THE (CO)NORMAL CYCLE AND CURVATURE MEASURES

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ABSTRACT. We survey the construction of a few applications of the normal cycle of a "reasonable" subset of an oriented Euclidean space.

CONTENTS

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Notation and conventions

- We set
$$\mathbb{N} := \mathbb{Z}_{>0}, \quad \mathbb{N}_0 := \mathbb{Z}_{\geq 0}.$$
- We denote by $\boldsymbol{\omega}_k$ the volume of the unit k-dimensional Euclidean ball
$$\boldsymbol{\omega}_k = \frac{\pi^{\frac{k}{2}}}{\Gamma\big(1+\frac{k}{2}\big)}.$$
We list below the values of $\boldsymbol{\omega}_n$ for small n.

n	0	1	2	3	4
$\boldsymbol{\omega}_n$	1	2	π	$\frac{4\pi}{3}$	$\frac{\pi^2}{2}$

- We denote by $\boldsymbol{\sigma}_{k-1}$ the "area" of the unit sphere $S^{k-1} \subset \mathbb{R}^k$,
$$\boldsymbol{\sigma}_{k-1} = k\boldsymbol{\omega}_k = \frac{2\pi^{\frac{k}{2}}}{\Gamma\big(\frac{k}{2}\big)}.$$
- For any set S, contained in some ambient space X, we denote by I_S the *indicator function* of S
$$I_S : X \to \{0,1\}, \quad I_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \notin S. \end{cases}$$
- Let $k \in \mathbb{N} \cup \{\infty\}$. For any C^k-manifold M we denote by $\mathrm{Vect}(M)$ the space of C^k -vector fields on M.
- We orient the boundary of an oriented manifold using the *outer-normal-first* convetion. We orient the total space of a fiber bundle using the *fiber-first* convention.

1. The (co)-normal bundle of submanifolds

1.1. **The set-up.** Let $\boldsymbol{V}$ be a finite dimensional Euclidean space. We set $N := \dim \boldsymbol{V}$, and we denote by $(-,-)$ the Euclidean inner product on $\boldsymbol{V}$, and by $|-|$ the associated norm.

We denote by $\boldsymbol{V}^*$ the dual of $\boldsymbol{V}$ and by $\langle -,-\rangle$ the natural bilinear pairing

$$\langle -,-\rangle : \boldsymbol{V}^* \times \boldsymbol{V} \to \mathbb{R},\ \ \langle \boldsymbol{\xi}, \boldsymbol{x}\rangle = \boldsymbol{\xi}(\boldsymbol{x}),\ \ \forall (\boldsymbol{\xi}, \boldsymbol{x}) \in \boldsymbol{V}^* \times \boldsymbol{V}.$$

The metric $(-,-)$ defines a lowering-the-indices isomorphism

$$-^{\downarrow} : \boldsymbol{V} \to \boldsymbol{V}^*,\ \ \langle \boldsymbol{u}^{\downarrow}, \boldsymbol{v}\rangle = (\boldsymbol{u}, \boldsymbol{v}),\ \ \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}.$$

Its inverse, the raising-the-indices isomorphism $\boldsymbol{V}^* \ni \boldsymbol{\xi} \to \boldsymbol{\xi}^{\uparrow} \in \boldsymbol{V}$ is defined by the equality

$$(\boldsymbol{\xi}^{\uparrow}, \boldsymbol{u}) = \langle \boldsymbol{\xi}, \boldsymbol{v}\rangle,\ \ \forall \boldsymbol{v} \in \boldsymbol{V}.$$

The duality isomorphism $\boldsymbol{V} \to \boldsymbol{V}^*$ induces an inner product $(-,-)$ on $\boldsymbol{V}^*$.

