THE GENERALIZED MAYER-VIETORIS PRINCIPLE AND SPECTRAL SEQUENCES

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ABSTRACT. This is my spin of the some stuff from from [1].

CONTENTS

1. DOUBLE COMPLEXES

Suppose that R is a commutative ring with 1. A *double complex* of R-modules is a bigraded R-module

$$E^{\bullet,\bullet}=\bigoplus_{p,q\geq 0}E^{p,q}$$

equipped with two morphisms

$$\delta_\epsilon:E^{\bullet,\bullet}\mapsto E^{\bullet,\bullet+1},\;\; d_h:E^{\bullet,\bullet}\mapsto E^{\bullet+1,\bullet}$$

satisfying the conditions

$$d_v^2=d_h^2=d_vd_h+d_hd_v=0. \tag{1.1}$$

The above equalities show that if we define $D:=d_v+d_h$, then $D^2=0$. We set

$$E^{p,q}=0 \text{ if } p<0 \text{ or } q<0.$$

The *total complex* associated to a double-complex $(E^{\bullet,\bullet},d_v,d_h)$ is the complex $(T^\bullet(E),D)$ where

$$T^n(E):=\bigoplus_{p+q=n}E^{p,q}.$$

Note that

$$D(E^{p,q})\subset E^{p+1,q}\oplus E^{p,q+1}\subset T^{p+q+1}(E).$$

We denote by $H^\bullet\big(T(E),\big)$ the cohomology of the total complex.

The *transpose* of a double compelx $(E^{\bullet,\bullet},d_h,d_v)$ is the double complex $(\widehat{E}^{\bullet,\bullet},\hat{d}_h,\hat{d}_v)$ where

$$\widehat{E}^{p,q}:=E^{q,p},\;\;\hat{d}_h=d_v,\;\;\hat{d}_v=d_h.$$

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Thus, the rows (respectively columns) of $\widehat{E}^{\bullet,\bullet}$ are the columns (respectively rows) of $E^{\bullet,\bullet}$. Note that the total complex of the transpose coincides with the total complex of the original double complex.

A morphism of double complexes $\varphi : E^{\bullet,\bullet} \to F^{\bullet,\bullet}$ is defined in the obvious fashion. Such a morphism induces a morphism between the associated total complexes

$$\varphi : T^{\bullet}(E) \to T^{\bullet}(F).$$

A complex of R-modules $(C^{\bullet}, d)$ can be viewed as a degenerate double complex $(C^{\bullet,\bullet}, d_v, d_h)$, where

$$C^{p,q} = \begin{cases} 0, & q > 0 \\ C^p, & q > 0, \end{cases}$$

$d_h = 0$, and $d_h : C^{p,0} \to C^{p+1,0}$ coincides with the coboundary operator d of $C^{\bullet}$. In particular we can speak of a morphism from a simple complex to a double complex.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow d_v & & \uparrow d_v & & \uparrow d_v & & \\
E^{0,1} & \xrightarrow{d_h} & E^{1,1} & \xrightarrow{d_h} & E^{2,1} & \xrightarrow{d_h} & \cdots \\
\uparrow d_v & & \uparrow d_v & & \uparrow d_v & & \\
E^{0,0} & \xrightarrow{d_h} & E^{1,0} & \xrightarrow{d_h} & E^{2,0} & \xrightarrow{d_h} & \cdots \\
\uparrow \varphi & & \uparrow \varphi & & \uparrow \varphi & & \\
C^{0} & \xrightarrow{d} & C^{1} & \xrightarrow{d} & C^{2} & \xrightarrow{d} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & & \cdots
\end{array}$$

A *resolution* of the complex $(C^{\bullet}, d)$ by a double complex $(E^{\bullet,\bullet}, d_v, d_h)$ is a morphism $\varphi : (C^{\bullet}, d) \to (E^{\bullet,\bullet}, d_v, d_h)$ such that the columns in the above diagram are exact. More precisely, this means that for any $n \geq 0$ we have a long exact sequence

$$0 \hookrightarrow C^n \xrightarrow{\varphi} E^{p,0} \xrightarrow{d_v} E^{n,1} \xrightarrow{d_v} E^{n,2} \xrightarrow{d_v} \cdots . \tag{1.2}$$

Finally we define a *quasi-isomorphism* of cochain complexes to be a morphism of cochain complexes such that induces isomorphisms in cohomology.

Theorem 1.1 (Approximation Theorem. Version 1)**.** *(a) Suppose that* $\varphi : (C^{\bullet}, d) \to (E^{\bullet,\bullet}, d_v, d_h)$ *is a* resolution *of the complex* $(C^{\bullet}, d)$ *by a double complex* $(E^{\bullet,\bullet}, d_v, d_h)$*. Then the induced map*

$$\varphi_* : H^{\bullet}(C) \to H^{\bullet}\big(T(E)\big)$$

is an isomorphism.

Proof. We represent the elements $x = (x_{p,q})_{p+q=n} \in T^n(E)$ as zig-zags; Figure 1. Such a zig-zag is a D-cocycle if

$$d_h x_{p,q} + d_v x_{p+1,q} = 0, \;\; \forall p + q = n.$$

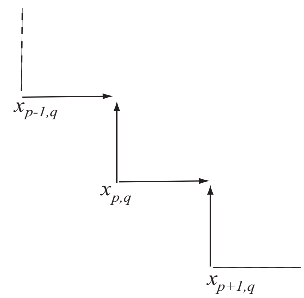


FIGURE 1. *A zig-zag.*

An element $x = (x_{p,q})_{p+q=n} \in T^n(E)$ is a D-coboundary if there exists

$$y = (y_{i,j})_{i+j=n-1} \in T^{n-1}(E)$$

such that

$$x_{p,q} = d_v y_{p,q-1} + d_h y_{p-1,q}, \quad \forall p+q=n.$$

Step 1. *The morphism $\varphi_* : H^\bullet(C) \to H^\bullet\big(T(E)\big)$ is surjective.* Suppose that $(x_{p,q})_{p+q=n} \in T^n(E)$ is a D-cocycle. We want to prove that it is D-cohomologous with a D-cocycle $y = (y_{p,q})_{p+q=n}$ such that

$$y_{p,q} = 0, \quad \forall q > 0, \tag{1.3a}$$

$$y_{n,0} = \varphi(u), \quad u \in C^n, \quad du = 0. \tag{1.3b}$$

For $u \in C^n$ we identify $\varphi(u)$ with the zig-zag depicted in Figure 2. Let us first observe that the

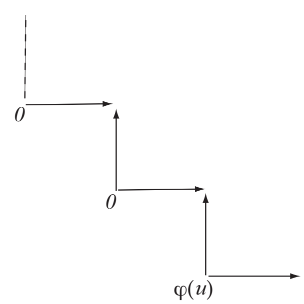


FIGURE 2. *A zig-zag representing $\varphi(u)$.*

condition $Dy = 0$ coupled with (1.3a) imply (1.3b). Indeed, these two conditions imply

$$d_v y_{n,0} = 0, \quad d_h y_{n,0} = 0.$$

From the exact sequence (1.2) and $d_v y_{n,0} = 0$, we deduce that there exists $u \in C^n$ such that $y_{n,0} = \varphi(u)$. On the other hand,

$$0 - d_h y_{n,0} = d_h \varphi(u) = \varphi(du).$$

since φ is injective we deduce $du = 0$. Thus it suffices to show that x is cohomologous with a cocycle y satisfying (1.3a). Note that this is equivalent to asking that

$$y_{p,q} = 0, \quad \forall p < n.$$

We will show inductively that for any $k \leq n$ there exists a D-cocycle $(y_{p,q})_{p+q=n}$ cohomologous to x such that

$$y_{p,q} = 0, \quad \forall p < k. \tag{C_k}$$

The condition (C_k) suggest the following object.

$$F^kT^n(E) := \{ y \in T^n(E);\;\; y_{p,q} = 0, \;\; \forall p < k \}.$$

Observe that $F^k(T^n(E)$ is a submodule of $T^n(E)$ and $DF^kT^n \subset F^kT^{n+1}$. Thus $(F^kT^\bullet, D)$ is a subcomplex of $(T^\bullet, D)$. Moreover

$$F^{k+1}T^\bullet \subset F^kT^\bullet, \;\; \forall k.$$

We say that the collection of subcomplexes $(F^kT^\bullet, D)_{k\geq 0}$ defines a *decreasing filtration* of the complex $(T^\bullet, D)$.

Note that the condition (C_0) is tautologically satisfied. We will show that $(C_k) \Rightarrow (C_{k+1})$. Suppose that $x = (x_{p,q})_{p+q=n}$ is a D-cocycle satisfying (C_k). Then

$$d_v x_{k,n-k} = 0$$

which implies that there exists $y_k \in E^{k,n-k-1}$ such that $x_{k,n-k} = d_v y_k$. Define the zig-zag $y = (y_{i,j})_{i+j=n-1} \in T^{n-1}(E)$ by setting

$$y_{i,j} = \begin{cases} 0, & i \neq k \\ y_k, & i = k. \end{cases}$$

Then the zig-zag $x - Dy$ is cohomologous to x and satisfies (C_{k+1}).

Step 2. *The morphism $\varphi_* : H^\bullet(C) \to H^\bullet\big(T(E)\big)$ is injective.* Let $u \in C^n$ such that $du = 0$ and $\varphi(u)$ is a D-coboundary. Thus there exists $y \in T^{n-1}E$ such that $Dy = \varphi(u)$, where $\varphi(u)$ is identified with a zig-zag as in Figure 2. We will show inductively that for any $k \leq n-1$ there exists $y' \in T^{n-2}(E)$ such that $z = y - Dy'$ satisfies the condition (C_k) i.e.

$$z_{p,q} = 0, \;\; \forall p < k.$$

Clearly y satisfies C_0. We will show that if y satisfies (C_k), $k < n-1$, and $Dy = \varphi(u)$, then there exists $y' \in T^{n-2}(E)$ such that $y - Dy'$ satisfies (C_{k+1}). Indeed if $Dy = \varphi(u)$ and y satisfies C_k with $k < n-1$, then $d_v y_{k,n-1-k} = 0$ and the exactness of (1.2) implies that there exists $y_k \in E^{k,n-2-k}$ such that

$$d_v y_k = y_{k,n-1-k}.$$

Now define

$$y' = (y'_{i,j})_{i+j=n-2} \in T^{n-2}(E), \;\; y'_{i,j} = \begin{cases} 0, & i \neq k \\ y_k, & i = k. \end{cases}$$

Thus we can find $z = (z_{p,q})_{p+q=n-1} \in T^{n-1}(E)$ such that

$$Dz = \varphi(u), \;\; z_{p,q} = 0, \;\; \forall p < n-1.$$

We deduce that

$$d_v z_{n-1,0} = 0, \;\; d_h z_{n-1,0} = \varphi(u).$$

From the first equality we deduce that there exists $v \in C^{n-1}$ such that $z_{n-1,0} = \varphi(v)$. Using this in the second equality above we deduce

$$\varphi(u) = d_h\varphi(v) = \varphi(dv).$$

Since φ is injective we conclude that $u = dv$, so that u represents 0 in cohomology. □

Example 1.2 (Another look at the Mayer-Vietoris sequence)**.** Suppose that M is a smooth manifold of dimension m and $\{U_0, U_1\}$ is an open cover of M. We can form the double complex $E^{\bullet,\bullet}$ where

$$E^{p,0} = \Omega^p(U_0 \sqcup U_1) = \Omega^p(U_0) \oplus \Omega^p(U_1), \;\; E^{p,1} = \Omega^p(U_0 \cap U_1), \;\; \forall p \geq 0,$$

$$E^{p,q} = 0, \;\; \forall q > 1,$$

the differential d_h is the exterior derivative $d : \Omega^p(X) \to \Omega^{p+1}(X)$, $X = U_0 \sqcup U_1, U_0 \cap U_1$, while

$$d_v : \Omega^p(U_0) \oplus \Omega^p(U_1) \to \Omega^p(U_0 \cap U_1), \;\; d(\alpha_0, \alpha_1) = (-1)^p \delta(\alpha_0, \alpha_1),$$

$$\delta(\alpha_0, \alpha_1) = \big(\alpha_1|_{U_0 \cap U_1} - \alpha_0|_{U_0 \cap U_1}\big).$$

$$\begin{array}{ccccccc}
0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots \\
\uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \\
\Omega^0(U_0 \cap U_1) & \xrightarrow[d]{} & \Omega^1(U_0 \cap U_1) & \xrightarrow[d]{} & \Omega^2(U_0 \cap U_1) & \xrightarrow[d]{} & \cdots \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle -\delta} & & \uparrow{\scriptstyle \delta} & & \\
\Omega^0(U_0 \sqcup U_1) & \xrightarrow[d]{} & \Omega^1(U_0 \sqcup U_1) & \xrightarrow[d]{} & \Omega^2(U_0 \sqcup U_1) & \xrightarrow[d]{} & \cdots
\end{array}$$

Observe that the total complex associated to this double complex can be identified with the cone of the morphism

$$\delta : \Omega^\bullet(U_0 \sqcup U_1) \to \Omega^\bullet(U_0 \cap U_1).$$

Thus, we have a long exact sequence

$$\cdots \to H^{q-1}(U_0 \cap U_1) \to H^q\big(T(E)\big) \to H^q(U_0) \oplus H^q(U_1) \to H^q(U_0 \cap U_1) \to \cdots . \quad (1.4)$$

On the other hand, the Mayer-Vietoris principle implies that the diagram below is anti-commutative and the columns are exact

$$\begin{array}{ccccccc}
0 & \xrightarrow{\;0\;} & 0 & \xrightarrow{\;0\;} & 0 & \xrightarrow{\;0\;} & \cdots \\
{\scriptstyle 0}\uparrow & & {\scriptstyle 0}\uparrow & & {\scriptstyle 0}\uparrow & & \\
\Omega^0(U_0\cap U_1) & \xrightarrow[d]{} & \Omega^1(U_0\cap U_1) & \xrightarrow[d]{} & \Omega^2(U_0\cap U_1) & \xrightarrow[d]{} & \cdots \\
{\scriptstyle \delta}\uparrow & & {\scriptstyle -\delta}\uparrow & & {\scriptstyle \delta}\uparrow & & \\
\Omega^0(U_0\sqcup U_1) & \xrightarrow[d]{} & \Omega^1(U_0\sqcup U_1) & \xrightarrow[d]{} & \Omega^2(U_0\sqcup U_1) & \xrightarrow[d]{} & \cdots \\
{\scriptstyle r}\uparrow & & {\scriptstyle r}\uparrow & & {\scriptstyle r}\uparrow & & \\
\Omega^0(M) & \xrightarrow[d]{} & \Omega^1(M) & \xrightarrow[d]{} & \Omega^2(M) & \xrightarrow[d]{} & \cdots \\
{\scriptstyle 0}\uparrow & & {\scriptstyle 0}\uparrow & & {\scriptstyle 0}\uparrow & & \\
0 & & 0 & & 0 & & \cdots
\end{array}$$

Using the Approximation Theorem 1.1 we deduce that $H^\bullet\big(T(E)\big)$ is isomorphic to $H^\bullet(M)$. Under this isomorphism the long exact sequence (1.4) becomes the long exact Mayer-Vietoris sequence in cohomology. □

2. THE GENERALIZED MAYER-VIETORIS PRINCIPLE

Suppose that $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ is an open cover of the smooth manifold M of dimension M. For any subset $A\subset\mathcal{A}$ we set

$$U_A := \bigcup_{\alpha\in A} U_\alpha$$

We define the *nerve* of the cover $\mathcal{U}$ to be the collection $\boldsymbol{N}(\mathcal{U})$ of finite nonempty subsets $A\subset\mathcal{A}$ such that $U_A\neq\emptyset$. The subsets $A\in\boldsymbol{N}(\mathcal{U})$ are called the *faces* of the nerve. We set

$$\boldsymbol{N}_q(\mathcal{U}) := \big\{ A\in\boldsymbol{N}(\mathcal{U});\;\; \#A = q+1 \,\big\} \subset \boldsymbol{N}(\mathcal{U}). \tag{2.1}$$

The elements of $\boldsymbol{N}_q(\mathcal{U})$ are called the *q-faces* of $\boldsymbol{N}(U)$. Observe that a nonempty subset of a face is another face of the nerve. Moreover, if $A\subset B$ are faces of $\boldsymbol{N}(U)$ then

$$U_A \supset U_B.$$

For every nonnegative integer q we set

$$\Delta_q := \{0,1,\dots,q\}.$$

A *q-simplex* of the nerve is a map

$$\sigma:\Delta_q\to\mathcal{A},\;\; \Delta_q\ni j\mapsto \sigma_j\in\mathcal{A},$$

whose range $\sigma(\Delta_q)$ is a face of the nerve. We denote by $\mathcal{S}_q(\mathcal{U})$ the set of q-simplices of the nerve. Note that for any $\sigma\in\mathcal{S}_q(\mathcal{U})$ we have $\dim\sigma(\Delta_q)\leq q$.

