial^2_{x \times} u(x) \partial^2_{x \times} u(x) \right)$$

$$= \sum_{n=1}^N \partial^2_{x \times} \psi_n(x) \partial^2_{x \times} \psi_n(x) = \frac{\partial^4 E(x, y)}{\partial x^i \partial x^j \partial y^k \partial y^l} \big|_{x=y=0}. \quad (2.26)$$

Similarly we have

$$\Omega(\xi_{ij}, \partial_{x_k}) = E\left( \partial^2_{x \times} u(p), \partial_{x_k} u(p) \right) = \frac{\partial^3 E(x, y)}{\partial x^i \partial x^j \partial y^k} \big|_{x=y=0}. \quad (2.27)$$
To identify $\Omega$ with an operator it suffices to observe that $(\partial_{x^k})$ is an orthonormal basis of $T_p M$, while the collection $\{ \xi_{ij} \}_{i,j \leq n} \subset S^\vee_m$, 
\[ \xi_{ij} = \begin{cases} 
\xi_{ij}, & i = j \\
\sqrt{2}\xi_{ij}, & i < j 
\end{cases} \]
is an orthonormal basis of $S^\vee_m$. If we denote by $\hat{E}_{ij}$ the dual orthonormal basis of $S_m$, then 
\[ \Omega \partial_{x^k} = \sum_{i,j \leq n} \Omega(\xi_{ij}, \partial_{x^k}) \hat{E}_{ij}. \]

**Remark 2.7.** If the metric $g$ coincides with the stochastic metric $\sigma$, then the covariance operator $\Omega$ is trivial. For a proof of this and of many other nice properties of the metric $\sigma$ we refer to [1, §12.2]. □

### 2.3. Zonal domains of spherical harmonics of large degree.

In the conclusion of this section we want to discuss an application of the above results to critical sets of random spherical harmonics.

Let $(M, g)$ be the unit round sphere $S^2$. The spectrum of the Laplacian on $S^2$ is 
\[ \lambda_n = n(n + 1), \quad n = 0, 1, 2, \ldots, \quad \dim \ker(\lambda_n - \Delta) = 2n + 1 = d_n. \]
The space $U_n = \ker(\lambda_n - \Delta)$ has a well known description: it consists of spherical harmonics, i.e., restrictions to $S^2$ of harmonic polynomials of degree $n$ in three variables. We want to describe the behavior of $N(U_n)$ as $n \to \infty$, where $U_n$ is equipped with the $L^2$-metric. In other words we want to find the expected number of critical points of a spherical harmonic of very large degree.

In this case the covariance kernel $E_n(p, q)$ of $U_n$ has a very simple description. More precisely, if $(\Psi_k)_{1 \leq k \leq 2n + 1}$ is an orthonormal basis of $U_n$, then the classical addition theorem, [25, §1.2] shows that 
\[ E_n(p, q) = \sum_k \Psi_k(p)\Psi_k(q) = \frac{2n + 1}{4\pi} P_n(p \cdot q), \quad \forall p, q \in S^2, \]
where $\cdot$ denotes the inner product in $\mathbb{R}^3$, and $P_n$ denotes the $n$-th Legendre polynomial, 
\[ P_n(t) = (-1)^n \frac{1}{2^n n!} \frac{d^n}{dt^n}(1 - t^2)^n. \]
In this case the stochastic metric $\sigma = \sigma_n$ is obviously $SO(3)$-invariant and it is a (constant) multiple of the round metric. In view of Remark 2.7 this implies that for any $p \in S^2$ the random variables 
\[ U_n \ni u \mapsto \text{Hess}_p(u, g) \quad \text{and} \quad U_n \ni u \mapsto d\text{u}(p) \]
are independent and we deduce that 
\[ N(U_n) = \frac{1}{2\pi} \int_{S^2} \frac{1}{J_g(p)} \left( \int_{U_n} \left| \det \text{Hess}_p(u, g) \right| \frac{e^{-\frac{1}{2}|u|^2}}{\left(2\pi\right)^{\frac{d_m}{2}} |u|} \right) \ dV_g(p). \]
Clearly, the integrand in the above formula is invariant with respect to the $SO(3)$-action on $S^2$ and we thus have 
\[ N(U_n) = \frac{2}{J_g(p_0)} \int_{U_n} \left| \det \text{Hess}_{p_0}(u, g) \right| \ d\gamma_n(u), \tag{2.28} \]
where $p_0$ a fixed (but arbitrary) point on $S^2$. To compute the term in the right-hand side of the above equality we use the equalities (2.25) and (2.26).
We deduce that at \( p \in \mathcal{C} \) is described by a smooth function
\[
\mathcal{C} \ni (x^1, x^2) \mapsto p(x^1, x^2) \in \mathbb{R}^3.
\]
Observe that the tangent vector \( \partial_{x^i} \), viewed as a vector in \( \mathbb{R}^3 \), corresponds with the derivative \( p_{x^i} := \partial_{x^i}p \) of the above function. At \( p_0 \) we have
\[
p_{x^i} \cdot p_{x^j} = \delta_{ij} \quad \text{and} \quad p_{x^i} \cdot p_0 = 0, \ \forall i, j. \tag{2.29}
\]
The arcs \( C_1 = \{ x^2 = 0 \} \) and \( C_2 = \{ x^1 = 0 \} \) are portions of great circles intersecting orthogonally at \( p_0 \). Note that \( x^1 \) is the arclength parameter along \( C_i, i = 1, 2 \). The vectors \( p_{x^i} \) are unit tangent vectors along these arcs. This shows that at \( p_0 \) we have
\[
p_{x^i \times x^i} = -p_0.
\]
Since the arcs \( C_1 \) and \( C_2 \) are planar their torsion is trivial and the Frenet formulæ imply that at \( p_0 \) we have
\[
p_{x^i \times x^j} = 0, \ \forall i \neq j.
\]
The last two equalities can be rewritten in compact form as
\[
p_{x^i \times x^j} = -\delta_{ij}p_0, \ \forall i, j \tag{2.30}
\]
We set
\[
\begin{align*}
\sigma_{(\partial_{x^j}, \partial_{x^k})} &= \partial_{x^j} \partial_{x^k} \mathcal{E}(p, q)|_{p=q=p_0} \\
&= \frac{2n + 1}{4\pi} \left( P_n'(p \cdot q)p_{x^j} \cdot q_{y^k} + P_n^{(2)}(p \cdot q)(p_{x^j} \cdot q)(p \cdot q_{y^k}) \right)_{p=q=p_0} \\
&= s_n \delta_{jk},
\end{align*}
\]
and
\[
J_g(p_0) = s_n. \tag{2.33}
\]
To compute \( \partial_{x^2 \times x^2 y^i y^j} \mathcal{E}(p, q) \) at \( p = q = p_0 \) we will use (2.29) and (2.30) to cut down the complexity of the final formula. We deduce that at \( p = q = p_0 \) we have
\[
\begin{align*}
\partial_{x^2 \times x^2 y^i y^j} \mathcal{E}_n(p, q) &= \frac{2n + 1}{4\pi} \left( P_n'(p \cdot q)p_{x^2 \times x^2} \cdot q_{y^i y^j} + P_n^{(2)}(p \cdot q)(p_{x^2 \times x^2} \cdot q)(p \cdot q_{y^i y^j}) \right)_{p=q} \\
&+ \frac{2n + 1}{4\pi} \left( P_n^{(2)}(p \cdot q)(p_{x^2} \cdot q_{y^i})(p_{x^2} \cdot q_{y^j}) + P_n^{(2)}(p \cdot q)(p_{x^2} \cdot q_{y^j})(p_{x^2} \cdot q_{y^i}) \right)_{p=q},
\end{align*}
\]
and thus
\[
\partial_{x^2 \times x^2 y^i y^j} \mathcal{E}_n(p, q)_{p=q} = (s_n + t_n) \delta_{ij} \delta_{kl} + t_n \left( \delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl} \right) \tag{2.34}
\]
Denote by \( d\Gamma_n \) the pushforward of the Gaussian measure \( d\gamma_n \) via the Hessian map
\[
U_n \ni u \mapsto \text{Hess}_{p_0}(u, g) \in S(T_{p_0}S^2) = S_2.
\]
We deduce from (2.34) that the covariance form \( \Sigma_n \) of \( d\Gamma_n \) satisfies the equality
\[
\Sigma_n = \Sigma_{a_n, b_n, c_n}, \ \ a_n = s_n + 3t_n, \ \ b_n = s_n + t_n, \ \ c_n = t_n,
\]
where \( \Sigma_{a,b,c} \) is defined by the conditions (B.2a) and (B.2b). Observe that \( a_n, b_n, c_n \) satisfy (B.4),
\[
a_n = b_n + 2c_n.
\]
As explained in Appendix B, this implies that $d\Gamma_n$ is $O(2)$-invariant. Set

$$a_n^* = \frac{a_n}{t_n}, \quad b_n^* = \frac{b_n}{t_n}, \quad c_n^* = \frac{c_n}{t_n},$$

and denote by $d\Gamma_n^*$ the Gaussian measure on $S_2$ with covariance matrix $\Sigma_{a_n^*, b_n^*, c_n^*}$. Using (A.7) we deduce that

$$\int_{S^2} |\det X| \, d\Gamma_n(X) = t_n \int_{S^2} |\det X| \, d\Gamma_n^*(X).$$

From (2.28) and (2.33) we now deduce

$$N(U_n) = \frac{2t_n}{s_n} \int_{S^2} |\det X| \, d\Gamma_n^*(X).$$

Observe that as $n \to \infty$ we have

$$\frac{2t_n}{s_n} \sim \frac{n^2}{4}, \quad a_n^* \sim 3, \quad b_n^* \sim 1, \quad c_n^* \sim 1,$$

so that

$$N(U_n) \sim \frac{n^2}{4} \int_{S^2} |\det X| \, d\Gamma_{3,1,1}(X),$$

where $d\Gamma_{3,1,1}(X)$ is the Gaussian measure on $S_2$ with covariance form $\Sigma_{3,1,1}$. More precisely (see (B.11))

$$d\Gamma_{3,1,1}(X) = \frac{1}{4(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{4} \left( \text{tr} X^2 - \frac{1}{4} (\text{tr} X)^2 \right)} \cdot \sqrt{2} \prod_{1 \leq i \leq j \leq 2} dx_{ij}.$$

In Appendix C we show that

$$\int_{S^2} |\det X| \, d\Gamma_{3,1,1}(X) = \frac{4}{\sqrt{3}},$$

and we deduce from (2.28) that

$$N(U_n) \sim \frac{n^2}{\sqrt{3}} \text{ as } n \to \infty.$$  \hspace{1cm} (2.38)

Let us observe that for $n$ very large, a typical spherical harmonic $u \in U_n$ is a Morse function on $S^2$ and $0$ is a regular value. The nodal set $\{u = 0\}$ is disjoint union of smoothly embedded circles. We denote by $D_u$ the set of connected components of the complement of the nodal set are called the nodal domains of $u$ and we denote $\delta(u)$ the cardinality of $D_u$. A result of Pleijel and Peetre, [4, 30, 32], shows that

$$\delta(u) \leq \frac{4}{j_0} n^2 \approx 0.692 n^2,$$ \hspace{1cm} (2.39)

where $j_0$ denotes the first positive zero of the Bessel function $J_0$.

We think of $\delta(u)$ as a random variable and we denote by $\delta_n$ its expectation,

$$\delta_n = \frac{1}{(2\pi)^{\dim U_n}} \int_{U_n} \delta(u) e^{-\frac{1}{2} |u|^2} \, du.$$  \hspace{1cm} (2.40)

Recently, Nazarov and Sodin [26], have proved that there exists a positive constant $a > 0$ such that

$$\delta_n \sim an^2 \text{ as } n \to \infty.$$  \hspace{1cm} (2.41)

Additionally, for large $n$, with high probability, $\delta(u)$ is close to $an^2$ (see [26] for a precise statement).

The equality (2.39) implies that

$$a \leq \frac{4}{j_0} \approx 0.692.$$  \hspace{1cm} (2.42)

We can improve this a little bit.
Denote by \( p(u) \) the number of local minima and maxima of \( u \), and by \( s(u) \) the number of saddle points. Then

\[
N(y) = p(u) + s(u), \quad p(u) - s(u) = \chi(S^2) = 2.
\]

This proves that

\[
p(y) = \frac{1}{2}(N(u) + 2).
\]

For every nodal region \( D \), we denote by \( p(u, D) \) the number of local minima and maxima\(^2\) of \( u \) on \( D \). Note that \( p(u, D) > 0 \) for any \( D \) and thus the number \( p(u) = \sum_{D \in D_u} p(u, D) \) can be viewed as a weighted count of nodal domains. We set

\[
p(U_n) := \frac{1}{(2\pi)^{\dim U_n}} \int_{U_n} e^{-\frac{1}{2}|u|^2} p(u) \, |du|.
\]

The equality (2.38) implies that

\[
p(U_n) \sim \frac{1}{2\sqrt{3}} n^2 \text{ as } n \to \infty.
\]

This shows that

\[
a \leq \frac{1}{2\sqrt{3}} \approx 0.288. \tag{2.40}
\]

**Remark 2.8.** Using (2.13) we deduce in a similar fashion a stochastic Gauss-Bonnet formula

\[
2 = \chi(S^2) = \frac{2}{s_n} \int_{U_n} \det \text{Hess}_{p_0}(u, g) \, d\gamma_n(u).
\]

In Appendix C we give a direct proof of this equality to test the correctness of the various normalization constants in the above computations. \qed

### 3. The Proof of Theorem 1.1

We fix an orthonormal basis of \( L^2(M, g) \) consisting of eigenfunctions \( \Psi_n \) of \( \Delta_g \),

\[
\Delta_g \Psi_n = \lambda_n \Psi_n, \quad n = 0, 1, \ldots, \quad \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots.
\]

The collection \( (\Psi_n)_{\lambda_n \leq L} \) is therefore an orthonormal basis of \( U_L \) so that the covariance kernel of the Gaussian field determined by \( U_L \) is

\[
\mathcal{E}_L(p, q) = \sum_{\lambda_n \leq L} \Psi_n(p) \Psi_n(q).
\]

This function is also known as the *spectral function* associated to the Laplacian. Observe that

\[
\int_M \mathcal{E}_L(p, p) \, |dV_g(p)| = \dim U_L.
\]

In the ground breaking work [20], L. Hörmander used the kernel of the wave group \( e^{it\sqrt{\Delta}} \) to produce refined asymptotic estimates for the spectral function. More precisely he showed (see [20] or [21, §17.5])

\[
\mathcal{E}_L(p, p) = \frac{\omega_m}{(2\pi)^m} L^m + O(L^{m-rac{1}{2}}) \quad \text{as } \quad L \to \infty, \tag{3.1}
\]

\(^2\)A simple application of the maximum principle shows that on each nodal domain, all the local extrema of \( y \) are of the same type: either all local minima or all local maxima. Thus \( p(u, D) \) can be visualized as the number of peaks of \( |u| \) on \( D \).
uniformly with respect to \( p \in M \). Above, \( \omega_m \) denotes the volume of the unit ball in \( \mathbb{R}^m \). This implies immediately the classical Weyl estimates

\[
\dim U_L \sim \frac{\omega_m}{(2\pi)^m} \text{vol}_g(M)L^m. \tag{3.2}
\]

Recently (2004), X. Bin \cite{5} used Hörmander’s approach to produce substantially refined asymptotic estimates of the behavior of the spectral function in a infinitesimal neighborhood of the diagonal.

