Eta invariants, spectral flows and finite energy Seiberg-Witten monopoles

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I first got in touch with B. Booss-Bavnbek and K. Wojciekowski at a crucial moment in my career. They have since been very supportive, both professionally and personally, and it has always been a pleasure interacting with them. I want to use this occasion to thank them for their continuous support and for the opportunity to include the present survey in this volume.

ABSTRACT. We survey the computation of the virtual dimensions of finite energy Seiberg-Witten moduli spaces on 4-manifolds bounding Seifert fibrations. We then present some implications of these computations to 3-dimensional topology.

Introduction

The discovery of the Seiberg-Witten equations in the fall of 1994 sent shockwaves in the low dimensional topology community. Besides simplifying many of the proofs relying on the Yang-Mills equations, the new theory opened previously unsuspected avenues.

From an analytical point of view, the Seiberg-Witten theory is much nicer than the Yang-Mills theory for various reasons: the gauge group of the new theory is abelian and the new equations have a built-in positivity which ensures the compactness of the space of gauge equivalence classes of its solutions.

From a geometric point of view, the Seiberg-Witten equations are more sensitive to the background geometry. This sensitivity is plainly manifested in the case of Kähler surfaces when one can *explicitly* list all solutions of these equations and derive highly nontrivial topological conclusions (see for example [**3, 24**]).

The Seiberg-Witten invariants are certain clever counts of solutions of the Seiberg-Witten equations. A very useful tool of investigation, pioneered in the study of Yang-Mills equations, is by cutting and pasting. This technique can be generically described as follows.

1991 *Mathematics Subject Classification.* 58G10, 58G18, 58G20, 58D27.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

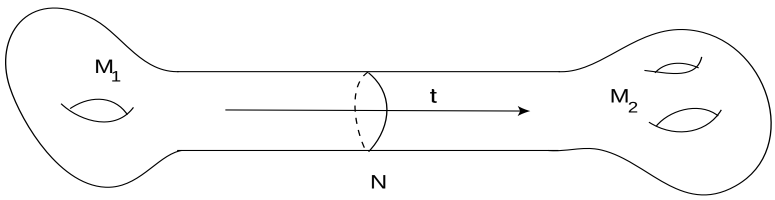

FIGURE 1. *Stretching the neck.*

Suppose M is closed, compact, oriented 4-manifold and $N \hookrightarrow M$ is an oriented hypersurface dividing M into two manifolds with boundary

$$M = M_1 \cup_N M_2.$$

Denote U a tubular neighborhood of $N \hookrightarrow M$

$$U \cong [-2,2] \times N$$

and fix a metric $\hat{g}$ on $M \setminus ([-1.5, 1.5] \times N)$ and a metric g on N. For each $L \gg 0$ we can equip M with a metric $\hat{g}_L$ which agrees with $\hat{g}$ on $M \setminus U$ and such that along $[-1,1] \times N$ we have

$$\hat{g}_L = L^2 dt^2 + g$$

where t denotes the longitudinal coordinate. Roughly speaking, we are stretching the tubular neighborhood until it becomes very, very long (see Figure 1).

The Seiberg-Witten equations depend on the metric $\hat{g}_L$ and, as $L \to \infty$, their solutions split into several parts corresponding to the manifolds with cylindrical ends in Figure 2. This is the cutting part of the program. The pasting part investigates how can one recover the original solutions from the "disintegrated" ones.

The solutions on these manifolds with cylindrical ends have special behavior near ∞, encoded by a *finite energy condition.* Before we carry out the cut-and-paste program for computing the Seiberg-Witten invariants we need to have a quite detailed understanding of the spaces of finite energy solutions. This is a very involved issue but fortunately, all the ideas required by such an endeavor were already developed in Yang-Mills theory by J. Morgan, T. Mrowka and D. Ruberman, [**15**].

The asymptotic behavior of these solutions is a key issue. The solutions have asymptotic limits but, for topological applications, we have to be much *more explicit* than that. This problem is made more complicated by the fact that the nature of these asymptotic limits is extremely sensitive to the geometry of N, a feature which was not present in the Yang-Mills case.

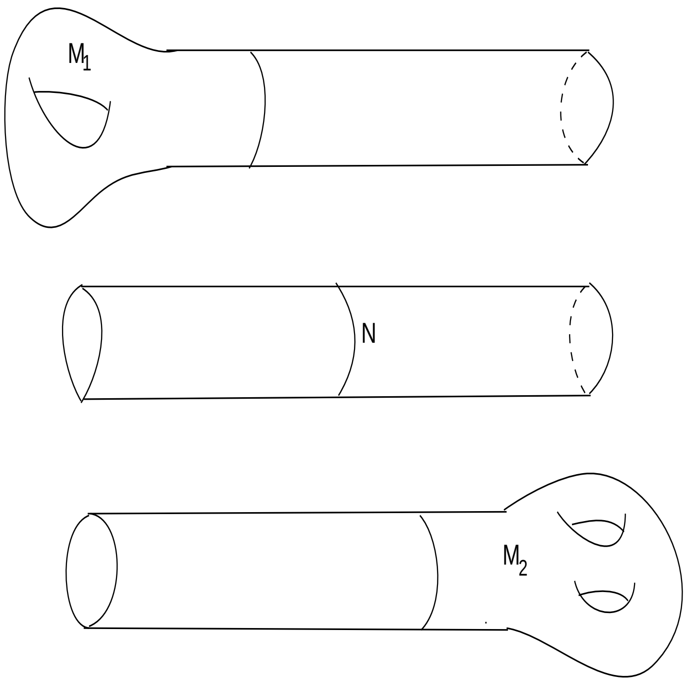

FIGURE 2. *Manifolds with cylindrical ends.*

Once the structure of the asymptotic limit set is settled, we can return to the original issue: what is the structure of the set of finite energy solutions. In particular we need to answer the following question

What is the virtual dimension of this space of solutions?

Recall that the virtual dimension is by definition the index of the linearization of the Seiberg-Witten equations on a manifold with cylindrical ends. The computation of this dimension is essentially an Atiyah-Patodi-Singer type problem with an added requirement: the results have to be *very explicit*.

In general the computation of an Atiyah-Patodi-Singer index can be a very demanding task even if the operators and the manifolds involved are reasonably nice. In the present case the difficulties are amplified by the nature of the asymptotic limits discussed above.

In the present paper we will outline the main steps involved in the computation of such dimension in some special, yet sufficiently large class of examples. The very explicit nature of our answers will allow us to derive several very surprising topological conclusions.

The only topological restriction we impose to our problem is that N should be a Seifert fibration, i.e. a 3-manifold equipped with a S^1-action which has only finite stabilizers. The reader could think of these manifolds as circle bundles over singular Riemann surfaces. As observed by W. Thurston, these manifolds are equipped with nice Riemann metrics, compatible in a very nice way with the fibration structure.

We gradually get rid of excess informational baggage by adiabatically collapsing these Seifert fibration onto their bases. This means that we deform the metric on

N so that the fibers become shorter and shorter. In the adiabatic limit all the irrelevant information disappears and we are left with enough information to be able to perform all the relevant computations.

The paper is divided in five parts. The first section is a fast paced survey of the technology developed in [**15**]. In particular, we formulate more concretely which are the main obstacles we need to overcome. Section 2 is a topological interlude in which we describe some fundamental topological and geometric features of Seifert manifolds. In particular, in this section we introduce the *adiabatic Dirac operators* and describe their eta invariants in terms of some classical arithmetic quantities (Dedekind sums). These Dirac operators arise naturally in the above adiabatic limit, whence their name. The third section is a survey of the outcome of the adiabatic analysis of [**17**]. More precisely, in this section we present a very detailed and explicit description of the set of all possible asymptotic limits. The forth section contains the virtual dimension formula. We collected the topological applications in a fifth section.

CONTENTS

1. Finite energy monopoles on 4-manifolds with cylindrical ends

The geometric background is a compact, oriented, Riemann 4-manifold $(M, \hat{g})$ with boundary $N := \partial M$ such that a tubular neighborhood U of N is isometric to a cylinder

$$(U, \hat{g}) \cong_{isom} \big((-1, 0] \times N, dt^2 + g \big), \quad g := \hat{g}\,|_N$$

where dt denotes the outer co-normal. We can form a non-compact manifold M_∞ by attaching the semi-infinite cylinder $[0, \infty) \times N$.

