Norman Steenrod once said that a topologist is “a person who sees no difference between a donut and a mug”. For this course, the fact that the surface of a donut happens to be an algebraic variety (elliptic curve to be precise) will be particularly relevant.

In general, the algebraic varieties are sets of solutions of polynomial equations and their structure has many special features. I will be interested mainly in their topological properties. The class of algebraic varieties may seem small, special and thus uninteresting. By the end of this course I hope to convince you of the contrary, and present a few interesting features of these special topological spaces.\(^1\) The course will consist of three parts.

1. **Picard-Lefschetz theory**
2. Singularities of curves and surfaces and their resolutions.
3. The Milnor fiber, the link and the vanishing cohomology of a singularity.

The Picard-Lefschetz theory is the complex analytic counterpart of the versatile Morse theory. Historically, Picard-Lefschetz theory came first, at the dawn of the 20th century, and had a dramatic impact on the development of modern algebraic topology. The recent years have witnessed an increased the interest in this old theory due mainly to the surprising results of S. Donaldson on the topology of symplectic manifolds.

Roughly speaking, Morse theory describes a procedure of reconstructing a smooth manifold \(M\) assuming we know a smooth function \(f : M \to \mathbb{R}\) and its behavior near its critical points. Picard-Lefschetz theory proposes a general reconstruction process for a complex analytic manifold \(M\) knowing a holomorphic function \(f : M \to \mathbb{C}\) and its behavior near its critical points. Picard-Lefschetz theory requires that all critical points of \(f : M \to \mathbb{C}\) are nondegenerate, just as Morse theory does. For this part of the course I plan to follow closely the modern approach in [4].

Naturally, one can ask what happens if we relax the above requirement and allow degenerate critical points as well. In this case one is lead to the study of singularities of hypersurfaces

\[
X = P^{-1}(0) \subset \mathbb{C}^n
\]

where \(P = P(z_1, \cdots, z_n)\) is a polynomial in \(n\)-complex variables such that \(0 \in X\) is a singular point. I will discuss two methods for studying the singularities of such hypersurfaces. Both are based on the same principle: investigate the singular object \(X\) by “approximating” it with a smooth one.

The first “approximation” technique is the so called resolution of singularities and I will discuss it only in low dimensions \(n = 2, 3\). The case \(n = 2\) leads to beautiful connections with knot theory, while the case \(n = 3\) leads to interesting and important examples of 3-manifolds. My presentation will be inspired from [2, 5].

Another, simpler way of approximating \(X\) is by deforming the defining equation \(P = 0\) to \(P = \varepsilon\). This process is called smoothing and the resulting hypersurface

\[
X_\varepsilon := \{\vec{z} \in \mathbb{C}^n; \ P(\vec{z}) = \varepsilon\}
\]

is called the Milnor fiber of the singularity. The odd-dimensional manifold \(N_\varepsilon := X \{|z| = \varepsilon\}\) is called the link of the singularity. When \(n = 2\), so that \(X\) is a complex curve, the link is a one-dimensional

\[^1\text{Even if you don’t plan to attend this class you ought to read the captivating brief historical survey “Singulaties” by Alan Durfee, xxx-Archive math.AG/9801123, to learn about some exciting piece of mathematics.}\]
submanifold of the sphere $S^3$, a.k.a. knot. The knots obtained as links of singularities are called algebraic.

The topologies of the Milnor fiber and the link of a singularity are nontrivially constrained by the complex geometry of the singularity. For example, J. Milnor has shown that amongst these links one can detect manifolds which “talk like a sphere, walk like a sphere” but they are not quite spheres: they are homeomorphic but not diffeomorphic to spheres.

As $\varepsilon \to 0$, some of the cycles of $X_\varepsilon$ collapse. These are called vanishing cycles and the subgroup they generate is called the vanishing (co)homology of the singularity. My presentation will be inspired from [1, 3, 6]. If time permits, I plan to say a few words about the connection between the asymptotics of certain oscillatory integrals and vanishing cohomology.

To enjoy this class you need to be familiar with basic facts of algebraic topology, especially singular (co)homology and very little analysis. I believe the things I will talk about will be of interest to topologists, algebraic and/or differential geometers alike. The algebraic varieties represent an excellent testing ground for many topological questions and ought to belong to the bag of examples of any topologist or geometer.

The class will be example-oriented because, as E.C. Zeeman once put it, “a good example is worth ten theorems”. This is quite an eclectic subject and there will be many special interesting things I won’t be able to cover in class. That is why, part of the requirement of this class will be to present a one hour informal talk on a subject of your choice from a list I will submit in class. In compiling this list I will try to satisfy all the backgrounds in class. E.g. a student pursuing a degree in algebraic geometry will be able to choose an elementary topic in this field which will be relevant to all of us in class.

References


TENTATIVE LIST OF TOPICS FOR MATH 658

I. Complex manifolds - Definitions and Examples
   1. The projective spaces.
   2. Complex submanifolds: implicit function theorem, critical and regular points, critical values.
   3. Algebraic manifolds.
   5. The tautological line bundle and the blow-up construction

II. Lefschetz theory
   1. Linear systems, pencils.
   2. Lefschetz pencils, examples.
   4. Morse lemma and Picard-Lefschetz formulæ

III. Invariants of critical points
   1. Mather-Tougeron sufficiency theorem.
   2. Milnor number
   3. Morsification and Milnor fibrations
   5. Resolutions of singularity (curves and surfaces only)