To formulate these estimates we set \( \lambda := L^{\frac{1}{2}} \). Fix a point \( p \) and normal coordinates \( x = (x^1, \ldots, x^m) \) at \( p \). Note that \( x(p) = 0 \). For any multi-indices \( \alpha, \beta \in \mathbb{Z}_+^m \) we set

\[
\mathcal{E}_{L^p}^{\alpha,\beta}(p) := \left. \frac{\partial^{\alpha+\beta} \mathcal{E}(x,y)}{\partial x^\alpha \partial y^\beta} \right|_{x=y=0}
\]

X. Bin proved that for any multi-indices \( \alpha, \beta \in \mathbb{Z}_+^m \) we have

\[
\mathcal{E}_{L^p}^{\alpha,\beta}(p) = C_m(\alpha, \beta) \lambda^{m+|\alpha|+|\beta|} + O\left( \lambda^{m+|\alpha|+|\beta|} \right)
\]

where

\[
C_m(\alpha, \beta) = \begin{cases} 
0, & \alpha - \beta \notin (2\mathbb{Z})^m \\
\frac{(-1)^{|\alpha|-|\beta|}}{(2\pi)^m} \int_{B^m} x^\alpha |dx|, & \alpha - \beta \in (2\mathbb{Z})^m,
\end{cases} \tag{3.4}
\]

and \( B^m \) denotes the unit ball

\[
B^m = \{ x \in \mathbb{R}^m; \ |x| = 1 \}.
\]

The estimates (3.3) are uniform in \( p \in M \). Using (A.6) we deduce (compare with (B.13) )

\[
\frac{1}{(2\pi)^m} \int_{B^m} x^\alpha |dx| = \frac{1}{(4\pi)^{\frac{m}{2}}} \frac{1}{\Gamma\left(1 + \frac{|\alpha|+|\beta|}{2}\right)} \int_{\mathbb{R}^m} x^\alpha e^{-|x|^2} \pi^\frac{m}{2} |dx|.
\]

We set

\[
K_m = C_m(\alpha, \alpha), \quad |\alpha| = 1,
\]

so that

\[
K_m = \frac{1}{(4\pi)^{\frac{m}{2}}} \frac{1}{\Gamma\left(2 + \frac{m}{2}\right)} \int_{\mathbb{R}^m} x_1 e^{-\frac{|x|^2}{\pi}} |dx| = \frac{1}{2(4\pi)^\frac{m}{2}} \frac{1}{\Gamma\left(2 + \frac{m}{2}\right)}.
\]

For any \( i \leq j \) define \( \alpha_{ij} \in \mathbb{Z}^m \) so that

\[
x^{\alpha_{ij}} = x_i x_j.
\]

For \( i \leq j \) and \( k \leq j \) we set

\[
C_m(i, j; k, \ell) = C_m(\alpha_{ij}, \alpha_{k\ell}) = \frac{1}{(4\pi)^{\frac{m}{2}}} \frac{1}{\Gamma\left(3 + \frac{m}{2}\right)} \int_{\mathbb{R}^m} x_i x_j x_k x_\ell e^{-\frac{|x|^2}{\pi}} |dx|.
\]

For \( i < j \) we have

\[
C_m(i, i; j, j) = \frac{1}{(4\pi)^{\frac{m}{2}}} \frac{1}{\Gamma\left(3 + \frac{m}{2}\right)} \int_{\mathbb{R}^m} x_i^2 x_j^2 e^{-\frac{|x|^2}{\pi}} |dx| = \frac{1}{4(4\pi)^{\frac{m}{2}}} \frac{1}{\Gamma\left(3 + \frac{m}{2}\right)} =: c_m.
\]

\[
C_m(i, j; i, j) = C_m(i, i; j, j),
\]

Finally

\[
C_m(i, i; i, i) = \frac{1}{(4\pi)^{\frac{m}{2}}} \frac{1}{\Gamma\left(3 + \frac{m}{2}\right)} \int_{\mathbb{R}^m} x_i^4 e^{-\frac{|x|^2}{\pi}} |dx| = \frac{3}{4(4\pi)^{\frac{m}{2}}} \frac{1}{\Gamma\left(3 + \frac{m}{2}\right)} = 3c_m,
\]
and
\[ C_m(i, j; k, \ell) = 0, \quad \forall k \leq \ell, \quad (i, j) \neq (k, \ell). \]

We denote by \( \sigma^L \) the stochastic metric on \( M \) determined by the sample space \( U_L, \quad L \gg 0 \). As explained in the previous section the covariance form of the random vector \( U_L \ni u \mapsto du(p) \in T_p M \) is \( \sigma^L_p \), and from (3.3) we deduce
\[
\begin{align*}
\sigma^L_p(\partial_{x^i}, \partial_{x^j}) &= \frac{\partial^2 \mathcal{E}_L(x, y)}{\partial x^i \partial y^j} |_{x=y=0} = K_m \lambda^{m+2} \delta_{ij} + O(\lambda^{m+1}) \\
&= K_m \lambda^{m+2} g_p(\partial_{x^i}, \partial_{x^j}) + O(\lambda^{m+1}) \quad \text{as} \quad L \to \infty, \quad \text{uniformly in } p. \tag{3.5}
\end{align*}
\]

In particular, if \( S^L_{du(p)} \) denotes the covariance operator of the random vector \( du(p) \), then we deduce from the above equality that
\[ S^L_{du(p)} = K_m \lambda^{m+2} 1_m + O(\lambda^{m+1}), \quad \text{uniformly in } p, \tag{3.6} \]
and invoking (2.23) we deduce
\[ J^L_g(p) = (\det S^L_{du(p)})^{\frac{1}{2}} = K_m^\frac{m}{2} \lambda^{\frac{m(m+2)}{2}} + O\left(\lambda^{\frac{m(m+2)}{2} - 1}\right), \quad \text{uniformly in } p. \tag{3.7} \]

Denote by \( \Sigma^L_{hp} \) the covariance form of the random matrix
\[ U_L \ni u \mapsto \nabla^2_p u \in S(T_p M) = S_m. \]

Using (2.26) and (3.3) we deduce
\[ \Sigma^L_{hp} = c_m \lambda^{m+4} \Sigma_{3,1,1} + O(\lambda^{m+3}), \quad \text{uniformly in } p, \tag{3.8} \]
where the positive definite, symmetric bilinear form \( \Sigma_{3,1,1} : S^\vee_m \times S^\vee_m \to \mathbb{R} \) is described by the equalities (B.2a) and (B.2b). We denote by \( \Gamma_{3,1,1} \) the centered Gaussian measure on \( S_m \) with covariance form \( \Sigma_{3,1,1} \).

The equality (2.27) coupled with (3.3) imply that the covariance operator \( \Omega^L_p \) satisfies
\[ \Omega^L_p = O(\lambda^{m+2}), \quad \text{uniformly in } p. \tag{3.9} \]

Using (3.6), (3.8) and (3.9) we deduce that the covariance operator \( \Xi^L_p \) defined as in (2.21) satisfies the estimate
\[ \Xi^L_p = c_m \lambda^{m+4} \hat{Q}_{3,1,1} + O(\lambda^{m+2}), \quad \text{as} \quad L \to \infty, \quad \text{uniformly in } p, \tag{3.10} \]
where \( \hat{Q}_{3,1,1} \) is the covariance operator associated to the covariance form \( \Sigma_{3,1,1} \) and it is described explicitly in (B.3). If we denote by \( d\Gamma^L \) the Gaussian measure on \( S_m \) with covariance operator \( \Xi^L_p \), we deduce that
\[
d\Gamma^L(Y) = \frac{1}{(2\pi)^{N_m} (\det \Xi^L_p)^{\frac{1}{2}}} \cdot 2^{\frac{1}{2}(m)} \prod_{i \leq j} dy_{ij},
\]
where
\[ N_m = \dim S_m = \frac{m(m+1)}{2}. \]

Let us observe that \( |dY| \) is the Euclidean volume element on \( S_m \) defined by the natural inner product on \( S_m, \quad (X, Y) = \text{tr}(XY) \). We set
\[ c_L := c_m \lambda^{m+4}, \quad Q^L_p = \frac{1}{c_L} \Xi^L_p. \]
We seek \( u, v, w \) for any real numbers \( u, v, w \).

§

We follow the strategy in [17]; see also [16, §1.5]. Recall first the classical equality

\[
\int_{\mathbb{R}} e^{-\left(at^2 + bt + c\right)} \, dt = \left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{\frac{\Delta}{4a}}, \quad \Delta = b^2 - 4ac, \quad a > 0,
\]

which follows from the well known identity

\[
a t^2 + b t + c = a \left( t - \frac{b}{2a} \right)^2 - \frac{\Delta}{4a}.
\]

For any real numbers \( u, v, w \), we have

\[
u t^2 + v \text{tr}(X + wt1_m)^2 = (u + mw^2)t^2 + 2vw \text{tr} X + v \text{tr} X^2
\]

\[
= a(u, v, w)t^2 + b(u, v, w)t + c(u, v, w).
\]

We seek \( u, v, w \) such that

\[
\frac{b^2 - 4ac}{4a} = -\frac{1}{4} \left( \text{tr} X^2 - \frac{1}{m+2}(\text{tr} X)^2 \right).
\]
We have
\[ \frac{b^2 - 4ac}{4a} = \frac{v^2w^2}{u + mw^2}(\text{tr} X)^2 - \frac{v}{u + mw^2} \text{tr} X^2. \]

This implies
\[ \frac{v}{u + mw^2} = \frac{1}{4}, \quad \frac{v^2w^2}{u + mw^2} = \frac{1}{4(m + 2)}. \]

We deduce
\[ vw^2 = \frac{1}{(m + 2)}, \quad v = \frac{1}{4}(u + mw^2) \iff u = 4v - mw^2. \]

Hence
\[ w^2 = \frac{1}{v(m + 2)}, \quad u = 4v - \frac{m}{v(m + 2)}. \]

We choose \( v = \frac{1}{2} \) so that
\[ w^2 = \frac{2}{(m + 2)}; \quad u = 2 - \frac{2m}{(m + 2)} = \frac{4}{m + 2}, \quad a(u, v, w) = 4v = 2, \]
\[ e^{-\frac{1}{4}(\text{tr} X^2 - \frac{1}{m+2}(\text{tr} X)^2)} = \left( \frac{2}{\pi} \right)^\frac{1}{2} \int_\mathbb{R} e^{-\frac{4t^2}{m+2}} e^{-\frac{1}{2} \text{tr}(X + t\sqrt{\frac{2}{m+2}})^2} dt \]
\[ (s = \sqrt{\frac{2}{m+2}}) \]
\[ = \left( \frac{2(m + 2)}{\pi} \right)^\frac{1}{2} \int_\mathbb{R} e^{-\frac{1}{2} \text{tr}(X + s1_m)^2} e^{-s^2} ds = \left( \frac{m + 2}{2} \right)^\frac{1}{2} \int_\mathbb{R} e^{-\frac{1}{2} \text{tr}(X-s1_m)^2} \cdot e^{-2s^2} \frac{1}{\sqrt{\pi}} ds. \]

Hence
\[ I_m = \frac{(m + 2)^\frac{3}{2}}{\sqrt{2\pi} (2\pi)^\frac{m(m+1)}{4} \mu_m} \int_\mathbb{R} \left( \int_{\mathbb{R}^m} \text{det} X |e^{-\frac{1}{2} \text{tr}(X-s1_m)^2}| |dX| \right) d\gamma(s) \]
\[ =: A_m \int_\mathbb{R} \left( \int_{\mathbb{R}^m} \text{det} (x1_m - Y) |e^{-\frac{1}{2} \text{tr} Y^2}| |dY| \right) d\gamma(x). \]

For any \( O(n) \) invariant function \( : \mathbb{R}^n \to \mathbb{R} \) we have a Weyl integration formula (see [2, 16, 23]),
\[ \int_{\mathbb{R}^n} f(X)|dX| = \frac{1}{Z_n} \int_{\mathbb{R}^n} f(\lambda)|\Delta_m(\lambda)| |d\lambda|, \]
where
\[ \Delta_n(\lambda) := \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i), \]
and the constant \( Z_n \) is defined by the equality
\[ Z_n = \frac{\int_{\mathbb{R}^n} e^{-\frac{1}{2} |\lambda|^2} |\Delta_m(\lambda)| |d\lambda|}{\int_{\mathbb{R}^n} e^{-\frac{1}{2} \text{tr} X^2} |dX|}. \]
The denominator is equal to \((2\pi)^{\frac{\dim S_n}{2}}\) while the numerator is given by ([23, §17.6])

\[
\int_{\mathbb{R}^n} e^{-\frac{1}{2}|\lambda|^2} |\Delta_m(\lambda)| \, d\lambda| = 2^{\frac{3n}{2}} \prod_{j=1}^{n} \Gamma\left(1 + \frac{j}{2}\right),
\]

so that

\[
Z_n = \frac{2^{\frac{3n}{2}} \prod_{j=1}^{n} \Gamma\left(1 + \frac{j}{2}\right)}{(2\pi)^{\frac{\dim S_n}{2}}}. \tag{4.1}
\]

Now observe that for any \(\lambda_0 \in \mathbb{R}\) we have

\[
f_m(\lambda_0) = \frac{1}{Z_m} \int_{\mathbb{R}^m} e^{-\frac{1}{2} \sum_{j=1}^{n} |\lambda_j - \lambda_0||\Delta_m(\lambda)|} \, d\lambda
\]

\[
= \frac{e^{\frac{1}{2} \sum_{i=1}^{m} \lambda_0^2}}{Z_m} \int_{\mathbb{R}^m} e^{-\frac{1}{2} \sum_{i=1}^{m} \lambda_i^2} |\Delta_{m+1}(\lambda_0, \lambda_1, \ldots, \lambda_m)| \, d\lambda_1 \cdots d\lambda_m
\]

\[
= \frac{e^{\frac{1}{2} \sum_{i=1}^{m} \lambda_0^2}}{Z_{m+1}} \frac{1}{Z_m} \int_{\mathbb{R}^m} e^{-\frac{1}{2} \sum_{i=1}^{m} \lambda_i^2} |\Delta_{m+1}(\lambda_0, \lambda_1, \ldots, \lambda_m)| \, d\lambda_1 \cdots d\lambda_m.
\]

The function \(\rho_n(x) = nR_n(x)\) is known in random matrix theory as the 1-point correlation function of the Gaussian orthogonal ensemble of symmetric \(n \times n\) matrices, [11, §4.4.1], [23, §4.2]. We conclude that

\[
I_m = \frac{A_m Z_{m+1}}{Z_m} \int_{\mathbb{R}} R_{m+1}(x) e^{\frac{x^2}{2}} \, dx = \frac{A_m Z_{m+1}}{Z_m} \int_{\mathbb{R}} R_{m+1}(x) \sqrt{\frac{2}{\pi}} e^{-\frac{3x^2}{2}} \, dx.
\]

We have

\[
\frac{Z_{m+1}}{Z_m} = \frac{2^{\frac{3}{2}} \Gamma\left(m + \frac{3}{2}\right)}{2^{\frac{m+1}{2}}},
\]

\[
\sqrt{\frac{2}{\pi}} \frac{A_m Z_{m+1}}{Z_m} = \frac{2^{\frac{3}{2}} \Gamma\left(m + \frac{3}{2}\right)}{2^{\frac{m+1}{2}} \times \sqrt{\frac{2}{\pi}} \times \frac{(m + 2)^\frac{3}{2}}{2^{\frac{3}{2}}(2\pi)^{\frac{m(m+1)}{4}} \sqrt{\mu_m}}}
\]

\[
= \frac{\Gamma\left(m + \frac{3}{2}\right)}{2^{m-1} \sqrt{\pi}} \times \frac{1}{2^{\frac{m(m+1)}{4}} + \frac{1}{4} (m + 2) m^{\frac{m-2}{2}}} = \frac{\Gamma\left(m + \frac{3}{2}\right)}{\sqrt{\pi}2^{m^2 + m - \frac{1}{2}} (m + 2)^{\frac{m-2}{2}}}
\]

We deduce

\[
I_m = \frac{\Gamma\left(m + \frac{3}{2}\right)}{\sqrt{\pi}2^{m^2 + m - \frac{1}{2}} (m + 2)^{\frac{m-2}{2}}} \int_{\mathbb{R}} R_{m+1}(x) e^{\frac{-3x^2}{2}} \, dx
\]

\[
= \frac{\Gamma\left(m + \frac{3}{2}\right)}{\sqrt{\pi}2^{m^2 + m - \frac{1}{2}} (m + 2)^{\frac{m-2}{2}} (m + 1) \int_{\mathbb{R}} \rho_{m+1}(x) e^{\frac{-3x^2}{2}} \, dx. \tag{4.2}
\]