Fix a connection[1] $\hat{\nabla}$ on TM compatible with $\hat{g}$. We do not require that $\hat{\nabla}$ is the Levi-Civita connection (henceforth distinguished by the symbol $\hat{\nabla}^g$) but, for technical reasons, we need to assume the torsion of $\hat{\nabla}$ is traceless and that along the neck U it has the form $\hat{\nabla} = dt \otimes \partial_t + \nabla$, $\nabla := \hat{\nabla}\,|_N$. Both $\hat{g}$ and $\hat{\nabla}$ admit natural extensions to M_∞.

Next, fix a $spin^c$ structure $\hat{\sigma}$ on M_∞ and denote by $\hat{\mathbb{S}}_\sigma = \hat{\mathbb{S}}_\sigma^+ \oplus \hat{\mathbb{S}}_\sigma^-$ the associated $\mathbb{Z}_2$-graded bundle of complex spinors. Once we choose a hermitian connection $\hat{A}$ on $\det(\hat{\sigma}) := \det \hat{\mathbb{S}}_\sigma^+$ the connection $\hat{\nabla}$ functorially induces a connection $\hat{\nabla}^A$ on $\hat{\mathbb{S}}_\sigma$. $\hat{\nabla}^A$ is compatible with the $\mathbb{Z}_2$-grading and the Clifford multiplication i.e.

$$\hat{\nabla}^A_X(\hat{\mathbf{c}}(\alpha)\hat{\psi}) = \hat{\mathbf{c}}(\hat{\nabla}_X \alpha)\hat{\psi} + \hat{\mathbf{c}}(\alpha)\hat{\nabla}^A_X\hat{\psi} \tag{1.1}$$

[1] In practice $\hat{\nabla}$ is chosen so that it is compatible with an additional geometric structure on M.

$\forall X \in \mathrm{Vect}\,(M_\infty)$, $\alpha \in \Omega^1(M_\infty)$, $\hat{\psi} \in \Gamma(\hat{\mathbb{S}}_\sigma)$ where $\hat{\mathbf{c}}$ denotes the Clifford multiplication by a differential form. We can now form the Dirac operator $\hat{\not{D}}_{\hat{A}} : \Gamma(\hat{\mathbb{S}}^+_\sigma) \to \Gamma(\hat{\mathbb{S}}^-_\sigma)$ defined as the composition

$$\Gamma(\hat{\mathbb{S}}^+_\sigma) \xrightarrow{\hat{\nabla}^A} \Gamma(T^*M(\infty) \otimes \hat{\mathbb{S}}^+_\sigma) \xrightarrow{\hat{\mathbf{c}}} \Gamma(\hat{\mathbb{S}}^-_\sigma).$$

A *4-dimensional* $(\hat{g}, \hat{\nabla})$*-monopole* is a pair $\hat{\mathsf{C}} := (\hat{\psi}, \hat{A})$ where $\hat{\psi} \in \Gamma(\hat{\mathbb{S}}^+_\sigma)$ and the hermitian connection $\hat{A}$ on $\det(\hat{\sigma})$ satisfy the differential equations

$$\left\{ \begin{array}{rcl} \hat{\not{D}}_{\hat{A}}\hat{\psi} & = & 0 \\ \hat{\mathbf{c}}(F^+_{\hat{A}}) & = & q(\hat{\psi}) \end{array} \right. . \tag{1.2}$$

Above, $q(\hat{\psi})$ denotes the traceless, symmetric endomorphism of $\hat{\mathbb{S}}^+_\sigma$ described by the correspondence

$$\Gamma(\hat{\mathbb{S}}^+_\sigma) \ni \hat{\phi} \mapsto \langle \hat{\phi}, \hat{\psi} \rangle \hat{\psi} - \frac{1}{2}|\hat{\psi}|^2 \hat{\phi} \in \Gamma(\hat{\mathbb{S}}_\sigma).$$

The monopole is said to have *finite energy* if

$$\int_0^\infty dt \int_N |\hat{\nabla}^A_t \hat{\psi}(t)|^2 + |i_{\partial_t} F_A(t)|^2 dv_g < \infty$$

where i_{∂_t} denotes the contraction with ∂_t and for any object $\hat{O}$ defined over $[0, \infty) \times N$ we denoted by $\hat{O}(t)$ its restriction to $\{t\} \times N$. Observe that these equations are invariant under the action of the infinite dimensional group $\mathfrak{G}_M := C^\infty(M_\infty, S^1)$.

Using the techniques of [**15**] one can show that if $\hat{\mathsf{C}}$ is a finite energy 4-dimensional monopole then modulo $\mathfrak{G}_M$ the restrictions $\hat{\mathsf{C}}(t)$ converge as $t \to \infty$ to an asymptotic limit $\mathsf{C}_\infty := (\psi_\infty, A_\infty)$. ψ_∞ is a section of $\mathbb{S}_\sigma := \hat{\mathbb{S}}^+_\sigma|_N$ and A_∞ is a hermitian connection on $\det(\sigma) := \det \mathbb{S}_\sigma$. The bundle $\mathbb{S}_\sigma$ can be identified with the bundle of complex spinors associated to the *spin*^{c} structure $\sigma := \hat{\sigma}\,|_N$ on N. We denote by $\mathbf{c}$ the Clifford multiplication on this bundle. The connection ∇ on TN and a hermitian connection A on $\det(\sigma)$ induce in a natural fashion a connection ∇^A on $\mathbb{S}_\sigma$ compatible as in (1.1) with the Clifford multiplication. In turn, ∇^A induces a Dirac operator[2] $D_A := \mathbf{c} \circ \nabla^A$. The asymptotic limit (ψ_∞, A_∞) is a *3-dimensional monopole* i.e. satisfies the differential equations

$$\left\{ \begin{array}{rcl} D_{A_\infty}\psi_\infty & = & 0 \\ \mathbf{c}(*_g F_{A_\infty}) & = & q(\psi_\infty) \end{array} \right. \tag{1.3}$$

The above equations are invariant under the action of the infinite dimensional group $\mathfrak{G}_N := C^\infty(N, S^1)$.

The 3-dimensional Seiberg-Witten equations (1.3) have a variational nature. Its solutions are critical points of the energy functional

$$\mathcal{E}(\psi, A) = \frac{1}{2}\int_N (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_N \langle D_A\psi, \psi \rangle dv_g$$

where A_0 is a fixed reference connection. Along a cylinder $[a, b] \times N$ the 4-dimensional Seiberg-Witten equations are gauge equivalent with a gradient flow equation

$$\dot{\mathsf{C}} = \nabla\mathcal{E}(\mathsf{C}), \;\; \mathsf{C} = \mathsf{C}(t) = (\psi(t), A(t)).$$

[2] The fact that the torsion of $\hat{\nabla}$ is traceless implies that D_A is symmetric.

This variational aspect is essentially responsible for the existence of asymptotic limits. The finite energy monopoles on a cylinder $\mathbb{R} \times N$ are called *tunnelings.* We see that the tunnelings are gradient flow trajectories which connect critical points. The information carried by tunnelings can be organized in a very elegant fashion, leading to the Seiberg-Witten-Floer (co)homology; see [**13**]. The knowledge of the virtual dimensions of the spaces of tunnelings is an important step in deciphering the structure of the Seiberg-Witten-Floer homology.

Denote by $\mathfrak{M}_M(\hat{\sigma})$ the space of finite energy 4-dimensional monopoles modulo the $\mathfrak{G}_M$-action. $\mathfrak{M}_N(\sigma)$ is defined similarly, using the equations (1.3) instead. The space $\mathfrak{M}_N(\sigma)$ is a compact metric space and we denote by $\mathfrak{M}^i_N$, $i = 1, 2, \dots$ its connected components.

The correspondence $\hat{\mathsf{C}} \mapsto \mathsf{C}_\infty$ defines a map

$$\mathbf{LIM} : \mathfrak{M}_M(\hat{\sigma}) \to \mathfrak{M}_N(\sigma).$$

Set $\mathfrak{M}^i_M(\hat{\sigma}) := \mathbf{LIM}^{-1}(\mathfrak{M}^i_N)$.