Observe that $\boldsymbol{V}^* \times \boldsymbol{V}$ can be identified with the cotangent bundle of $\boldsymbol{V}$ and, as such, it is equipped with a canonical 1-form $\alpha \in \Omega^1(\boldsymbol{V}^* \times \boldsymbol{V})$. If we choose linear coordinates (x^i) on $\boldsymbol{V}$ with dual coordinates (ξ_i) on $\boldsymbol{V}^*$, then

$$\alpha = \sum_{i=1}^N \xi_i dx^i$$

We denote by ω the canonical symplectic form on $\boldsymbol{V}^* \times \boldsymbol{V}$

$$\omega = d\alpha = \sum_{i=1}^N d\xi_i \wedge dx^i.$$

We orient $\boldsymbol{V}^* \times \boldsymbol{V}$ using the volume form

$$\frac{(-1)^{\frac{N(N-1)}{2}}}{N!}\omega^{\wedge N} = d\xi_1 \wedge \cdots \wedge d\xi_N \wedge dx^1 \wedge \cdots \wedge dx^N.$$

We denote by $S(\boldsymbol{V}^*)$ the unit sphere in $\boldsymbol{V}^*$, and by $S(\boldsymbol{V})$ the unit sphere in $\boldsymbol{V}$.

1.2. **The (co)normal bundle of a submanifold.** Suppose that $X \subset \boldsymbol{V}$ is a submanifold. The *conormal subbundle* of X is the bundle $T^*_X\boldsymbol{V} \to X$ given by

$$\boxed{T^*_X\boldsymbol{V} := \big\{ (\boldsymbol{\xi}, \boldsymbol{x}) \in \boldsymbol{V}^* \times X;\ \ \langle \boldsymbol{\xi}, \boldsymbol{v}\rangle = 0,\ \ \forall \boldsymbol{v} \in T_{\boldsymbol{x}}X \big\}}.$$

The projection $T^*_X\boldsymbol{V} \to X$ is induced from the natural projection $\boldsymbol{V}^* \times X \to X$. The *normal bundle* of X in $\boldsymbol{V}$, denoted by $T_X\boldsymbol{V}$ is the image of $T^*_X\boldsymbol{V}$ via the isometry

$$T^*\boldsymbol{V} \subset \boldsymbol{V}^* \times X \ni (\boldsymbol{\xi}, \boldsymbol{x}) \mapsto (\boldsymbol{\xi}_{\uparrow}, \boldsymbol{x}) \in \boldsymbol{V} \times X = TV. \tag{1.1}$$

More precisely,

$$\boxed{T_X\boldsymbol{V} = \big\{ (\boldsymbol{v}, \boldsymbol{x}) \in \boldsymbol{V} \times \boldsymbol{V};\ \ \boldsymbol{x} \in X,\ \ \boldsymbol{v} \perp T_{\boldsymbol{x}}X \big\}}.$$

Proposition 1.1. *Suppose that X is a smooth submanifold of $\boldsymbol{V}$, not necessarily orientable.*

(i) *The conormal bundle $T^*_X\boldsymbol{V}\to X$ is an* exact Lagrangian *submanifold of $\boldsymbol{V}^*\times\boldsymbol{V}$, i.e., the restriction of α to $T^*_X\boldsymbol{V}$ is trivial.*

(ii) *An orientation on $\boldsymbol{V}$ induces canonically an orientation on the total space $T^*_X\boldsymbol{V}$.*

Proof. (i) Indeed, if $t\mapsto\big(\boldsymbol{\xi}(t),\boldsymbol{x}(y)\big)$, is a smooth path on $T^*_X\boldsymbol{V}$, then

$$\alpha\big(\dot{\boldsymbol{\xi}}(t)\oplus\dot{\boldsymbol{x}}(y)\big)=\sum_i\xi_i(t)\dot{x}^i(t)=\big\langle\,\boldsymbol{\xi}(t),\dot{\boldsymbol{x}}(t)\,\big\rangle=0.$$

(ii) To prove the second statement, it suffices to construct a natural orientation on the total space of the normal bundle T_XV. Consider the geodesic map

$$\mathbf{Exp}:T\boldsymbol{V}=\boldsymbol{V}\times\boldsymbol{V}\to\boldsymbol{V},\ \ \mathbf{Exp}(\boldsymbol{v},\boldsymbol{x})=\boldsymbol{v}+\boldsymbol{x}.$$

It defines a diffeomorphism from an open neighborhood $\mathcal{U}$ of the zero section in $T_X\boldsymbol{V}$ onto an open neighborhood U of $X\in\boldsymbol{V}$. The orientation on $\boldsymbol{V}$ defines an orientation on $\mathcal{U}$ that extends to an orientation on $T^*_X\boldsymbol{V}$.