To proceed further we use as guide Wigner’s theorem, [2, Thm.2.1.1] stating that the sequences of probability measures,

\[
\tilde{\rho}_n(x) := \frac{1}{\sqrt{n}} \rho_n(\sqrt{n}x),
\]
From the Christoffel-Darboux formula [35, Eq. (5.5.9)] we deduce that the covariances of our random matrices differ by a factor from those in [2]. Our conventions agree with those in [23].

\[
\rho(x) = \frac{1}{\pi} \left\{ \sqrt{2 - x^2}, \quad |x| \leq \sqrt{2} \\
0, \quad |x| > \sqrt{2}.
\right.
\]

We deduce

\[
\int_R \rho_n(x)e^{-\frac{3x^2}{2}} dx = n \int_R \rho_n(\sqrt{n}s)e^{-\frac{3n^2}{2}} |ds| = \frac{n^3}{2} \int e^{-\frac{3n^2}{2}} \rho_n(s)ds
\]

\[
= \left( \frac{2\pi}{3} \right)^\frac{1}{2} n \int_R \frac{(3n)^\frac{1}{2} e^{-\frac{3n^2}{2}}}{(2\pi)^\frac{1}{2}} \cdot \rho_n(s)ds.
\]

We observe that the Gaussian measures \( w_n(s)ds \) converge to the Dirac delta measure concentrated at the origin. The 1-point correlation function \( \rho_n(x) \) can be expressed explicitly in terms of Hermite polynomials, [23, Eq. (7.2.32) and §A.9],

\[
\rho_n(x) = \sum_{k=0}^{n-1} \psi_k(x)^2 + \left( \frac{n}{2} \right)^\frac{1}{2} \int_R \varepsilon(x-t)\psi_n(t)dt + \alpha_n(x),
\]

where

\[
\psi_n(x) = \frac{1}{(2^n n! \sqrt{\pi})^\frac{1}{2}} e^{-\frac{x^2}{2n}} H_n(x), \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}),
\]

\[
\alpha_n(x) = \left\{ \begin{array}{ll} 
0, & n \in \mathbb{Z}, \\
\psi_n(x)/\int_\mathbb{R} \psi_n(x)dx, & n \in 2\mathbb{Z} + 1,
\end{array} \right.
\]

and

\[
\varepsilon(x) = \left\{ \begin{array}{ll} 
\frac{1}{2}, & x > 0 \\
0, & x = 0, \\
-\frac{1}{2}, & x < 0.
\end{array} \right.
\]

From the Christoffel-Darboux formula [35, Eq. (5.5.9)] we deduce

\[
\pi^\frac{1}{2} e^{x^2} \sum_{k=0}^{n-1} \psi_k(x)^2 = \sum_{k=1}^{n-1} \frac{1}{2^k k!} H_k(x)^2 = \frac{1}{2^{n(n-1)!}} \left( H_n'(x)H_{n-1}(x) - H_n(x)H_n'(x) \right)
\]

Using the recurrence formula \( H_n'(x)H_{n-1}(x) - H_n(x)H_n'(x) = H_n'(x)^2 - H_n(x)H_{n+1}(x) \)

and

\[
k_n(x) = \frac{e^{-x^2}}{2^n(n-1)! \pi^\frac{1}{2}} \left( H_n^2(x) - H_n(x)H_{n+1}(x) \right).
\]

We set

\[
\overline{k}_n(x) := k_n\left( \frac{\sqrt{n}x}{\sqrt{n}} \right), \quad \overline{\rho}_n(x) := \frac{k_n\left( \sqrt{n}x \right)}{\sqrt{n}},
\]

The difference between our definition (4.3) of the semicircle measure and the one in [2, Thm. 2.1.1] is due to the fact that the covariances of our random matrices differ by a factor from those in [2]. Our conventions agree with those in [23].
so that

$$\tilde{\rho}_n(x) = \tilde{k}_n(x) + \tilde{\ell}_n(x).$$

The integrals entering into the definition of $\ell_n$ in (4.4) can be given a more explicit description using the generating function, [35, Eq. (5.5.6)],

$$\sum_{n=0}^{\infty} H_n(x) \frac{T^n}{n!} = e^{2Tx-T^2}.$$

Using the refined asymptotic estimates for Hermite polynomials [15] and [35, Thm.8.22.9, Thm.8.91.3] we deduce that the terms $\ell_n$ are asymptotically negligible, i.e.,

$$\lim_{n \to \infty} \int_{\mathbb{R}} \tilde{\ell}_n(s) w_n(s) ds = 0,$$

so that

$$\lim_{n \to \infty} \int_{\mathbb{R}} (\tilde{\rho}_n(s) - \rho(x)) w_n(s) ds = \lim_{n \to \infty} \int_{\mathbb{R}} (\tilde{k}_n(s) - \rho(x)) w_n(s) ds = 0.$$

On the other hand,

$$\lim_{n \to \infty} \int_{\mathbb{R}} \rho(s) w_n(s) ds = \rho(0) = \frac{\sqrt{2}}{\pi}.$$ 

Using the above equality in (4.2) and (4.4) we deduce that

$$I_m \sim \frac{2\Gamma(m + \frac{3}{2})}{\pi^{2m^2 + m - 1}(m + 2)^{\frac{m-2}{2}}}$$

as $m \to \infty$.

We now invoke Stirling’s formula to conclude that

$$\Gamma(m + \frac{3}{2}) \sim \sqrt{2\pi} \left(m + \frac{3}{2}\right)^{m+\frac{1}{2}} e^{-m - \frac{3}{2}} \sim \sqrt{2\pi m^{m+\frac{1}{2}} e^{\frac{3}{2}}}. $$

We have

$$(m + 2)^{\frac{m-2}{2}} = m^{\frac{m-2}{2}} \left(1 + \frac{2}{m}\right)^{\frac{m-2}{2}} \sim m^{\frac{m-2}{2}} e.$$ 

Thus

$$\log I_m \sim \frac{m-3}{2} \log m \sim \frac{m}{2} \log m, \quad \text{as } m \to \infty. \quad (4.6)$$
Form (1.2) we deduce that
\[
\log C(m) = \log I_m + \frac{m}{2} \log 4\pi + \log \Gamma \left(1 + \frac{m}{2}\right) - \frac{m}{2} \log(m + 4).
\]
Stirling’s formula and (4.6) imply that
\[
\log C(m) \sim \log I_m \text{ as } m \to \infty.
\]
This proves (1.3).

5. The variance of the number of critical points of a random trigonometric polynomial

A random trigonometric polynomial of degree \(\leq \nu\) is a sample function of the gaussian process
\[
\eta_\nu(t) = \frac{1}{\sqrt{2\pi}} a_0 + \frac{1}{\sqrt{\pi\nu}} \sum_{m=1}^{\nu} (a_m \cos mt + b_m \sin mt),
\]
where \(a_m, b_m\) are independent normally distributed random variables with mean 0 and variance 1. The statistics of the critical set of a random sample function of the above process is identical to the statistics of the zero set of a random sample function of the stationary gaussian process
\[
\xi_\nu(t) = \frac{d\eta_\nu(t)}{dt} = \frac{1}{\sqrt{\pi\nu}} \sum_{m=1}^{\nu} (-ma_m \sin mt + mb_m \cos mt).
\]
Equivalently, consider the gaussian process
\[
\Phi_\nu(t) = \frac{1}{\sqrt{\pi\nu^3}} \sum_{m=1}^{\nu} \left( mc_m \cos \left(\frac{t}{\nu}\right) + md_m \sin \left(\frac{t}{\nu}\right) \right),
\]
where \(c_m, d_m\) are independent random variables with identical standard normal distribution, and the random variable
\[
Z_\nu := \text{the number of zeros of } \Phi_\nu(t) \text{ in the interval } [-\pi\nu, \pi\nu].
\]
Note that the expectation of \(Z_\nu\) is precisely the expected number of critical points of a random trigonometric polynomial in the space
\[
\text{span}\{\cos(m\theta), \sin(m\theta) ; \ m = 1, \ldots, \nu\},
\]
equipped with the inner product
\[
(u, v) = \int_0^{2\pi} u(\theta)v(\theta)d\theta.
\]
The covariance function of \(\Phi_\nu\) is
\[
R_\nu(t) = \frac{1}{\pi\nu^3} \sum_{m=1}^{\nu} m^2 \cos \left(\frac{mt}{\nu}\right).
\]
We let \(E(\zeta)\), and respectively \(\text{var}(\zeta)\), denote the expectation, and respectively the variance, of a random variable \(\zeta\). The Rice formula, [10, Eq. (10.3.1)], implies that
\[
E(Z_\nu) = 2\nu \left(\frac{\lambda_2(\nu)}{\lambda_0(\nu)}\right)^{\frac{1}{2}},
\]
where
\[
\lambda_0(\nu) = R_\nu(0) = \frac{1}{\pi\nu^3} \sum_{m=1}^{\nu} m^2 \quad \text{and} \quad \lambda_2(\nu) = -R_\nu''(0) = \frac{1}{\pi\nu^5} \sum_{m=1}^{\nu} m^4.
\]
This proves that

$$E(Z_ν) \sim 2n \sqrt{\frac{3}{5}}$$

as \( n \to \infty \).

The following is the main result of this section.

**Theorem 5.1.** Set

$$\bar{\lambda}_0 := \lim_{\nu \to \infty} \lambda_0(\nu) = \frac{1}{3}, \quad \bar{\lambda}_2 := \lim_{\nu \to \infty} \lambda_2(\nu) = \frac{1}{5}.$$

Then for any \( t \in \mathbb{R} \) the limit \( \lim_{\nu \to \infty} R_\nu(t) \) exists, it is equal to

$$R_\infty(t) := \frac{1}{\nu^3} \int_0^\nu \tau^2 \cos \tau d\tau = \int_0^1 \lambda^2 \cos(\lambda t) d\lambda, \quad \forall t \in \mathbb{R},$$

and

$$\lim_{\nu \to \infty} \frac{1}{\nu} \text{var}(Z_\nu) = \delta_\infty := \frac{2}{\pi} \int_{-\infty}^{\infty} \left( f_\infty(t) - \frac{\bar{\lambda}_2}{\lambda_0} \right) dt + 2 \sqrt{\frac{\bar{\lambda}_2}{\lambda_0}},$$

where

$$f_\infty(t) = \frac{(\bar{\lambda}_0^2 - R_\infty^2)\lambda_0 - \lambda_0(R_\nu^\prime)^2}{(\bar{\lambda}_0^2 - R_\infty^2)^{1/2}} \left( \sqrt{1 - \rho^2} + \rho_\infty \arcsin \rho_\infty \right),$$

and

$$\rho_\infty = \frac{R_\nu^\prime(\bar{\lambda}_0^2 - R_\infty^2) + (R_\nu^\prime)^2 R_\infty}{(\bar{\lambda}_0^2 - R_\infty^2)\lambda_2 - \lambda_0(R_\nu^\prime)^2}. $$

Moreover, the constant \( \delta_\infty \) is positive.\(^4\)

**Proof.** We follow a strategy inspired from [19]. The variance of \( Z_\nu \) can be computed using the results in [10, §10.6]. We introduce the gaussian field

$$\Psi(t_1, t_2) := \begin{bmatrix} \Phi_\nu(t_1) \\ \Phi_\nu(t_2) \\ \Phi_\nu^\prime(t_1) \\ \Phi_\nu^\prime(t_2) \end{bmatrix}.$$  

Its covariance matrix depends only on \( t = t_2 - t_1 \). We have (compare with [19, Eq. (17)])

$$\Xi(t) := \begin{bmatrix} \lambda_0 & R_\nu(t) & 0 & R_\nu^\prime(t) \\ R_\nu(t) & \lambda_0 & -R_\nu^\prime(t) & 0 \\ 0 & -R_\nu^\prime(t) & \lambda_2 & -\nu_\nu^\prime(t) \\ R_\nu^\prime(t) & 0 & -R_\nu^\prime(t) & \lambda_2 \end{bmatrix} =: \begin{bmatrix} A & B \\ B^\dagger & C \end{bmatrix}. $$

As explained in [31], to apply [10, §10.6] we only need that \( \Xi(t) \) is nondegenerate. This is established in the next result whose proof can be found in Appendix D.

**Lemma 5.2.** The matrix \( \Xi(t) \) is nonsingular if and only if \( t \not\in 2\nu\mathbb{Z} \).

\( \square \)

For any vector

$$\mathbf{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$$

\( ^4 \) Numerical experiments indicate that \( \delta_\infty \approx 0.35 \).
we set

\[ p_{t_1, t_2}(x) := \frac{1}{4\pi^2 (\det \Xi)^{1/2}} e^{-\frac{1}{2} (\Xi^{-1} x, x)}. \]

Then, the results in [9, §10.6] show that

\[ E(Z_\nu^2) - E(Z_\nu) = \int_{I_\nu \times I_\nu} \left( \int_{\mathbb{R}^2} |y_1 y_2| \cdot p_{t_1, t_2}(0, 0, y_1, y_2) |dy_1 dy_2| \right) |dt_1 dt_2|. \quad (5.2) \]

As in [19] we have

\[ \Xi(t)^{-1} = \begin{bmatrix} * & * \\ * & \Omega^{-1} \end{bmatrix}, \Omega = C - B^\dagger A^{-1} B. \]

More explicitly,

\[
\Omega = C - B^\dagger A^{-1} B \\
= \begin{bmatrix} \lambda_2 & -R''_\nu(t) \\ -R''_\nu(t) & \lambda_2 \end{bmatrix} - \frac{1}{\lambda_0^2 - R'_\nu(t)^2} \begin{bmatrix} 0 & -R'_\nu(t) \\ R'_\nu(t) & 0 \end{bmatrix} \begin{bmatrix} -R'_\nu(t) & 0 \\ 0 & R'_\nu(t) \end{bmatrix} \begin{bmatrix} \lambda_0 & -R'_\nu(t) \\ -R'_\nu(t) & \lambda_0 \end{bmatrix} = \mu \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix},
\]

where

\[
\mu = \mu_\nu = (\lambda_0^2 - R^2_\nu)\lambda_2 - \lambda_0 (R'_\nu)^2 \lambda_2^2 - R'_\nu (\lambda_0^2 - R^2_\nu)\lambda_2 - \lambda_0 (R'_\nu)^2.
\]

We want to emphasize, that in the above equalities the constants \( \lambda_0 \) and \( \lambda_2 \) do depend on \( \nu \), although we have not indicated this in our notation.