As explained in the introduction, surgery problems involving the Seiberg-Witten invariants require a detailed understanding of the local structure of the spaces $\mathfrak{M}^i_M$. This local structure is easier to describe when the component $\mathfrak{M}^i_N$ satisfies an additional condition

Additional assumption *The component $\mathfrak{M}^i_N$ is non-degenerate.*

To explain the exact meaning of this assumption we need to introduce some notations. For every pair $\mathsf{C} = (\phi, A) \in \mathfrak{M}_N$ we denote by $d(\mathsf{C})$ the dimension of the stabilizer[3] of C with respect to the action of $\mathfrak{G}_N$. Also we define the *stabilized Hessian* at C as the following formally self-adjoint, first order, partial differential operator

$$\tilde{\mathfrak{H}}_{\mathsf{C}} : \Gamma(\mathbb{S}_\sigma \oplus \mathbf{i}(\Lambda^1 + \Lambda^0)T^*N) \to \Gamma(\mathbb{S}_\sigma \oplus \mathbf{i}(\Lambda^1 + \Lambda^0)T^*N)$$

$$\tilde{\mathfrak{H}}_{\mathsf{C}} \left[\begin{array}{c} \dot{\psi} \\ \mathbf{i}\dot{a} \\ \mathbf{i}f \end{array} \right] = \left[\begin{array}{cccc} D_A\dot{\psi} & & + & \mathbf{c}(\mathbf{i}\dot{a})\phi - \mathbf{i}f\phi \\ & -\mathbf{i} * d\dot{a} + \mathbf{i}df & + & \dot{q}(\phi, \dot{\psi}) \\ & \mathbf{i}d^*\dot{a} & + & \mathbf{i}\mathfrak{Im}\langle \phi, \dot{\psi}\rangle \end{array} \right]$$

where $\dot{q}(\phi, \dot{\psi}) := \frac{d}{dt}|_{t=0}\, q(\phi + t\dot{\psi})$. Also, we will denote by $\tilde{\mathfrak{H}}^0_{\mathsf{C}}$ the operator obtained by setting $\phi = 0$ in the above description of $\tilde{\mathfrak{H}}_{\mathsf{C}}$. Observe that the difference $\mathcal{P}_{\mathsf{C}} := \tilde{\mathfrak{H}}_{\mathsf{C}} - \tilde{\mathfrak{H}}^0_{\mathsf{C}}$ is a zeroth order operator.

The component $\mathfrak{M}^i_N$ is non-degenerate if $\mathfrak{M}^i_N$ is a smooth manifold and for every $\mathsf{C} \in \mathfrak{M}^i_N$ we have the equality

$$\dim \ker \tilde{\mathfrak{H}}_{\mathsf{C}} = \dim \mathfrak{M}^i_N + d(\mathsf{C}).$$

Given the nondegeneracy assumption, the techniques of [**15**] imply that, if nonempty, the moduli space $\mathfrak{M}^i_M(\hat{\sigma})$ is generically a smooth manifold. Denote by $d(\hat{\sigma}, i)$ its virtual (expected) dimension. This dimension can be expressed in terms of $d_i := \dim \mathfrak{M}^i_N$ and the Atiyah-Patodi-Singer index (over $\mathbb{R}$) of a certain first order operator on M

$$\hat{\mathcal{O}}_{\hat{\mathsf{C}}} : \Gamma(\hat{\mathbb{S}}_\sigma \oplus \mathbf{i}\Lambda^1 T^*M) \to \Gamma(\hat{\mathbb{S}}^-_\sigma \oplus \mathbf{i}(\Lambda^2_+ + \Lambda^0)T^*M), \;\; \hat{\mathsf{C}} = (\hat{\phi}, \hat{A}) \in \mathfrak{M}^i_M(\hat{\sigma})$$

[3] This stabilizer is either the trivial group or S^1 so that $d(\mathsf{C}) \in \{0, 1\}$.

The operator $\hat{\mathcal{O}}_{\hat{\mathsf{C}}}$ can be written as a sum $\hat{\mathcal{O}} = \hat{\mathcal{N}}_{\hat{\mathsf{C}}} + \mathcal{T}_{\hat{\mathsf{C}}}$. $\hat{\mathcal{N}}_{\hat{\mathsf{C}}}$ is the direct sum of the operators

$$\hat{\not{\mathfrak{D}}}_{\hat{A}} : \Gamma(\hat{\mathbb{S}}_\sigma^+) \to \Gamma(\hat{\mathbb{S}}_\sigma^-) \text{ and } \mathbf{ASD} := (d_+ + d^*) : \mathbf{i}\Omega^1(M) \to \mathbf{i}(\Omega^2_+ + \Omega^0)(M).$$

(Observe that **ASD** is the **a**nti-**s**elf**d**uality operator.) $\mathcal{T}_{\hat{\mathsf{C}}}$ is a zeroth order operator which along the neck U has the form $-G \circ \mathcal{P}_{\mathsf{C}}$, $\mathsf{C} = \mathbf{LIM}(\hat{\mathsf{C}}) = (\phi, A) \in \mathfrak{M}^i_N$ and G is the bundle morphism obtained by evaluating the symbol of $\hat{\mathcal{N}}_{\hat{\mathsf{C}}}$ at dt.

More precisely, we have (see [**18**] and [**19**])

$$d(\hat{\sigma}, i) = d_i + i_{APS}(\hat{\mathcal{O}}_{\hat{\mathsf{C}}}). \tag{1.4}$$

A simple excision trick shows that (see [**18**])

$$i_{APS}(\hat{\mathcal{O}}_{\hat{\mathsf{C}}}) = i_{APS}(\hat{\mathcal{N}}_{\hat{\mathsf{C}}}) - SF(\tilde{\mathfrak{H}}^0_{\mathsf{C}} \to \tilde{\mathfrak{H}}_{\mathsf{C}})$$

where by $SF(A \to B)$ we denote the spectral flow of the path $t \mapsto (1-t)A + tB$, $t \in [0,1]$. Hence

$$\begin{aligned} d(\hat{\sigma}, i) &= i_{APS}(\hat{\mathcal{N}}_{\hat{\mathsf{C}}}) - SF(\tilde{\mathfrak{H}}^0_{\mathsf{C}} \to \tilde{\mathfrak{H}}_{\mathsf{C}}) + d_i \\ &= 2i_{APS}(\hat{\not{\mathfrak{D}}}_{\hat{A}}) + i_{APS}(\mathbf{ASD}) - SF(\tilde{\mathfrak{H}}^0_{\mathsf{C}} \to \tilde{\mathfrak{H}}_{\mathsf{C}}) + d_i \end{aligned} \tag{1.5}$$

where $i_{APS}(\hat{\not{\mathfrak{D}}}_{\hat{A}})$ denotes the Atiyah-Patodi-Singer index *over* $\mathbb{C}$ of $\hat{\not{\mathfrak{D}}}_{\hat{A}}$.

This is still not the most convenient description because when trying to apply the Atiyah-Patodi-Singer theorem to the operator $\hat{\not{\mathfrak{D}}}_{\hat{A}}$ we are faced with another problem. Namely, the local index density of $\hat{\not{\mathfrak{D}}}_{\hat{A}}$ is difficult to compute since this *spin^c* Dirac operator *does not originate from the Levi-Civita connection.* It was constructed starting with a possible non-symmetric connection $\hat{\nabla}$. To fix this problem denote by $\hat{\mathfrak{D}}_{\hat{A}}$ the Dirac operator $\Gamma(\hat{\mathbb{S}}_\sigma^+) \to \Gamma(\hat{\mathbb{S}}_\sigma^-)$ induced by the *Levi-Civita connection* $\hat{\nabla}^g$ and the hermitian connection $\hat{A}$. Using again an excision trick we deduce

$$\begin{aligned} d(\hat{\sigma}, i) = 2i_{APS}(\hat{\mathfrak{D}}_{\hat{A}}) + i_{APS}(\mathbf{ASD}) - 2SF(\hat{\mathfrak{D}}_{\hat{A}} \to \hat{\not{\mathfrak{D}}}_{\hat{A}}) \\ -SF(\tilde{\mathfrak{H}}^0_{\mathsf{C}} \to \tilde{\mathfrak{H}}_{\mathsf{C}}) + d_i. \end{aligned} \tag{1.6}$$

The local index density of $\hat{\mathfrak{D}}_{\hat{A}}$ is

$$\rho_{dir}(\hat{A}) = -\frac{1}{24}p_1(\hat{\nabla}^g) + \frac{1}{8}c_1(\hat{A})^2$$

and the local density of **ASD** is

$$\rho_{asd} = -\frac{1}{2}\mathbf{e}(\hat{\nabla}^g) - \frac{1}{6}p_1(\hat{\nabla}^g)$$

where $\mathbf{e}(\hat{\nabla}^g)$ denotes the Euler form of TM constructed starting from $\hat{\nabla}^g$ using the Chern-Weil correspondence. We then have

$$i_{APS}(\mathbf{ASD}) = \int_M \rho_{asd} - \frac{1}{2}(b_0(N) + b_1(N) - \eta_{sign}(g))$$

where $\eta_{sign}(g)$ denotes the eta invariant of the odd signature operator on (N, g).