□

☞ *In the sequel we will assume that $\boldsymbol{V}$ is equipped with an orientation.*

The 1-form α defines a contact form when restricted to the cotangent unit sphere bundle $S(T^*\boldsymbol{V})=S(\boldsymbol{V}^*)\times\boldsymbol{V}$, i.e., the restriction of $\alpha\wedge(d\alpha)^{\wedge(N-1)}$ to $S(T^*\boldsymbol{V})$ is a volume form.

Via the isometry $\boldsymbol{V}^*\to\boldsymbol{V}$ described in (1.1) it defines a contact form η on the tangent unit sphere bundle $S(T\boldsymbol{V})=S(\boldsymbol{V})\times\boldsymbol{V}$. More precisely, if $(\boldsymbol{v},\boldsymbol{x})\in S(\boldsymbol{V})\times X$ and $(\dot{\boldsymbol{v}},\dot{\boldsymbol{x}})\in T_{(\boldsymbol{v},\boldsymbol{x})}S(T\boldsymbol{V})$, then

$$\eta_{(\boldsymbol{v},\boldsymbol{x})}\big(\dot{\boldsymbol{v}},\dot{\boldsymbol{x}}\big)=(\boldsymbol{v},\dot{\boldsymbol{x}}).$$

We denote by $S(T_X\boldsymbol{V})$ the normal unit sphere bundle of X in $\boldsymbol{V}$, i.e.

$$S(T_X\boldsymbol{V}):=T_X\boldsymbol{V}\cap\big(\,S(\boldsymbol{V})\times\boldsymbol{V}\,\big)=\big\{\,(\boldsymbol{v},\boldsymbol{x})\in\boldsymbol{V}\times X;\ \ |\boldsymbol{v}|=1,\ \boldsymbol{v}\perp T_{\boldsymbol{x}}X\,\big\}.$$

Note that $S(T_X\boldsymbol{V})$ is the boundary of the normal unit disk bundle

$$B(T_X\boldsymbol{V}):=T_X\boldsymbol{V}\cap\big(\,B(\boldsymbol{V})\times\boldsymbol{V}\,\big)=\big\{\,(\boldsymbol{v},\boldsymbol{x})\in\boldsymbol{V}\times X;\ \ |\boldsymbol{v}|\leq 1,\ \boldsymbol{v}\perp T_{\boldsymbol{x}}X\,\big\}.$$

The natural orientation on $B(T_X\boldsymbol{V})$ induces an orientation on its boundary $S(T_X\boldsymbol{V})$ via the *outer-normal-first* convention. Proposition 1.1 shows that the restriction of η on $T_X\boldsymbol{V}$ is trivial. Hence η restricts to a trivial form on $S(T_X\boldsymbol{V})$. Since

$$\dim S(T\boldsymbol{V})=2N-1,\ \ \dim S(T_X\boldsymbol{V})=N-1,$$

we deduce the following result.

Proposition 1.2. *The normal unit sphere bundle $S(T_X\boldsymbol{V})$ is a Legendrian submanifold of $S(T\boldsymbol{V})$.* □

Let us observe that the normal bundle is obtained from the normal unit tangent bundle via a coning construction. More precisely, $T_X\boldsymbol{V}$ is the image of $[0,\infty)\times S(T_X\boldsymbol{V})$ via the map

$$\mathcal{C}:[0,\infty)\times S(\boldsymbol{V})\times\boldsymbol{V}\to\boldsymbol{V}\times\boldsymbol{V},\;\;(t,\boldsymbol{v},\boldsymbol{x})\mapsto(t\boldsymbol{v},\boldsymbol{x}).$$

1.3. **Morse theoretical properties of the conormal bundle.** As is well-known, the differential of any smooth function $f:\boldsymbol{V}\to\mathbb{R}$ defines a Lagrangian submanifold of T^*V

$$\boxed{\Gamma_{df}=\Big\{\,(\,df(\boldsymbol{v}),\boldsymbol{v}\,)\in\boldsymbol{V}^*\times\boldsymbol{V},\;\;\boldsymbol{v}\in\boldsymbol{V}\,\Big\}}.$$

The projection

$$\Gamma_{df}\ni\big(\,df(\boldsymbol{v}),\boldsymbol{v}\,\big)\mapsto\boldsymbol{v}\in\boldsymbol{V}$$

is a diffeomorphism and thus, the orientation on $\boldsymbol{V}$ induces an orientation on Γ_{df}.