**Remark 5.3.** The nondegeneracy of \( \Xi \) implies that \( \mu_\nu(t) \neq 0 \) and \( |\rho_\nu(t)| < 1 \), for all \( t \not\in 2\pi\nu \mathbb{Z} \). \( \square \)

We obtain as in [19, Eq. (24)]

\[
\det \Xi = \det A \cdot \det \Omega = \mu^2 (\lambda_0^2 - R^2_\nu) (1 - \rho^2), \quad \Omega^{-1} = \frac{1}{\mu(1 - \rho^2)} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.
\]

We can now rewrite the equality (5.2) as \((t = t_2 - t_1)\)

\[
E(Z^2_\nu) - E(Z_\nu) = \int_{I_\nu \times I_\nu} \left( \int_{\mathbb{R}^2} |y_1 y_2| e^{-\frac{y^2 + 2y_1 y_2 + y^2_2}{2(1 - \rho^2)}} |dy_1 dy_2| \frac{4\pi^2}{d_1 d_2} \right) \frac{\mu |dt_1 dt_2|}{\sqrt{(\lambda_0^2 - R^2_\nu) (1 - \rho^2)}}
\]

From [6, Eq. (A.1)] we deduce that

\[
\int_{\mathbb{R}^2} |y_1 y_2| e^{-\frac{y^2 + 2y_1 y_2 + y^2_2}{2(1 - \rho^2)}} |dy_1 dy_2| \frac{4\pi^2}{d_1 d_2} = \frac{1 - \rho^2}{\pi^2} \left( 1 + \frac{\rho}{\sqrt{1 - \rho^2}} \arcsin \rho \right).
\]

Hence

\[
E(Z^2_\nu) - E(Z_\nu) = \frac{1}{\pi^2} \int_{I_\nu \times I_\nu} \frac{\mu}{(\lambda_0^2 - R^2_\nu)^2} \left( \sqrt{1 - \rho^2} + \rho \arcsin \rho \right) |dt_1 dt_2|
\]
\[
= \frac{1}{\pi^2} \int_{\mathcal{I}_t \times \mathcal{I}_\nu} \frac{(\lambda_2^2 - R_s^2 - \lambda_0(R_s^2)^2)}{(\lambda_2^2 - R_s^2)^2} \left( \sqrt{1 - \rho^2} + \rho \arcsin \rho \right) |dt_1dt_2|.
\]

The function \( f_\nu(t) = f_\nu(t_2 - t_1) \) is doubly periodic with periods \( 2\pi \nu, 2\pi \nu \) and we conclude that
\[
E([Z_\nu]_2) := E(Z_\nu^2) - E(Z_\nu) = \frac{2\nu}{\pi} \int_{-\pi\nu}^{\pi\nu} f_\nu(t)dt.
\]

We conclude that
\[
\text{var}(Z_\nu) = E([Z_\nu]_2) + E(Z_\nu) - \left( E(Z_\nu) \right)^2 = E([Z_\nu]_2) - \left[ E(Z_\nu) \right]^2
\]
\[
= \frac{2\nu}{\pi} \int_{-\pi\nu}^{\pi\nu} \left( f_\nu(t) - \frac{\lambda_2}{\lambda_0} \right) dt + 2\nu \sqrt{\frac{\lambda_2}{\lambda_0}}.
\]

To complete the proof of Theorem 5.1 we need to investigate the integrand in (5.4). This requires a detailed understanding of the behavior of \( R_\nu \) as \( \nu \to \infty \). It is useful to consider more general sums of the form
\[
A_{\nu,r}(t) = \frac{1}{\nu^{r+1}} \sum_{m=1}^{\nu} m^r \cos \frac{mt}{\nu}, \quad B_{\nu,r}(t) = \frac{1}{\nu^{r+1}} \sum_{m=1}^{\nu} m^r \sin \frac{mt}{\nu}, \quad r \geq 1.
\]

Note that if we set \( z := \cos \frac{mt}{\nu} + i \sin \frac{mt}{\nu} \). We have
\[
A_{\nu,r}(t) + iB_{\nu,r}(t) = \frac{1}{\nu^{r+1}} \sum_{m=1}^{\nu} m^r z^m.
\]

Observe that
\[
R_\nu^{(k)}(t) = -\frac{1}{\pi \nu^{k+3}} \text{Re} \left( \frac{1}{\nu^{k+2}} C_{\nu,k+2}(t) \right).
\]

We set
\[
A_{\infty,r}(t) = \frac{1}{\nu^{r+1}} \int_0^t \tau^r \cos \tau d\tau, \quad B_{\infty,r}(t) := \frac{1}{\nu^{r+1}} \int_0^t \tau^r \sin \tau d\tau.
\]

Observe that \( R_\nu = A_{\nu,2} \) and \( R_\infty = A_{\infty,2} \). We have the following result.

**Lemma 5.4.**

\[
|A_{\nu,r}(t) - A_{\infty,r}(t)| + |B_{\nu,r}(t) - B_{\infty,r}(t)| = O \left( \frac{\max(1, t)}{\nu} \right), \quad \forall t \geq 0,
\]

where, above and in the sequel, the constant implied by the \( O \)-symbol is independent of \( t \) and \( \nu \). In particular
\[
\lim_{\nu \to \infty} \frac{1}{\nu^{r+1}} C_{\nu,r}(t) = C_r(t) := \frac{1}{t^{r+1}} \int_0^t \tau^r e^{\tau r} d\tau, \quad \forall t \geq 0.
\]

**Proof.** We have
\[
A_\nu(t) = \frac{1}{\nu^{r+1}} \sum_{m=1}^{\nu} m^r \cos \left( \frac{mt}{\nu} \right) = \frac{1}{\nu^{r+1}} \sum_{m=1}^{\nu} \left( \frac{mt}{\nu} \right)^r \cos \left( \frac{mt}{\nu} \right) \cdot \left( \frac{t}{\nu} \right).
\]
The term $S_\nu(t)$ is a Riemann sum corresponding to the integral
\[ \int_0^t f(\tau)d\tau, \quad f(\tau) := \tau^r \cos \tau, \]
and the subdivision
\[ 0 < \frac{t}{\nu} < \cdots < \frac{(\nu - 1)t}{\nu} < t. \]
of the interval $[0, t]$. A simple application of the mean value theorem implies that there exist points
\[ \theta_m \in \left[ \frac{(m - 1)t}{\nu}, \frac{mt}{\nu} \right] \]
such that
\[ \int_0^t f(\tau)d\tau = \sum_{m=1}^\nu f(\theta_m) \frac{t}{m}. \]
We deduce that
\[ \int_0^t f(\tau)d\tau - S_\nu(t) = \frac{t}{\nu} \sum_{m=1}^\nu \left( f(\theta_m) - f \left( \frac{mt}{\nu} \right) \right) \]
Now set
\[ M(t) := \sup_{0 \leq \tau \leq t} |f'(\tau)|. \]
Observe that
\[ M(t) = \begin{cases} O(t^{r-1}), & 0 \leq t \leq 1 \\ O(t^r), & r > 1. \end{cases} \]
We deduce
\[ \left| S_\nu(t) - \int_0^t f(\tau)d\tau \right| \leq M(t) \cdot \frac{t^2}{\nu}. \]
This, proves the $A$-part of (5.7). The $B$-part is completely similar. \(\square\)

We need to refine the estimates (5.7). Recall that $[m]_r := m(m - 1) \cdots (m - r + 1)$, $r \geq 1$. We will express $C_{\nu,r}(t)$ in terms of the sums
\[ D_{\nu,r}(t) := \sum_{m=1}^\nu [m]_r z^m = z^r \frac{d^r}{dz^r} \left( \sum_{k=1}^\nu z^k \right) = z^r \frac{d^r}{dz^r} \left( \frac{z - z^{\nu+1}}{1 - z} \right) = z^r \frac{d^r}{dz^r} \left( \frac{1 - z^{\nu+1}}{1 - z} \right). \]
Using the classical formula, [34, §1.4, Eq. (24d)],
\[ m^r = \sum_{k=1}^r S(r, k)[m]_k, \]
where $S(r, k)$ are the Stirling numbers of the second kind, we deduce,
\[ C_{\nu,r}(\zeta) = \sum_{k=1}^r S(r, k)D_{\nu,k}(\zeta) = \sum_{k=1}^r S(r, k)z^r \frac{d^k}{dz^k} \left( \frac{1 - z^{\nu+1}}{1 - z} \right). \]
Lemma 5.5. Set $\theta := \frac{t}{2\nu}$, and $f(\theta) = \frac{\sin \theta}{\theta}$. Then
\[ \left( \frac{t^{r+1}}{\nu^{r+1}} D_{\nu,r}(t) \right) = i^r r! \left( \frac{2 \sin \left( \frac{(\nu+1)t}{2\nu} \right)}{f(\theta)^{r+1}} \cdot e^{-i \theta} - e^{i \theta} \sum_{j=1}^r i^{1-j} \left( \begin{array}{c} \nu+1 \\ j \end{array} \right) b_j \cdot \left( \frac{e^{i\theta}}{f(\theta)} \right)^{r+1-j} \right) \]
Proof. We have

\[ D_{\nu,r}(t) = z^r \sum_{j=0}^{r} \binom{r}{j} \frac{d^j}{dz^j} (1 - z^{\nu+1}) \frac{d^{r-j}}{dz^{r-j}} (1 - z)^{-1} \]

\[ = r! \frac{z^r (1 - z^{\nu+1})}{(1 - z)^{r+1}} - \sum_{j=1}^{r} \binom{r}{j} [\nu + 1]_j (r - j)! \frac{z^\nu (1 + r - j)}{(1 - z)_1^{r-j}} \]

\[ = r! \frac{z^r (1 - z^{\nu+1})}{(1 - z)^{r+1}} - z^\nu r! \sum_{j=1}^{r} \binom{\nu + 1}{j} \left( \frac{z}{1 - z} \right)^{r+1-j} . \]

Using the identity

\[ 1 - e^{it\alpha} = \frac{2}{t} \sin \left( \frac{\alpha}{2} \right) e^{\frac{it\alpha}{2}} \]

we deduce

\[ \frac{z}{1 - z} = \frac{ie^{it\frac{\pi}{2}}}{2 \sin \left( \frac{t}{2\nu} \right)} , \]

and

\[ D_{\nu,r}(t) = i^r r! \frac{\sin \left( \frac{(\nu+1)t}{2\nu} \right) e^{\frac{(\nu+1+2\nu)t}{2\nu}}}{\sin^r \left( \frac{t}{2\nu} \right)} e^{\frac{(r+1)it}{2\nu}} \sum_{j=1}^{r} i^{r+1-j} \binom{\nu + 1}{j} \left( \frac{e^{it\theta}}{2 \sin \theta} \right)^{1+r-j} . \]

Multiplying both sides of the above equality by \( \left( \frac{t}{2\nu} \right)^{r+1} \) we get (5.10).

Lemma 5.5 coupled with the fact that the function \( f(\theta) \) is bounded on \([0, \frac{\pi}{2}]\) yield the following estimate.

\[ \frac{t^{r+1}}{\nu^{r+1}} D_{\nu,r}(t) = O(1), \; \forall \nu, \; 0 \leq t \leq \pi \nu . \] (5.11)

Using (5.11) and the identity \( S(r, 1) = 1 \) in (5.9) we deduce that there exists \( K = K_r > 0 \) such that for any \( \nu > 0 \) and any \( t \in [0, \pi \nu] \) we have

\[ \left| \frac{t^{r+1}}{\nu^{r+1}} \left( C_{\nu,r}(t) - D_{\nu,r}(t) \right) \right| \leq K_r \sum_{j=0}^{r-1} \frac{t^{r+1}}{\nu^{r+1}} D_{\nu,j}(t) \leq K_r \sum_{j=0}^{r-1} \left( \frac{t}{\nu} \right)^{r-j} \leq K_r \frac{t^r}{\nu} , \]

so that

\[ \left| \frac{1}{\nu^{r+1}} \left( C_{\nu,r}(t) - D_{\nu,r}(t) \right) \right| \leq K_r \frac{1}{\nu^{r+1}} \] (5.12)

Using Lemma 5.5 we deduce

\[ \lim_{\nu \to \infty} \frac{t^{r+1}}{\nu^{r+1}} \Re D_{\nu,r}(t) = I_r(t) := i^r r! \left( 2 \sin \left( \frac{t}{2} \right) \cdot e^{\frac{it\pi}{2}} - e^{it} \sum_{j=1}^{r} i^{1-j} \frac{t^j}{j!} \right) . \] (5.13)

uniformly for \( t \) on compacts. The estimate (5.12) implies that

\[ \lim_{\nu \to \infty} \frac{t^{r+1}}{\nu^{r+1}} \Re D_{\nu,r}(t) = I_r(t) . \]

We have the following crucial estimate whose proof can be found in Appendix D.
Lemma 5.6. For every $r \geq 0$ there exists $C_r > 0$ such that for any $\nu > 0$ we have
\[
\left| \frac{1}{\nu^{r+1}} D_{\nu,r}(t) - \frac{1}{\nu^{r+1}} I_r(t) \right| \leq C_r \frac{t^{r+1} - 1}{\nu t^r (t-1)}, \quad \forall 0 < t \leq \pi \nu. \tag{5.14a}
\]

Using Lemma 5.6 in (5.12) we deduce
\[
\left| \frac{1}{\nu^{r+1}} C_{\nu,r}(t) - \frac{1}{\nu^{r+1}} I_r(t) \right| \leq C_r \frac{t^{r+1} - 1}{\nu t^r (t-1)}, \quad \forall 0 < t \leq \pi \nu. \tag{5.14a}
\]
\[
I_r(t) = t^{r+1} C_r(t) = \int_0^t \tau e^{i\tau} d\tau. \tag{5.14b}
\]

Using (5.7) and (5.14a) we deduce that for any nonnegative integer $r$ there exists a positive constant $K = K_r > 0$ such that
\[
|C_{\nu,r}(t) - C_r(t)| \leq K \frac{t^{r+1} - 1}{\nu t^r (t-1)}. \tag{5.15}
\]

Coupling the above estimates with (5.7) we deduce
\[
C_{\nu,r}(t) = C_r(t) + O \left( \frac{1}{\nu} \right), \quad \forall 0 \leq t \leq \nu, \tag{5.16}
\]
where the constant implied by the symbol $O$ depends on $k$, but it is independent of $\nu$. The last equality coupled with (5.6) implies that
\[
R_{\nu}^{(k)}(t) = R_{\nu}^{(k)}(t) + O \left( \frac{1}{\nu} \right), \quad \forall 0 \leq t \leq \nu. \tag{5.17}
\]

We deduce that, for any $t > 0$ we have
\[
\lim_{\nu \to \infty} f_{\nu}(t) = f_{\infty}(t)
\]
where $f_{\nu}$ is the function defined in (5.3), while
\[
f_{\infty}(t) = \frac{(\bar{\lambda}_0^2 - R_{\infty}^2) \bar{\lambda}_2 - \bar{\lambda}_0 (R_{\infty}^r) ^2}{(\bar{\lambda}_0^2 - R_{\infty}^2)^2} \left( \sqrt{1 - \rho_{\infty}^2} + \rho_{\infty} \arcsin \rho_{\infty} \right), \tag{5.18}
\]
where
\[
\bar{\lambda}_0 = \lambda_0(\infty) = \lim_{\nu \to \infty} \lambda_0(\nu) = \frac{1}{3}, \quad \bar{\lambda}_2 = \lim_{\nu \to \infty} \lambda_2(\nu) = \frac{1}{5},
\]
\[
\rho_{\infty}(t) = \lim_{\nu \to \infty} \rho_{\nu}(t) = \frac{R_{\nu}''(\bar{\lambda}_0^2 - R_{\infty}^2) + (R_{\nu}''^r R_{\infty}^2)}{(\bar{\lambda}_0^2 - R_{\infty}^2) \bar{\lambda}_2 - \bar{\lambda}_0 (R_{\infty}^r)^2}. \tag{5.19c}
\]

We have the following result whose proof can be found in Appendix D.

Lemma 5.7.\[
|R_{\infty}(t)| < R_{\infty}(0), |R_{\infty}'(t)| < |R_{\infty}'(0)|, \quad \forall t > 0, \tag{5.19a}
\]
\[
R_{\infty}(t), \ R_{\infty}'(t), \ R_{\infty}''(t) = O \left( \frac{1}{t} \right) \quad \text{as} \quad t \to \infty. \tag{5.19b}
\]
\[
(\bar{\lambda}_0^2 - R_{\infty}^2) \bar{\lambda}_2 - \bar{\lambda}_0 (R_{\infty}^r)^2 > 0, \quad \forall t > 0. \tag{5.19c}
\]
\[
R_{\infty}(t) = \frac{1}{3} - \frac{1}{10} t^2 + \frac{1}{168} t^4 + O(t^6), \quad R_{\infty}'(t) = -\frac{1}{5} t + \frac{1}{42} t^3 + O(t^5), \quad \text{as} \quad t \to 0. \tag{5.19d}
\]
\[\square\]
We set \( \delta(t) := \max(t, 1), \quad t \geq 0. \)

We find it convenient to introduce new functions
\[
G_\nu(t) := \frac{1}{R_\nu(0)} R_\nu(t) = \frac{1}{\lambda_0(\nu)} R_\nu(t), \quad H_\nu(t) = \frac{1}{R_\nu''(0)} R_\nu''(t) = -\frac{1}{\lambda_2(\nu)} R_\nu''(t).
\]

Using these notations we can rewrite (5.19c) as
\[
(1 - G^2_{\infty}) - \frac{\lambda_0}{\lambda_2}(G'_\infty)^2 > 0, \quad \forall t > 0.
\]

The equalities (5.19d) imply that
\[
\eta(t) = \frac{3}{4373} t^4 + O(t^6), \quad \forall |t| \ll 1.
\]

Then
\[
f_\nu(t) = \frac{\lambda_2(\nu)}{\lambda_0(\nu)} \times C_\nu(t) \times \left( \sqrt{1 - \rho^2_\nu} + \rho_\nu \arcsin \rho_\nu \right),
\]

where
\[
C_\nu(t) := \frac{(1 - G_\nu(t)^2) - \frac{\lambda_0(\nu)}{\lambda_2(\nu)} G'_\nu(t)^2}{(1 - G_\nu(t)^2)^{3/2}},
\]

and
\[
\rho_\nu = \frac{R''_\nu(\lambda_0(\nu)^2 - R^2_\nu) + (R'_\nu)^2 R''_\nu}{(\lambda_0(\nu)^2 - R^2_\nu) \lambda_2(\nu) - \lambda_0(R''_\nu)^2} = -H_\nu(1 - G^2_\nu) + \frac{\lambda_0(\nu)}{\lambda_2(\nu)} (G'_\nu)^2 G_\nu.
\]

**Lemma 5.8.** Let \( \kappa \in (0, 1). \) Then
\[
C_\nu(t) = O(t), \quad \forall 0 \leq t \leq \nu^{-\kappa},
\]

where the constant implied by \( O \)-symbol is independent of \( \nu \) and \( t \in [0, \nu^{-\kappa}] \), but it could depend on \( \kappa \).