From the Gaußß-Bonnet-Chern formula (on manifolds with boundary) we deduce

$$\int_M \mathbf{e}(\hat{\nabla}^g) = \chi(M)$$

and the Atiyah-Patodi-Singer index theorem implies ([**1**])

$$\frac{1}{3}\int_M p_1(\hat{\nabla}^g) - \eta_{sign}(g) = \tau(M) := \text{the signature of } M.$$

Hence

$$i_{aps}(\mathbf{ASD}) = -\frac{1}{2}(\chi(M) + \tau(M)) - \frac{b_0(N) + b_1(N)}{2}.$$

To compute the Atiyah-Patodi-Singer index of $\hat{\mathfrak{D}}_{\hat{A}}$ we need to introduce the Dirac operator $\mathfrak{D}_A$ on $\mathbb{S}_\sigma \to N$ induced by the Levi-Civita connection ∇^g and A. Then we have

$$\begin{aligned} i_{APS}(\hat{\mathfrak{D}}_{\hat{A}}) &= \int_M \rho_{dir}(\hat{A}) - \frac{1}{2}(\dim_{\mathbf{C}}\ker\mathfrak{D}_A + \eta(\mathfrak{D}_A)) \\ &= \frac{1}{8}\int_M c_1(\hat{A})^2 - \frac{1}{2}(\dim_{\mathbf{C}}\ker\mathfrak{D}_A + \eta(\mathfrak{D}_A)) \\ &\quad -\frac{1}{8}\left(\int_M \frac{1}{3}p_1(\hat{\nabla}^g) - \eta_{sign}(g)\right) - \frac{1}{8}\eta_{sign}(g) \end{aligned}$$

$(\mathbf{F}(g, A) := 4\eta(\mathfrak{D}_A) + \eta_{sign}(g))$

$$= \frac{1}{8}\int_M c_1(\hat{A})^2 - \frac{1}{8}\tau(M) - \frac{1}{8}(4\dim_{\mathbf{C}}\ker\mathfrak{D}_A + \mathbf{F}(g, A)).$$

Putting all the above together we deduce

$$\begin{aligned} d(\hat{\sigma}, i) = \frac{1}{4}\int_M c_1(\hat{A})^2 - \frac{1}{4}(2\chi + 3\tau) - \frac{1}{4}\mathbf{F}(g, A) - \dim_{\mathbf{C}}\ker\mathfrak{D}_A \\ -2SF(\mathfrak{D}_A \to D_A) - SF(\tilde{\mathfrak{H}}^0_{\mathsf{C}} \to \tilde{\mathfrak{H}}_{\mathsf{C}}) + d_i \end{aligned} \tag{1.7}$$

Thus, the computation of these virtual dimensions in a specific case depends on our ability of solving the following problems.

Problem 1 *Describe the moduli spaces $\mathfrak{M}^i_N$ as explicitly as possible. In particular, describe which connected components are non-degenerate.*

Problem 2 *Compute the quantities* $\mathbf{F}(g, A)$, $\dim_{\mathbf{C}}\ker\mathfrak{D}_A$ *and* $SF(\mathfrak{D}_A \to D_A)$ *for every* $\mathsf{C} = (\phi, A) \in \mathfrak{M}^i_N$.

Problem 3 *Compute* $SF(\tilde{\mathfrak{H}}^0_{\mathsf{C}} \to \tilde{\mathfrak{H}}_{\mathsf{C}})$, $\forall \mathsf{C} \in \mathfrak{M}^i_N$.

The main difficulty in solving each of these problems is their extreme sensitivity upon the geometry (N, g). In the remaining part of this survey we will describe one quite general instance when all of these problems can be solved explicitly, leading to interesting topological consequences. More precisely we will consider the case when N is a Seifert fibered manifold.

2. Seifert manifolds

The Seifert manifold generalize the concept of circle bundle over a Riemann surface. They can be regarded as circle bundle over 2-dimensional orbifolds. For a detailed presentation of this concept we refer to [**20**] and the references therein.

A compact oriented 2-orbifold is described by a smooth compact surface Σ together with an additional finite collection of *singularity data*

$$\{(p_i, \alpha_i) \in \Sigma \times \mathbb{Z}_+; \;\; i = 1, \dots, m\}.$$

The points p_i are assumed to be distinct and possessing neighborhoods homeomorphic to a cyclic quotient $\mathbf{D}/\mathbb{Z}_{\alpha_i}$, $\alpha_i > 1$ where

$$\mathbf{D} := \{|z| < 1\} \subset \mathbb{C}.$$

The underlying topological space of this orbifold , i.e. the surface Σ with no reference to the singularity data, is called the *support* of the orbifold and will be denoted by $|\Sigma|$.

A very constructive method of defining the Seifert fibration is via the concept of *line V-bundle*. Away from the singular points p_i, a line V-bundle $L \to \Sigma$ is a genuine complex line bundle. Near p_i the line bundle is regarded as the quotient of a $\mathbb{Z}_{\alpha_i}$-equivariant line bundle $\tilde{L} \to \mathbf{D}$. These equivariant line bundles are described by an integer $\beta_i \in [1, \alpha_i)$ where β_i describes the action of $\mathbb{Z}_{\alpha_i}$ in the fiber of $\tilde{L}$ over 0. Each line V-bundle has a degree $\deg(L) \in \mathbb{Q}$ satisfying

$$\deg(L) - \sum_i \frac{\beta_i}{\alpha_i} \in \mathbb{Z}.$$

When L is the quotient of a *smooth* line bundle $\tilde{L} \to \tilde{\Sigma}$ over a *smooth* Riemann surface $\tilde{\Sigma}$, equivariant under the action of a finite group G, then

$$\deg(L) = \frac{1}{\#G} \deg(\tilde{L}).$$

The *smooth* line bundle over $|\Sigma|$ of degree $\deg(L) - \sum_i \frac{\beta_i}{\alpha_i}$ is called the *desingularization* of L and is denoted by $|L|$.

Each 2-orbifold Σ with singularity data $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$ has a *canonical line V-bundle* K_Σ with singularity data $(\alpha_1 - 1, \dots, \alpha_m - 1)$ and degree

$$\deg K_\Sigma = 2g - 2 - \sum_i (1 - \frac{1}{\alpha_i})$$

where g denotes the genus of $|\Sigma|$. Observe that $|K_\Sigma| \cong K_{|\Sigma|}$.

As in the smooth case one can define the tensor product of line V-bundles and we have

$$\deg(L_1 \otimes L_2) = \deg(L_1) + \deg(L_2).$$

We denote by $\mathrm{Pic}^t(\Sigma)$ the set of isomorphism classes of line V-bundles over Σ. Then $\mathrm{Pic}^t(\Sigma)$ is an abelian group with respect to the tensor product. We have the following more precise result.

PROPOSITION 2.1. *Define*

$$\tau = \tau_{\vec{\alpha}} : \mathrm{Pic}^t(\Sigma) \to \mathbb{Q} \oplus \mathbb{Z}_{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_{\alpha_n}$$

by

$$L(\vec{\beta}) \mapsto (\deg L, \beta_1(\mod \alpha_1), \dots, \beta_n(\mod \alpha_n))$$

and $\delta : \mathbb{Q} \oplus \mathbb{Z}_{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_{\alpha_n}$ *by*

$$(c, \beta_1(\mod \alpha_1), \dots, \beta_n(\mod \alpha_n)) \mapsto \left(c - \sum_i \frac{\beta_i}{\alpha_i}\right) (\mod \mathbb{Z}).$$

Then

$$0 \to \mathrm{Pic}^t(\Sigma) \xrightarrow{\tau} \mathbb{Q} \oplus \mathbb{Z}_{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_{\alpha_n} \xrightarrow{\delta} \mathbb{Q}/\mathbb{Z} \to 0$$

is a short exact sequence of abelian groups.