Suppose that we are given a smooth function $f:\boldsymbol{V}\to\mathbb{R}$ and a compact smooth submanifold $X\subset\boldsymbol{V}$. We can associate to these objects two oriented Lagrangian submanifolds Γ_{-df} and $T^*_X\boldsymbol{V}$.[1]

Let us observe that if the point $(\boldsymbol{\xi},\boldsymbol{x})$ belongs to both the above Lagrangians if and only if $\boldsymbol{x}\in X$ and the restriction of f to X has a critical point at $\boldsymbol{x}$. In fact, we can say a bit more.

Proposition 1.3 (R. MacPherson)**.** *Let*

$$\hat{\boldsymbol{p}}:=(\boldsymbol{\xi},\boldsymbol{x})\in(T^*_X\boldsymbol{V})\cap\Gamma_{-df}.$$

Then the following hold.

(i) *The conormal bundle $T^*_X\boldsymbol{V}$ intersects Γ_{-df} transversally at $\hat{\boldsymbol{p}}$ inside $\boldsymbol{V}^*\times\boldsymbol{V}$ if and only if $\boldsymbol{x}$ is a* nondegenerate *critical point of $f\big|_X$.*

(ii) *If $T^*_X\boldsymbol{V}$ intersects Γ_{-df} transversally at $\hat{\boldsymbol{p}}$ inside $\boldsymbol{V}^*\times\boldsymbol{V}$, then the local intersection number of these two submanifolds at $\hat{\boldsymbol{p}}$ is*

$$\boxed{i\big(\,(T^*_X\boldsymbol{V})\bullet\Gamma_{-df},\,\hat{\boldsymbol{p}},\,\boldsymbol{V}^*\times\boldsymbol{V}\,\big)=(-1)^{\lambda(f|_X,\boldsymbol{x})}},$$

where $\lambda(f|_X,\boldsymbol{x})$ is the Morse index of $f|_X$ at $\boldsymbol{x}$.

(iii) *If the restriction of f to X is a Morse function, then the intersection number of $(T^*_X\boldsymbol{V})$ with Γ_{-df} (in this order) is equal to $\chi(X)$, the Euler characteristic of X*

$$\boxed{(T^*_X\boldsymbol{V})\bullet\Gamma_{-df}=(-1)^N\Gamma_{-df}\bullet(T^*_X\boldsymbol{V})=\chi(X)}.$$

□

Let us mention a special case of the above fact. Suppose that $\xi\in S(\boldsymbol{V}^*)$. The graph of its differential is

$$G_\xi=\{\xi\}\times\boldsymbol{V}\subset S(\boldsymbol{V}^*)\times\boldsymbol{V}$$

Note that G_ξ intersects $T^*_X\boldsymbol{V}$ transversally in $\boldsymbol{V}^*\times\boldsymbol{V}$ if and only if it intersects $S(T^*_X\boldsymbol{V})$ transversally inside $S(\boldsymbol{V}^*)\times\boldsymbol{V}$. We deduce that, if $\hat{\boldsymbol{p}}=(\xi,\boldsymbol{x})\in G_\xi\cap S(T^*_X\boldsymbol{V})$ is a

[1]The choice of Γ_{-df} instead Γ_{df} will simplify various signs that will appear in the sequel.

transverse intersection point, then the local intersection number of $(T^*_X\boldsymbol{V})$ and G_ξ at $\hat{\boldsymbol{p}}$ in $S(\boldsymbol{V}^*)\times\boldsymbol{V}$ is

$$\boxed{i\big(\,S(T^*_X\boldsymbol{V})\bullet G_\xi,\,\hat{\boldsymbol{p}},\,S(\boldsymbol{V}^*)\times\boldsymbol{V}\,\big)=(-1)^{\lambda(-\xi|_X,\boldsymbol{x})}}. \tag{1.2}$$

1.4. **Weyl's tube formula.** Suppose that $X\subset\boldsymbol{V}$ is a compact, connected submanifold of $\boldsymbol{V}$. We set

$$m:=\dim X,\ \ c:=N-m=\operatorname{codim}X>0.$$

The *tube of radius* r around X is the set

$$\boxed{\mathbb{T}_r(X)=\big\{\,\boldsymbol{p}\in\boldsymbol{V}:\ \ \operatorname{dist}(\boldsymbol{p},\boldsymbol{V})\leq r\,\big\}}.$$

Weyl's tube formula describes the volume of $\mathbb{T}_r(X)$ for $r\ll 1$ in terms of *intrinsic* geometric invariants of X.