We will often describe a q-simplex σ as a an ordered string of elements in $\mathcal{A}$

$$\sigma = (\sigma_0, \dots, \sigma_q).$$

We set

$$U_\sigma = U_{\sigma_0 \cdots \sigma_q} := U_{\sigma(\Delta_q)} = \bigcap_{j=0}^{q} U_{\sigma_j}.$$

If $\sigma : \Delta_q \to \mathcal{A}$ is a q-simplex of $\boldsymbol{N}(U)$, and $j \in \Delta_q$ then we denote by $\partial_j \sigma$ the $(q-1)$-simplex defined by the composition

$$\Delta_{q-1} \xrightarrow{\varphi_j} \Delta_q \to \mathcal{A}$$

where $\varphi_j : \Delta_{q-1} \to \Delta_q$ is the unique strictly increasing function whose range is $\Delta_q \setminus \{j\}$. More explicitly, if $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_q)$, then $\partial_j \sigma$ is described by the sequence

$$(\sigma_0, \dots, \sigma_{j-1}, \sigma_{j+1}, \dots, \sigma_q).$$

We have thus constructed a map $\partial_j : \mathcal{S}_q(\mathcal{U}) \to \mathcal{S}_{q-1}(\mathcal{U})$

Define

$$C^q(\mathcal{U}, \Omega^p) := \prod_{\sigma \in \mathcal{S}_q(\mathcal{U})} \Omega^p(U_\sigma).$$

An element $\omega \in C^q(\mathcal{U}, \Omega^p)$ is therefore a collection $(\omega_\sigma)_{\sigma \in \mathcal{S}_q(\mathcal{U})}$ of p-forms $\omega_\sigma = \omega_{\sigma_0 \cdots \sigma_q} \in \Omega^p(U_\sigma)$. For convenience, we write

$$\langle \omega, \sigma \rangle := \omega_\sigma, \;\; \forall \omega \in C^q(\mathcal{U}, \Omega^p), \;\; \sigma \in \mathcal{S}_q(\mathcal{U}).$$

Define

$$\delta : C^{q-1}(\mathcal{U}, \Omega^p) \to C^q(\mathcal{U}, \Omega^p),$$

by setting for $\omega \in C^{q-1}(\mathcal{U}, \Omega^p)$ and $\sigma \in \mathcal{S}_q(\mathcal{U})$

$$\langle \delta\omega, \sigma \rangle := \sum_{j=0}^{q} (-1)^j \langle \omega, \partial_j \sigma \rangle |_{U_\sigma}.$$

More explicitly, if $\sigma = (\sigma_0, \dots, \sigma_q)$ then

$$(\delta\omega)_{\sigma_0 \cdots \sigma_q} = \sum_{j=0}^{q} (-1)^j \omega_{\sigma_0 \cdots \hat{\sigma}_j \cdots \sigma_q} |_{U_\sigma}.$$

Example 2.1. Let us explicitly describe

$$\delta : C^0(\mathcal{U}, \Omega^p) \to C^1(\mathcal{U}, \Omega^p) \;\text{ and }\; \delta : C^1(\mathcal{U}, \Omega^p) \to C^2(\mathcal{U}, \Omega^p).$$

An element $\omega \in C^0(\mathcal{U}, \Omega^p)$ is a collection of p-forms $\omega_\alpha \in \Omega(U_\alpha)$. The differential $\delta\omega$ is the collection of p-forms

$$(\delta\omega)_{\alpha\beta} \in \Omega^p(U_{\alpha\beta})$$

given by the equalities

$$(\delta\omega)_{\alpha\beta} = \omega_\beta|_{U_{\alpha\beta}} - \omega_\alpha|_{U_{\alpha\beta}}.$$

Observe that $\delta\omega = 0$ if and only if there exists a *globally defined* $\tilde{\omega} \in \Omega^p(M)$ such that

$$\omega_\alpha = \tilde{\omega}|_{U_\alpha}, \;\; \forall \alpha \in \mathcal{A}.$$

An element of $\omega \in C^1(\mathcal{U}, \Omega^p)$ is a collection of p-forms $\omega_{\sigma_0\sigma_1} \in \Omega^p(U_{\sigma_0\sigma_1})$. Then $\delta\omega$ is a collection of forms $\eta_{\sigma_0\sigma_1\sigma_2} \in \Omega^p(U_{\sigma_0\sigma_1\sigma_2})$ defined by

$$\eta_{\sigma_0\sigma_1\sigma_2} = \omega_{\sigma_1\sigma_2}|_{U_{\sigma_0\sigma_1\sigma_2}} - \omega_{\sigma_0\sigma_2}|_{U_{\sigma_0\sigma_1\sigma_2}} + \omega_{\sigma_0\sigma_1}|_{U_{\sigma_0\sigma_1\sigma_2}}.$$

Less precisely, we can write

$$\eta_{\sigma_0\sigma_1\sigma_2} = \omega_{\sigma_1\sigma_2} - \omega_{\sigma_0\sigma_2} + \omega_{\sigma_0\sigma_1} \text{ on } U_{\sigma_0\sigma_1\sigma_2}.$$

□

Proposition 2.2.

$$\delta^2 = 0.$$

Proof. For clarity we will show only that the composition

$$C^0(\mathcal{U}, \Omega^p) \xrightarrow{\delta} C^2(\mathcal{U}, \Omega^p) \xrightarrow{\delta} C^2(\mathcal{U}, \Omega^p)$$

is trivial. Let $\omega \in C^0(\mathcal{U}, \Omega^p)$. Thus ω is a collection of forms $\omega_\alpha \in \Omega^p(U_\alpha)$, $\alpha \in \mathcal{A}$. Then for any $\sigma = (\sigma_0, \sigma_1) \in \mathcal{S}_1(\mathcal{U})$ we have

$$(\delta\omega)_{\sigma_0\sigma_1} = \omega_{\sigma_1} - \omega_{\sigma_0} \text{ on } U_{\sigma_0\sigma_1}.$$

Next, using the computations in Example 2.1 we deduce that for any $\sigma = (\sigma_0, \sigma_1, \sigma_2) \in \mathcal{S}_2(\mathcal{U})$ we have

$$\begin{aligned}(\delta^2\omega)_{\sigma_0\sigma_1\sigma_2} &= (\delta\omega)_{\sigma_1\sigma_2} - (\delta\omega)_{\sigma_0\sigma_2} + (\delta\omega)_{\sigma_0\sigma_1} \text{ on } U_\sigma \\ &= (\omega_{\sigma_2} - \omega_{\sigma_1}) - (\omega_{\sigma_2} - \omega_{\sigma_0}) + (\omega_{\sigma_1} - \omega_{\sigma_0}) \text{ on } U_\sigma \\ &= 0.\end{aligned}$$

□

We have thus constructed a cochain complex $\big(C^\bullet(\mathcal{U}, \Omega^p), \delta\big)$ called the *Čech complex* of the cover $\mathcal{U}$ with coefficients in (the presheaf)[1] Ω^p.

Remark 2.3. The Čech complexes can be replaced by a smaller homotopy equivalent complexes. Define $C^q_{\mathrm{alt}}(\mathcal{U}, \Omega^p)$ to be the subspace of $C^q(\mathcal{U}), \Omega^p)$ consisting of collection $(\omega_\sigma)_{\sigma\in\mathcal{S}_q(\mathcal{U})}$ such that, for any permutation $\varphi : \Delta_q \to \Delta_q$ and any $\sigma \in \mathcal{S}_q(\mathcal{U})$ we have

$$\omega_{\sigma\circ\varphi} = \epsilon(\varphi)\omega_\sigma,$$

where $\epsilon(\varphi)$ denotes the sign of the permutation φ. Note that this implies that if $\sigma : \Delta_q \to \mathcal{A}$ is a q-simplex which is not injective, then $\omega_\sigma = 0$. One can show (see [11]) that $\delta : C^\bullet_{\mathrm{alt}}(\mathcal{U}, \Omega^p) \subset C^{\bullet+1}_{\mathrm{alt}}(\mathcal{U}, \Omega^p)$ so that $\big(C^\bullet_{\mathrm{alt}}(\mathcal{U}, \Omega^p), \delta\big)$ is a subcomplex of $\big(C^\bullet(\mathcal{U}, \Omega^p), \delta\big)$.

On the other hand, if we fix a total order $\prec$ on $\mathcal{A}$ we can define $\vec{\mathcal{S}}_q(\mathcal{U}) \subset \mathcal{S}_q(\mathcal{U})$ to consist of simplices $\sigma : \Delta_q \to \mathcal{A}$ such that $\sigma_i \prec \sigma_j$, $\forall i < j$. We the define

$$\vec{C}^q(\mathcal{U}, \Omega^p) = \prod_{\sigma\in\vec{\mathcal{S}}_q(\mathcal{U})} \Omega^p(U_\sigma).$$

We define δ in a similar way and we obtain a new complex $\big(\vec{C}^\bullet(\mathcal{U}), \delta\big)$. The natural projection

$$\pi : \prod_{\sigma\in\mathcal{S}_q(\mathcal{U})} \Omega^p(U_\sigma) \to \prod_{\sigma\in\vec{\mathcal{S}}_q(\mathcal{U})} \Omega^p(U_\sigma)$$

induces a morphism of complexes

$$\pi : \big(C^\bullet(\mathcal{U}, \Omega^p), \delta\big) \to \big(\vec{C}^\bullet(\mathcal{U}), \delta\big).$$

[1] We will explain later the meaning of the term presheaf.

Moreover, the restriction of π to the subcomplex $(C^\bullet_{\rm alt}(\mathcal{U},\Omega^p),\delta)$ induces an *isomorphism* of complexes

$$\pi : \big(\, C^\bullet_{\rm alt}(\mathcal{U},\Omega^p),\delta\big) \to \big(\vec{C}^\bullet(\mathcal{U}),\delta\,\big).$$

One can show (see [3, VI.6] or [11, p. 214]) that the inclusion

$$\big(\, C^\bullet_{\rm alt}(\mathcal{U},\Omega^p),\delta\,\big) \to \big(\, C^\bullet(\mathcal{U},\Omega^p),\delta\,\big)$$

is a homotopy equivalence of complexes. This implies that the morphism

$$\pi : \big(\, C^\bullet(\mathcal{U},\Omega^p),\delta\,\big) \to \big(\, \vec{C}^\bullet(\mathcal{U}),\delta\,\big),$$

is a homotopy equivalence of complexes. In the sequel, we will often prefer to work with the equivalent complexes $\big(\, \vec{C}^\bullet(\mathcal{U}),\delta\,\big)$ or $\big(\, C^\bullet_{\rm alt}(\mathcal{U},\Omega^p),\delta\,\big)$. □

Form now the double complex $E^{\bullet,\bullet}=E^{\bullet,\bullet}_{\mathcal{U}}$

$$E^{p,q}_{\mathcal{U}} = C^q(\mathcal{U},\Omega^p),$$

where $d_h : C^q(\mathcal{U},\Omega^p)\to C^q(\mathcal{U},\Omega^{p+1})$ is given by the exterior derivative

$$C^q(\mathcal{U},\Omega^p)\ni(\omega_\sigma)_{\sigma\in\mathcal{S}_q(\mathcal{U})}\mapsto(d\omega_\sigma)_{\sigma\in\mathcal{S}_q(\mathcal{U})}\in C^q(\mathcal{U},\Omega^{p+1}),$$

and $d_v : C^q(\mathcal{U},\Omega^p)\to C^{q+1}(\mathcal{U},\Omega^p)$ is given by $\delta_\epsilon=\delta_\epsilon:=(-1)^p\delta$. Since $d\delta=\delta d$ we deduce that the conditions (1.1) are satisfied. We will refer to the double complex $(E^{\bullet,\bullet}_{\mathcal{U}},d,\pm\delta)$ as the *Čech-DeRham complex*.

Observe that we have a morphisms of complexes (see (1.4))

$$r : (\Omega^\bullet(M),d)\to(E^{\bullet,\bullet},d_h,\delta_\epsilon)$$

given by

$$\Omega^p(M)\ni\omega\mapsto(\omega|_{U_\alpha})_{\alpha\in\mathcal{A}}\in C^0(\mathcal{U},\Omega^p).$$

Theorem 2.4 (Generalized Mayer-Vietoris Principle)**.** *The morphism of complexes*

$$r : (\Omega^\bullet(M),d)\to(E^{\bullet,\bullet},d_h,\delta_\epsilon)$$

is a resolution of the DeRham complex. More explictly, this means that the columns in the diagram (2.2) are exact.

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle -\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^1(\mathcal{U},\Omega^0) & \xrightarrow[d]{} & C^1(\mathcal{U},\Omega^1) & \xrightarrow[d]{} & C^1(\mathcal{U},\Omega^2) & \xrightarrow[d]{} & \cdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle -\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^0(\mathcal{U},\Omega^0) & \xrightarrow[d]{} & C^0(\mathcal{U},\Omega^1) & \xrightarrow[d]{} & C^0(\mathcal{U},\Omega^2) & \xrightarrow[d]{} & \cdots \\
\uparrow{\scriptstyle r} & & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle r} & & \\
\Omega^0(M) & \xrightarrow[d]{} & \Omega^1(M) & \xrightarrow[d]{} & \Omega^2(M) & \xrightarrow[d]{} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & & \cdots
\end{array}
\tag{2.2}
$$

Proof. We have to show that for any $p \leq \dim M$ the sequence below is exact.

$$0 \to \Omega^p(M) \xrightarrow{r} C^0(\mathcal{U},\Omega^p) \xrightarrow{\delta} C^1(\mathcal{U},\Omega^p) \xrightarrow{\delta} \cdots .$$

We regard the above sequence as a cochain complex and we will show that the identity morphism is homotopic to the trivial morphism. Thus we set

$$C^{-1}(\mathcal{U},\Omega^p) := \Omega^p(M), \;\; C^q(\mathcal{U},\Omega^p) = 0, \;\; q < -1,$$

we will produce linear maps

$$K : C^q(\mathcal{U},\Omega^p) \to C^{q-1}(\mathcal{U},\Omega^p)$$

such that for any $q \geq -1$ and any $\omega \in C^q(\mathcal{U},\Omega^p)$ we have

$$\omega = K\delta\omega + \delta K\omega$$

where for uniformity we set $r = \delta$.