The line bundle with singularity data $\vec{\beta} = (\beta_1, \dots, \beta_m)$ and rational degree c will be denoted by $L(c, \vec{\beta})$.

A Seifert manifold is by definition the unit sphere V-bundle of a complex line V-bundle over a 2-orbifold. More precisely, consider the 2-orbifold $\Sigma(g, \vec{\alpha})$ and the line V-bundle

$$L(c, \vec{\beta}) \to \Sigma$$

such that if $\beta_i \neq 0$ then $g.c.d.(\beta_i, \alpha_i) = 1$. The unit sphere bundle $S(L)$ of L is a Seifert manifold we will denote by $N(g; b; \vec{\alpha}, \vec{\beta})$ where $b = \deg |L|$. The collection $(g; b; \vec{\alpha}, \vec{\beta})$ is known as the *(normalized) Seifert invariant.* Σ is known as the base of N. Observe that N is equipped with an infinitesimally free S^1-action. One can show (see [**20**]) that any 3-manifold endowed with such an action can be presented as the S^1-bundle of a line V-bundle.

The basic topological invariants of a Seifert manifold are known (see [**10**]). We include only one fact relevant in the sequel.

THEOREM 2.2. *If $N = N(g; b, \vec{\alpha}, \vec{\beta})$) (m singular points) then*

$$H^2(N, \mathbb{Z}) \cong \big(\mathrm{Pic}^t(\Sigma)/\mathbb{Z}[L]\big) \oplus \mathbb{Z}^{2g}.$$

Moreover, the projection $\pi : N \to \Sigma$ pulls back line V-bundles on Σ to genuine smooth *line bundles on N and the torsion subgroup* $\mathrm{Pic}^t/\mathbb{Z}[L]$ *of $H^2(N, \mathbb{Z})$ can be identified with the image of the morphism*

$$\mathrm{Pic}^t(\Sigma) \xrightarrow{\pi^*} \{Smooth\ line\ bundles\ on\ N\} \xrightarrow{c_1} H^2(N, \mathbb{Z}).$$

DEFINITION 2.3. *Let $N \xrightarrow{\pi} \Sigma$ denote a Seifert fibration over a 2-orbifold Σ such that $\ell = \deg(N) \neq 0$. The* canonical representative *of a line bundle $\tilde{L} \to N \in \pi^*(\mathrm{Pic}^t(\Sigma))$ is the V-line bundle $L = L(c; \gamma)$ uniquely determined by the requirements*

$$\pi^*(L) \cong \tilde{L} \ \text{ and } \ \rho(\tilde{L}) := \frac{\deg K_\Sigma - 2c}{2\ell} \in [0, 1).$$

In the sequel we fix a 2-orbifold $\Sigma = \Sigma(g, m, \vec{\alpha}, \vec{x})$ of genus g and with singularity data $\{(x_1, \alpha_1), \dots, (\alpha_m, x_m)\}$ and a line V-bundle

$$\mathbb{L}_0 = \mathbb{L}_0(\ell, \vec{\beta}) \xrightarrow{\pi} \Sigma.$$

such that $0 < \beta_i < \alpha_i$ and $g.c.d.(\alpha_i, \beta_i) = 1$, $\forall i = 1, \dots, m$. ℓ is the rational degree of $\mathbb{L}_0$. Denote by N the associated Seifert manifold $N = S(\mathbb{L}_0)$. We orient it as the boundary of a complex manifold following the convention

$$outer\ normal\ \wedge\ \mathbf{or}\,(boundary) = \mathbf{or}\,(manifold).$$

As explained in [**23**], the manifold N admits a locally homogeneous Riemann metric g_N. The natural S^1 action preserves such a metric. We denote by ζ the infinitesimal generator of this action. ζ is a nowhere vanishing Killing vector field. This metric induces V-metric g_Σ on the base which we normalize so that $\mathrm{vol}\,(\Sigma) = \pi$.

g_Σ has constant sectional curvature. By eventually rescaling the metric g_N in the ζ direction we can assume $|\zeta|_{g_N} \equiv 1$. Now denote by φ the *global angular form* defined as the g_N-dual of ζ. As shown in [**16**] and [**17**] we have a fundamental identity

$$d\varphi = -2\ell * \varphi. \tag{2.1}$$

Here we want to stress a subtle point. The above equality describes a special type of interaction between the S^1-action and the given (eventually rescaled) Thurston metric. For example, it implies that the orbits of the S^1-action are geodesics. A given 3-manifold may have different Seifert structures but only *one type of locally homogeneous metrics*. Some of Seifert structures may not interact with the metric as required by (2.1) but in [**17**] we showed that at least one of these Seifert structures interacts with the metric as above. We will call these structures *geometric Seifert structures*. For example, a lens space admits infinitely many Seifert structures but, as shown in [**21**], *only two* (!!!) of them are geometric. In the sequel we will deal exclusively with geometric Seifert structures.

Denote by $Cl(T^*N)$ the bundle of Clifford algebras generated by $T^*N, g_N)$. The bundle Λ^*T^*N is naturally a bundle of $Cl(T^*N)$ modules (see [**2**]) and we denote by

$$\mathbf{c} : Cl(T^*N) \to \mathrm{End}\,(\Lambda^*T^*N)$$

the corresponding Clifford multiplication. The symbol map

$$\sigma : Cl(T^*N) \to \Lambda T^*N, u \mapsto \mathbf{c}(u)\cdot \mathbf{1}$$

is a bundle isomorphism with inverse known as the *quantization map*. This allows us to define an action of ΛT^*N on itself by

$$\Lambda T^*N \to \mathrm{End}\,(\Lambda^*T^*N),\ \ \omega \mapsto \mathbf{c}(\sigma^{-1}(\omega)).$$

For simplicity we continue to denote this map with $\mathbf{c}$. We call the resulting operation the Clifford multiplication by a form.

Let $\langle\varphi\rangle^\perp$ denote the orthogonal complement of the real line sub-bundle of T^*N spanned by φ. As shown in [**17**] the bundle $\langle\varphi\rangle^\perp$ is $\mathbf{c}(*\varphi)$-invariant. The bundle $\langle\varphi\rangle^\perp$ equipped with the (almost) complex structure $-\mathbf{c}(*\varphi)$ will be called the *canonical line bundle* of N and will be denoted by $\mathfrak{K}$. We have the isomorphism

$$\mathfrak{K} \cong \pi^* K_\Sigma.$$

In [**17**], §2.2 we showed that $\mathfrak{K}$ determines a canonical *spin*^{c} structure on N with associated bundle of spinors

$$\mathbb{S}_{can} \cong \mathfrak{K}^{-1} \oplus \underline{\mathbb{C}} \tag{2.2}$$

where $\underline{\mathbb{C}}$ denotes generically the trivial complex line bundle. This allows us to identify the space $\mathrm{Spin}^c(N)$ of *spin*^{c} structures on N with the topological Picard group, $\mathrm{Pic}^t(N)$. The bundle of spinors corresponding to the line bundle $L \to N$ is

$$\mathbb{S}_L = \mathbb{S}_{can} \otimes L \cong L\otimes \mathfrak{K}^{-1} \oplus L. \tag{2.3}$$

As above, we get a Clifford multiplication map $\mathbf{c} : \Lambda^* T^* N \to \mathrm{End}\,(\mathbb{S}_L)$. We "orient" it using the conventions of [**4**]

$$\mathbf{c}(dv_N) = -\mathbf{id}.$$

On TN there are two natural g_N-compatible connections. The first one is the Levi-Civita connection ∇^{LC}. The other connection ∇^∞, called the *adiabatic connection* in [**17**] and [**18**] is described as follows. Using the decomposition

$$T^*N = \langle \varphi \rangle \oplus \langle \varphi \rangle^\perp \cong \langle \varphi \rangle \oplus \pi^* T\Sigma$$

we define ∇^∞ as the direct sum of the trivial connection on φ and the pullback of the Levi-Civita connection of $T\Sigma$ on $\langle \varphi \rangle$.