For $r>0$ sufficiently small the tube $\mathbb{T}_r(X)$ is a domain in $\boldsymbol{V}$ with smooth boundary and the map

$$\mathbf{Exp}_\nu:[0,\infty)\times S(T_X\boldsymbol{V})\to V,\ \ \mathbf{Exp}_\nu(t,\boldsymbol{v},\boldsymbol{x})=\boldsymbol{x}+t\boldsymbol{v}.$$

induces a diffeomorphism of $(0,r]\times S(T_X\boldsymbol{V})$ onto $\mathbb{T}_r(X)\setminus X$. Denote by $\Omega_{\boldsymbol{V}}$ the canonical volume form on $\boldsymbol{V}$ defined by by the Euclidean metric and the orientation on $\boldsymbol{V}$. We deduce that

$$\operatorname{vol}\big(\,\mathbb{T}_r(X)\,\big)=\int_{[0,r]\times S(T_X\boldsymbol{V})}\mathbf{Exp}_\nu^*\,\Omega_{\boldsymbol{V}}.$$

To proceed further we need to understand a bit better the N-form

$$\mathbf{Exp}_\nu^*\,\Omega_{\boldsymbol{V}}\in\Omega^N([0,\infty)\times S(\boldsymbol{V})\times\boldsymbol{V}).$$

Choose an oriented orthonormal basis $(\boldsymbol{e}_i)$ of $\boldsymbol{V}$. We obtain Euclidean coordinates v^i,x^j, $1\leq i,j\leq N$ on $T\boldsymbol{V}=\boldsymbol{V}\times\boldsymbol{V}$. Then

$$\begin{gathered}\Omega_{\boldsymbol{V}}=dx^1\wedge\cdots\wedge dx^N,\\ \mathbf{Exp}_\nu^*\,\Omega_{\boldsymbol{V}}=\big(\,d(tv^1)+dx^1\,\big)\wedge\cdots\wedge\big(\,d(tv^N)+dx^N\,\big)\\ =t^N(dv^1\wedge\cdots\wedge dv^N)\big|_{S(\boldsymbol{V})\times\boldsymbol{V}}+dt\wedge\sum_{i=0}^{N-1}t^{N-1-i}\kappa_i,\end{gathered}\tag{1.3}$$

where

$$\kappa_0,\kappa_1,\dots,\kappa_{N-1}\in\Omega^{N-1}\big(\,S(\boldsymbol{V})\times\boldsymbol{V}\,\big). \tag{1.4}$$

are independent of t. Let us point out that the form κ_i has degree i in the variables $dx^1,\dots,dx^N$. Note that

$$(dv^1\wedge\cdots\wedge dv^N)\big|_{S(\boldsymbol{V})\times\boldsymbol{V}}=0.$$

We have

$$\operatorname{vol}\big(\,\mathbb{T}_r(X)\,\big)=\int_{[0,r]\times S(T_X\boldsymbol{V})}dt\wedge\sum_{i=0}^{N-1}t^{N-1-i}\kappa_i=\sum_{i=0}^{N-1}\frac{r^{N-i}}{N-i}\int_{S(\,T_X\boldsymbol{V})}\kappa_i$$

$$= \sum_{i=0}^{N-1} \boldsymbol{\omega}_{N-i} r^{N-i} \int_{S(T_X \boldsymbol{V})} \frac{1}{\boldsymbol{\omega}_{N-i}(N-i)} \kappa_i = \sum_{i=0}^{N-1} \boldsymbol{\omega}_{N-i} r^{N-i} \int_{S(T_X \boldsymbol{V})} \frac{1}{\boldsymbol{\sigma}_{N-1-i}} \kappa_i,$$

where we recall that $\boldsymbol{\omega}_m$ denotes the volume of the unit ball in $\mathbb{R}^m$ $\boldsymbol{\sigma}_d = (d+1)\omega_{d+1}$ denotes the “area” of the d-dimensional unit sphere in $\mathbb{R}^{d+1}$.