To construct K we fix a partition of unity $(\rho_\alpha)_{\alpha\in\mathcal{A}}$ subordinated to the cover $\mathcal{U}$. For $q \geq 1$ and $\omega = (\omega_\sigma)_{\sigma\in\mathcal{S}_q(\mathcal{U})} \in C^q(\mathcal{U},\Omega^p)$ we define $K\omega \in C^{q-1}(\mathcal{U},\Omega^p)$ to be the collection $(K_\tau\omega)_{\tau\in\mathcal{S}_{q-1}(\mathcal{U})} \in C^{q-1}(\mathcal{U},\Omega^p)$, where

$$K_{\tau_0\tau_1\cdots\tau_{q-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha\tau_0,\cdots\tau_{q-1}}.$$

When $\omega = (\omega_\alpha)_{\alpha\in\mathcal{A}} \in C^0(\mathcal{U},\Omega^p)$, then we define $K\omega \in \Omega^p(M)$ by the formula

$$K\omega = \sum_\alpha \rho_\alpha\omega_\alpha.$$

Note that for $q \geq 0$, any $\omega \in C^q(\mathcal{U},\Omega^p)$, $\sigma \in \mathcal{S}_q(\mathcal{U})$ we have

$$(\delta K\omega)_\sigma = \sum_{i=0}^q (-1)^i K_{\partial_i\sigma}\omega$$

$$= \sum_i (-1)^i \sum_\alpha \rho_\alpha \omega_{\alpha\sigma_0\cdots\hat{\sigma}_i\cdots\sigma_q}$$

On the other hand

$$K_\sigma(\delta\omega) = \sum_\alpha \rho_\alpha (\delta\omega)_{\alpha\sigma_0\cdots\sigma_q} = \sum_\alpha \rho_\alpha\omega_\sigma - \sum_{i=0}^{q}(-1)^i \sum_\alpha \rho_\alpha \omega_{\alpha\sigma_0\cdots\hat{\sigma}_i\cdots\sigma_q} = \omega_\sigma - (\delta K\omega)_\sigma.$$

When $\omega \in C^{-1}(\mathcal{U}, \Omega^p)$ the identity

$$\omega = \delta K\omega + K\delta\omega$$

is immediate. □

The Approximation Theorem 1.1 implies the following result.

Corollary 2.5. *Let $\mathcal{U}$ be an open cover of the smooth manifold M. Then the DeRham cohomology of M is isomorphic to the cohomology of the Čech-DeRham complex.* □

We can improve a bit this result. For $p \geq 0$ define

$$\mathcal{K} : C^q(\mathcal{U}, \Omega^p) \to C^{q-1}(\mathcal{U}, \Omega^p), \;\; \mathcal{K} := (-1)^p K,$$

where K is the homotopy operator constructed in the proof of Theorem 2.4. Then

$$\mathcal{K}\delta_\epsilon + \delta_\epsilon\mathcal{K} = \mathbb{1}. \tag{2.3}$$

Lemma 2.6. *For $i \geq 1$ we have*

$$[\delta_\epsilon, (d\mathcal{K})^i] := \delta_\epsilon(d\mathcal{K})^i - (d\mathcal{K})^i\delta_\epsilon = -(d\mathcal{K})^{i-1}d. \tag{2.4}$$

Proof. In any associative algebra A the commutator $a \mapsto [x, a] := xa - ax$ behaves like a derivation

$$[x, ab] = [x, a]b + a[x, b].$$

We deduce that

$$[\delta_\epsilon, (d\mathcal{K})^i] = [\delta_\epsilon, (d\mathcal{K})](d\mathcal{K})^{i-1} + (d\mathcal{K})^{i-1}[\delta_\epsilon, (d\mathcal{K})].$$

We have

$$[\delta_\epsilon, d\mathcal{K}] = \delta_\epsilon d\mathcal{K} - d\mathcal{K}\delta_\epsilon \overset{(d\delta_\epsilon + \delta_\epsilon d = 0)}{=} -d(\delta_\epsilon\mathcal{K} + \mathcal{K}\delta_\epsilon) \overset{(2.3)}{=} -d.$$

Hence

$$[\delta_\epsilon, (d\mathcal{K})^i] = -\underbrace{d(d\mathcal{K})^{i-1}}_{=0} - (d\mathcal{K})^{i-1}d.$$

□

Suppose that $\omega \in T^pC^\bullet(\mathcal{U}, \Omega^\bullet)$ then we can write

$$\omega = \sum_{i=0}^{p} \omega_i \;\; \omega_i \in C^i(\mathcal{U}, \Omega^{p-i})$$

and

$$D\omega = \sum_{j=0}^{p+1} \eta_j, \;\; \eta_j = d_h\omega_j + \delta_\epsilon\omega_{j-1} \in C^j(\mathcal{U}, \Omega^{p+1-j}),$$

and, following [1, Prop. 9.5] we write

$$f(\omega) = \sum_{i=0}^{p}(-d\mathcal{K})^i\omega_i - \sum_{j=1}^{p+1}\mathcal{K}(-d\mathcal{K})^{j-1}\eta_j. \tag{2.5}$$

This can be simplified a bit. We have

$$\begin{aligned} f(\omega) &= \sum_{i=0}^{p}(-d\mathcal{K})^i\omega_i - \sum_{j=1}^{p+1}\mathcal{K}(-d\mathcal{K})^{j-1}d\omega_j - \sum_{j=1}^{p+1}\mathcal{K}(-d\mathcal{K})^{j-1}\delta_\epsilon\omega_{j-1} \\ &= \sum_{i=0}^{p}(-d\mathcal{K})^i\omega_i - \sum_{j=1}^{p+1}\mathcal{K}[\delta_\epsilon,(-d\mathcal{K})^j]\omega_i - \sum_{j=1}^{p+1}\mathcal{K}(-d\mathcal{K})^{j-1}\delta_\epsilon\omega_{j-1}. \\ &= \sum_{i=0}^{p}(-d\mathcal{K})^i\omega_i - \sum_{j=1}^{p+1}\mathcal{K}\delta_\epsilon(-d\mathcal{K})^j\omega_j + \sum_{j=1}^{p+1}\mathcal{K}(-d\mathcal{K})^j\delta_\epsilon\omega_j - \sum_{j=1}^{p+1}\mathcal{K}(-d\mathcal{K})^{j-1}\delta_\epsilon\omega_{j-1} \\ &= \underbrace{\sum_{i=0}^{p}(-d\mathcal{K})^i\omega_i - \sum_{j=1}^{p+1}\mathcal{K}\delta_\epsilon(-d\mathcal{K})^j\omega_j}_{I} + \underbrace{\sum_{j=1}^{p+1}\mathcal{K}(-d\mathcal{K})^j\delta_\epsilon\omega_j - \sum_{i=0}^{p}\mathcal{K}(-d\mathcal{K})^i\delta_\epsilon\omega_i}_{II} \\ &= \underbrace{\omega_0 + \sum_{i=1}^{p}(\mathbb{1}-\mathcal{K}\delta_\epsilon)(-d\mathcal{K})^i\omega_i}_{I} - \mathcal{K}\delta_\epsilon\omega_0 \\ &= \delta_\epsilon\mathcal{K}\omega_0 + \sum_{i=1}^{p}\delta_\epsilon\mathcal{K}(-d\mathcal{K})^i\omega_i. \end{aligned}$$

Hence

$$f(\omega) = \delta_\epsilon\mathcal{K}\left(\sum_{i=0}^{p}(-d\mathcal{K})^i\omega_i\right). \tag{2.6}$$

Observe that $f(\omega) \in C^0(\mathcal{U},\Omega^p)$. In fact $\eta := f(\omega)$ lies in the image of $r : \Omega^p(M) \to C^0(\mathcal{U},\Omega^p)$. Indeed $\delta\eta = \delta_\epsilon\eta = 0$, which means that the collection $\{\eta_\alpha \in \Omega^p(U_\alpha)\}$ satisfies $\eta_\alpha = \eta_\beta$ on $U_{\alpha\beta}$.

Proposition 2.7 (Collating Formula)**.** *The morphism $f : C(\mathcal{U},\Omega) \to T^\bullet C(\mathcal{U},\Omega)$ commutes with $D = d_h + \delta_\epsilon$ so it is a morphism of complexes. Moreover, it is chain homotopic to the identity, where the homotopy operator*

$$L : T^pC^\bullet(\mathcal{U},\Omega^\bullet) \to T^{p-1}C^\bullet(\mathcal{U},\Omega^\bullet)$$

is given by

$$L\omega_i = \sum_{j<i}\underbrace{\mathcal{K}(-d\mathcal{K})^{i-j-1}\omega_i}_{\in C^j(\mathcal{U},\Omega^{p-j-1})},\ \ \forall\omega_i \in C^i(\mathcal{U},\Omega^{p-i}).$$

Proof. (a) Let us first show that $fD = df$. Let

$$\omega = \sum_{i=0}^{p}\omega_i,\ \ \omega_i \in C^i(\mathcal{U},\Omega^{p-i}),\ \ D\omega = \sum_{j=0}^{p+1}\eta_j,\ \ \eta_j \in C^j(\mathcal{U},\Omega^{p+1-j}).$$

Using (2.5) we deduce

$$f(D\omega) = \sum_{i=0}^{p+1}(-d\mathcal{K})^j\eta_j.$$

On the other hand,

$$Df(\omega)=df(\omega)=d\omega_0-\sum_{j=1}^{p+1}d\mathcal{K}(-d\mathcal{K})^{j-1}=\eta_0+\sum_{j=1}^{p+1}(-d\mathcal{K})^j\eta_j=f(D\omega).$$

Let $\omega_i\in C^i(\mathcal{U},\Omega^{p-i})$. Then

$$f(\omega_i)\stackrel{(2.6)}{=}\delta_\epsilon\mathcal{K}(-d\mathcal{K})^i\omega_i.$$

Next, we observe that

$$\begin{aligned}DL\omega_i&=\sum_{j<i}d\mathcal{K}(-d\mathcal{K})^{i-j-1}\omega_i+\sum_{j<i}\delta_\epsilon\mathcal{K}(-d\mathcal{K})^{i-j-1}\omega_i\\&=-\sum_{j<i}(-d\mathcal{K})^{i-j}\omega_i+\sum_{j<i}\delta_\epsilon\mathcal{K}(-d\mathcal{K})^{i-j-1}\omega_i,\\LD\omega_i&=Ld\omega_i+L\delta_\epsilon\omega_i=\sum_{j<i}\mathcal{K}(-d\mathcal{K})^{i-j-1}d\omega_i+\sum_{k\leq i}\mathcal{K}(-d\mathcal{K})^{i-k}\delta_\epsilon\omega_i\end{aligned}$$

Using (2.4) we deduce

$$(-d\mathcal{K})^{i-k}\delta_\epsilon=\delta_\epsilon(-d\mathcal{K})^{i-k}-(-d\mathcal{K})^{i-k-1}d$$

On the other hand (2.3) implies

$$\mathcal{K}\delta_\epsilon(-d\mathcal{K})^{i-k}=(-d\mathcal{K})^{i-k}-\delta_\epsilon\mathcal{K}(-d\mathcal{K})^{i-k},$$

so that

$$\mathcal{K}(-d\mathcal{K})^{i-k}\delta_\epsilon=(-d\mathcal{K})^{i-k}-\delta_\epsilon\mathcal{K}(-d\mathcal{K})^{i-k}-\mathcal{K}(-d\mathcal{K})^{i-k-1}d.$$

Hence

$$\begin{aligned}LD\omega_i&=\sum_{j<i}\mathcal{K}(-d\mathcal{K})^{i-j-1}d\omega_i+\sum_{k\leq i}(-d\mathcal{K})^{i-k}\omega_i-\sum_{k\leq i}\delta_\epsilon\mathcal{K}(-d\mathcal{K})^{i-k}\omega_i\\&\quad-\sum_{k\leq i}\mathcal{K}(-d\mathcal{K})^{i-k-1}d\omega_i\\&=\sum_{k\leq i}(-d\mathcal{K})^{i-k-1}\omega_i-\sum_{k\leq i}\delta_\epsilon\mathcal{K}(-d\mathcal{K})^{i-k}\omega_i.\end{aligned}$$

We deduce

$$\begin{aligned}DL\omega_i+LD\omega_i&=-\sum_{j<i}(-d\mathcal{K})^{i-j}\omega_i+\sum_{k<i}\delta_\epsilon\mathcal{K}(-d\mathcal{K})^{i-k-1}\omega_i\\&\quad+\sum_{k\leq i}(-d\mathcal{K})^{i-k-1}\omega_i-\sum_{k\leq i}\delta_\epsilon\mathcal{K}(-d\mathcal{K})^{i-k}\omega_i\\&=\omega_i-\delta_\epsilon\mathcal{K}(-d\mathcal{K})^i\omega_i=\omega_i-f(\omega_i).\end{aligned}$$

□

Corollary 2.8. *Suppose that*

$$\omega=\sum_{i=0}^{p}\omega_i,\quad\Omega^i=C^i(\mathcal{U},\Omega^{p-i})$$

is a Čech-DeRham cocycle. Then its is cohomologous with the DeRham cocycle

$$f(\omega)=\sum_{i=0}^{p}(-d\mathcal{K})^i\omega^i.$$

□

For any $q \geq 0$, we set

$$C^q(\mathcal{U}, \mathbb{R}) := \prod_{\sigma \in \mathcal{S}_q(\mathcal{U})} \mathbb{R}.$$

In other words, and element $r \in C^q(\mathcal{U}, \mathbb{R})$ is a collection of real numbers

$$\underline{r} = (r_{\sigma_0 \dots \sigma_q})_{(\sigma_0, \dots, \sigma_q) \in \mathcal{S}_q(\mathcal{U})}.$$

As before we have a linear map

$$\delta : C^q(\mathcal{U}, \mathbb{R}) \to C^{q+1}(\mathcal{U}, \mathbb{R}), \quad (\delta \underline{r})_{\sigma_0 \dots \sigma_{q+1}} = \sum_{i=0}^{q+1} (-1)^i r_{\sigma_0 \dots \hat{\sigma}_i \dots \sigma_{q+1}}.$$

This operator is a coboundary operator, i.e., $\delta^2 = 0$. We obtain a Cochain complex $(C^\bullet(\mathcal{U}, \mathbb{R}), \delta)$ called the *Čech complex of the cover* $\mathcal{U}$ *with coefficients in* $\mathbb{R}$. We denote by $H^\bullet(\mathcal{U}, \mathbb{R})$ its cohomology, and we will refer to it as the *Čech cohomology of the cover* $\mathcal{U}$ *with coefficients in* $\mathbb{R}$.