The line bundle $\mathfrak{K}^{-1}$ comes with a natural hermitian connection induced by pullback from the Levi-Civita connection on the base. Thus a connection on $\det \mathbb{S}_L \cong L^2 \oplus \mathfrak{K}^{-1}$ can be specified by indicating a connection on L. Fix such a connection A. Using the connection ∇^{LC} we obtain a Dirac operator $\mathfrak{D}_A$ on $\mathbb{S}_L$ while the adiabatic connection ∇^∞ induces a different Dirac operator, D_A. We will call D_A the *adiabatic Dirac operator*. These two Dirac operators are related by the equality

$$D_A = \mathfrak{D}_A + \frac{\ell}{2}. \tag{2.4}$$

D_A can be decomposed as

$$D_A = Z_A + T_A$$

where, with respect to the decomposition (2.3), the operators Z and T have the matrix descriptions

$$Z_A = \begin{bmatrix} \mathbf{i}\nabla^A_\zeta & 0 \\ 0 & -\mathbf{i}\nabla^A_\zeta \end{bmatrix} \tag{2.5}$$

and

$$T_A = \begin{bmatrix} 0 & {}^\flat\bar{\partial}_A \\ {}^\flat\bar{\partial}^*_A & 0 \end{bmatrix}. \tag{2.6}$$

We refer to [**17**] for the exact definitions of Z and T. It suffices to say that Z uses only derivatives along the fiber direction while T uses only derivatives along horizontal directions. Moreover Z and T interact in an especially nice manner:

$$\{Z, T\} := ZT + TZ = 0. \tag{2.7}$$

The above equality is responsible for many dramatic simplifications.

To formulate our next results we must first define the notion of *Dedekind-Rademacher sum*. For co-prime integers α, β such that $\alpha > 0$ and $x, y \in \mathbb{R}$ set following [**22**]

$$s(\beta, \alpha; x, y) := \sum_{r=1}^{\alpha} \left(\left(x + \beta\frac{r+y}{\alpha}\right)\right)\left(\left(\frac{r+y}{\alpha}\right)\right).$$

where

$$((x)) := \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

Despite their apparent complicated description these sums enjoy certain features which make them computationally friendly; see [**19**] and [**22**] for details.

The following results were proven in [**19**].

PROPOSITION 2.4. *Consider a line V-bundle $L = L(c, \vec{\gamma}) \to \Sigma = \Sigma(g, m; \vec{\alpha}, \vec{x})$ equipped with a hermitian connection A. Denote by $\tilde{L}$ and resp. $\tilde{A}$ the pullbacks of L and A to $N = S(\mathbb{L}_0)$.*

If $\ell = \deg \mathbb{L}_0 \neq 0$ then the eta invariant $\eta_{L,A}$ of the Dirac operator $D_{\tilde{A}}$ on $\mathbb{S}_{\tilde{L}}$ is given by

$$\eta_{L,A} = \frac{\ell}{6} - 2S(\vec{\beta}, \vec{\alpha}; \vec{\gamma}) - d(\vec{\beta}, \vec{\alpha}; \vec{\gamma}) \tag{2.8}$$

where

$$\begin{aligned} 2S(\vec{\beta}, \vec{\alpha}, \vec{\gamma}) &= \sum_{i=1}^{m} (\, s(\beta_i, \alpha_i; \gamma_i/\alpha_i, 0) - s(-\beta_i, \alpha_i; \gamma_i/\alpha_i, 0)\,) \\ &= 2\sum_{i=1}^{m} s(\beta_i, \alpha_i; \gamma_i/\beta_i, 0). \end{aligned} \tag{2.9}$$

$q_i\beta_i \equiv 1$ *mod* α_i *and*

$$d(\vec{\beta}, \vec{\alpha}; \vec{\gamma}) = \sum_{i=1}^{m} \left(\left(\frac{q_i\gamma_i}{\alpha_i}\right)\right).$$

PROPOSITION 2.5. *Let $\tilde{L} \in \pi^*\mathrm{Pic}^t(\Sigma)$ be a line bundle on N whose first Chern class c_1 is torsion. Denote by $L = L(c, \vec{\gamma}) \to \Sigma$ its canonical representative and equip $\tilde{L}$ with a connection $\tilde{B}$ of the form*

$$\tilde{B} = \tilde{B}_0 + \mathbf{i}\rho\varphi$$

where $\rho = \rho(\tilde{L}) = \frac{\deg K - 2c}{2\ell}$ and $\tilde{B}_0$ is the pullback of a constant curvature connection B_0 on L. Denote by $\eta_L = \eta_{\tilde{L},\tilde{B}}$ the eta invariant of the associated adiabatic Dirac operator $D_{\tilde{B}}$ on $\mathbb{S}_{\tilde{L}}$.
(a) If $\rho = 0$ then

$$\eta_L = \frac{\ell}{6} - 2S(\vec{\beta}, \vec{\alpha}, \vec{\gamma}) - d(\vec{\beta}, \vec{\alpha}, \vec{\gamma})$$

(b) If $\rho \in (0, 1)$ then

$$\begin{aligned} \eta_L &= \frac{1}{2}(\deg K - \deg|K|)(1 - 2\rho) - \ell\rho(1 - \rho) + \frac{\ell}{6} + m\rho \\ &\quad -2\sum_{i=1}^{m} s(\beta_i, \alpha_i; \frac{\gamma_i + \beta_i\rho}{\alpha_i}, -\rho) - \sum_{i=1}^{m} F_\rho(\alpha_i, \beta_i, \gamma_i). \end{aligned} \tag{2.10}$$

where for $\rho \in (0, 1)$ and $\beta q \equiv 1$ mod α we defined

$$F_\rho(\alpha, \beta, \gamma) := \{\frac{q\gamma + \rho}{\alpha}\}.$$

We know two ways of proving Propositions 2.4 and 2.5. The first, more "brutal", method is based on the adiabatic results of Bismut-Cheeger-Dai, [**4, 5, 6**] and is discussed in great detail in [**18**]. The second method uses the adiabatic limit in a mild manner, but speculates in a clever way the identity (2.7). A similar idea was employed in [**4**] but in our special context we can be far more explicit. For details we refer to [**18, 19**]. This method allows the explicit description of the *whole eta functions* of the adiabatic operators in terms of the Riemann-Hurwitz function.

By now the reader may wonder why study the above Dirac operators in the first place. The motivation comes again from Seiberg-Witten theory.

3. Seiberg-Witten monopoles on Seifert fibrations

Suppose N is a Seifert fibration determined by the V-line bundle $\mathbb{L}_0 = L(\ell, \vec{\beta})$ of rational degree $\ell \neq 0$ over the V-surface $\Sigma(g, \vec{\alpha})$, $S^1 \hookrightarrow N \xrightarrow{\pi} \Sigma$ equipped with the metric described in the previous section. As background g-compatible connection on N we choose the adiabatic connection ∇^∞. We want to describe the (g, ∇^∞) monopoles on N. Recall that a monopole $\mathsf{C} = (\phi, A)$ is said to be *reducible* id $d(\mathsf{C}) > 0$ or equivalently, $\phi \equiv 0$. Otherwise the monopole is called *irreducible.*

The (g, ∇^∞)-monopoles were explicitly described in [**16**] and [**17**]. here are the relevant facts.

Fact 1. If $c_1(L)$ is not torsion then there are no monopoles corresponding to this *spin*^{c} structure.