**Proof.** Observe that for \( t \in [0, \nu^{-\kappa}] \) we have
\[
G_\nu(t) = 1 - \frac{\lambda_2(\nu)}{2 \lambda_0(\nu)} t^2 + O(t^4), \quad G_\nu'(t) = -\frac{\lambda_2(\nu)}{\lambda_0(\nu)} t + O(t^3)
\]

so that
\[
(1 - G_\nu(t)^2)^{-3/2} = \left( \frac{\lambda_2(\nu)}{\lambda_0(\nu)} t \right)^{-3} \times (1 + O(t)),
\]

and
\[
(1 - G_\nu(t)^2) - \frac{\lambda_0(\nu)}{\lambda_2(\nu)} (G_\nu') = O(t^4)
\]

so that \( C_\nu(t) = O(t). \)

**Lemma 5.9.** Let \( \kappa \in (0, 1). \) Then
\[
C_\nu = C_\infty \times \left( 1 + O \left( \frac{(G'_\infty)^2}{\nu \eta(t)} + \frac{1}{\nu \gamma(t) \delta(t)} + \frac{1}{\nu \delta(t) \eta(t)} + \frac{1}{\nu^2 \eta(t)} \right) \right),
\]

(5.22)
and
\[ C_\nu(t) = C_\infty(t) + O \left( \frac{(G_\infty')^2}{\nu \gamma(t)^{3/2}} + \frac{\eta(t)}{\nu \delta(t) \gamma(t)^{3/2}} + \frac{1}{\nu \delta(t) \gamma(t)^{3/2}} + \frac{1}{\nu^2 \gamma(t)^{3/2}} \right), \quad (5.23) \]
where
\[ \eta(t) := (1 - G_\nu^2) - \frac{\lambda_0}{\lambda_2} (G_\nu')^2 \quad \text{and} \quad \gamma(t) = 1 - G_\infty(t)^2. \]

**Proof.** Observe that
\[ G_\nu^2 = \left( G_\infty + O \left( \frac{1}{\nu} \right) \right)^2 = G_\infty^2 + O \left( \frac{1}{\nu \delta(t)} \right), \quad (5.24) \]
so that
\[ 1 - G_\infty(t)^2(t) = \left( 1 - G_\infty(t)^2 \right) \left( 1 + O \left( \frac{1}{\nu \delta(t) \gamma(t)} \right) \right). \]
For \( t > \nu^{-\kappa} \) we have
\[ \frac{1}{\nu \delta(t) \gamma(t)} = O \left( \frac{1}{\nu^{1-\kappa}} \right) = o(1) \quad \text{uniformly in } t > \nu^{-\kappa} \text{ as } \nu \to \infty. \]

Hence
\[ (1 - G_\nu^2)^{-3/2} = (1 - G_\infty^2)^{-3/2} \left( 1 + O \left( \frac{1}{\nu \delta(t) \gamma(t)} \right) \right), \quad \gamma(t) := 1 - G_\infty^2(t). \quad (5.25) \]

Next observe that
\[ \lambda_0(\nu) = \frac{B_3(\nu + 1)}{3 \nu^3} = \frac{1}{3} + \nu^{-1} + O(\nu^{-2}), \quad \lambda_2(\nu) = \frac{B_5(\nu + 1)}{5 \nu^5} = \frac{1}{5} + \nu^{-1} + O(\nu^{-2}) \quad (5.26) \]
and
\[ \frac{\lambda_0(\nu)}{\lambda_2(\nu)} = \frac{\frac{1}{3} + \nu^{-1} + O(\nu^{-2})}{\frac{1}{5} + \nu^{-1} + O(\nu^{-2})} = \frac{5}{3} - \frac{10}{3} \nu^{-1} + O(\nu^{-2}). \]

Using (5.24) and the above estimate we deduce
\[ (1 - G_\nu(t)^2)^{-3/2} \left( \frac{\lambda_0(\nu)}{\lambda_2(\nu)} \right) (G_\nu')^2 = (1 - G_\infty^2) - \frac{\lambda_0}{\lambda_2} (G_\infty')^2 + \frac{10}{3 \nu} (G_\infty')^2 + O \left( \frac{1}{\nu \delta(t)} \right) + O \left( \frac{1}{\nu^2} \right), \quad (5.27) \]

Using (5.25) we deduce (5.22). The estimate (5.23) follows from (5.22) by invoking the definitions of \( \gamma(t) \) and \( \eta(t) \). \( \square \)

**Lemma 5.10.** Let \( \kappa \in (0, \frac{1}{3}) \). Then
\[ \rho_\nu = \rho_\infty + O \left( \frac{1}{\nu \eta(t)} + \frac{(G_\infty')^2}{\nu \eta(t) \delta(t)} + \frac{1}{\nu \delta(t)^2} + \frac{1}{\nu \delta(t)^2 \eta(t)} + \frac{1}{\nu^2 \eta(t) \delta(t)} \right), \quad \forall t > \nu^{-\kappa}, \quad (5.28) \]
where the constant implied by \( O \)-symbol is independent of \( \nu \) and \( t > \nu^{-\kappa} \), but it could depend on \( \kappa \).
Proof. The estimates (5.19b), (5.19d) and (5.21) imply that for \( t > \nu^{-\kappa} \) we have
\[
\frac{(G'_\infty)^2}{\nu\eta(t)} + \frac{1}{\nu\delta(t)\eta(t)} + \frac{1}{\nu^2\eta(t)} = O\left(\frac{1}{\nu^{1-4\kappa}}\right) = o(1),
\]
uniformly in \( t > \nu^{-\kappa} \). We conclude from (5.27) that
\[
\left((1 - G_\nu(t)^2) - \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G'_\nu)^2\right)^{-1} = \left((1 - G_\infty^2) - \frac{\lambda_0}{\lambda_2}(G'_\infty)^2\right)^{-1}
\]
\[
\times \left(1 + O\left(\frac{(G'_\infty)^2}{\nu\eta(t)} + \frac{1}{\nu\delta(t)\eta(t)} + \frac{1}{\nu^2\eta(t)}\right)\right).
\]
(5.29)

Since \( H_\nu = H_\infty + O(\nu^{-1}) \) we deduce
\[
-H_\nu(1 - G_\nu)^2 + \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G'_\nu)^2G_\nu = -H_\infty(1 - G_\infty)^2 + \frac{\lambda_0}{\lambda_2}(G'_\infty)^2G_\infty + O\left(\frac{1}{\nu}\right)
\]
Recalling that
\[
\rho_\nu = \frac{-H_\nu(1 - G_\nu^2) + \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G'_\nu)^2G_\nu}{(1 - G_\infty^2) - \frac{\lambda_0}{\lambda_2}(G'_\infty)^2}
\]
and \(-H_\infty(1 - G_\infty)^2 + \frac{\lambda_0}{\lambda_2}(G'_\infty)^2G_\infty = O(\delta^{-1})\) we see that (5.28) follows from (eq: est22). \(\square\)

Consider the function
\[
A(u) = \sqrt{1 - u^2} + u \arcsin u, \quad |u| \leq 1.
\]
Observe that
\[
\frac{dA}{du} = O(1), \quad \forall |u| \leq 1, \quad (5.30a)
\]
\[
\frac{dA}{du} = \arcsin u = O(u) \text{ as } u \to 0. \quad (5.30b)
\]
Now fix an exponent \( \kappa \in (0, \frac{1}{4}) \). We discuss separately two cases.

1. If \( \nu^k < t < \pi\nu \). Then in this range we have
\[
\frac{1}{\delta} \rho_\infty = O(t^{-1}), \quad \rho_\nu = \rho_\infty + o(1), \quad \frac{1}{\eta(t)} = O(1),
\]
and using (5.30b) and (5.28) we deduce
\[
A(\rho_\nu) = A(\rho_\infty) + A'(\rho_\infty)(\rho_\nu - \rho_\infty) + O\left( (\rho_\nu - \rho_\infty)^2 \right) = A(\rho_\infty) + O\left(\frac{1}{\nu t\nu} + \frac{1}{\nu^2}\right). \quad (5.31)
\]

2. \( \nu^{-\kappa} < t < \nu^\kappa \). The equality (5.21) shows that in this range we have
\[
\frac{1}{\eta(t)} = O(\nu^{-4\kappa}), \quad \frac{1}{\delta(t)} = O(1)
\]
so that and (5.28) implies that
\[
\rho_\nu - \rho_\infty = O\left(\frac{1}{\nu^{1-4\kappa}}\right).
\]
Using (5.30a) we deduce
\[
A(\rho_\nu) - A(\rho_\infty) = O\left(\frac{1}{\nu^{1-4\kappa}} + \frac{1}{\nu^{1-4\kappa}t\delta(t)} + \frac{1}{\nu^{2-8\kappa}t\delta(t)}\right), \quad (5.32)
\]
To prove (5.1) we need to prove the following equality.

Then using (5.26) we deduce that

\[ q_\nu - q_\infty = \frac{6}{5} \nu^{-1} + O(\nu^{-2}), \quad q_\nu = q_\infty \left( 1 + 2\nu^{-1} + O(\nu^{-2}) \right). \]

Using (5.23) and (5.32) we deduce

\[ (q_\nu - q_\infty)(C_\infty A(\rho_\infty) - 1) + q_\infty \Delta_\nu. \]

To prove (5.1) we need to prove the following equality.

\[ (q_\nu - q_\infty) \int_0^{\pi \nu} \left( C_\infty(t)A(\rho_\infty(t)) - 1 \right) dt, \quad q_\infty \int_0^{\pi \nu} \Delta_\nu(t) dt = o(1) \text{ as } \nu \to \infty. \quad (5.33) \]

We can dispense easily of the first integral above since \( C_\infty(t)A(\rho_\infty(t)) - 1 \) is absolutely integrable on \([0, \infty)\) and \( q_\nu - q_\infty = O(\nu^{-1}) \).

The second integral requires a bit of work. More precisely, we will show the following result.

**Lemma 5.11.** If \( 0 < \kappa < \frac{1}{2} \), then

\[ \int_{\nu^{-k}}^{\nu^k} \Delta_\nu(t) dt, \quad \int_{\nu^{-k}}^{\nu^k} \Delta_\nu(t) dt, \quad \int_{\nu^{-k}}^{\nu^k} \Delta_\nu(t) dt = o(1) \text{ as } \nu \to \infty. \quad (5.34) \]

**Proof.** We will discuss each of the three cases separately.

1. \( 0 < t < \nu^{-k} \). The easiest way to prove that \( \int_0^{\nu^{-k}} \Delta_\nu(t) dt \to 0 \) is to show that

\[ C_\nu(t)A(\rho_\nu(t)) = O(1), \quad 0 < t < \nu^{-k}. \]

This follows using Lemma 5.8 and observing that the function \( A(u) \) is bounded.

2. \( \nu^{-k} < t < \nu^k \). In this range we have

\[ \frac{1}{\eta} = O(\nu^{-4\kappa}), \quad \frac{1}{\gamma(t)} = O(\nu^{-2\kappa}), \quad \frac{1}{\delta(t)} = O(1), \quad G'_\infty(t) = O(1). \]

Using (5.23), (5.32) we deduce

\[ C_\nu(t) = C_\infty(t) + O \left( \frac{1}{\nu^{1-4\kappa}} \right), \quad A(\rho_\nu) = A(\rho_\infty) + O \left( \frac{1}{\nu^{1-4\kappa}} \right). \]

Hence

\[ \Delta_\nu = O \left( \frac{1}{\nu^{1-4\kappa}} \right) \quad \text{and} \quad \int_{\nu^{-k}}^{\nu^k} \Delta_\nu(t) dt = o(1). \]

3. \( \nu^k < t < \pi \nu \). For these values of \( t \) we have

\[ \eta(t), \quad \frac{1}{\eta(t)} = O(1), \quad \frac{1}{\delta(t)} = O(t^{-1}). \]

Using (5.23) and (5.31) we deduce

\[ C_\nu(t) = C_\infty(t) + O \left( \frac{1}{\nu t} + \frac{1}{\nu^2} \right), \quad A(\rho_\nu) = A(\rho_\infty) + O \left( \frac{1}{\nu t} + \frac{1}{\nu^2} \right). \]
so that
\[ \Delta_\nu(t) = O\left(\frac{1}{\nu^2} + \frac{1}{\nu^2} \right), \quad \int_{\nu^{-\infty}}^{\nu^\infty} \Delta_\nu(t) \, dt = O\left(\frac{1}{\nu} + \frac{\log \nu}{\nu}\right) = o(1). \]

The fact that \( \delta_\infty \) defined as in (5.1) is positive follows by arguing exactly as in [19, §3.2]. This completes the proof of Theorem 5.1.

**Remark 5.12.** The proof of Lemma 5.11 shows that for any \( \varepsilon > 0 \) we have
\[ \text{var}(Z_\nu) = \nu \delta_\infty + O(\nu^\varepsilon) \quad \text{as} \quad \nu \to \infty. \] (5.35)
Numerical experiments suggest that \( \delta_\infty \approx 0.35. \)

**APPENDIX A. GAUSSIAN MEASURES AND GAUSSIAN RANDOM FIELDS**

For the reader’s convenience we survey here a few basic facts about Gaussian measures. For more details we refer to [7]. A Gaussian measure on \( \mathbb{R} \) is a Borel measure \( \gamma_{m,\sigma} \) of the form
\[ \gamma_{m,\sigma}(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx. \]
The scalar \( m \) is called the *mean* while \( \sigma \) is called the *standard deviation*. We allow \( \sigma \) to be zero in which case \( \gamma_{m,0} = \delta_m \) = the Dirac measure on \( \mathbb{R} \) concentrated at \( m \).