Note that $C^q(\mathcal{U}, \mathbb{R})$ is naturally a subspace of $C^q(\mathcal{U}, \Omega^0)$. Indeed, an element $(r_{\sigma_0, \dots, \sigma_q}) \in C^q(\mathcal{U}, \mathbb{R})$ can be viewed as a collection of constant functions $r_{\sigma_0 \dots \sigma_q} : U_{\sigma_0 \dots \sigma_q} \to \mathbb{R}$. Thus, the Čech complex $C^\bullet(\mathcal{U}, \mathbb{R})$ is a subcomplex of $C^\bullet(\mathcal{U}, \Omega^0)$. In particular, the inclusion

$$(C^\bullet(\mathcal{U}, \mathbb{R}), \delta_\epsilon) \stackrel{i}{\hookrightarrow} (C^\bullet(\mathcal{U}, \Omega^0), \delta_\epsilon)$$

leads to a morphism to the transpose $\widehat{E}_{\mathcal{U}}^{\bullet,\bullet}$ of the Čech-DeRham double complex $E_{\mathcal{U}}^{\bullet,\bullet}$.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^0(\mathcal{U}, \Omega^1) & \xrightarrow[-\delta]{} & C^1(\mathcal{U}, \Omega^1) & \xrightarrow[-\delta]{} & C^2(\mathcal{U}, \Omega^1) & \xrightarrow[-\delta]{} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^0(\mathcal{U}, \Omega^0) & \xrightarrow[\delta]{} & C^1(\mathcal{U}, \Omega^0) & \xrightarrow[\delta]{} & C^2(\mathcal{U}, \Omega^0) & \xrightarrow[\delta]{} & \cdots \\
\uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \\
C^0(\mathcal{U}, \mathbb{R}) & \xrightarrow[\delta]{} & C^1(\mathcal{U}, \mathbb{R}) & \xrightarrow[\delta]{} & C^2(\mathcal{U}, \mathbb{R}) & \xrightarrow[\delta]{} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & & \cdots
\end{array} \tag{2.7}$$

The Poincaré lemma implies that when $\mathcal{U}$ is a *good cover* then the columns in the above diagram are exact. Indeed, for any $\sigma \in \mathcal{S}_q(\mathcal{U})$ the overlap U_σ is diffeomorphic to $\mathbb{R}^m$ so that the sequence below is exact

$$0 \to \mathbb{R} \hookrightarrow \Omega^0(U_\sigma) \xrightarrow{d} \Omega^1(U_\sigma) \xrightarrow{d} \Omega^2(U_\sigma) \xrightarrow{d} \cdots.$$

The Approximation Theorem 1.1 coupled with Corollary 2.5 imply the following result.

Corollary 2.9. *If* $\mathcal{U}$ *is a* good *cover of* M*, then the cohomology of the Čech complex complex is isomorphic to the DeRham cohomology of* M*, i.e.,* $H^q(\mathcal{U}, \mathbb{R}) \cong H^q_{DR}(M)$*,* $\forall q \geq 0$.

Corollary 2.10. *Suppose that* $\mathcal{U}$ *is a* finite, good *cover of* $\mathcal{M}$. *Set*

$$c_k(\mathcal{U}) := \#\boldsymbol{N}_q(\mathcal{U}),$$

where $\boldsymbol{N}_q(\mathcal{U})$ *is defined by (2.1). Then*

$$\chi(M) = \sum_{q\geq 0}(-1)^q c_1(\mathcal{U}).$$

Proof. Observe that

$$c_q(\mathcal{U}) = \dim \vec{C}_q(\mathcal{U}, \mathbb{R})$$

so that

$$\sum_{q\geq 0}(-1)^q c_q(\mathcal{U}) = \sum_{q\geq 0}(-1)^q \dim H^q(\mathcal{U}, \mathbb{R}) = \sum_{q\geq 0}(-1)^q \dim H^q_{DR}(M).$$

□

Suppose that $\mathcal{U}$ is a good cover. Then a Čech n-cocycle is a collection of numbers $\underline{r} = (r_\sigma)_{\sigma\in\mathcal{S}_n(\mathcal{U})}$ satisfying the cocycle condition

$$\sum_{i=0}^{n+1}(-1)^i r_{\sigma_0\ldots\hat{\sigma}_i\ldots\sigma_{n+1}} = 0, \;\; \forall(\sigma_0, \ldots, \sigma_{n+1}) \in \mathcal{S}_{n+1}(\mathcal{U}).$$

According to the above corollary, this cocycle determines a DeRham cohomology class $[\underline{r}] \in H^n_{DR}(M)$. Using the Collating Formula we can describe explicitely a degree n closed form representing this class. More precisely

$$\omega = (-d\mathcal{K})^n \underline{r}. \tag{2.8}$$

Example 2.11. A Čech cochain is a collection $\underline{r}$ of real numbers $r_{\alpha\beta}$, one number for any pair (α, β) such that $U_{\alpha\beta} \neq \emptyset$. It is a Čech cocycle if for any triplet (α, β, γ) such that $U_{\alpha\beta\gamma} \neq \emptyset$ we have

$$r_{\beta\gamma} - r_{\alpha\gamma} + r_{\alpha\beta} = 0. \tag{2.9}$$

If in the above equality we choose $\alpha = \beta = \gamma$ we deduce that $r_{\alpha\alpha} = 0$, $\forall\alpha \in \mathcal{A}$. If next we choose $\alpha = \gamma$ we deduce $r_{\beta\alpha} = -r_{\alpha\beta}$. Thus (2.9) is equivalent to

$$r_{\alpha\beta} = -r_{\beta\alpha}, \;\; r_{\gamma\alpha} = r_{\gamma\beta} + r_{\beta\alpha}, \;\; \forall(\alpha, \beta, \gamma) \in \mathcal{S}_2(\mathcal{U}). \tag{2.10}$$

If we choose a partition of unity $(\rho_\alpha)_{\alpha\in\mathcal{A}}$ subordinated to $\mathcal{U}$ then we have then $\eta := \mathcal{K}\underline{r} = K\underline{r} \in C^0(\mathcal{U}, \Omega^0)$ is defined by

$$\eta_\alpha = \sum_\beta \rho_\alpha r_{\alpha\beta}.$$

Then $(-d\mathcal{K})\underline{r}$ is represented by the collection of forms $\omega_\alpha := -d\eta_\alpha \in \Omega^1(U_\alpha)$. The cocycle condition (2.10) implies that

$$\omega_\alpha = \omega_\beta \;\text{ on } U_{\alpha\beta}$$

so that there exists a closed 1-form $\omega \in \Omega^1(M)$ such that $\omega|_{U_\alpha} = \omega_\alpha$, $\forall\alpha \in \mathcal{A}$. The DeRham cohomology class determined by ω corresponds to the Čech cohomology class determined by $\underline{r}$ via the above isomorphism

$$H^\bullet(\mathcal{U}, \mathbb{R}) \to H^\bullet_{DR}(M).$$

□

Remark 2.12. (a) The Čech complex $C^\bullet(\mathcal{U},\mathbb{R})$ has homotopy equivalent descriptions $C^\bullet_{\mathrm{alt}}(\mathcal{U},\mathbb{R})$ and $\vec{C}^\bullet(\mathcal{U},\mathbb{R})$ constructed following the same prescriptions as the ones in Remark 2.3. Recall that the definition of $\vec{C}^\bullet(\mathcal{U},\mathbb{R})$ uses a fixed (but arbitrary) linear order on $\mathcal{A}$. In the remainder of this remark we will assume we have fixed such a linear order $\prec$ on $\mathcal{A}$.

(b) To an open cover $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ we can associate a triangulated space $|\boldsymbol{N}(\mathcal{U})|$ called the *geometric realization of the nerve* which is constructed as follows. The space has one vertex v_α for each $\alpha\in\mathcal{A}$. Two vertices v_α, v_β are joined by an edge if and only if $\{\alpha,\beta\}\in\boldsymbol{N}(\mathcal{U})$, i.e., $U_{\alpha\beta}\neq\emptyset$. Three vertices $v_\alpha, v_\beta, v_\gamma$ span a triangle if and only if $\{\alpha,\beta,\gamma\}\in\boldsymbol{N}(\mathcal{U})$ etc.

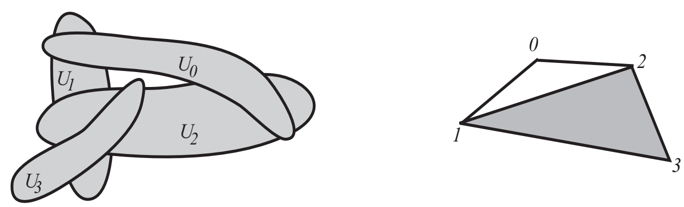


FIGURE 3. *An open cover and the geometric realization of its nerve.*

Formally and more precisely, this triangulated space is defined as follows. Denote by $\mathbb{R}^{\mathcal{A}}$ the vector space of functions $f:\mathcal{A}\to\mathbb{R}$ such that $f(\alpha)=0$ for all but finitely many $\alpha\in\mathcal{A}$. Note that for any finite subset $S\subset\mathcal{A}$ the space $\mathbb{R}^S$ is contained in $\mathbb{R}^{\mathcal{A}}$. We define a topology on $\mathbb{R}^{\mathcal{A}}$ by declaring a subset $X\subset\mathbb{R}^{\mathcal{A}}$ open if and only if for any finite subset $S\subset\mathcal{A}$ the intersection $X\cap\mathbb{R}^S$ is open in the finite dimensional Euclidean space $\mathbb{R}^S$. For $\alpha\in\mathcal{A}$ we denote by $\boldsymbol{v}_\alpha\in\mathbb{R}^{\mathcal{A}}$ the function defined by

$$\boldsymbol{v}_\alpha(\beta)=\begin{cases}0, & \beta\neq\alpha\\ 1, & \beta=\alpha.\end{cases}$$

For a finite subset $S\subset\mathcal{A}$ of cardinality $q+1>0$ we denote by $\Delta_S\subset\mathbb{R}^{\mathcal{A}}$ the convex hull of the set $\{v_s;\;\; s\in S\}$. The set Δ_S is an affine q-dimensional simplex. Thus, $\Delta_{\alpha,\beta}$ is the line segment joining the points $\boldsymbol{v}_\alpha,\boldsymbol{v}_\beta$, $\Delta_{\alpha,\beta,\gamma}$ is the triangle spanned by the points $\boldsymbol{v}_\alpha,\boldsymbol{v}_\beta,\boldsymbol{v}_\gamma$, etc. The triangulated space associated to $\mathcal{U}$ is then

$$|\boldsymbol{N}(\mathcal{U})| = \bigcup_{S\in\boldsymbol{N}(\mathcal{U})}\Delta_S\subset\mathbb{R}^{\mathcal{A}}.$$

Any partition of unity (ρ_α) subordinated to $\mathcal{U}$ defines a map

$$\mathcal{R}: M\to|\boldsymbol{N}(\mathcal{U})|,\;\; M\ni x\mapsto\sum_{\alpha\in\mathcal{A}}\rho_\alpha(x)\boldsymbol{v}_\alpha$$

The map is well defined because for any $x\in M$ the number $\rho_\alpha(x)$ is zero for all but finitely many α's. If we set

$$S_x := \bigl\{\alpha\in\mathcal{A};\;\;\rho_\alpha(x)\neq 0\bigr\},$$

then we deduce that $S_x\in\boldsymbol{N}(\mathcal{U})$ and $\mathcal{R}(x)\in\Delta_{S_x}\subset|\boldsymbol{N}(\mathcal{U})|$. The map $\mathcal{R}$ is continuous and moreover, we have the following result. For a particularly nice proof we refer to [12, §5].

Theorem 2.13 (Borsuk-Weil)**.** *If* $\mathcal{U}$ *is a* good cover*, then the map* $\mathcal{R}: M\to|\boldsymbol{N}(\mathcal{U})|$ *is a* ***homotopy equivalence****.*

The complex $\vec{C}^\bullet(\mathcal{U},\mathbb{R})$ is the *simplicial* cochain complex associated to the triangulated space $|\boldsymbol{N}(\mathcal{U})|$. This is isomorphic to the singular cohomology of $|\boldsymbol{N}(\mathcal{U})|$ and the Borsuk-Weil theorem implies that it is also isomorphic to the singular cohomology of M with real coefficients. If we now invoke Corollary 2.9, then we deduce that the DeRham cohomology of M is isomorphic to the singular cohomology of M with real coefficients. □

3. Presheaves and sheaves

Suppose that X is a topological space and R is a commutative ring with 1. We denote by $\boldsymbol{Mod}_R$ the category of R-modules. We denote by $\boldsymbol{Open}(X)$ the collection of open subsets of X. For any family of open sets $\mathcal{U} \subset \boldsymbol{Open}(X)$ we set

$$\mathcal{O}_{\mathcal{U}} := \bigcup_{U \in \mathcal{U}} U.$$

We organize $\boldsymbol{Open}(X)$ as a category whose morphisms are the inclusions $U \hookrightarrow V$.

Definition 3.1. A *presheaf* of R-modules on X is contravariant functor $\mathcal{S} : \boldsymbol{Open}(X) \to \boldsymbol{Mod}_R$. We denote by $\boldsymbol{PSh}_R(X)$ the collection of presheaves of R-modules on X □

In other words a presheaf $\mathcal{S}$ is a correspondence that associates to each $U \in \boldsymbol{Open}(X)$ an R-module $\mathcal{S}(U)$, and to each inclusion $U \hookrightarrow V$ a "restriction" morphism

$$r_{UV} = r^{\mathcal{S}}_{UV} : \mathcal{S}(V) \to \mathcal{S}(U)$$

such that

$$r_{UU} = \mathbb{1}_{\mathcal{S}(U)}, \;\; r_{UW} = r_{UV} \circ r_{VW}$$

for all $U, V, W \in \boldsymbol{Open}(X)$, $U \subset V \subset W$. When no confusion is possible, for $U, V \in \boldsymbol{Open}(X)$, $U \subset V$, and $s \in \mathcal{S}(V)$ we will set

$$s|_U := r_{UV}(s)$$

and we will refer to $s|_U$ as the restriction of s to U.

The module $\mathcal{S}(U)$ is called the module of sections of $\mathcal{S}$ over U. Often one uses the alternate notation

$$\Gamma(\mathcal{U}, \mathcal{S}) := \mathcal{S}(U).$$

Example 3.2. (a) For any R-module G we denote by $\underline{G}_X$ the *constant presheaf* of R-module on X determined by G. For $U \in \boldsymbol{Open}(X)$ the group of sections of $\underline{G}$ over U is the group locally constant maps $f : U \to G$. Thus, if U is connected, then we have an isomorphism

$$\Gamma(U, \underline{G}) \cong G.$$

(b) Denote by C^0_X the presheaf of vector spaces defined by

$$\Gamma(U, C^0_X) := \big\{ \text{continuous functions } U \to \mathbb{R} \big\}.$$

(c) Denote by $C^0_{X,b}$ the presheaf of vector spaces defined by

$$\Gamma(U, C^0_{X,b}) := \big\{ \text{bounded continuous functions } U \to \mathbb{R} \big\}.$$

(d) If M is a smooth manifold, then we denote by Ω^p_M the presheaf of vector spaces defined by

$$\Gamma(U, \Omega^p_M) = \Omega^p(U).$$

(e) Suppose that M, N are smooth manifolds and $f : N \to M$ is a smooth map. Then we have a presheaf H^q_f on M defined by

$$\Gamma(U, H^q_f) := H^q_{DR}\big(f^{-1}(U) \big).$$

(f) if $\mathcal{S} \in \boldsymbol{PSh}_R(X)$ and $U \in \boldsymbol{Open}(X)$, then $\mathcal{S}|_U$ denotes the presheaf on U given by

$$\mathcal{S}|_U(V) = \mathcal{S}(V), \;\; \forall V \in \boldsymbol{Open}(U).$$

We say that $\mathcal{S}|_U$ is the restriction of $\mathcal{S}$ to U. □

Definition 3.3. Let $\mathcal{S}_0, \mathcal{S}_1 \in \boldsymbol{PSh}_R(X)$. A morphisms of presheaves $\varphi : \mathcal{S}_0 \to \mathcal{S}_1$ is a collection of morphisms of R-modules

$$\varphi_U : \mathcal{S}_0(U) \to \mathcal{S}_1(U), \;\; U \in \boldsymbol{Open}(X),$$

such that for any $U, V \in \boldsymbol{Open}(X)$, $U \subset V$ we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{S}_0(V) & \xrightarrow[\varphi_V]{} & \mathcal{S}_1(V) \\
{\scriptstyle r^{\mathcal{S}_0}_{UV}}\Big\downarrow & & {\scriptstyle r^{\mathcal{S}_1}_{UV}}\Big\downarrow \\
\mathcal{S}_1(U) & \xrightarrow[\varphi_V]{} & \mathcal{S}_1(U)
\end{array}$$

□

Definition 3.4. (a) A presheaf $\mathcal{S} \in \boldsymbol{PSh}_R(X)$ is called a *sheaf* if it satisfies the following two conditions.