Assume now that $c_1(L) = \kappa \in \mathrm{Pic}^t(\Sigma)/\mathbb{Z}[\mathbb{L}_0]$ and define

$$\tilde{R}_\kappa = \{E \in \mathrm{Pic}^t(\Sigma)\ ;\ 0 < |\deg E - \frac{1}{2}\deg K_\Sigma| \leq \frac{1}{2}\deg K_\Sigma,\ \ E \equiv \kappa \bmod \mathbb{Z}[\mathbb{L}_0]\}.$$

For each $E \in \tilde{R}_\kappa$ set $\nu(E) = \deg E - \frac{1}{2}\deg K_\Sigma$. We will often identify $\tilde{R}_\kappa$ with its image in $\mathbb{Q}$ via ν. Now set

$$R_\kappa^- = \{E \in \tilde{R}_\kappa\ ;\ \nu(E) < 0,\ \ \deg|E| \geq 0\}$$

$$R_\kappa^+ = \{E \in \tilde{R}_\kappa\ ;\ \nu(E) > 0,\ \ \deg|K_\Sigma - E| \geq 0\}$$

and

$$R_\kappa = R_\kappa^- \cup R_\kappa^+.$$

Fact 2. Any irreducible monopole (ϕ, A) is gauge equivalent to the pullback of a pair $(\tilde{\phi}, B)$ where B is a connection in a V-line bundle $E \to \Sigma$ in R_κ and $\tilde{\phi} = \tilde{\phi}_- \oplus \tilde{\phi}_-$ is a section of $C^\infty(K^{-1} \otimes E \oplus E)$. The connection B defines holomorphic structures in $K^{-1} \otimes E$ and E. $\tilde{\phi}_-$ is an anti-holomorphic section of $K^{-1} \otimes E$ while $\tilde{\phi}_+$ is a holomorphic section of E. Moreover, exactly one of $\tilde{\phi}_-$ or $\tilde{\phi}_+$ is zero according to the identity:

$$\frac{1}{4\pi}\int_\Sigma (|\tilde{\phi}_-|^2 - |\tilde{\phi}_+|^2)\, dv = \nu(E) \neq 0.$$

Thus $\tilde{\phi}_+ = 0$ iff $\nu(E) > 0$ and $\tilde{\phi}_- = 0$ iff $\nu(E) < 0$. Pairs $(\tilde{\phi}_- \oplus \tilde{\phi}_+, B)$ as above are known as *vortex pairs* on Σ corresponding to the V-line bundle E. If $\nu(E) < 0$ we say we have a *holomorphic vortex on E* while if $\nu(E) > 0$ we say we have an *anti-holomorphic vortex* on E.

The irreducible part (mod $\mathfrak{G}_N$), denoted by $\mathfrak{M}^*$ consists of $\#R_\kappa$ components

$$\mathfrak{M}^* = \bigcup_{E\in R_\kappa} \mathfrak{M}_{\kappa,E} = \bigcup_{n\in\nu(R_\kappa)} \mathfrak{M}_{\kappa,n}.$$

The component $\mathfrak{M}_n = \mathfrak{M}_{\kappa,n}$ corresponding to a choice $\nu(E) = n < 0$ is diffeomorphic to a symmetric product of $\deg|E|$ copies of Σ. If $n = \nu(E) > 0$ the moduli space is isomorphic to a symmetric product of $\deg|K - E|$ copies of Σ. Each component is non-degenerate.

Fact 3. The components of the space of reducible monopoles are in an one-to-one correspondence with $\tilde{R}_\kappa$. The component corresponding to $E \in \tilde{R}_\kappa$ consists of pairs $(0, A)$ where A is a connection of the form

$$A = A_0 + \mathbf{i}\rho(E)\varphi$$

where A_0 is the pullback of a constant curvature connection on the canonical representative of E.

Modulo $\mathfrak{G}_N$ they form a space homeomorphic to the Jacobian $J(|\Sigma|)$ which is a $2g$-dimensional torus. If $\rho(E) \neq 0$ or $g = 0$ the associated reducible component is non-degenerate.

We see that the Dirac operators discussed in the previous section are precisely those arising when describing the (g, ∇^∞)-monopoles.

4. Virtual dimensions

We can now explain how to solve **Problem1,2,3** in the special case when N is a Seifert manifold equipped with the metric g_r and the background connection is the adiabatic connection ∇^∞. Observe first that the solution to **Problem 1** is contained in the previous section.

Problem 2 requires some more work. The adiabatic analysis in [**17**] allows one to deduce that $\ker \mathfrak{D}_A = 0$ and determine explicitly the spectral flow $SF(\mathfrak{D}_A \to D_A)$. Observe that

$$(1-t)\mathfrak{D}_A + tD_A = \mathfrak{D}_t := \mathfrak{D}_A + \frac{t\ell}{2}.$$

Invoking again the adiabatic analysis in [**17**] we conclude that $\ker \mathfrak{D}_t = 0$ for $t \in [0, 1)$ so the only contribution to the spectral flow arises for $t = 1$ and we have

$$SF(\mathfrak{D}_t) = \frac{1 + \operatorname{sign}(\ell)}{2} \dim_{\mathbf{C}} \ker D_A.$$

The kernel of D_A can be described explicitly using the decomposition $D_A = Z_A + T_A$.

To compute the eta invariant of $\mathfrak{D}_A$ we will use variational formulæ for the eta invariants.

Set $h_t := \dim_{\mathbf{C}} \ker \mathfrak{D}_t$ and $\xi_t := \frac{1}{2}(h_t + \eta(\mathfrak{D}_t))$. We denote by $[\xi_t]$ the image of ξ_t in $\mathbb{R}/\mathbb{Z}$. Then $[\xi_t]$ depends smoothly on t and we have the following result

$$\xi_1 - \xi_0 = SF(\mathfrak{D}_t) + \int_0^1 \frac{d[\xi_t]}{dt} dt.$$

The continuous variation $\frac{d}{dt}[\xi_t]$ is a local quantity. More precisely using Thm. 1.13.2 in [**12**] we deduce that

$$\frac{d[\xi_t]}{dt} = -\frac{\ell}{2\sqrt{\pi}} a_2(\mathfrak{D}_t^2)$$

where for every $u \in [0,1]$ and $j \geq 0$ the density $a_j(\mathfrak{D}_u^2)$ is determined from the heat kernel asymptotics

$$Tr(\exp(-t(\mathfrak{D}_u^2) \sim \sum_{j\geq 0} a_j(\mathfrak{D}_u^2) t^{(j-3)/2}, \;\; t \to 0.$$

The term a_2 is explicitly described in [**12**] (watch out the curvature conventions there) and, after some elementary manipulations we conclude

$$\frac{d[\xi_t]}{dt} = -\frac{\ell}{16\pi^2}\int_N (\frac{s(g)}{6} - \ell^2 t^2) dv_g.$$

The scalar curvature $s(g)$ is explicitly computed in [**17**] and the end result is

$$\int_0^1 \frac{d[\xi_t]}{dt} dt = -\frac{\ell}{12}(\ell^2 - \chi)$$

where $\chi := -\deg K_\Sigma$. Thus the eta invariant of $\mathfrak{D}_A$ is explicitly computable. The eta invariant $\eta_{sign}(g)$ was computed in [**21**]. This solves **Problem 2**.

Problem 3 is analytically the most complicated one. The main reason is that the eigenvalues changing signs in the family $(1-t)\tilde{\mathfrak{H}}_{\mathbb{C}}^0 + t\tilde{\mathfrak{H}}_{\mathbb{C}}$ do so in a non-transverse manner and detecting them requires a delicate perturbation analysis *à la* [**8**]. For details we refer to [**18**].

REMARK 4.1. The virtual dimension problem was solved by entirely different methods in [**16**] in the special case when $M_\infty \cong \mathbb{R} \times N$. Although their final formulæ look quite different from ours, numerical experimentations show perfect agreement.

5. Applications

The above computations have nontrivial applications to low dimensional topology. For the reader's convenience we include here a brief description of the Froyshov invariant of a rational homology sphere. For more details we refer to the original source, [**9**].

Suppose N is rational homology sphere equipped with a Riemann metric g and σ is a *spin*^{c} structure on N. Denote by $\mathbb{S}_\sigma$ the bundle of complex spinors associated to σ and set $\det\sigma = \det\mathbb{S}_\sigma$. The metric g is said to be *good* iff the following hold.

- The irreducible solutions of $SW(g,\sigma)$ are nondegenerate for all σ.
- If $\theta = (\psi = 0, A_\sigma)$ is the reducible solution of $SW(g,\sigma)$ then $\ker \mathfrak{D}_{A_\sigma} = 0$ where $\mathfrak{D}_{A_\sigma}$ denotes the Dirac operator on $\mathbb{S}_\sigma$ coupled with the *flat* connection A_σ on $\det\sigma$.
- If nonempty, the spaces of gradient flow lines (of the 3-dimensional Seiberg-Witten energy functional) which connect irreducible solutions form smooth moduli spaces of the correct dimension.

For any irreducible solution α of $SW(g,\sigma)$ denote by $i(\alpha,\theta)$ the virtual dimension of the space of tunnelings (= connecting gradient flow lines) from α to θ. Define $m = m(g,\sigma)$ as the smallest nonnegative integer such that there are no tunnelings $\alpha \to \theta$ with $i(\alpha,\theta) = 2m+1$. Now set

$$\mathbf{Froy}(N,g,\sigma) := 8m(g,\sigma) + 4\eta(\mathfrak{D}_{A_\sigma}) + \eta_{sign}(g)$$

where $\eta_{sign}(g)$ denotes the eta invariant of the odd-signature operator on N determined by the metric g. Observe that $4\eta(\mathfrak{D}_{A_\sigma}) + \eta_{sign}(g)$ is precisely the quantity $\mathbf{F}(g, A)$ of **Problem 2**.