Suppose that \( V \) is a finite dimensional vector space. A Gaussian measure on \( V \) is a Borel measure \( \gamma \) on \( V \) such that, for any \( \xi \in V^\vee \), the pushforward \( \xi_* \gamma \) is a Gaussian measure on \( \mathbb{R} \),
\[ \xi_* \gamma = \gamma_{m(\xi),\sigma(\xi)}. \]
One can show that the map \( V^\vee \ni \xi \mapsto m(\xi) \in \mathbb{R} \) is linear, and thus can be identified with a vector \( m_\gamma \in V \) called the *barycenter* or expectation of \( \gamma \) that can be alternatively defined by the equality
\[ m_\gamma = \int_V v \, d\gamma(v). \]
Moreover, there exists a nonnegative definite, symmetric bilinear map
\[ \Sigma : V^\vee \times V^\vee \to \mathbb{R} \]
such that
\[ \sigma(\xi)^2 = \Sigma(\xi, \xi), \quad \forall \xi \in V^\vee. \]
The form \( \Sigma \) is called the *covariance form* and can be identified with a linear operator \( S : V^\vee \to V \) such that
\[ \Sigma(\xi, \eta) = \langle \xi, S\eta \rangle, \quad \forall \xi, \eta \in V^\vee, \]
where \( \langle - , - \rangle : V^\vee \times V \to \mathbb{R} \) denotes the natural bilinear pairing between a vector space and its dual. The operator \( S \) is called the *covariance operator* and it is explicitly described by the integral formula
\[ \langle \xi, S\eta \rangle = \Lambda(\xi, \eta) = \int_V \langle \xi, v - m_\gamma \rangle \langle \eta, v - m_\gamma \rangle \, d\gamma(v). \]

The Gaussian measure is said to be *nondegenerate* if \( \Sigma \) is nondegenerate, and it is called *centered* if \( m = 0 \). A nondegenerate Gaussian measure on \( V \) is uniquely determined by its covariance form and its barycenter.
Example A.1. Suppose that $U$ is an $n$-dimensional Euclidean space with inner product $\langle -,- \rangle$. We use the inner product to identify $U$ with its dual $U^\vee$. If $A : U \to U$ is a symmetric, positive definite operator, then

$$d\gamma_A(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det A}}e^{-\frac{1}{2}(A^{-1}u,u)}|du|$$

(A.1)

is a centered Gaussian measure on $U$ with covariance form described by the operator $A$. \qed

If $V$ is a finite dimensional vector space equipped with a Gaussian measure $\gamma$ and $L : V \to U$ is a linear map then the pushforward $L_*\gamma$ is a Gaussian measure on $U$ with barycenter

$$m_{L,*}\gamma = L(m_\gamma)$$

and covariance form

$$\Sigma_{L,*}\gamma : U^\vee \times U^\vee \to \mathbb{R}$$

given by

$$\Sigma_{L,*}\gamma(\eta,\eta) = \Sigma_\gamma(L^\vee \eta,L^\vee \eta), \ \forall \eta \in U^\vee,$$

where $L^\vee : U^\vee \to V^\vee$ is the dual (transpose) of the linear map $L$. Observe that if $\gamma$ is nondegenerate and $L$ is surjective, then $L_*\gamma$ is also nondegenerate.

Suppose $(\mathcal{S}, \mu)$ is a probability space. A Gaussian random vector on $(\mathcal{S}, \mu)$ is a (Borel) measurable map

$$X : \mathcal{S} \to V, \ V \text{ finite dimensional vector space}$$

such that $X_*\mu$ is a Gaussian measure on $V$. We will refer to this measure as the associated Gaussian measure, we denote it by $\gamma_X$ and we denote by $\Sigma_X$ (respectively $S_X$) its covariance form (respectively operator),

$$\Sigma_X(\xi_1,\xi_2) = E\left(\langle \xi_1, X - E(X) \rangle \langle \xi_2, X - E(X) \rangle\right).$$

Note that the expectation of $\gamma_X$ is precisely the expectation of $X$. The random vector is called nondegenerate, respectively centered, if the Gaussian measure $\gamma_X$ is such.

Suppose that $X_j : \mathcal{S} \to V_j, j = 1, 2$, are two centered Gaussian random vectors such that the direct sum

$$X_1 \oplus X_2 : \mathcal{S} \to V_1 \oplus V_2$$

is also a centered Gaussian random vector with associated Gaussian measure

$$\gamma_{X_1 \oplus X_2} = p_{X_1 \oplus X_2}(x_1,x_2)|dx_1dx_2|.$$

We obtain a bilinear form

$$\text{cov}(X_1, X_2) : V_1^\vee \times V_2^\vee \to \mathbb{R}, \ \text{cov}(X_1, X_2)(\xi_1, \xi_2) = \Sigma(\xi_1, \xi_2),$$

called the covariance form. The random vectors $X_1$ and $X_2$ are independent if and only if they are uncorrelated, i.e.,

$$\text{cov}(X_1, X_2) = 0.$$

We can form the random vector $E(X_1|X_2)$, the conditional expectation of $X_1$ given $X_2$. If $X_1$ and $X_2$ are independent then $E(X_1|X_2) = E(X_1)$, while at the other extreme we have $E(X_1|X_2)$.

To find a formula for $E(X_1|X_2)$ in general we fix Euclidean metrics $\langle -,- \rangle_j$ on $V_j$. We can then identify $\text{cov}(X_1, X_2)$ with a linear operator $\text{Cov}(X_1, X_2) : V_2 \to V_1$, via the equality

$$E\left(\langle \xi_1, X_1 \rangle \langle \xi_2, X_2 \rangle\right) = \text{cov}(X_1, X_2)(\xi_1, \xi_2) = \langle \xi_1, \text{Cov}(X_1, X_2)\xi_2^\dagger \rangle, \ \forall \xi_1 \in V_1^\vee, \ \xi_2 \in V_2^\vee.$$
where $\xi_2^\dagger \in V_2$ denotes the vector metric dual to $\xi_2$. The operator $\text{Cov}(X_1, X_2)$ is called the covariance operator of $X_1, X_2$. If $X_2$ is nondegenerate, then we have the regression formula

$$E(X_1|X_2) = \text{Cov}(X_1, X_2)S_{X_2}^{-1}(X_2 - E(X_2)) + E(X_1).$$

(A.2)

To prove this we follow the elegant argument in [3, Prop. 1.2]. Observe first that it suffices to prove that it suffices to prove the equality in the special case when $E(X_1) = 0$ and $E(X_2) = 0$, which is what we will assume in the sequel. We seek a linear operator $C : V_2 \to V_1$ such that the random vector $Y = X_1 - CX_2$ is independent of $X_2$. If such an operator exists then

$$E(X_1|X_2) = E(Y|X_2) + E(CX_2|X_2) = E(Y) + CX_2 = CX_2.$$

Since the random vector $X_1 - CX_2$ is Gaussian the operator $C$ must satisfy the constraint

$$\text{cov}(X_1 - CX_2, X_2) = 0 \iff E\{\text{cov}(X_1 - CX_2, X_2)\} = \text{Cov}(X_1, X_2) - \text{Cov}(CX_2, X_2)$$

To find $C$ we note that

$$\langle \xi_1, \text{Cov}(CX_2, X_2)\xi_2^\dagger \rangle = E\{\langle \xi_1, CX_2\rangle \langle \xi_2, X_2\rangle\} = E\{(C^\dagger \xi_1, X_2)\langle \xi_2, X_2\rangle\} = \Sigma_{X_2}(C^\dagger \xi_1, \xi_2) = \langle \xi_1, C\Sigma_{X_2}\xi_2 \rangle.$$

Hence identifying $V_2$ with $V_2^\dagger$ via the Euclidean metric $\langle \cdot, \cdot \rangle_{V_2}$, we can regard $\Sigma_{X_2}$ as a linear, symmetric nonnegative operator $V_2 \to V_2$, and we deduce

$$\text{Cov}(CX_2, X_2) = C\Sigma_{X_2} = \text{Cov}(X_1, X_2),$$

which shows that

$$C = \text{Cov}(X_1, X_2)\Sigma_{X_2}^{-1}.$$ (A.3)

The conditional probability density of $X_1$ given that $X_2 = x_2$ is the function

$$p(x_1|X_2 = x_2)(x_1) = \frac{p_{X_1|X_2}(x_1, x_2)}{\int_{V_1} p_{X_1|X_2}(x_1, x_2)|dx_1|}.$$ 

For a measurable function $f : V_1 \to \mathbb{R}$ the conditional expectation $E(f(X_1)|X_2 = x_2)$ is the (deterministic) scalar

$$E(f(X_1)|X_2 = x_2) = \int_{V_1} f(x_1)p(X_1|X_2 = x_2)(x_1)|dx_1|.$$

Again, if $X_2$ is nondegenerate, then we have the regression formula

$$E(f(X_1)|X_2 = x_2) = E\{f(Y + CX_2)\}$$

where $Y : \mathbb{S} \to V_1$ is a Gaussian vector with

$$E(Y) = E(X_1) - CE(X_2), \quad \Sigma_Y = S_{X_1} - \text{Cov}(X_1, X_2)S_{X_2}^{-1}\text{Cov}(X_2, X_1),$$ (A.5)

and $C$ is given by (A.3).

Let us point out that if $X : \mathbb{S} \to U$ is a Gaussian random vector and $L : U \to V$ is a linear map, then the random vector $LX : \mathbb{S} \to V$ is also Gaussian. Moreover

$$E(LX) = LE(X), \quad \Sigma_{LX}(\xi, \xi) = \Sigma_X(L^\dagger \xi, L^\dagger \xi), \quad \forall \xi \in V^\vee,$$

where $L^\dagger : V^\vee \to U^\vee$ is the linear map dual to $L$. Equivalently,

$$S_{LX} = LS_XL^\dagger.$$

A random field (or function) on a set $T$ is a map

$$\xi : T \times (\mathbb{S}, \mu) \to \mathbb{R}, \quad (t, s) \mapsto \xi_t(s)$$

such that
• \((S, \mu)\) is a probability space, and
• for any \(t \in T\) the function \(\xi_t : S \to \mathbb{R}\) is measurable, i.e., it is a random variable.

Thus, a random field on \(T\) is a family of random variables \(\xi_t\) parameterized by the set \(T\). For simplicity we will assume that all these random variables have finite second moments. For any \(t \in T\) we denote by \(\mu_t\) the expectation of \(\xi_t\). The covariance function or kernel of the field is the function \(C_\xi : T \times T \to \mathbb{R}\) defined by

\[
C_\xi(t_1, t_2) = \mathbb{E}( (\xi_{t_1} - \mu_{t_1})(\xi_{t_2} - \mu_{t_2}) ) = \int_S (\xi_{t_1}(s) - \mu_{t_1})(\xi_{t_2}(s) - \mu_{t_2}) \, d\mu(s).
\]

The field is called Gaussian if for any finite subset \(F \subset T\) the random vector
\[
S \in s \mapsto (\xi_t(s))_{t \in F} \in \mathbb{R}^F
\]
is a Gaussian random vector. Almost all the important information concerning a Gaussian random field can be extracted from its covariance kernel.

Here is a simple method of producing Gaussian random fields on a set \(T\). Choose a finite dimensional space \(U\) of real valued functions on \(T\). Once we fix a Gaussian measure \(d\gamma\) on \(U\) we obtain tautologically a random field

\[
\xi : T \times U \to \mathbb{R}, \quad (t, u) \mapsto \xi_t(u) = u(t).
\]

This is a Gaussian field since for any finite subset \(F \subset T\) the random vector
\[
\Xi : U \to \mathbb{R}^F, \quad u \mapsto (u(t))_{t \in F}
\]
is Gaussian because the map \(\Xi\) is linear and thus the pushforward \(\Xi_*d\gamma\) is a Gaussian measure on \(\mathbb{R}^F\). For more information about random fields we refer to [1, 3, 18].

In the conclusion of this section we want to describe a few simple integral formulas.

**Proposition A.2.** Suppose \(V\) is an Euclidean space of dimension \(N\), \(f : U \to \mathbb{R}\) is a locally integrable, positively homogeneous function of degree \(k \geq 0\), and \(A : U \to U\) is a positive definite symmetric operator. Denote by \(B(U)\) the unit ball of \(V\) centered at the origin, and by \(S(U)\) its boundary. Then the following hold

\[
\frac{1}{\pi \frac{N}{2} (k + N)} \int_{S(U)} f(u)|dA(u)| = \frac{1}{\pi \frac{N}{2}} \int_{B(U)} f(u)|du| = \frac{1}{\Gamma(1 + k + \frac{N}{2})} \int_U f(u) \frac{e^{-|u|^2}}{\pi \frac{N}{2}} |du|.
\]

(A.6)

\[
\int_U f(u)d\gamma_A(u) = t^k \int_U f(u)d\gamma_{tA}(u) \quad \forall t > 0,
\]

(A.7)

where \(d\gamma_A\) is the Gaussian measure defined by (A.1).

**Proof.** We have

\[
\int_{B(U)} f(u)|du| = \int_0^1 t^{k+N-1} \left( \int_{S(U)} f(u)|dA(u)| \right) = \frac{1}{k + N} \int_{S(U)} f(u)|dA(u)|.
\]

On the other hand

\[
\frac{1}{\pi \frac{N}{2}} \int_U f(u)e^{-|u|^2} |du| = \frac{1}{\pi \frac{N}{2}} \left( \int_0^\infty t^{k+N-1}e^{-t^2} dt \right) \int_{S(U)} f(u)|dA(u)|
\]

\[
= \frac{1}{2\pi \frac{N}{2}} \Gamma \left( \frac{k + N}{2} \right) \int_{S(U)} f(u)|dA(u)| = \frac{k + N}{2\pi \frac{N}{2}} \Gamma \left( \frac{k + N}{2} \right) \int_{B(U)} f(u)|du|.
\]
\[ = \frac{1}{\pi^2} \Gamma \left(1 + \frac{k + N}{2}\right) \int_{B(U)} f(u) |du|. \]

This proves (A.6). The equality (A.7) follows by using the change in variables \( u = t^2 v \).

**APPENDIX B. GAUSSIAN RANDOM SYMMETRIC MATRICES**

We want to describe in some detail a 3-parameter family of centered Gaussian measures on \( S_m \), the vector space of real symmetric \( m \times m \) matrices, \( m > 1 \).

For any \( 1 \leq i \leq j \) define \( \xi_{ij} \in S_m^\vee \) so that for any \( A \in S_m \)

\[ \xi_{ij}(A) = a_{ij} = \text{the } (i,j)\text{-entry of the matrix } A. \]

The collection \( (\xi_{ij})_{1 \leq i \leq j \leq m} \) is a basis of the dual space \( S_m^\vee \). We denote by \( (E_{ij})_{1 \leq i \leq j} \) the dual basis of \( S_m \). More precisely, \( E_{ij} \) is the symmetric matrix whose \((i,j)\) and \((j,i)\) entries are 1 while all the other entries are equal to zero. For any \( A \in S_m \) we have

\[ A = \sum_{i \leq j} \xi_{ij}(A) E_{ij}. \]

The space \( S_m \) is equipped with an inner product

\[ (\cdot, \cdot) : S_m \times S_m \to \mathbb{R}, \quad (A, B) = \text{tr}(AB), \quad \forall A, B \in S_m. \]

This inner product is invariant with respect to the action of \( \text{SO}(m) \) on \( S_m \). We set

\[ \hat{E}_{ij} := \begin{cases} E_{ij}, & i = j \\ \frac{1}{\sqrt{2}} E_{ij}, & i < j. \end{cases} \]

The collection \( (\hat{E}_{ij})_{i \leq j} \) is a basis of \( S_m \) orthonormal with respect to the above inner product. We set

\[ \hat{\xi}_{ij} := \begin{cases} \xi_{ij}, & i = j \\ \sqrt{2} \xi_{ij}, & i < j. \end{cases} \]

The collection \( (\hat{\xi}_{ij})_{i \leq j} \) the orthonormal basis of \( S_m^\vee \) dual to \( (\hat{E}_{ij}) \).

To any numbers \( a, b, c \) satisfying the inequalities

\[ a - b, \quad c, \quad a + (m - 1)b > 0. \quad (B.1) \]

we will associate a centered Gaussian measure \( \Gamma_{a,b,c} \) on \( S_m \) uniquely determined by its covariance form

\[ \Sigma = \Sigma_{a,b,c} : S_m^\vee \times S_m^\vee \to \mathbb{R} \]

defined as follows:

\[ \Sigma(\xi_{ii}, \xi_{ii}) = a, \quad \Sigma(\xi_{ii}, \xi_{ij}) = b, \quad \forall i \neq j; \quad (B.2a) \]

\[ \Sigma(\xi_{ij}, \xi_{ij}) = c, \quad \Sigma(\xi_{ij}, \xi_{k\ell}) = 0, \quad \forall i < j, k \leq \ell, \quad (i,j) \neq (k,\ell). \quad (B.2b) \]

To see that \( \Sigma_{a,b,c} \) is positive definite if \( a, b, c \) satisfy (B.1) we decompose \( S_m^\vee \) as a direct sum of subspaces

\[ S_m^\vee = D_m \oplus O_m, \]

\[ D_m = \text{span} \{ \xi_{ii}; \ 1 \leq i \leq m \}, \quad O_m = \text{span} \{ \xi_{ij}; \ 1 \leq i < j \leq m \}, \quad \text{dim } O_m = \binom{m}{2}. \]

With respect to this decomposition, and the corresponding bases of these subspaces the matrix \( Q_{a,b,c} \) describing \( \Sigma_{a,b,c} \) with respect to the basis \( (\xi_{ij}) \) has a direct sum decomposition

\[ Q_{a,b,c} = G_m(a, b) \oplus c I_{\binom{m}{2}}, \]
where $G_m(a, b)$ is the $m \times m$ symmetric matrix whose diagonal entries are equal to $a$ while all the off diagonal entries are all equal to $b$.