($\mathbf{S_1}$) For any family $\mathcal{U} \subset \boldsymbol{Open}(X)$ the morphism

$$\mathcal{S}(\mathcal{O}_{\mathcal{U}}) \to \prod_{U \in \mathcal{U}} \mathcal{S}(U), \;\; \mathcal{S}(\mathcal{O}_{\mathcal{U}}) \ni s \mapsto (s|_U)_{U \in \mathcal{U}} \in \prod_{U \in \mathcal{U}} \mathcal{S}(U)$$

is injective.

($\mathbf{S_2}$) For any family $\mathcal{U} \subset \boldsymbol{Open}(X)$ and any collection $s_U \in \mathcal{S}(U)$, $U \in \mathcal{U}$ such that

$$(s_U)|_{U \cap V} = (s_V)|U \cap V, \;\; \forall U, V \in \mathcal{U}$$

there exists a section $s \in \mathcal{S}(\mathcal{O}_{\mathcal{U}})$ such that

$$s|_U = s_U, \;\; \forall U \in \mathcal{U}.$$

We denote by $\boldsymbol{Sh}_R(X) \subset \boldsymbol{PSh}_R\, X)$ the full[2] subcategory of R-modules on X. □

Example 3.5. The presheaves in Example 3.2(a),(b), (d) are sheaves. The presheaves (c), (e) are not always sheaves. Also, the restriction of a sheaf to an open subset is a sheaf on that open subset. □

Suppose that $\mathcal{S} \in \boldsymbol{PSh}_R(X)$ and $x \in X$. Denote by $\boldsymbol{Open}_x(X)$ the collection of all open subset of X that contain x. The collection of modules

$$\big\{ \mathcal{S}(U_x); \;\; U_x \in \boldsymbol{Open}_x(X) \big\}$$

together with the family of restriction morphisms r_{UV}, $U, V \in \boldsymbol{Open}_x(X)$, $U \subset V$ is an inductive family. We set

$$\mathcal{S}_x := \varinjlim_{U \in \boldsymbol{Open}_x(X)} \mathcal{S}(U).$$

We refer to the R-module $\mathcal{S}_x$ the *stalk* of $\mathcal{S}$ at x. More explicitely, $\mathcal{S}_x$ is the set of equivalence classes of an equivalence relation $\sim_x$ on the disjoint union of the modules $\mathcal{S}(U)$, $U \in \boldsymbol{Open}_x(X)$: for $U_0, U_1 \in \boldsymbol{Open}_x(X)$ we declare the sections $s_0 \in \mathcal{S}(U_0)$ and $s_1 \in \mathcal{S}(U_1)$ to be equivalent if there exists $U_2 \subset U_0 \cap U_1$, $U_2 \in \boldsymbol{Open}_x(X)$ such that

$$s_0|_{U_2} = s_1|_{U_2}.$$

[2]The attribute *full* signify that morphisms between sheaves $\mathcal{S}_0, \mathcal{S}_1$ are the same as morphisms between $\mathcal{S}_0, \mathcal{S}_1$ viewed as presheaves.

The equivalence class of a section $s \in \mathcal{S}(U)$, $U \in \boldsymbol{Open}_x(X)$ is denoted by $[s]_x$ and it is called the *germ of s at x*.

The *completion* of a presheaf $\mathcal{S} \in \boldsymbol{PSh}_R(X)$ is the presheaf $\widehat{\mathcal{S}} \in \boldsymbol{PSh}_R(X)$, where for any $U \in \boldsymbol{Open}(X)$ a section of $\widehat{\mathcal{S}}$ over U is defined to be a collection of germs $(\gamma_x)_{x\in U}$ satisfying the following conditions.

- $\gamma_x \in \mathcal{S}_x, \forall x \in U$.
- For any $x \in U$, there exists $V \in \boldsymbol{Open}_x(U)$ and $s \in \mathcal{S}(V)$ such that

$$\gamma_y = [s]_y, \ \ \forall y \in V.$$

Observe that

$$\mathcal{S}(U) \subset \widehat{\mathcal{S}}(U), \ \ \forall U \in \boldsymbol{Open}(X).$$

We have the following elementary result whose proof is left to the reader.

Proposition 3.6. *(a) The completion of a presheaf is a sheaf.*
(b) If $\mathcal{S}$ is a sheaf, then $\widehat{\mathcal{S}} = \mathcal{S}$. □

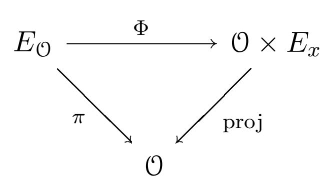
Due to the above result the completion $\widehat{\mathcal{S}}$ is also known as the *sheaf associated to the presheaf* $\mathcal{S}$ of the *sheafification* of $\mathcal{S}$.

Definition 3.7. (a) A sheaf $\mathcal{S}$ of Abelian groups on a topological space X is called *constant* if there exists a group G such that $\mathcal{S}$ is isomorphic to the sheaf $\underline{G}_X$. The sheaf $\underline{G}_X$ is called the *constant sheaf on X with stalk G*.
(b) A sheaf $\mathcal{S}$ on a topological space is called locally constant if for any point $x \in X$ there exists an open neighborhood U_x of x such that the restriction of $\mathcal{S}$ to U_x is a constant sheaf. □

Example 3.8. Suppose that $\pi : E \to M$ is a smooth fiber bundle with fiber F over the smooth, connected m-dimensional manifold. For $n \geq 0$ consider the presheaf H^n_π,

$$\Gamma(U, H^n_\pi) = H^n\big(\pi^{-1}(U)\big)$$

For any $x \in M$ there exists an open neighborhood $\mathcal{O}$ of x diffeomeorphic to $\mathbb{R}^m$ such that the restriction of E over $\mathcal{O}$ is a trivializable fiber bundle. In other words, there exists a diffeomorphism $\Phi : E_{\mathcal{O}} \to \mathcal{O} \times F$ such that the diagram below is commutative.



In particular for any $U \in \boldsymbol{Open}(\mathcal{O})$ we have

$$\Gamma(U, H^n_\pi) \cong H^n(\mathcal{O} \times E_x) \cong \bigoplus_{j=0}^{n} H^j(U) \otimes H^{n-j}(E_x).$$

We deduce that the stalk at x of H^n_π is the vector space $H^n_{\pi,x}$

$$H^n_{\pi,x} = \bigoplus_{j=0}^{n} \Big(\varinjlim_{U \in \boldsymbol{Open}_x(\mathcal{O})} H^j(U) \Big) \otimes H^{n-j}(E_x) = H^n(E_x).$$

We denote by $\widehat{H}^n_\pi$ the completion of this presheaf.

Suppose that $\mathcal{O}$ is an open subset of M such that E is trivilizable over $\mathcal{O}$ so that we can assume that $E_{\mathcal{O}}$ is the trivial fiber bundle $\mathcal{O} \times F \to \mathcal{O}$. We want to prove that the restriction of $\widehat{H}^n_\pi$ to $\mathcal{O}$ is a constant sheaf. Let $U \in \boldsymbol{Open}(\mathcal{O})$. A section of $\widehat{H}^n_\pi$ over an open set U is a collection of cohomology classes $\omega_x \in H^n(E_x) = H^n(\{x\} \times F)$ such that for any $x \in U$ there exists $V_x \in \boldsymbol{Open}_x(U)$ and $\omega_{V_0} \in H^n(E_{V_x})$ such that

$$\omega_{V_x}|_{E_y} = \omega_y \in H^n(F), \;\; \forall y \in V_x.$$

We want to prove that the map

$$U \ni x \mapsto \omega_x \in H^n(F)$$

is locally constant. Fix $x_0 \in U$ and let $V_0 = V_{x_0}$ as above. We will show that the map

$$V_0 \ni y \mapsto \omega_y \in H^n(F)$$

is locally constant.

A cohomology class $\omega_{V_0} \in H^n(E_V)$ has a unique decomposition

$$\omega_{V_0} = \sum_{i=0}^n \alpha^i \boxtimes \beta^{n-i}, \;\; \alpha^i \in H^i(V_0), \;\; \beta^{n-i} \in H^{n-i}(V_0).$$

Observe that α^0 is represented by a locally constant function $V_0 \to \mathbb{R}$. For $y \in V_0$ we have

$$\omega_y = \omega_{V_0}|_{E_y} = \omega_{V_0}|_{y \times F} = \alpha^0_V(y)\beta^n_V.$$

We deduce that the restriction of the sheaf $\widehat{H}^n_\pi$ to the trivializing open set $\mathcal{O}$ is isomorphic to the constant sheaf with stalk $H^n(F)$.

Example 3.9. Suppose that X is path connected and locally path connected. Denote by $\pi : \widetilde{X} \to X$ the universal cover of X. For any $U \in \boldsymbol{Open}(X)$ we set

$$\widetilde{U} := \pi^{-1}(U) \in \boldsymbol{Open}(\widetilde{X}).$$

Fix $x_0 \in X$. Then there exists a free right action of $\pi_1(X, x_0)$ on $\widetilde{X}$ and a natural homeomorphism $\widetilde{X}/\pi_1(X, x_0) \to X$.

Fix an R-module M and a group morphism

$$\mu : \pi_1(X, x_0) \to \operatorname{Aut}_R(M), \;\; \pi_1(X, x_0) \ni \gamma \mapsto \mu_\gamma \in \operatorname{Aut}_R(M).$$

For any $U \in \boldsymbol{Open}(X)$ we denote by $\mathcal{C}_\mu(U, M)$ the space of locally constant maps $f : \widetilde{U} \to M$ satisfying the *equivariance condition*

$$f(\tilde{x} \cdot \gamma^{-1}) = \mu_\gamma\big(f(\tilde{x})\big), \;\; \forall \tilde{x} \in \widetilde{U}, \;\; \gamma \in \pi_1(X, x_0).$$

The collection $\mathcal{C}_\mu(U, M)$, $U \in \boldsymbol{Open}(X)$ defines a locally constant sheaf $\underline{M}_{X,\mu} \in \boldsymbol{Sh}_R(X)$. The morphism μ is called the monodromy of this sheaf. □

The following important result states that any locally constant sheaf can be constructed using the method in Example 3.9. For a proof we refer to [7, IV.9] or [10].

Theorem 3.10. *Suppose that X is a path connected and locally path connected space and $\mathcal{S} \in \boldsymbol{Sh}_R(X)$ is a locally constant sheaf. Fix $x_0 \in X$ and set $M := \mathcal{S}_{x_0} \in \boldsymbol{Mod}(R)$. Then there exists a group morphism*

$$\mu : \pi_1(X, x_0) \to \operatorname{Aut}_R(M)$$

such that the locally constant sheaf $\underline{M}_{X,\mu}$ is isomorphic to $\mathcal{S}$. □

Corollary 3.11. *Suppose that M is a connected and simply connected smooth manifold. Then any locally constant sheaf on M is constant.* □

Definition 3.12. Suppose that $\pi : \mathbf{\Sigma} \to M$ is a smooth fiber bundle over the connected m-dimensional manifold with fiber the sphere S^n. An *orientation* of this fiber bundle is a collection of cohomology classes $\{\omega_x \in H^n(\mathbf{\Sigma}_x);\;\; x \in M\}$ satisfying the following properties.

($\mathbf{O_1}$) There exists an open cover $\mathfrak{U}$ of M and cohomology classes $\omega_U \in H^n(\mathbf{\Sigma}_U)$, $U \in \mathfrak{U}$ such that for any $U \in \mathfrak{U}$ and any $x \in U$ we have

$$\omega_U|_{\mathbf{\Sigma}_x} = \omega_x.$$

($\mathbf{O_2}$) $\omega_x \neq 0$, $\forall x \in M$.

The sphere bundle is called *orientable* if it admits an orientation. □

The discussion in Example 3.8 shows that an orientation of an n-sphere bundle $\pi : \mathbf{\Sigma} \to M$ is a nowhere vanishing section of the sheaf $\widehat{H}^n_\pi$.

Proposition 3.13. *Suppose that $\pi : \mathbf{\Sigma} \to M$ is a smooth n-sphere bundle. Then the following statements are equivalent.*

(a) *The sphere bundle is orientable.*
(b) *The sheaf $\widehat{H}^n_\pi$ is constant with stalk $\mathbb{R}$.*

Proof. Clearly (b) ⇒ (a) so it suffices to prove the reverse implication. Suppose that $\underline{\omega} = (\omega_x)_{x\in M}$ is a nowhere vanishing section of $\widehat{H}^n_\pi$. Then there exist an open cover $\mathfrak{U}$ of M and cohomology classes $\omega_U \in H^n(\mathbf{\Sigma}_U)$ such that for any $U \in \mathfrak{U}$ and any $x \in U$ we have

$$\omega_x = \omega_U|_{\mathbf{\Sigma}_x}.$$

We can also assume that any $U \in \mathfrak{U}$ is connected and the restriction $\pi : \mathbf{\Sigma}_U \to U$ is trivial.

For any $V \in \boldsymbol{Open}(X)$ we have a natural morphism

$$\phi_V : \Gamma(V, \underline{\mathbb{R}}_V) \to \Gamma(V, \widehat{H}^n_\pi)$$

that associates to a locally constant function $f : V \to \mathbb{R}$ the section $f \cdot \underline{\omega}|_V \in \Gamma(V, \widehat{H}^n_\pi)$. Concretely , $f \cdot \underline{\omega}|_V$ is given by the collection

$$f(x)\omega_x \in H^n(\mathbf{\Sigma}_x),\;\; x \in V.$$

To see that $f \cdot \underline{\omega}|_V$ is indeed a section of $\widehat{H}^n_\pi$ over V we look at the open cover $\mathfrak{U}_V = \{U \cap V\}_{U\in\mathfrak{U}}$. For any U in $\mathfrak{U}$ and any connected component W of $U \cap V$ the restriction of f to W is a constant function f_W. Then for any $y \in W$ we have

$$f(y)\omega_y = (f_W\omega_U)|_{\mathbf{\Sigma}_y}$$

This proves that the collection ϕ_V defines a morphism of sheaves $\phi : \underline{\mathbb{R}}_M \to \widehat{H}^n_\pi$ such that ϕ_V is injective for any $V \in \boldsymbol{Open}(M)$. To prove that ϕ_V is surjective consider a section $\underline{\eta} = (\eta_x)_{x\in V} \in \Gamma(V, \widehat{H}^n_\pi)$. Then for any $x \in V$ there exists $f(x) \in \mathbb{R}$ such that

$$\eta_x = f(x)\omega_x.$$

Arguing as in Example 3.8 we deduce that the map $V \ni x \mapsto f(x) \in \mathbb{R}$ is locally constant. This proves that ϕ is an isomorphism. □

The above proposition coupled with Corollary 3.11 have the following consequence.

Corollary 3.14. *Any smooth sphere bundle over a simply connected manifold is orientable.* □

Example 3.15. Suppose that $\pi : E \to M$ is a real vector bundle of rank $(n+1)$ over the smooth connected m-dimensional manifold. By fixing a metric on E we obtain a sphere bundle $\boldsymbol{\Sigma}(E) \to M$ whose fiber over $x \in M$ consists of the unit vectors in E_x. We say that $\boldsymbol{\Sigma}(E)$ is the unit sphere bundle associated to the metric vector bundle E.