In [**9**] it was shown the quantity

$$\mathbf{Froy}(N, \sigma) := \inf\{\mathbf{Froy}(N, g, \sigma);\ g \text{ is good}\}$$

is *finite*. Now define the Froyshov invariant of N by

$$\mathbf{Froy}(N) := \max_{\sigma} \mathbf{Froy}\,(N, \sigma).$$

To explain the relevance of this invariant in topology we need to introduce another, arithmetic invariant.

Consider a negative definite integer quadratic form q defined on a lattice Λ. Set $\Lambda^\sharp := \mathrm{Hom}(\Lambda, \mathbb{Z})$. The quadratic form induces a morphism

$$I_q : \Lambda \to \Lambda^\sharp$$

and since q is nondegenerate the sublattice $I_q(\Lambda)$ has finite index δ_q in $\Lambda^\sharp$. q induces a *rational* quadratic form $q^\sharp$ on $\Lambda^\sharp$ defined by the equality

$$q^\sharp(\xi_1, \xi_2) := \frac{1}{\delta_q}\langle\, \xi_1\,,\, I_q^{-1}(\delta_q \xi_2)\,\rangle$$

where $\langle\, ?\,,\, ?\,\rangle : \Lambda^\sharp \times \Lambda \to \mathbb{Z}$ denotes the natural pairing. A vector $\xi \in \Lambda^\sharp$ is called *characteristic* if

$$\langle \xi, x\rangle \equiv q(x, x) \mod 2, \ \ \forall x \in \Lambda.$$

We define the *Elkies invariant* of q by the equality

$$\Theta(q) := \mathrm{rank}(q) + \max\{q^\sharp(\xi, \xi);\ \xi \text{ characteristic vector of } q\}$$

Note that if q is an even, negative definite form then

$$\Theta(q) = \mathrm{rank}(q) \tag{5.1}$$

since in this case $\xi = 0$ is a characteristic vector. A result of Elkies ([**7**]) states that if q is a negative definite, *unimodular* quadratic form then $\Theta(q) \geq 0$ *with equality if and only if q is diagonal.*

THEOREM 5.1. **(Froyshov, [9])** *If X is a smooth, oriented, negative definite 4-manifold bounding the rational homology sphere N then*

$$\Theta(q_X) \leq \mathbf{Froy}(N)$$

where q_X denotes the intersection form of X. In particular, if N is an integral homology sphere and $\mathbf{Froy}(N) \leq 0$*, then q_X must be diagonal.*

A large source of examples of homology spheres is the class of Brieskorn spheres. More precisely, for every vector $\vec{a} = (a_1, a_2, a_3) \in \mathbb{Z}_+^3$ with mutually coprime entries define

$$Z(\vec{a}) = \{\sum_i z_i^{a_i} = 0\} \subset \mathbb{C}^3.$$

The Brieskorn sphere $\Sigma(\vec{a})$ is defined as the intersection of $Z(\vec{a})$ with the unit sphere in $\mathbb{C}^3$. It is both a homology sphere and a Seifert manifold. For example, $\Sigma(2, 3, 5)$ is the celebrated Poincaré sphere. Froyshov has shown in [**9**] that

$$\mathbf{Froy}(\Sigma(2, 3, 5)) = 8.$$

In [**19**] we describe an algorithm which produces upper bounds for all $\mathbf{Froy}(\Sigma(\vec{a}))$. Surprisingly, in each concrete computation we have performed we could also show by ad-hoc methods that these bounds are optimal. For example, we have shown that

$$\mathbf{Froy}(\Sigma(2,3,6k+1)) = 0, \ \ \mathbf{Froy}(\Sigma(2,3,6k-1)) = 8.$$

The second equality generalizes the earlier result of Froyshov. The first equality implies that any negative definite manifold which bounds $\Sigma(2,3,6k+1)$ must have diagonalizable intersection form.

As explained in [**19**], one can use Donaldson's theorem on negative definite smooth 4-manifolds to independently verify this conclusion for all Brieskorn spheres which bound a smooth contractible 4-manifold. $\Sigma(2,3,13)$ and $\Sigma(2,3,25)$ are such examples. On the other hand, the Brieskorn spheres $\Sigma(2,3,6k+1)$ with k odd *do not bound smooth contractible manifolds* (because they have nontrivial Rohlin invariant) so this conclusion is beyond the reach of Donaldson's theorem.

We believe the above identities seem to be special cases of a deeper phenomenon. Let us first mention that the isolated singularity of $Z(\vec{a})$ can be resolved leading to a minimal smooth complex surface $R(\vec{a})$ which bounds $\Sigma(\vec{a})$. The intersection form $q_{\vec{a}}$ of $R(\vec{a})$ is negative definite and thus, by Froyshov's theorem

$$\Theta(q_{\vec{a}}) \leq \mathbf{Froy}\ (\Sigma(\vec{a}). \tag{5.2}$$

In [**19**] we have shown that $\Theta(q_{\vec{a}}) \geq \mathbf{Froy}\ (\Sigma(\vec{a})$ for many infinite families of $\vec{a}$'s such as

$$\vec{a} \in \{(2,3,6k\pm 1), (2,4k+1,4k+3), (3,3k+1,3k+2)\}.$$

In fact, numerical experiments indicate that in (5.2) we should have equality for all $\vec{a}'$s. To place this fact into some conceptual perspective we need to mention a conjecture of N. Elkies.

Define an equivalence relation " $\sim$ " on the space of unimodular negative definite forms by setting

$$q_1 \sim q_2 \Longleftrightarrow q_1 \oplus \text{diagonal negative definite} \cong q_2 \oplus \text{diagonal negative definite}.$$

Denote by $[q]$ the equivalence class of q and by $\mathcal{N}$ the set of equivalence classes of " $\sim$ ". Since $\Theta(q \oplus \text{diagonal}) = \Theta(q)$ we deduce that Θ defines a map

$$\Theta : \mathcal{N} \to \mathbb{Z}_+.$$

N. Elkies conjectured in [**7**] that for every $k \in \mathbb{Z}_+$ the level sets

$$\mathcal{N}^k = \{[q];\ \ \Theta([q]) \leq k\}$$

is finite. (This is known to be true for $k \leq 24$; see [**11**].) Loosely speaking, this conjecture says that Θ should be viewed as a measure of complexity of an unimodular, negative definite intersection form q: if $\Theta(q) \leq k$ there can only be finitely many choices for q. Observe that Θ achieves its minimum on the simplest intersection form, the diagonal form.

Using Theorem 5.1 we can now interpret the equality $\Theta(q_{\vec{a}}) = \mathbf{Froy}\ (\Sigma(\vec{a}))$ by saying that *the minimal resolution $R(\vec{a})$ is "the most complicated" negative definite smooth* 4*-manifold which bounds $\Sigma(\vec{a})$!!!*

We believe this to be the case for arbitrary isolated singularities whose links are homology spheres. In particular, we want to formulate the following weaker conjecture.

Conjecture *Suppose the integral homology sphere N is the link of an isolated complex singularity whose minimal resolution has diagonalizable intersection form. Then any negative definite* 4*-manifold which bounds N must have diagonalizable intersection form.*

Finally, let us mention that in [**20**] we have used the techniques outlined in the present survey to investigate the invariants of lens spaces obtained from Seiberg-Witten theory. This is a difficult subject since on a rational homology sphere all conceivable monopole counts are metric dependent . In order to obtain genuine *topological* invariants, we have to correct these counts by other metric dependent quantities, such as eta invariants. We were pleasantly surprised to see that a refined version of Meng-Taubes theorem, [**14**], continues to hold, at least for lens spaces. More precisely, in [**20**] we showed that the Seiberg-Witten theory determines the Milnor-Reidemeister-Turaev torsion and the Casson-Walker invariant of a lens space.

Lens spaces are also links of isolated singularities. In [**20**] we showed that for many infinite families of lens spaces, their Froyshov invariants are equal to the Elkies' invariants of their minimal resolutions.

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