The the spectrum of $G_m(a, b)$ consists of two eigenvalues: $(a - b)$ with multiplicity $(m - 1)$ and the simple eigenvalue $a - b + mb$. Indeed, if $C_m$ denotes the $m \times m$ matrix with all entries equal to 1, then $G_m(a, b) = (a - b)\mathbb{I}_m + bC_m$. The matrix $C_m$ has rank 1 and a single nonzero eigenvalue equal to $m$ with multiplicity 1. This proves that $Q_{a, b, c}$ is positive definite since its spectrum is positive. We denote by $d\Gamma_{a, b, c}$ the centered Gaussian measure on $S_m$ with covariance form $\Sigma_{a, b, c}$.

Since $S_m$ is equipped with an inner product we can identify $\Sigma_{a, b, c}$ with a symmetric, positive definite bilinear form on $S_m$. We would like to compute the matrix $\hat{Q} = \hat{Q}_{a, b, c}$ that describes $\Sigma_{a, b, c}$ with respect to the orthonormal basis $(\hat{E}_{ij})_{1 \leq i \leq j}$. We have

$$\hat{Q}(\hat{E}_{ii}, \hat{E}_{ii}) = Q(\xi_{ii}, \xi_{ii}) = a, \quad \hat{Q}(\hat{E}_{ii}, \hat{E}_{jj}) = b, \quad \forall i \neq j,$$

$$\hat{Q}(\hat{E}_{ij}, \hat{E}_{ij}) = Q(\xi_{ij}, \xi_{ij}) = 2Q(\xi_{ij}, \xi_{ij}) = 2c, \quad \forall i < j,$$

Thus

$$\hat{Q}_{a, b, c} = G_m(a, b) \oplus 2c\mathbb{I}_{\binom{m}{2}}. \tag{B.3}$$

If $| - |_{a, b, c}$ denotes the Euclidean norm on $S_m$ determined by $\Sigma_{a, b, c}$ then for

$$A = \sum_{i \leq j} a_{ij}E_{ij} = \sum_{i} a_{ii}\hat{E}_{ii} + \sqrt{2}\sum_{i < j} a_{ij}\hat{E}_{ij},$$

we have

$$|A|_{a, b, c}^2 = a \sum_{i} a_{ii}^2 + 2b \sum_{i < j} a_{ii}a_{jj} + 4c \sum_{i < j} a_{ij}^2$$

$$= (a - b - 2c) \sum_{i} a_{ii}^2 + b \left( \sum_{i} a_{ii} \right)^2 + 2c \left( \sum_{i} a_{ii}^2 + 2 \sum_{i < j} a_{ij}^2 \right)$$

$$= (a - b - 2c) \sum_{i} a_{ii}^2 + b(\text{tr} A)^2 + 2c \text{tr} A^2.$$

Observe that when

$$a - b = 2c \tag{B.4}$$

we have

$$|A|_{a, b, c}^2 = b(\text{tr} A)^2 + 2c \text{ tr} A^2 \tag{B.5}$$

so that the norm $| - |_{a, b, c}$ and the Gaussian measure $d\Gamma_{a, b, c}$ are $O(m)$-invariant. Let us point out that the space $S_m$ equipped with the Gaussian measure $d\Gamma_{2, 0, 1}$ is the well known GOE, the Gaussian orthogonal ensemble.

To obtain a more concrete description of $\Gamma_{a, b, c}$ we first identify $\Sigma_{a, b, c}$ with a symmetric operator

$$\hat{Q}_{a, b, c} : S_m \rightarrow S_m.$$

Using (B.3) we deduce that

$$\hat{Q}_{a, b, c} = G(a, b) \oplus 2c\mathbb{I}_{\binom{m}{2}}.$$

Observe that

$$\det \hat{Q}_{a, b, c} = (a - b)(a + (m - 1)b)^{m-1}(2c)^{\binom{m}{2}}, \tag{B.6}$$

and

$$\hat{Q}_{a, b, c}^{-1} = \hat{Q}_{a', b', c'} = G_m(a', b') \oplus 2c'\mathbb{I}_{\binom{m}{2}}, \tag{B.7}$$
where \(2c' = \frac{1}{2c}\) and the real numbers \(a', b'\) are determined from the linear system
\[
\begin{align*}
a' - b' &= \frac{1}{a-b} \\
a' + (m-1)b' &= \frac{1}{a+(m-1)b}.
\end{align*}
\] (B.8)

We then have
\[
d\Gamma_{a,b,c}(X) = \frac{1}{(2\pi)^{m(m+1)/2} (\det \tilde{Q}_{a,b,c})^{1/2}} \cdot e^{-\frac{1}{4c} (\tilde{Q}_{a,b,c}^{-1}X,X) 2^{1/2} (m) \prod_{i\leq j} dx_{ij}},
\] (B.9)

where
\[
(\tilde{Q}_{a,b,c}^{-1}X,X) = \left( a' - b' - \frac{1}{2c} \right) \sum x_{ii}^2 + b'(\text{tr } X)^2 + \frac{1}{2c} \text{tr } X^2.
\] (B.10)

The special case \(b = c > 0, a = 3c\) is particularly important for our considerations. We denote by \((-,-,c)\) and respectively \(d\Gamma_c\) the inner product and respectively the Gaussian measure on \(S_m\) corresponding to the covariance form \(\Sigma_{3c,c,c}\).

If we set \(\tilde{Q}_c := \tilde{Q}_{3c,c,c}\) then we deduce from (B.7) that
\[
\tilde{Q}_c^{-1} = \tilde{Q}_{a',b',c'} = G_m(a',b') \oplus \frac{1}{2c} \mathbb{I}_{\binom{m}{2}},
\]

where
\[
\begin{align*}
a' - b' &= \frac{1}{2c} = 2c' \\
a' + (m-1)b' &= \frac{1}{(m+2)c}.
\end{align*}
\]

We deduce
\[
m b' = \frac{1}{(m+2)c} - \frac{1}{2c} = \frac{m}{2c(m+2)} 
\Rightarrow b' = \frac{1}{2c(m+2)}.
\]

Note that the invariance condition (B.4) \(a' - b' = 2c'\) is automatically satisfied so that
\[
(\tilde{Q}_c^{-1}X,X) = \frac{1}{2c} \text{tr } X^2 - \frac{1}{2c(m+2)} (\text{tr } X)^2.
\]

Using (B.6) and (B.9) we deduce
\[
d\Gamma_c(X) = \frac{1}{(2\pi c)^{m(m+1)/4} \sqrt{\mu_m}} \cdot e^{-\frac{1}{4c} \left( \text{tr } X^2 - \frac{1}{m+2} (\text{tr } X)^2 \right) \prod_{i\leq j} |dx_{ij}|},
\] (B.11)

where
\[
\mu_m := 2^{\binom{m}{2}} (m+2)^{m-1}.
\] (B.12)

The inner product \((-,-)\) has the alternate description
\[
(A,B)_c = I_c(A,B) := 4c \int_{\mathbb{R}^m} (Ax,x)(Bx,x) \frac{e^{-|x|^2}}{\pi^{m/2}} |dx|
\] (B.13)

To verify (B.13) it suffices to show that
\[
I_c(E_{ii},E_{ii}) = 3c, \quad I_c(E_{ii},E_{jj}) = c, \quad I_c(E_{ij},E_{ij}) = 4c, \quad \forall 1 \leq i < j \leq m,
\]
\[
I_c(E_{ij},E_{k\ell}) = 0, \quad \forall 1 \leq i < j \leq m, \quad k \leq \ell, \quad (i,j) \neq (k,\ell).
\]
To achieve this we need to use the classical identity
\[ \int_{\mathbb{R}^m} x_1^{2p_1} \cdots x_m^{2p_m} e^{-|x|^2} \, dx = \prod_{k=1}^{m} \int_{\mathbb{R}} x_k^{2p_k} e^{-t^2} \, dx_k = \prod_{k=1}^{m} \Gamma \left( p_k + \frac{1}{2} \right). \]

Observe that
\[ \int_{\mathbb{R}^m} (E_{ij}(x, x))(E_{jj}(x, x)) \frac{e^{-|x|^2}}{\pi^2} |dx| = \int_{\mathbb{R}^m} x_i^2 x_j^2 \frac{e^{-|x|^2}}{\pi^2} |dx| = \pi^{-m} \Gamma \left( \frac{1}{2} \right)^m \frac{1}{4} \prod_{i<j} |\Gamma \left( \frac{3}{2} \right)|. \]

Next, if \( i < j \) we have
\[ \int_{\mathbb{R}^m} (E_{ij}(x, x))(E_{jj}(x, x)) \frac{e^{-|x|^2}}{\pi^2} |dx| = 4 \int_{\mathbb{R}^m} x_i^2 x_j^2 \frac{e^{-|x|^2}}{\pi^2} |dx| = 1. \]

Finally, if \( i < j, k \leq \ell \) and \( (i, j) \neq (k, \ell) \), then the quartic polynomial
\( (E_{ij}(x, x))(E_{k\ell}(x, x)) \)

is odd with respect to a reflection \( x_p \mapsto -x_p \) for some \( p = \{i, j, k, \ell\} \) and thus
\[ \int_{\mathbb{R}^m} (E_{ij}(x, x))(E_{k\ell}(x, x)) \frac{e^{-|x|^2}}{\pi^2} |dx| = 0. \]

**APPENDIX C. SOME GAUSSIAN INTEGRALS**

The proof of (2.37). We want to find the value of the integral
\[ I = \frac{1}{4(2\pi)^{3/2}} \int_{S^2} |\det X| e^{-\frac{1}{4} \left( \operatorname{tr} X^2 - \frac{1}{2} (\operatorname{tr} X)^2 \right)} \cdot \sqrt{2} \prod_{1 \leq i \leq j \leq 2} dx_{ij}. \]

We first make the change in coordinates
\[ x_{11} = x + y, \quad x_{22} = x - y, \quad x_{12} = z. \]

Then
\[ \det X = x^2 - y^2 - z^2, \quad \operatorname{tr} X = 2x, \quad \operatorname{tr} X^2 = 2(x^2 + y^2 + z^2). \]

Hence
\[ I = \frac{1}{2(2\pi)^{3/2}} \int_{\mathbb{R}^3} |x^2 - y^2 - z^2| e^{-\frac{1}{4} (x^2 + y^2 + z^2)} \sqrt{2} |dxdydz| \]
\[ (x = \sqrt{2}u) \]
\[ = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} |2u^2 - y^2 - z^2| e^{-\frac{1}{4} (u^2 + y^2 + z^2)} |dudydz| \]
\[ = \frac{2}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} |2u^2 - y^2 - z^2| e^{-(u^2 + y^2 + z^2)} |dudydz|. \]

We now make the change in variables \( y = r \cos \theta, \quad y = r \sin \theta, \quad r > 0, \theta \in [0, 2\pi) \) and we deduce
\[ I = \frac{2}{\pi^{3/2}} \int_{0}^{\infty} \int_{0}^{2\pi} \left( \int_{0}^{2\pi} |2u^2 - r^2| e^{-u^2 + r^2} \, d\theta \right) r \, dr \, du \]
\[ = \frac{8\pi}{\pi^{3/2}} \int_{0}^{\infty} \int_{0}^{2\pi} |2u^2 - r^2| e^{-(u^2 + r^2)} \, r \, dr \, du. \]
We now make the change in variables

\[ u = t \sin \varphi, \quad r = t \cos \varphi, \quad t > 0, \quad 0 \leq \varphi \leq \frac{\pi}{2}, \]

and we conclude

\[ I = \frac{8}{\pi^2} \left( \int_0^\infty e^{-t^2} t^4 dt \right) \left( \int_0^\frac{\pi}{2} \left| 3 \sin^2 \varphi - 1 \right| \cos \varphi d\varphi \right) \]

\[(t = \sqrt{s}, x = \sin \varphi)\]

\[= \frac{4}{\pi^2} \left( \int_0^\infty e^{-s \frac{3}{2} ds} \right) \left( \int_0^1 \left| 3x^2 - 1 \right| dx \right)
= \frac{4}{\pi^2} \times \Gamma \left( \frac{5}{2} \right) \times \frac{4}{3\sqrt{3}} = \frac{4}{\sqrt{3}} \quad \square \]

**Proposition C.1.** Consider the Gaussian measure \( \Gamma_n \) on \( S_2 \) with covariance form \( \Sigma_{a_n, b_n, c_n} \), where

\[ a_n = s_n + 3t_n, \quad b_n = s_n + t_n, \quad c_n = t_n, \quad s_n, t_n > 0. \]

Then

\[ I_n := \frac{2}{s_n} \int_{S_2} \det X d\Gamma_n(X) = 2. \quad (C.1) \]

**Proof.** Using (B.9) we deduce

\[ d\Gamma_n(X) = \frac{1}{(2\pi)^\frac{3}{2} (\det Q_{a_n, b_n, c_n})^{\frac{1}{2}}} e^{-\frac{1}{2} \left( \tilde{Q}_{a_n, b_n, c_n} X, X \right)} \sqrt{2} \prod_{1 \leq i \leq j \leq 2} dx_{ij}, \]

where \( \det \tilde{Q}_{a_n, b_n, c_n} \) \( (B.6) \)

\[ = 2(a_n - b_n)(a_n + b_n)c_n = 8t_n^2(s_n + 2t_n) =: \delta_n, \]

and, according to (B.8), the parameters \( a'_n, b'_n \) are determined by the linear system

\[
\begin{cases}
  a'_n - b'_n &= \frac{1}{a_n - b_n} \\
  a'_n + b'_n &= \frac{1}{a_n + b_n}.
\end{cases}
\]

We deduce that

\[ b'_n = -\frac{b_n}{a_n^2 - b_n^2} \]

and since \( a_n - b_n = 2c_n \) we have

\[ \left( a'_n - b'_n - \frac{1}{2c_n} \right) = 0. \]

Hence

\[ \left( \tilde{Q}_{a_n, b_n, c_n} X, X \right) = \frac{1}{2c_n} \tr X^2 - \frac{b_n}{a_n^2 - b_n^2} \left( \tr X \right)^2 = \frac{1}{2t_n} \left( \tr X^2 - \frac{s_n + t_n}{2(s_n + 2t_n)} \left( \tr X \right)^2 \right). \]