Let us observe that the vector bundle E is orientable if and only if the associated sphere bundle $\boldsymbol{\Sigma}(E) \to M$ is orientable. [1, Prop.11.2]. □

4. THE ČECH COHOMOLOGY OF PRESHEAVES

Let X be a topological space, R a commutative ring with 1 and $\mathcal{F}$ a presheaf of R-module on X. Suppose that $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ is an open cover of X. For any nonnegative integer q we set

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{\sigma\in\mathcal{S}_q(\mathcal{U})} \mathcal{S}(U_\sigma).$$

Thus, the elements of $C^q(\mathcal{U}, \mathcal{F})$ are collection of sections

$$\underline{f} = (f_\sigma)_{\sigma\in\mathcal{S}_q(\mathcal{U})},\ \ f_\sigma \in \mathcal{F}(U_\sigma).$$

Define

$$\delta : C^q(\mathcal{U}, \mathcal{S}\mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F}),$$

where for any $\underline{f} = (f_\sigma)_{\sigma\in\mathcal{S}_q(\mathcal{U})} \in C^q(\mathcal{U}, \mathcal{F})$, and any $\tau \in \mathcal{S}_{q+1}(\mathcal{U})$ we have

$$(\delta\underline{f})_\tau = \sum_{j=0}^{q+1} (-1)^j f_{\tau_0\cdots\hat{\tau}_j\cdots\tau_{q+1}}|_{U_\tau}.$$

Arguing as before we deduce that $\delta^2 = 0$. The resulting complex is called the *Čech complex* of the cover $\mathcal{U}$ with coefficients in the presheaf $\mathcal{F}$.

An open cover $\mathcal{V} = (V_i)_{i\in I}$ is called a *refinement* of the open cover $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ if there exists a *refinement map* $\phi : I \to \mathcal{A}$ such that

$$V_i \subset U_{\phi(i)},\ \ \forall i \in I.$$

Observe that if $A \in \boldsymbol{N}(\mathcal{V})$, then $\phi(S) \in \boldsymbol{N}(\mathcal{U})$ so that for any $\sigma \in \mathcal{S}_q(\mathcal{V})$ the composition $\phi\circ\sigma$ belongs to $\mathcal{S}_q(\mathcal{U})$ refinement map induces a map $\phi : \boldsymbol{N}(\mathcal{V}) \to \boldsymbol{N}(\mathcal{U})$ and a morphism of complexes

$$\phi^* : \big(\, C^\bullet(\mathcal{U}, \mathcal{F}), \delta\,\big) \to \big(\, C^\bullet(\mathcal{V}, \mathcal{F}), \delta\,\big),$$

defined as follows. Given $(\underline{f} \in C^q(\mathcal{U}, \mathcal{F})$ and $\sigma \in \mathcal{S}_q(\mathcal{V})$ we have

$$(\phi^*\underline{f})_\sigma = f_{\phi\circ\sigma}.$$

We thus obtain a morphism

$$\phi^* : H^\bullet(\mathcal{U}, \mathcal{F}) \to H^\bullet(\mathcal{V}, \mathcal{F}).$$

Lemma 4.1. *Suppose that $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ and $\mathcal{V} = (V_i)_{i\in I}$ are two open covers of X and $\phi, \psi : I \to \mathcal{A}$ are two refinement maps. Then the induced morphisms*

$$\phi^*, \psi^* : \big(\, C^\bullet(\mathcal{U}, \mathcal{F}), \delta\,\big) \to \big(\, C^\bullet(\mathcal{V}, \mathcal{F}), \delta\,\big),$$

are chain homotopic, i.e., there exists $K : C^\bullet(\mathcal{U}, \mathcal{F}) \to C^{\bullet-1}(\mathcal{V}, \mathcal{F})$ such that

$$\psi^* - \phi^* = \delta K + K\delta. \tag{4.1}$$

In particular,

$$\phi^*, \psi^* : H^\bullet(\mathcal{U}, \mathcal{F}) \to H^\bullet(\mathcal{V}, \mathcal{F})$$

are identical

Proof. For $\underline{f} \in C^q(\mathcal{U}, \mathcal{F})$ define

$$(K\underline{f})_{i_0 \cdots i_{q-1}} = \sum_{r=0}^{q-1} (-1)^r f_{\phi(i_0)\cdots\phi(i_r)\psi(i_r)\cdots\psi(i_{q-1})}|_{V_{i_0\cdots i_{q-1}}}.$$

An elementary computation left to the reader shows that K defined as above satisfies (4.1) . □

We denote by $\boldsymbol{Cov}(X)$ the collection of all open covers of X. This is equipped with a partial order $\prec$ where we declare $\mathcal{U} \prec \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$. This is a *directed set*, i.e., for any $\mathcal{U}_0, \mathcal{U}_1 \in \boldsymbol{Cov}(X)$ there exists $\mathcal{U}_2 \in \boldsymbol{Cov}(X)$ such that $\mathcal{U}_0, \mathcal{U}_1 \prec \mathcal{U}_2$. (It suffices to take $\mathcal{U}_2 = (U_0 \cap U_1)_{U_0 \in \mathcal{U}_0, U_1 \in \mathcal{U}_1}$.)

Lemma 4.1 shows that for any $\mathcal{U}, \mathcal{V} \in \boldsymbol{Cov}(X)$ such that $\mathcal{U} \prec \mathcal{V}$ there exists a morphism

$$\Phi_{\mathcal{V}\mathcal{U}} : H^\bullet(\mathcal{U}, \mathcal{F}) \to H^\bullet(\mathcal{V}, \mathcal{F})$$

satisfying

$$\Phi_{\mathcal{W}\mathcal{U}} = \Phi_{\mathcal{W}\mathcal{V}} \circ \Phi_{\mathcal{V}\mathcal{U}}, \ \ \forall \mathcal{U} \prec \mathcal{V} \prec \mathcal{W}.$$

We can thus define the direct (inductive) limit

$$\check{H}^\bullet(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^\bullet(\mathcal{U}, \mathcal{F}).$$

We will refer to $\check{H}^\bullet(X, \mathcal{F})$ as the *Čech cohomology* of X with coefficients in the presheaf $\mathcal{F}$.

Example 4.2. Suppose that $\mathcal{F}$ is a *sheaf*. Then an element $\underline{f} \in \check{H}^0(X, \mathcal{F})$ is represented by a collection

$$\underline{f}_{\mathcal{U}} = (f_U)_{U \in \mathcal{U}}, \ \ \mathcal{U} \in \boldsymbol{Cov}(X), \ \ f_U \in \Gamma(U, \mathcal{F}),$$

satisfying the cocycle condition

$$f_U|_{U \cap V} = f_V|_{U \cap V}, \ \ \forall U, V \in \mathcal{U}.$$

The sheaf axioms show that this collection defines a unique section $[f] \in \Gamma(X, \mathcal{F})$ that is independent of the representative $\underline{f}_{\mathcal{U}}$ of $\underline{f}$. This shows that when $\mathcal{F}$ is a sheaf we have a natural isomorphism

$$\check{H}^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F}).$$

□

For any presheaf $\mathcal{F} \in \boldsymbol{PSh}_R(X)$ and any $U \in \boldsymbol{Open}(X)$ we have natural morphisms

$$\mathcal{F}(U) \to \widehat{\mathcal{F}}(U)$$

compatible with the restriction morphisms. We thus obtain a natural morphism

$$\check{H}^\bullet(X, \mathcal{F}) \to \check{H}^\bullet(X, \widehat{\mathcal{F}})$$

We have the following useful fact, [6, Thm. 2.9.1]

Theorem 4.3. *If X is a paracompact Hausdorff space then the natural morphism*

$$\check{H}^\bullet(X, \mathcal{F}) \to \check{H}^\bullet(X, \widehat{\mathcal{F}})$$

is an isomorphism. In particular $H^0(X, \mathcal{F}) \cong \Gamma(X, \widehat{\mathcal{F}})$. □

Observe that if $\mathcal{S}_0, \mathcal{S}_1 \in \boldsymbol{PSh}(X)$ the we can define their dirrect sum $\mathcal{S}_0 \oplus \mathcal{S}_1$ to be the presheaf

$$\boldsymbol{Open}(X) \ni U \mapsto \Gamma(U, \mathcal{S}_0 \oplus \mathcal{S}_1) := \mathcal{S}_0(U) \oplus \mathcal{S}_1(U) \in \boldsymbol{Mod}(R).$$

Theorem 4.4. *Suppose that V is a finite dimensional real vector space and M is a smooth connected manifold of dimension m. Then for any good cover $\mathcal{U}$ of M we have*

$$H^\bullet(\mathcal{U}, \underline{V}) \cong \check{H}^k(M, \underline{V}) \cong H^k_{DR}(M) \otimes V, \quad \forall k \geq 0. \tag{4.2}$$

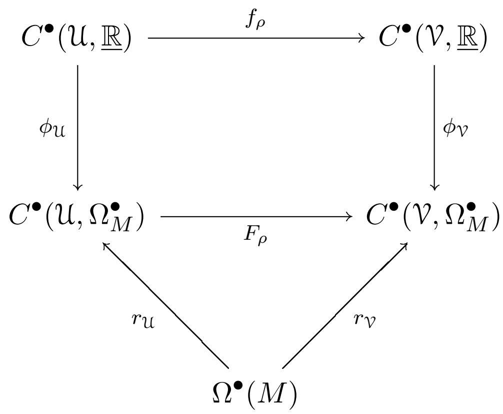
Proof. Suppose first that $\dim V$ is one-dimensional so that $\underline{V} = \underline{\mathbb{R}}$. Then for any open cover $\mathcal{U} \in \boldsymbol{Cov}(M)$ the complex $C^\bullet(\mathcal{U}, \underline{\mathbb{R}})$ coincides with the Čech complex $C^\bullet(\mathcal{U}, \mathbb{R})$ defined in Section 2. The natural morphism

$$\phi_{\mathcal{U}} : C^\bullet(\mathcal{U}, \underline{\mathbb{R}}) \to C^\bullet(\mathcal{U}, \Omega^\bullet_M)$$

is a quasi-isomorphism, i.e., it induces an isomorphism in cohomology. Suppose that $\mathcal{V}$ is another good cover of M such that $\mathcal{U} \prec \mathcal{V}$. Fix a refinement map $\rho : \mathcal{V} \to \mathcal{U}$. We then get morphisms of complexes

$$f_\rho : C^\bullet(\mathcal{U}, \underline{\mathbb{R}}) \to C^\bullet(\mathcal{V}, \underline{\mathbb{R}}) \quad F_\rho : C^\bullet(\mathcal{U}, \Omega^\bullet_M) \to C^\bullet(\mathcal{V}, \Omega^\bullet_M),$$

that fit in commutative diagrams



where $r_{\mathcal{U}}$ and $r_{\mathcal{V}}$ are the morphisms that appear in the Generalized Mayer-Vietoris principle, Theorem 2.4.

The morphsims $r_{\mathcal{U}}$ and $r_{\mathcal{V}}$ are quasi-isomorphisms so that F_ρ is a quasi-isomorphisms. Since $\phi_{\mathcal{U}}$ and $\phi_{\mathcal{V}}$ are quasi-isomorphisms we deduce that f_ρ is a quasi-isomorphism. Thus, the canonical morphism

$$\Phi_{\mathcal{V}\mathcal{U}} : H^\bullet(\mathcal{U}, \underline{\mathbb{R}}) \to H^\bullet(\mathcal{V}, \underline{\mathbb{R}})$$

is an isomorphism for any two good covers $\mathcal{U} \prec \mathcal{V}$. Since for any open cover $\mathcal{W}$ we can find a good cover $\mathcal{U} \succ \mathcal{W}$ we deduce that for any good cover $\mathcal{U}$ the natural morphism

$$H^\bullet(\mathcal{U}, \underline{\mathbb{R}}) \to \check{H}^\bullet(M, \underline{\mathbb{R}})$$

is an isomorphism. Corollary 2.9 implies that $H^\bullet(\mathcal{U}, \underline{\mathbb{R}}) \cong H^\bullet_{DR}(M)$. This proves (4.2) in the case when $\dim V = 1$.

More generally

$$\check{H}^\bullet(M, \underline{\mathbb{R}}^n) \cong \underbrace{\check{H}^\bullet(M, \underline{\mathbb{R}}) \oplus \cdots \oplus \check{H}^\bullet(M, \underline{\mathbb{R}})}_{n}$$
$$\cong \check{H}^\bullet(M, \underline{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{R}^n \cong H^\bullet_{DR}(M) \otimes_{\mathbb{R}} \mathbb{R}^n.$$

□

For a more in-depth presentation of Čech cohomology and its relationship with the cohomology of sheaves we refer to [2, Chap. 4] or [5].

5. Spectral sequences

Consider a gain a double complex of R-modules

$$\Bigl(\bigoplus_{p,q\geq 0} E^{p,q}, d_h, d_v\Bigr)$$

with associated total complex $(T^\bullet(E), D)$, where

$$T^n(E) = \bigoplus_{p+q=n} E^{p,q}, \quad D = d_h + d_v.$$

We would like to describe a way of computing the cohomology of $(T^\bullet(E), D)$. We need to introduce some terminology.

Definition 5.1. (a) A *filtration* of an R-module M is a collection of submodules

$$\cdots F^p M \supset F^{p+1} M \supset \cdots$$

such that

$$M = \bigcup_{p\in\mathbb{Z}} F^p M.$$

The filtration is called *bounded* if there exist integers $t < b$ such that

$$F^t M = M, \quad F^b M = 0.$$

(b) A *filtered* module is a module equipped with a filtration. The *graded* module associated to a filtered module $(M, F^\bullet)$ is

$$\mathbf{Gr}^\bullet(M) := \bigoplus_{p\in\mathbb{Z}} F^p M / F^{p+1} M.$$

(c) A *morphism of filtered modules* $(M, F^\bullet)$, $(N, F^\bullet)$ is a morphism of modules $\phi : M \to N$ such that

$$\phi\bigl(F^p M\bigr) \subset F^p N, \quad \forall p \in \mathbb{Z}.$$

(d) A *filtration* of a graded module

$$K^\bullet = \bigoplus_{n\in\mathbb{Z}} K^n$$

is a collection of filtration $F^\bullet K^n$ on each of the factors K^n. The filtration is called *regular* if for any $n \in \mathbb{Z}$ the filtration of K^n is bounded, i.e., there exist integers $t(n) < b(n)$ such that

$$F^{t(n)} K^n = K^n, \quad F^{b(n)} K^n = 0.$$

We set

$$\mathbf{Gr}^{p,n}(K^\bullet, F) := F^p K^n / F^{p+1} K^n.$$

(e) A *filtration* of a cochain complex $(K^\bullet, d)$ is a filtration $F^p K^\bullet$ of the graded module $K^\bullet$ such that

$$dF^p K^\bullet \subset F^p K^{n+1}, \quad \forall p, n \in \mathbb{Z}$$

A filtration on a cochain complex $(K^\bullet, d)$ induces a filtration on the cohomology $H^\bullet(K^\bullet, d)$. More precisely $F^p H^n(K)$ is defined to be the image of $H^n(F^p K^\bullet)$ in $H^n(K^\bullet)$. □

Example 5.2. The total complex $(T^\bullet(E), D)$ is equipped with two regular filtrations. The first filtration $F_I^\bullet$ is defined by defined by

$$F_I^p T^n = \sum_{i\geq p} E^{i,n-i}. \tag{5.1}$$

The filtration is regular since

$$F_I^{-1}T^n = T^n, \;\; F_I^{n+1}T^n = 0,$$

so that we can take $t(n) = -1$, $b(n) = n+1$. The second filtration $F_{II}^{\bullet}$is defined by

$$F_{II}^p T^n = \bigoplus_{j\geq p} E^{n-j,j}.$$

□

Theorem 5.3. *Consider a cochain complex $(K^\bullet, d)$ equipped with a* regular filtration $F^\bullet K^\bullet$. *Then there exists a* spectral sequence *converging to the cohomology of $(K^\bullet, d)$, i.e., there exists a sequence of bigraded complexes $(E_r^{\bullet,\bullet}, d_r)_{r\geq 0}$ satisfying the following properties.*

$(\mathbf{SS_1})$ *The coboundary operator d_r has bidegree $(1-r, r)$, i.e.,*

$$d_r(E_r^{p,q}) \subset E_r^{p-(r-1),q+r}, \;\; \forall p,q,r.$$

$(\mathbf{SS_2})$ *For any $r \geq 0$ there exists an isomorphism of bigraded modules*

$$\alpha_r : E_{r+1}^{\bullet,\bullet} \to H(E_r^{\bullet,\bullet}, d_r)$$

$(\mathbf{SS_3})$ *For any $p,q \in \mathbb{Z}$ there exists $r = r(p,q) \geq 0$ such that, for any $r, s \geq r(p,q)$ we have*

$$E_r^{p,q} = E_s^{p,q} \cong F^p H^{p+q}(K,d)/F^{p+1}H^{p+q}(K,d) = \mathbf{Gr}^p\, H^{p+q}(K,d).$$

We denote by $E_\infty^{p,q}$ the group $E_r^{p,q}$, $r \geq r(p,q)$.