To proceed further we observe that

$$\int_{S(T_X \boldsymbol{V})} \kappa_i = 0, \quad \forall i > \dim X = N - c.$$

Making the change in variables $i = m - j = N - c - j$ we deduce

$$\boxed{\operatorname{vol}\big(\mathbb{T}_r(X)\big) = \sum_{j=0}^{m} \boldsymbol{\omega}_{c+j} r^{c+j} \underbrace{\int_{S(T_X \boldsymbol{V})} \frac{1}{\boldsymbol{\sigma}_{c-1+j}} \kappa_{m-j}}_{=:\mu_{m-j}(X)} = \sum_{j=0}^{m} \boldsymbol{\omega}_{c+j} r^{c+j} \mu_{m-j}(X)}.$$

Theorem 1.4 (Weyl's tube formula)**.** *Suppose that $X \subset \boldsymbol{V}$ is a compact m-dimensional* submanifold. *Set $c = N - m = \operatorname{codim} X$. Denote by g the induced Riemann metric on X and by $|dV_g|$ the associated volume density. Then the following hold.*

(i) *For $r > 0$ sufficiently small* $\operatorname{vol}\big(\boldsymbol{T}_r(X)\big)$ *is a* polynomial of degree $\leq N$ in r,

$$\boxed{\operatorname{vol}\big(\mathbb{T}_r(X)\big) = \sum_{j=0}^{m} \mu_{m-j}(X) \boldsymbol{\omega}_{c+j} r^{c+j}}. \tag{1.5}$$

(ii) $\mu_{m-k}(X) = 0$ *if k is odd.*

(iii) *If $k = 2h \leq m$, then*

$$\boxed{\mu_{m-2h}(X) = \int_X P_h(R) |dV_g|},$$

where $P_h(R)$ is a universal *homogeneous polynomial of degree h in the entries of the Riemann curvature tensor R of the induced metric g.*

Weyl's original proof [32] is based on a clever use of his theory of invariants. For modern renditions of Wey's original proof we refer to [13, 20]. For a proof that does not rely on invariant theory we refer to [1].

The polynomials P_h are *very explicit*. To describe them we follow the very elegant approach in [1].

Recall that the Riemann curvature tensor at a point $\boldsymbol{x} \in X$ is an element $R(\boldsymbol{x})$ of the vector space $\operatorname{End}_-(T_{\boldsymbol{x}}X) \otimes \Lambda^2 T^*_{\boldsymbol{x}}X$, where $\operatorname{End}_-(T_{\boldsymbol{x}}X)$ denotes the vector space of skew-symmetric endomorphisms of $T_{\boldsymbol{x}}X$.

We have a canonical isomorphism

$$\operatorname{End}_-(T_{\boldsymbol{x}}X) \ni A \mapsto \omega_A \in \Lambda^2 T^*_{\boldsymbol{x}}X,$$

where

$$\omega_A(X, Y) = g(AX, Y), \quad \forall X, Y \in T_{\boldsymbol{x}}X.$$

Thus, we can view $R(\boldsymbol{x})$ as an element of the vector space $\Lambda^2 T^*_{\boldsymbol{x}} X \otimes \Lambda^2 T^*_{\boldsymbol{x}} X$. At this point we need to digress a bit to discuss some linear algebra constructions.

Digression 1.5. Suppose that $\boldsymbol{U}$ is a finite dimensional real vector space. For $0 \leq p, q \leq \dim \boldsymbol{U}$ we set

$$\Lambda^{p,q}(\boldsymbol{U}) := \Lambda^p \boldsymbol{U} \otimes \Lambda^q \boldsymbol{U}.$$

We have a natural product

$$\wedge : \Lambda^{p,q}(\boldsymbol{U}) \otimes \Lambda^{p',q'}(\boldsymbol{U}) \to \Lambda^{p+p',q+q'}(\boldsymbol{U})$$

uniquely determined by the requirement

$$(\alpha \otimes \beta) \wedge (\alpha' \otimes \beta') = (\alpha \wedge \alpha') \otimes (\beta \wedge \beta'),$$

$\forall \alpha \in \Lambda^p \boldsymbol{U}$, $\alpha' \in \Lambda^{p'} \boldsymbol{U}$, $\beta \in \Lambda^q \boldsymbol{U}$, $\beta' \in \Lambda^{q'} \boldsymbol{U}$.