Using the change in variables \( X := \sqrt{t_n} X \) we deduce

\[ I_n := \frac{2t_n^\frac{3}{2}}{s_n} \times \frac{1}{8\pi^\frac{3}{2} (s_n + 2t_n)^\frac{3}{2}} \int_{S^2} \det X e^{-\frac{1}{4} \left( \tr X^2 - \frac{s_n + t_n}{2(s_n + 2t_n)} \left( \tr X \right)^2 \right)} \sqrt{2} \prod_{1 \leq i \leq j \leq 1} dx_{ij} \]

\[=: I_n \quad (C.2) \]
We make the change in coordinates
\[ x_{11} = x + y, \quad x_{22} = x - y, \quad x_{12} = z. \]
Then
\[ \det X = x^2 - y^2 - z^2, \quad \text{tr} \, X = 2x, \quad \text{tr} \, X^2 = 2(x^2 + y^2 + z^2). \]

\[ \text{tr} \, X^2 - \frac{s_n + t_n}{2(s_n + 2t_n)}(\text{tr} \, X)^2 = \frac{4(s_n + 2t_n)(x^2 + y^2 + z^2) - 4(s_n + t_n)x^2}{2(s_n + 2t_n)} = \frac{4(s_n + 2t_n)(y^2 + z^2) + 4t_n x^2}{2(s_n + 2t_n)}, \]
and
\[ J_n = 2\sqrt{2} \int_{\mathbb{R}^3} \left( x^2 - (y^2 + z^2) \right) e^{-\frac{(s_n + 2t_n)(x^2 + y^2 + z^2)}{2(s_n + 2t_n)}} \, dx dy dz \]
\[ = 2\sqrt{2}(s_n + 2t_n)^{\frac{3}{2}} \int_{\mathbb{R}^3} \left( x^2 - (y^2 + z^2) \right) e^{-\frac{(s_n + 2t_n)(y^2 + z^2) - t_n x^2}{(s_n + 2t_n)}} \, dx dy dz \]
\[ (x := x\sqrt{2}, \quad y := y\sqrt{2}, \quad z := z\sqrt{2}) \]
\[ = 16(s_n + 2t_n)^{\frac{3}{4}} \int_{\mathbb{R}^3} \left( \frac{x^2 - y^2 + z^2}{t_n} \right) e^{-\frac{(x^2 + y^2 + z^2)}{s_n + 2t_n}} \, dx dy dz. \]
We now use the identity
\[ \int_{\mathbb{R}^3} u^2 e^{-(u^2 + v^2 + w^2)} \, du dv dw = \left( \int_{\mathbb{R}} u^2 e^{-u^2} \, du \right) \times \left( \int_{\mathbb{R}} e^{-t^2} \, dt \right)^2 = \pi \]
\[ = 2\pi \int_{0}^{\infty} u^2 e^{-u^2} \, du = \pi \int_{0}^{\infty} s^2 e^{-s^2} \, ds = \frac{\pi^3}{2}. \]
Hence
\[ J_n = \frac{16(s_n + 2t_n)^{\frac{3}{4}}}{t_n^{\frac{1}{4}}} \times \frac{3\pi^3}{2} \left( \frac{1}{t_n} - \frac{2}{s_n + 2t_n} \right) = 8\pi^\frac{3}{2}(s_n + 2t_n)^{\frac{3}{4}} \times \frac{s_n}{t_n(s_n + 2t_n)}. \]
The using the last equality in (C.2) we obtain (C.1). \( \square \)

APPENDIX D. SOME ELEMENTARY ESTIMATES

Proof of Lemma 5.2. Consider the complex valued random process
\[ \mathcal{F}_\nu(t) := \frac{1}{\sqrt{\nu}} \sum_{m=1}^{\nu} m z_m \, e^{imt}, \quad z_m = c_m - i d_m. \]
The covariance function of this process is
\[ \mathcal{R}_\nu(t) = \mathbb{E}(\mathcal{F}_\nu(t) \bar{\mathcal{F}}_\nu(0)) = \frac{1}{\nu^2} \sum_{m=1}^{\nu} m^2 \, e^{- \nu m}. \]
Observe that $\text{Re} \mathcal{F}_\nu = \Phi_\nu$. Note that the spectral measure of the process $\mathcal{F}_\nu$ is

$$d\sigma_\nu = \frac{1}{\pi \nu^3} \sum_{m=1}^\nu m^2 \delta_{m-\nu},$$

where $\delta_{m-\nu}$ denotes the Dirac measure on $\mathbb{R}$ concentrated at $t_0$. We form the covariance matrix of the gaussian vector valued random variable

$$\begin{bmatrix} \mathcal{F}_\nu(0) \\ \mathcal{F}_\nu(t) \\ \mathcal{F}_\nu'(0) \\ \mathcal{F}_\nu'(t) \end{bmatrix} \rightarrow \mathcal{X}_\nu(t) = \begin{bmatrix} \mathbb{R}_\nu(0) & \mathbb{R}_\nu(t) & \mathbb{R}_\nu''(0) & \mathbb{R}_\nu''(t) \\ \mathbb{R}_\nu(t) & \mathbb{R}_\nu(0) & -i\mathbb{R}_\nu'(t) & -i\mathbb{R}_\nu''(t) \\ \mathbb{R}_\nu''(0) & i\mathbb{R}_\nu'(t) & -\mathbb{R}_\nu''(0) & -\mathbb{R}_\nu''(t) \\ i\mathbb{R}_\nu'(t) & -\mathbb{R}_\nu''(0) & -\mathbb{R}_\nu''(t) & -\mathbb{R}_\nu''(t) \end{bmatrix}.$$  

Observe that $\text{Re} \mathcal{X}(t) = \text{Re} \Xi(t)$. If we let

$$\vec{z} = \begin{bmatrix} u_0 \\ v_0 \\ u_1 \\ v_1 \end{bmatrix} \in \mathbb{C}^4$$

Then, as in [9, Eq. (10.6.1)] we have

$$\langle \mathcal{X}_\nu \vec{z}, \vec{z} \rangle = \frac{1}{\pi \nu^3} \sum_{m=1}^\nu m^2 \left| \left( u_0 + v_0 e^{im/\nu} \right) + \frac{im}{\nu} \left( u_1 + v_1 e^{im/\nu} \right) \right|^2$$

We see that

$$\langle \mathcal{X}_\nu \vec{z}, \vec{z} \rangle = 0 \iff \left( u_0 + v_0 e^{im/\nu} \right) + \frac{im}{\nu} \left( u_1 + v_1 e^{im/\nu} \right) = 0, \ \forall m = 1, \ldots, \nu. \quad (D.1)$$

We see that if the linear system (D.1) has a nontrivial solution $\vec{z}$, then the complex $4 \times 4$ matrix

$$A_\nu(t) := \begin{bmatrix} 1 & \zeta & 1 & \zeta \\ 1 & \zeta^2 & 2 & \zeta^2 \\ 1 & \zeta^3 & 3 & 3\zeta^3 \\ 1 & \zeta^4 & 4 & 4\zeta^4 \end{bmatrix}, \ \zeta = e^{\frac{\nu}{\nu}},$$

must be singular, i.e., $\det A_\nu(t) = 0$. We have

$$\det A_\nu(t) = \det \begin{bmatrix} 1 & \zeta & 1 & \zeta \\ 0 & \zeta^2 - \zeta & 2\zeta^2 - z & \zeta^2 - z \\ 0 & \zeta^3 - \zeta & 2\zeta^3 - z & \zeta^3 - z \\ 0 & \zeta^4 - z & 4\zeta^4 - z & 4\zeta^4 - z \end{bmatrix} = \zeta^2 \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \zeta - 1 & 1 & 2\zeta - 1 \\ 0 & \zeta^2 - 1 & 2 & 3\zeta^2 - 1 \\ 0 & \zeta^3 - 1 & 3 & 4\zeta^3 - 1 \end{bmatrix}$$

$$= \zeta^2 \det \begin{bmatrix} \zeta - 1 & 1 & 2\zeta - 1 \\ \zeta^2 - 1 & 2 & \zeta^2 - 1 \\ \zeta^3 - 1 & 3 & \zeta^3 - 1 \\ \zeta^4 - 1 & 4 & \zeta^4 - 1 \end{bmatrix} = \zeta^2 \det \begin{bmatrix} \zeta - 1 & 1 & \zeta \\ \zeta^2 - 1 & 2 & \zeta^2 \\ \zeta^3 - 1 & 3 & \zeta^3 \\ \zeta^4 - 1 & 4 & \zeta^4 \end{bmatrix}$$

$$= \zeta^3 \det \begin{bmatrix} \zeta - 1 & 1 & 1 \\ \zeta^2 - 1 & 2 & \zeta^2 \\ \zeta^3 - 1 & 3 & \zeta^3 \end{bmatrix} = \zeta \det \begin{bmatrix} \zeta - 1 & 1 & 0 \\ \zeta^2 - 1 & 2 & 2\zeta^2 - 2 \\ \zeta^3 - 1 & 3 & 3\zeta^3 - 3 \end{bmatrix}$$

$$= \zeta^3 \det \begin{bmatrix} \zeta - 1 & 1 & 0 \\ \zeta^2 - 2\zeta + 1 & 0 & 2\zeta^2 - 2 \\ \zeta^3 - 3\zeta + 2 & 0 & 3\zeta^3 - 3 \end{bmatrix} = \zeta^3 \det \begin{bmatrix} \zeta - 1 & 1 & 0 \\ (\zeta - 1)^2 & 0 & 2(\zeta - 1)(\zeta + 1) \\ \zeta^3 - 3\zeta + 2 & 0 & 3\zeta^3 - 3 \end{bmatrix}.$$
Proof of Lemma 5.6. Recall that \( \theta := \frac{t}{2\nu}, f(\theta) := \frac{\sin \theta}{\theta} \). By (5.10) we have

\[
\frac{t^{r+1}}{\nu^r t^{r+1}} D_{\nu, r}(t) = r! \left( \frac{2 \sin \left( \frac{(\nu+1)t}{2\nu} \right)}{f(\theta)^{r+1}} \cdot e^{i \frac{t^{(\nu+1)}}{\nu^2}} - e^{i \theta} \sum_{j=1}^{r} i^{r-j} \frac{(\nu+1)}{\nu^3} t^j \cdot \left( \frac{e^{i \theta}}{f(\theta)} \right)^{r+1-j} \right).
\]

Using (5.13) we deduce that

\[
\left| \frac{t^{r+1}}{\nu^r t^{r+1}} D_{\nu, r}(t) - \frac{1}{\nu} I_r(t) \right| \leq 2r! t^{r+1} \left| \frac{\sin \left( \frac{(\nu+1)t}{2\nu} \right)}{f(\theta)^{r+1}} e^{i \theta} - \sin \left( \frac{t}{2} \right) \right| + r! \sum_{j=1}^{r} \theta \left( e^{i \theta} \right)^{r+1-j} \left( \frac{e^{i \theta}}{f(\theta)} \right)^{r+1-j} - \frac{1}{j!}.
\]  

(D.2)

In the sequel we will use Landau’s symbol \( O \). The various implied constants will be independent of \( \nu \). Throughout we assume \( 0 < t \leq \pi \nu \). For \( 0 < \theta < \frac{\pi}{2} \) and \( 0 \leq j \leq r \) we have

\[
e^{i \theta} = 1 + O(\theta), \quad \frac{(\nu+1)}{\nu^j} = 1 + O\left( \frac{1}{\nu} \right), \quad \sin \left( \frac{(\nu+1)t}{2\nu} \right) = \sin \left( \frac{t}{2} \right) + O(\theta),
\]

\[
\frac{e^{i \theta}}{f(\theta)} = \frac{\theta (\cos \theta + i \sin \theta)}{\sin \theta} = 1 + O(\theta).
\]

Hence

\[
\left| \frac{\sin \left( \frac{(\nu+1)t}{2\nu} \right)}{f(\theta)^{r+1}} e^{i \theta} - \sin \left( \frac{t}{2} \right) \right| = O(\theta), \quad \text{(D.3)}
\]

while for any \( 1 \leq j \leq r \) we have

\[
\left| \frac{(\nu+1)}{\nu^j} \left( \frac{e^{i \theta}}{f(\theta)} \right)^{r+1-j} - \frac{1}{j!} \right| = O \left( \frac{1}{\nu} + \theta \right). \quad \text{(D.4)}
\]

Using (D.3) and (D.4) in (D.2) we deduce that

\[
\left| \frac{t^{r+1}}{\nu^r t^{r+1}} D_{\nu, r}(t) - \frac{1}{\nu} I_r(t) \right| = O \left( \theta + \left( \theta + \frac{1}{\nu} \right) \sum_{j=1}^{r-1} t^j \right) = O \left( \frac{1}{\nu} \sum_{j=1}^{r+1} t^j \right).
\]

Hence

\[
\left| \frac{1}{\nu^r t^{r+1}} D_{\nu, r}(t) - \frac{1}{t^{r+1}} I_r(t) \right| = O \left( \frac{1}{\nu t} \frac{1 - t^{r+1}}{1-t} \right).
\]
Proof of Lemma 5.7. We have

$$R^{(k)}_{\infty}(t) = \pm \frac{1}{t^{k+3}} \int_0^t \tau^{k+2} u(\tau) d\tau, \quad u(\tau) = \begin{cases} \sin \tau, & k \in 1 + 2\mathbb{Z}; \\ \cos \tau, & k \in 2\mathbb{Z}. \end{cases}$$

Note that

$$|R^{(k)}_{\infty}(0)| = \frac{1}{k+3} = \frac{1}{t^{k+3}} \int_0^t \tau^{k+2} d\tau.$$

The inequality (5.19a) now follows from the inequality $|u(\tau)| \leq 1, \forall \tau$.

For any positive integer $r$ we denote by $j_r$ the $r$-th jet at 0 of a one-variable function. We can rewrite (5.13) as follows:

$$\frac{1}{t^r} I_r(t) = r! \left( 2 \sin \left( \frac{t}{2} \right) e^{it} - ie^{it} \sum_{j=1}^{r} \frac{(-it)^j}{j!} \right) = r! \left( 2 \sin \left( \frac{t}{2} \right) e^{it} - ie^{it} \cdot j_r(e^{-it} - 1) \right)$$

$$= r! \left( 2 \sin \left( \frac{t}{2} \right) e^{it} + ie^{it} - ie^{it} \cdot j_r(e^{-it}) \right)$$

$$= r! \left( -i(e^{it} - e^{-it}) e^{it} + ie^{it} - ie^{it} \cdot j_r(e^{-it}) \right) = ir! \left( 1 - e^{it} \cdot j_r(e^{-it}) \right)$$

Hence

$$\text{Re} \left( \frac{1}{t^r} I_r(t) \right) = \text{Im} \left( e^{it} \cdot j_r(e^{-it}) \right) \quad \text{and} \quad \frac{1}{t^{r+1}} I_r(t) = O(t^{-1}) \quad t \to \infty.$$

This proves (5.19b).

The spectral measure

$$d\sigma_{\nu} = \frac{1}{\pi \nu^3} \sum_{m=1}^{\nu} m^2 \delta_{\frac{m}{\nu}}$$

of the process $\mathcal{F}_\nu$ converges weakly as $\nu \to \infty$ to the measure

$$d\sigma_\infty = \frac{1}{\pi} \chi_{[0,1]} t^2 dt,$$

where $\chi_{[0,1]}$ denotes the characteristic function of $[0,1]$. Indeed, an argument identical to the one used in the proof of Lemma 5.4 shows that for every continuous bounded function $f : \mathbb{R} \to \mathbb{R}$ we have

$$\lim_{\nu \to \infty} \int_{\mathbb{R}} f(t) d\sigma_{\nu}(t) = \int_{\mathbb{R}} f(t) d\sigma_\infty.$$

The complex valued stationary Gaussian process $\mathcal{F}_{\infty}$ on $\mathbb{R}$ with spectral measure $d\sigma_{\infty}$ has covariance function

$$R_{\infty} = \frac{1}{\pi} \int_0^1 i^2 e^{it} dt.$$

Note that $\text{Re} R_{\infty} = R_{\infty}$. The results in [9, §10.6] show that the covariance matrix

$$\mathcal{X}_\infty = \begin{bmatrix} R_{\infty}(0) & R_{\infty}'(0) & R_{\infty}'(t) \\ R_{\infty}(t) & R_{\infty}(0) & R_{\infty}'(0) \\ R_{\infty}'(0) & R_{\infty}'(t) & R_{\infty}'(0) \end{bmatrix} = \lim_{\nu \to \infty} \mathcal{X}_{\nu},$$

is nondegenerate. The equality $\det \text{Re} \mathcal{X}_{\infty}(t) \neq 0, \forall t \in \mathbb{R}$ implies as in Remark 5.3 that $\mu_{\infty}(t) \neq 0$, $|\rho_{\infty}(t)| < 1, \forall t \in \mathbb{R}$, where

$$\mu_{\infty} = \frac{(\lambda_0^2 - R_{\infty}^2) \lambda_2 - \lambda_0 R_{\infty}}{\lambda_0^2 - R_{\infty}^2}, \quad \rho_{\infty} = \frac{R_{\infty}''(\lambda_0^2 - R_{\infty}^2) + (R_{\infty}')^2 R_{\infty}}{(\lambda_0^2 - R_{\infty}^2) \lambda_2 - \lambda_0 R_{\infty}}.$$
This proves (5.19c). The equality (5.19d) follows from the Taylor expansion of $R_\infty$. \hfill \Box

\section*{References}

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556-4618.

E-mail address: nicolaescu.1@nd.edu

URL: http://www.nd.edu/~lnicolae/