Remark 5.4. We can visualize $(E_r^{p,q}, d_r)$ as a grid of modules and morphisms of modules; see Figure 4. The spectral sequence is an atlas of such grids, where the grid $E_0^{\bullet,\bullet}$ is the opening page of the atlas, $E_1^{\bullet,\bullet}$ is on the next page of the atlas etc.

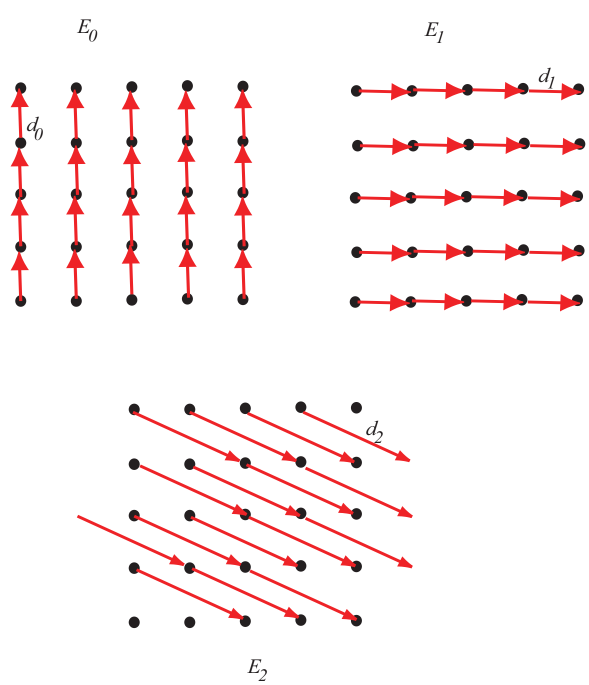


FIGURE 4. *The sheets E_0, E_1, E_2 of a spectral sequence*

□

Proof. To construct the spectral sequence we follow the approach in [4, III.7.5]. For $r \geq 0$ we set

$$\begin{aligned} Z_r^{p,q} &:= \text{elements in } x \in F^pK^{p+q} \text{ such that } dx \in F^{p+r}K^{p+q+1} \\ &= F^pK^{p+q} \cap d^{-1}\big(F^{p+r}K^{p+q+1}\big), \end{aligned} \tag{5.2}$$

$$B_r^{p,q} := F^{p+1}K^{p+q} + d(F^{p-(r-1)}K^{p+q-1}), \tag{5.3}$$

$$E_r^{p,q} := \frac{Z_r^{p,q}}{Z_r^{p,q} \cap B_r^{p,q}} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-(r-1),q+r-2}}. \tag{5.4}$$

Note that

$$Z_{r+1}^{p,q} \subset Z_r^{p,q}, \quad \forall r \geq 0, \quad \forall p,q.$$

Since $d^2 = 0$ we deduce

$$d(Z_r^{p,q}) \subset Z_r^{p+r,q-(r-1)}$$

On the other hand,

$$\begin{aligned} d(B_r^{p,q} \cap Z_r^{p,q}) &\subset (dF^{p+1}K^{p+q}) \cap Z_r^{p+r,q-(r-1)} \\ &= dF^{(p+r)-(r-1)}K^{(p+r)+(q-(r-1))-1} \cap Z_r^{p+r,q-(r-1)} \subset B_r^{p+r,q-(r-1)} \cap Z_r^{p+r,q-(r-1)} \end{aligned}$$

so that d induces an operator

$$d_r : E_r^{p,q} \to E_r^{p+r,q-(r-1)}$$

satisfying $d_r^2 = 0$. Thus $(E_r^{\bullet,\bullet}, d_r)$ is a bi-graded complex satisfying the condition $\mathbf{SS_1}$.

To prove $\mathbf{SS_2}$, i.e., the existence of a natural isomorphism

$$\alpha_r : E_{r+1}^{\bullet,\bullet} \to H(E_r^{\bullet,\bullet}, d_r)$$

We begin by constructing isomorphisms

$$\frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-(r-1),q+r-2}} \to Z(E_r^{p,q}) \tag{5.5a}$$

$$\frac{dZ_r^{p-r,q+r-1} + Z_{r-1}^{p+1,q-1}}{Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-(r-1),q+r-2}} \to B(E_r^{p,q}). \tag{5.5b}$$

To construct the morphism (5.5a) we observe that the denominator of the left-hand side of (5.5a) coincides with the denominator in the definition of $E_r^{p,q}$ while numerator of the left-hand side of (5.5a) satisfies

$$Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1} \subset Z_r^{p,q}$$

so that the left-hand side of (5.5a) is a subgroup of $E_r^{p,q}$. To prove that it is actually contained in $Z(E_r^{p,q})$ it suffices to show that

$$d\big(Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}\big) \subset \underbrace{Z_{r-1}^{p+r+1,q-r} + dZ_{r-1}^{p+1,q-1}}_{\text{the denominator of } E_r^{p+r,q-r+1}}$$

which follows from the definitions. Thus the morphism (5.5a) is well defined and injective. Similarly one shows that the morphism (5.5b) is well defined and injective. The surjectivity of (5.5b) is immediate. The surjectivity of (5.5a) is only slightly more complicated, but not particularly revealing so we refer for details to [4, p. 204].

We deduce that we have an isomorphism

$$\frac{Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}}{dZ_r^{p-r,q+r-1} + Z_{r-1}^{p+1,q-1}} \to H(E_r^{p,q}).$$

To conclude, it suffices to show that the right-hand side above is isomorphic to $E_{r+1}^{p,q}$. Indeed, we have

$$\frac{Z_{r+1}^{p,q}+Z_{r-1}^{p+1,q-1}}{dZ_r^{p-r,q+r-1}+Z_{r-1}^{p+1,q-1}}\cong\frac{Z_{r+1}^{p,q}}{Z_{r+1}^{p,q}\cap\big(\,dZ_r^{p-r,q+r-1}+Z_{r-1}^{p+1,q-1}\,\big)}$$

Next we observe that

$$dZ_r^{p-r,q+r-1}\subset Z_{r+1}^{p,q},\;\; Z_{r+1}^{p,q}\cap Z_{r-1}^{p+1,q-1}=Z_r^{p+1,q-1}.$$

Hence

$$Z_{r+1}^{p,q}\cap\big(\,dZ_r^{p-r,q+r-1}+Z_{r-1}^{p+1,q-1}\,\big)=dZ_r^{p-r,q+r-1}+Z_r^{p+1,q-1}.$$

This proves $\mathbf{SS_2}$.

Finally, let us prove $\mathbf{SS_3}$. Fix p,q, set $n:=p+q$. Note that

$$\mathbf{Gr}^p\,H^n(K)=\frac{F^pZ^n(K)}{F^pZ^n(K)\cap\big(\,F^{p+1}K^n+B^n(K)\,\big)},$$

where

$$F^pZ^n=F^pK^n\cap\ker\big(\,K^n\stackrel{d}{\to}K^{n+1}\,\big),\;\; F^pB^n(K)=F^pK^n\cap\operatorname{Im}\big(\,K^{n-1}\stackrel{d}{\to}K^n\,\big).$$

If $r>b(n)-p$, then $F^{p+r}K^n=0$, and we deduce that

$$Z_r^{p,q}=F^pZ^n(K).$$

If $r-1>p-t(n-1)$, then

$$F^{p-(r-1)}K^{p+q-1}=K^{p+q-1}.$$

This shows that if $r>\max\{b(n)-p,p-t(n-1)+1\}$, then

$$Z_r^{p,q}\cap\big(\,F^{p+1}K^{p+q}+d(F^{p-(r-1)}K^{p+q-1})\,\big)=F^pZ^n(K)\cap\big(\,F^{p+1}K^n+B^n(K)\,\big).$$

□

Remark 5.5. (a) The E_0-sheet of the spectral sequence has a simple description. More precisely (5.2), (5.3) and (5.4) imply that

$$E_0^{p,q}=\frac{F^pK^{p+q}}{F^{p+1}K^{p+q}}=\mathbf{Gr}^p\,K^{p+q}.$$

(b) When $(T^\bullet,D)$ is the total complex of a double complex $(C^{\bullet,\bullet},d_h,d_v)$ and F is the first filtration F_I defined in (5.1) then the construction of the spectral sequence can be given a somewhat more intuitive description.

We introduce a formal variable t and represent the elements of T^n as "polynomials"

$$x=\sum_{j=0}^n x_jt^j,\;\; x_j\in C^{i,n-j}.$$

We set $D_t:=d_v+td_h$. The condition $x\in F^pT^n$ can now be rewritten using the analyts' big-O notation as $x=O(t^p)$.

To compute $\mathbf{Gr}^p\,H^n(T)$ we need to compute

$$F^pZ^n(T,D)\mod F^{p+1}Z^n(T)+F^pB^n(T).$$

Note that $x\in F^pZ^n(T)$ if and only if

$$x\in T^n,\;\; x=O(t^p),\;\; D_tx=O(t^{n+1}).$$

We construct a class $x \in \mathbf{Gr}^p H^n(T)$ via successive approximations. We find these approximations by solving the systems of linear equations

$$D_t y = O(t^{p+r}), \;\; y \in T^n, \;\; y = O(t^p), \;\; r = 0, 1, 2, \dots \tag{S_r}$$

modulo $F^{p+1}Z^n(T) + F^pB^n(T)$. We do this successively, seeking solutions of S_{r+1} within the smaller space of solutions of S_r. Note that $Z_r^{p,n-p}$ can be identified with the space of solutions of S_r. We declare a y of S_r to be trivial if either

$$y = O(t^{p+1}), \tag{5.6}$$

or

$$\exists z \in T^{n-1} : \;\; z = O\big(t^{p-(r-1)}\big), \;\; y = D_t z. \tag{5.7}$$

Note that the space of trivial solutions of S_r is contained in $F^{p+1}Z^n(T) + F^pB^n(T)$ and coincides with it if $r > p + 1$. We have

$$E_r^{p,n-p} = \frac{\text{solutions of } S_r}{\text{trivial solutions of } S_r}.$$

The complex $(E_0^{\bullet,\bullet}, d_0)$ has a simple description. It is the complex $(C^{\bullet,\bullet}, d_v)$. We deduce that

$$E_1^{p,q} = H^q(C^{p,\bullet}, d_v) =: H_v^q(C^{p,\bullet}),$$

i.e., $E_1^{\bullet,\bullet}$ is determined by the cohomology of the complexes determined by the columns of the double complex.

The differential $d_1 : H_v^q(C^{p,\bullet}) \to H_v^q(C^{p+1,\bullet})$ is the morphism induced by the horizontal differential d_h. Therefore, the E_2 term consists of the cohomology of the complexes determined by the rows of E_1,

$$E_2^{p,q} = H^p\big(\, H_v^q(C^{\bullet,\bullet}), d_h\,\big) =: H_h^p H_v^q(C^{\bullet,\bullet}). \qquad \square$$

Suppose that $(C^\bullet, d_C, F_C^\bullet)$ and $(K^\bullet, d_K, F_K^\bullet)$ are two filtered cochain complexes with regular filtrations, and $\phi : (C^\bullet, d_C, F_C^\bullet) \to (K^\bullet, d_K, F_K^\bullet)$. Then ϕ induces morphisms of bigraded complexes

$$\phi_r : (E_r^{\bullet,\bullet}(C), d_r) \to (E_r^{\bullet,\bullet}(K), d_r), \;\; r = 0, 1, 2, \dots$$

$$\mathbf{Gr}(\phi) : \mathbf{Gr}^\bullet H^\bullet(C) \to \mathbf{Gr}^\bullet H^\bullet(K).$$

Theorem 5.6 (Approximation theorem. Version 2)**.** *Suppose that $\phi : (C^\bullet, d_C, F_C^\bullet) \to (K^\bullet, d_K, F_K^\bullet)$ is a morphism of complexes equipped with **regular** filtrations. Suppose that for some $r_0 \geq 0$ the induced morphism*

$$\phi_{r_0} : (E_{r_0}(C), d_{r_0}) \to (E_{r_0}(K), d_{r_0})$$

is an isomorphism of bigraded complexes. Then the following hold.

(a) *For any $r \geq r_0$ the morphism $\phi_r : (E_r^{\bullet,\bullet}(C), d_r) \to (E_r^{\bullet,\bullet}(K), d_r)$ is an isomorphism of complexes.*

(b) *The induced morphism*

$$\phi_* : H^\bullet(C) \to H^\bullet(K)$$

is an isomorphism.

Proof. Part (a) is obvious. To prove (b) we will show that for any n the induced morphism

$$\phi : H^n(C) \to H^n(K)$$

is an isomorphism. Since the filtrations on C^n and K^n are both finite there exist integers $t < b$ such that

$$F^tC^n = C^n, \;\; F^tK^n = K^n, \;\; F^bC^n - 0 = F^bK^n.$$

Thus

$$\mathbf{Gr}^{\bullet} H^n(C) = \bigoplus_{p=t}^{b} \mathbf{Gr}^p H^n(C), \mathbf{Gr}^{\bullet} H^n(K) = \bigoplus_{p=t}^{b} \mathbf{Gr}^p H^n(K).$$

There exists $r > 0$ such that for any integer p, $t \le p \le b$ we have

$$E_r^{p,n-p}(C) \cong \mathbf{Gr}^p H^n(C), \quad E_r^{p,n-p}(K) \cong \mathbf{Gr}^p H^n(K).$$

Thus the morphism ϕ induces isomorphisms

$$\mathbf{Gr}^p(\phi) : \mathbf{Gr}^p H^n(C) \to \mathbf{Gr}^p H^n(K), \quad \forall p = t, \dots, b.$$

Now observe that we have a collection of commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{p+1}H^n(C) & \longrightarrow & F^pH^n(C) & \longrightarrow & \mathbf{Gr}^p H^n(C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F^{p+1}(\phi)} & & \downarrow{\scriptstyle F^p(\phi)} & & \downarrow{\scriptstyle \mathbf{Gr}^p(\phi)} & & \\ 0 & \longrightarrow & F^{p+1}H^n(C) & \longrightarrow & F^pH^n(C) & \longrightarrow & \mathbf{Gr}^p H^n(C) & \longrightarrow & 0 \end{array}$$

From the above discussion we deduce that $\mathbf{Gr}^p(\phi)$ is an isomorphism. The five-lemma shows that

$$F^{p+1}(\phi) \text{ isomorphism} \Rightarrow F^p(\phi) \text{ isomorphism}.$$

When $p = b$ the morphism $F^{p+1}(\phi)$ isomorphism is trivially an isomorphism since

$$F^{b+1}H^n(C) = F^{b+1}H^n(K) = 0.$$

We deduce successively that $F^t(\phi)$ is an isomorphism.