If, additionally, the space $\boldsymbol{U}$ is equipped with an inner product g, this inner product induces inner products g_{Λ^p} on the exterior power $\Lambda^p \boldsymbol{U}$. In turn this induces traces

$$\operatorname{tr} : \Lambda^{p,p}(\boldsymbol{U}) \to \mathbb{R}, \;\; \operatorname{tr}\big(\, \alpha \otimes \beta \,\big) = g_{\Lambda^p}\big(\, \alpha, \beta \,\big). \qquad \square$$

Returning to the polynomials P_h that appear in Theorem 1.4(ii), they can be expressed in terms of the traces defined above. We have $R(\boldsymbol{x}) \in \Lambda^{2,2}(T^*_{\boldsymbol{x}} X)$ so

$$R(\boldsymbol{x})^{\wedge h} \in \Lambda^{2h,2h}(T^*_{\boldsymbol{x}} X)$$

Then (see [1])

$$\boxed{P_h\big(\, R(\boldsymbol{x}) \,\big) = \frac{1}{(2\pi)^h h!} \operatorname{tr}\big(\, -R(\boldsymbol{x}) \,\big)^{\wedge h}}. \tag{1.6}$$

For example, we deduce that (see [20])

$$\boxed{\mu_m(X) = \operatorname{vol}_m(X), \;\; \mu_{m-2}(X) = \frac{1}{4\pi} \int_X s_g(\boldsymbol{x}) \, |dV_g(\boldsymbol{x})|},$$

where s_g is the scalar curvature of the induced metric $g$on X. Thus, up to a multiplicative constant, $\mu_{m-2}(X)$ is the Hilbert-Einstein functional of the metric g.

When $m = \dim X$ is even, $m = 2m_0$, then the Gauss-Bonnet theorem implies that

$$\boxed{\mu_0(X) = \frac{1}{(2\pi)^{m_)} m_0!} \int_X \operatorname{tr}\big(\, -R(\boldsymbol{x}) \,\big)^{\wedge m} |dV_g(\boldsymbol{x})| = \chi(X)}.$$

Note that above no orientability assumption on X was imposed.

The quantities $\mu_k(X)$, $k = 0, 1, \dots, N-1$ are called the *curvature measures* or the *Lipschitz-Killing curvatures* of the *submanifold* X.

Example 1.6. Suppose that $\Sigma \subset \boldsymbol{V}$ is a compact, orientable 2-dimensional submanifold of $\boldsymbol{V}$ of genus g. In this case Weyl's tube formula reads

$$\boxed{\operatorname{vol}_N\big(\, \mathbb{T}_r(\Sigma) \,\big) = \operatorname{vol}_2(\Sigma) \boldsymbol{\omega}_{N-2} r^{N-2} + \chi(\Sigma) \boldsymbol{\omega}_N r^N}.$$

If $M \subset \boldsymbol{V}$ is a compact 3-dimensional submanifold of $\boldsymbol{V}$, then

$$\boxed{\operatorname{vol}_N\big(\, \mathbb{T}_r(M) \,\big) = \boldsymbol{\omega}_{N-3} r^{N-3} \operatorname{vol}_3(M) + \frac{\omega_{N-1} r^{N-1}}{4\pi} \int_M s_g(\boldsymbol{x}) \, |dV_g(\boldsymbol{x})|}. \qquad \square$$

Remark 1.7. Intuitively, the first order approximation for the volume of the tube $\mathbb{T}_r(X)$ is the volume of X multiplied by the volume of a normal ball of dimension c and radius r. The curvature measures can be viewed as describing higher order corrections. □