□

The first version of the Approximation theorem is a special case of Theorem 5.6

Corollary 5.7. *Suppose that $(C^{\bullet}, d)$ is a cochain complex and $\phi : (C^{\bullet}, d) \to (K^{\bullet,\bullet}, d_h, d_v)$ is a resolution of the complex $C^{\bullet}$ by the double complex $K^{\bullet,\bullet}$. Then ϕ induces isomorphisms*

$$H^{\bullet}(C, d) \to H^{\bullet}(T(K), D).$$

Proof. We regard $C^{\bullet}$ as a double complex in a trivial way and we equip both $C^{\bullet}$ and $K^{\bullet,\bullet}$ with the first filtrations F_I. The morphism ϕ is a morphism of filtered complexes. The fact that ϕ is resolution is equivalent with the fact that the induced morphism

$$\phi_1 : E_1^{p,q}(C) \to E_1^{p,q}(K)$$

is an isomorphism. Now invoke Theorem 5.6. □

Suppose that $(K^{\bullet}, d)$ is a cochain complex of algebras such that d is an odd derivation, i.e.,

$$\forall x_r \in K^r, \quad x_s \in K^s : \quad d(x_r \cdot x_s) = (dx_r) \cdot x_s + (-1)^r x_r \cdot dx_s.$$

A filtration $F^{\bullet}$ on $K^{\bullet}$ is said to be compatible with the algebra structure if

$$F^pK^{\bullet}) \cdot (F^qK^{\bullet}) \subset F^{p+q}K^{\bullet}, \quad \forall p, q \in \mathbb{Z}.$$

An inspection of the proof of Theorem 5.3 yields the following result.

Corollary 5.8. *Suppose that $(K^\bullet, d, F^\bullet)$ is a complex of R-algebras equipped with regular filtration compatible with the algebra structure and such that d is an odd derivation. Then for any $r \geq 0$ the bi-graded complex $(E_r^{\bullet,\bullet}, d_r)$ has an induced algebra structure satisfying the following condition: $\forall m, n, p, q$, $\forall x \in E_r^{m,n}$, $\forall y \in E_r^{p,q}$ we have*

$$x \cdot y \in E_r^{m+p,n+q}, \;\; d_r(xy) = (d_r x) \cdot y + (-1)^{m+n} x \cdot (d_r y). \tag{5.8}$$

□

Example 5.9 (A product on the Čech-DeRham double complex)**.** Suppose that M is a smooth manifold of dimension m. Consider the Čech-DeRham double complex $(C^{\bullet,\bullet}(\mathcal{U}, \Omega_M, d_h, d_v)$ associated to an open cover $\mathcal{U} \in \boldsymbol{Cov}(M)$,

$$C^{p,q}(\mathcal{U}, \Omega_M) := C^q(\mathcal{U}, \Omega_M^p), \;\; d_h = d, \;\; d_v = (-1)^p \delta : C^q(\mathcal{U}, \Omega_M^p) \to C^{q+1}(\mathcal{U}, \Omega_M^p)$$

We define a product

$$C^{p,q}(\mathcal{U}, \Omega_M) \times C^{p',q'}(\mathcal{U}, \Omega_M) \ni (\underline{\omega}, \underline{\omega}') \mapsto \underline{\omega} * \underline{\omega}' \in C^{p+p',q+q'}(\mathcal{U}, \Omega_M)$$

where for any $\sigma \in \mathcal{S}_{q+q'}(\mathcal{U})$ we have

$$\big(\underline{\omega} * \underline{\omega}'\big)_{\sigma_0 \dots \sigma_{q+q'}} = \omega_{\sigma_0 \dots \sigma_q}|_{U_\sigma} \wedge \omega'_{\sigma_q \dots \sigma_{q+q'}}|_{U_\sigma}.$$

A simple computation shows that

$$\delta(\underline{\omega} * \underline{\omega}') = \delta\underline{\omega} * \underline{\omega}' + (-1)^q \underline{\omega} * \delta\underline{\omega}'$$

and

$$d(\underline{\omega} * \underline{\omega}') = d\underline{\omega} * \underline{\omega}' + (-1)^p \underline{\omega} * d\underline{\omega}'$$

Now modify the product $*$ to a product

$$\cup : C^{p,q}(\mathcal{U}, \Omega_M) \times C^{p',q'}(\mathcal{U}, \Omega_M) \to C^{p+p',q+q'}(\mathcal{U}, \Omega_M),$$

$$\underline{\omega} \cup \underline{\omega}' = (-1)^{\nu(p,q,p'q')} \underline{\omega} * \underline{\omega}' \in C^{p+p',q+q'}(\mathcal{U}, \Omega_M)$$

where the sign $(-1)^{\nu(p,q,p'q')}$ is such that the operator $D = d_h + d_v$ is an odd derivation with respect to $\cup$. This means that

$$(-1)^{p+p'+\nu(p,p',q,q')} = (-1)^{p+\nu(p,p',q+1,q')} = (-1)^{p+p'+\nu(p,p',q,q'+1)}$$

$$(-1)^{\nu(p,p',q,q')} = (-1)^{\nu(p+1,p',q,q')} = (-1)^{q+\nu(p,p'+1,q,q')}.$$

The first string of equalities shows that $\nu(p, p', q, q') \bmod 2$ must be independent of $q' \bmod 2$. The second string of equalities that $\nu(p, p', q, q') \bmod 2$ must be independent of $p \bmod 2$. We see that the choice

$$\nu(p, p', q, q') = qp'$$

will do the trick.

The inclusions

$$(C^\bullet(\mathcal{U}, \mathbb{R}), \delta) \to C^\bullet(\mathcal{U}, \Omega_M^\bullet) \hookleftarrow (\Omega^\bullet(M), D)$$

are morphisms of complexes of algebras. We deduce that if $\mathcal{U}$ is a good cover, then we have an ismorphisms of algebras

$$H^\bullet(\mathcal{U}, \mathbb{R}) \cong H^\bullet(M).$$

□

6. The Leray-Serre spectral sequence

Suppose $\pi : E \to M$ is a smooth fiber bundle over the smooth connected m-dimensional manifold M with fiber F. We will make the following simplifying assumption.

The fiber has finite dimensional DeRham cohomology and the sheaves $\widehat{H}^q_\pi$ are constant, $\forall q \geq 0$, where $\widehat{H}^q_\pi$ is the sheaf associated to the $H^q_\pi \in \boldsymbol{PSh}_{\mathbb{R}}(M)$ is the presheaf

$$\boldsymbol{Open}(M) \ni U \mapsto H^q(U).$$

Fix a good open cover $\mathcal{U} \in \boldsymbol{Cov}(M)$ and denote by $\widetilde{\mathcal{U}} \in \boldsymbol{Cov}(E)$ the open cover $(\pi^{-1}(U))_{U\in\mathcal{U}}$. Consider the transpose of the double complex Čech-DeRham double complex associated to $\widetilde{\mathcal{U}}$,

$$\widehat{C}^{p,q}(\widetilde{\mathcal{U}}, \Omega_E) = C^p(\widetilde{\mathcal{U}}, \Omega^q_E),$$

with differentials

$$\widehat{d}_v = d, \;\; \widehat{d}_h = (-1)^q\delta : C^p(\widetilde{\mathcal{U}}, \Omega^q_M) \to C^{p+1}(\widetilde{\mathcal{U}}, \Omega^q_E)$$

The E_1-term of the spectral sequence determined by the first filtration of this double complex is

$$E_1^{p,q} = \prod_{\sigma\in\mathcal{S}_p(\widetilde{\mathcal{U}})} H^q(E_{U_\sigma}).$$

Now observe that the correspondence $\mathcal{U} \ni U \mapsto \pi^{-1}(U) \in \widetilde{U}$ gives an isomorphisms of nerves $\boldsymbol{N}(\mathcal{U}) \to \boldsymbol{N}(\widetilde{\mathcal{U}})$. We deduce

$$\prod_{\sigma\in\mathcal{S}_p(\widetilde{\mathcal{U}})} H^q(E_{U_\sigma}) = \prod_{\sigma\in\mathcal{S}_p(\mathcal{U})} H^q(E_{U_\sigma}) = C^p(\mathcal{U}, \widehat{H}^q_\pi).$$

From Theorem 4.4 we deduce

$$E_2^{p,q} = H^p(\mathcal{U}, \widehat{H}^q_\pi) \cong H^p(M, \widehat{H}^q_\pi) \cong H^p(M) \otimes H^q(F).$$

Note that $E_2^{\bullet,\bullet}$ has a product has an structure of $\mathbb{R}$-algebra satisfying (5.8).

Example 6.1 (The Gysin sequence)**.** Suppose that $\pi : E \to M$ is an orientable sphere bundle, with fiber S^n, $n \geq 0$.

Proposition 3.13 implies that the sheaves $\widehat{H}^q_\pi$ are constant. The E_2-term of the Leray-Serre spectral sequence is

$$E_2^{p,q} = H^p(M) \otimes H^q(S^n) \cong \begin{cases} H^p(M), & q = 0, n \\ 0, & q \neq 0, n. \end{cases}.$$

This implies that

$$(E_2, d_2) = \cdots = (E_{n+1}, d_{n+1})$$

The E_{n+1}-term has the form in Figure 5.

Observe that for any $r \geq n+2$ we have

$$E_r = H(E_{n+1}, d_{n+1})$$

We have

$$E_\infty^{p,n} \cong \ker(d_{n+1} : E^{p,n} \to E^{p+n+1,0}) \tag{6.1a}$$

$$E_\infty^{p,0} \cong \frac{E_{n+1}^{p,0}}{d_{n+1}E^{p-n-1,n}}, \tag{6.1b}$$

and

$$E_\infty^{p,q} = 0, \;\; q \neq 0, n.$$

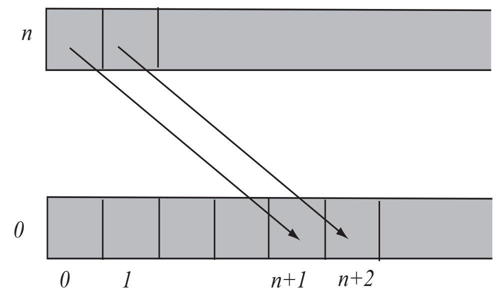


FIGURE 5. *The sheets E_{n+1}-term of the Leray-Serre spectral sequence of an S^n-bundle.*

Denote by $x \in H^n(S^n)$ the cohomology class uniquely determined by the requirement

$$\int_{S^n} x = 1.$$

We regard it as an element of $E_\infty^{0,n}$. Using the product structure on E_{n+1} we deduce that the multiplication by x defines an isomorphism

$$-\cup x : E_\infty^{0,n} \to E^{p,n}.$$

so that very element in $E_\infty^{p,n}$ is a linear combination of elements of the form $\omega \cup x$, $\omega \in H^p(M)$. Note that d_{n+1} acts trivially on $E_\infty^{p,0}$ and since d_{n+1} is an odd derivation we deduce

$$d(\omega \cup x) = \pm\omega \cup d_{n+1}x, \ \ \forall \omega \in H^p(M),$$

If we set

$$\boldsymbol{e} := d_{n+1}x \in H^{n+1}(M)$$

we deduce that the morphism

$$d_{n+1} : H^p(M) \cong E_\infty^{p,n} \to E^{p+n+1,0} = H^{p+n+1}(M)$$

has the form

$$\omega \mapsto \pm\omega \cup \boldsymbol{e}.$$

We deduce that

$$\mathbf{Gr}^\bullet\, H^k(E) = E_\infty^{k-n,n} \oplus E_\infty^{k,0}.$$

Thus we have a short exact sequence

$$0 \to E_\infty^{k,0} \to H^k(E) \to E_\infty^{k-n,n} \to 0$$

which in view of (6.1a) and (6.1b) translate into the long exact sequence

$$\cdots \to E_\infty^{k-n-1,n} \stackrel{d_{n+1}}{\longrightarrow} E_\infty^{k,0} \to H^k(E) \to E_\infty^{k-n,n} \stackrel{d_{n+1}}{\longrightarrow} E_\infty^{k+1,0} \to \cdots$$

or equivalently

$$\cdots \to H^{k-n-1}(M) \stackrel{\cup \boldsymbol{e}}{\longrightarrow} H^k(M) \stackrel{\pi^*}{\to} H^k(E) \to H^{k-n}(M) \stackrel{\cup \boldsymbol{e}}{\longrightarrow} H^{k+1}(M) \to \cdots$$

This is called the *Gysin sequence* associated to the oriented sphere bundle $E \to M$. The class $\boldsymbol{e} \in H^{n+1}(M)$ is called the *Euler class* of the sphere bundle.

As a special case consider the Hopf bundle $\pi : S^{2n+1} \to \mathbb{CP}^n$ with fiber S^1. Recall its definition.

Consider the unit sphere

$$S^{2n+1} := \big\{ \vec{z} \in \mathbb{C}^{n+1};\ \ |\vec{z}| = 1 \big\}.$$

We have a natural map $\pi : S^{2n+1} \to \mathbb{CP}^n$ that associates to $\vec{z} \in S^{2n+1}$ the one-dimensional complex subspace $L_{\vec{z}}$ spanned by $\vec{z}$. Note that if $\vec{z}, \vec{\zeta} \in S^{2n+1}$ then

$$L_{\vec{z}} = L_{\vec{\zeta}} \Longleftrightarrow \exists \lambda \in \mathbb{C},\ \ |\lambda| = 1 :\ \ \lambda \vec{z} = \vec{\zeta}.$$

Since $\mathbb{CP}^n$ is simply connected we deduce that the locally constant sheaves $\widehat{H}^k_\pi$ are actually constant. We can apply the above discussion. We denote by $\boldsymbol{e} \in H^2(\mathbb{CP}^n)$ the Euler class of the Hopf fibration. The Gysin sequence yields the isomorphisms

$$\mathbb{R} = H^0(\mathbb{CP}^n) \xrightarrow{\cup \boldsymbol{e}} H^2(\mathbb{CP}^n) \xrightarrow{\cup \boldsymbol{e}} \cdots \xrightarrow{\cup \boldsymbol{e}} H^{2n}(\mathbb{CP}^n)$$
$$H^1(\mathbb{CP}^n) \xrightarrow{\cup \boldsymbol{e}} H^3(\mathbb{CP}^n) \xrightarrow{\cup \boldsymbol{e}} \cdots \xrightarrow{\cup \boldsymbol{e}} H^{2n-1}(\mathbb{CP}^n).$$
$$H^1(\mathbb{CP}^n) \xrightarrow{\pi^*} H^1(S^{2n+1}) = 0.$$

This shows that the cohomology algebra $H^\bullet(\mathbb{CP}^n)$ is truncated polynomial algebra $\mathbb{R}[\boldsymbol{e}]/(\boldsymbol{e}^{n+1})$, $\deg \boldsymbol{e} = 2$. □

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