Definition 1.8. For *any* Riemann manifold (X, g), $\dim X = m$, not necessarily compact or embedded in a Euclidean space and any nonnegative integer h such that $2h \leq m$, we set

$$\boxed{\mu_{m-2h}(X, g) = \frac{1}{(2\pi)^h h!} \int_X \operatorname{tr}\big(-R_g(\boldsymbol{x})\big)^{\wedge h} |dV_g(\boldsymbol{x})|}, \tag{1.7}$$

whenever the above integrals are well defined. Above R_g denotes the Riemann curvature tensor of the metric g. □

Remark 1.9. Suppose that (X, g) is an m-dimensional Riemann manifold such the integral in the right-hand-side of (1.7 is well defined. Then, any open set $U \subset X$ is itself a Riemann manifold and $\mu_{m-2h}(U)$ is well defined. Moreover, the correspondence $U \mapsto \mu_{m-2h}(U)$ satisfies the inclusion-exclusion principle, i.e.,

$$\mu_{m-2h}\big(U \cup V\big) = \mu_{m-2h}\big(U\big) + \mu_{m-2h}\big(V\big) - \mu_{m-2h}\big(U \cap V\big),$$

for any open sets $U, V \subset X$. This last equality justifies the attribute *measure* attached to the μ_k-s.

Note also that

$$\mu_k(X, \lambda^2 g) = \lambda^k \mu_k(X, g), \;\; \forall \lambda > 0.$$

Thus, if we measure the distance in *meters*, then $\mu_k(X, g)$ is measured in *meters*^{k}. □

□

Our main goal is to explain how to define the curvature measure of singular subsets of a Euclidean space $\boldsymbol{V}$. The key to this is a simple formula we proved along the way.

Recall that we have defined the forms

$$\kappa_0, \kappa_1, \dots, \kappa_{N-1} \in \Omega^{N-1}\big(S(\boldsymbol{V}) \times \boldsymbol{V}\big),$$

via the equality

$$\partial_t \lrcorner \Big(\big(d(tv^1) + dx^1\big) \wedge \cdots \wedge \big(d(tv^N) + dx^N\big)\Big) = \sum_{i=0}^{N-1} t^{N-1-i}\kappa_i.$$

Then

$$\boxed{\mu_{m-j}(X) = \frac{1}{\boldsymbol{\sigma}_{N-1-(m-j)}} \int_{S(T_X\boldsymbol{V})} \kappa_{m-j}, \;\; \forall j = 1, \dots, m}. \tag{1.8}$$

Definition 1.10. We will refer to the forms

$$\kappa_0, \dots, \kappa_{N-1} \in \Omega^{N-1}\big(S(\boldsymbol{V}) \times \boldsymbol{V}\big)$$

as the *canonical forms* associated to the ambient space $\boldsymbol{V}$. Sometimes, when we want to emphasize the dependence on $\boldsymbol{V}$ we will write $\kappa_j^{\boldsymbol{V}}$ or κ_j^N instead of κ_j. They form a basis of the space of $SO(\boldsymbol{V})$ invariant $(N-1)$-forms on $S(T\boldsymbol{V}) = S(\boldsymbol{V}) \times \boldsymbol{V}$. □

The form κ_0 is sometimes referred to as the *global angular form* or *Gauss curvature form*, [24]

Example 1.11. Let us look at some low dimensional examples.

A. $N = 2$.

$$\begin{gathered}\big(\, d(tv^1) + dx^1\,\big) \wedge \big(\, d(tv^2) + dx^2\,\big) = \big(\, v^1dt + tdv^1 + dx^1\,\big) \wedge \big(\, v^2dt + tdv^2 + dx^2\,\big)\\ = t^2dx^1 \wedge dx^2 + tdt \wedge \underbrace{\big(- v^2dv^1 + v^1dv^2\big)}_{=:\kappa_0} + dt \wedge \underbrace{\big(v^1dx^2 - v^2dx^1\big)}_{=:\kappa_1}.\end{gathered}$$

To understand the forms κ_i better it is convenient to introduce polar coordinates in the (v^1, v^2)-plane,

$$v^1 = r\cos\theta,\ \ v^2 = r\sin\theta.$$

Then

$$\begin{gathered}dv^1 = \cos\theta dr - r\sin\theta\, d\theta,\ \ dv^2 = \sin\theta dr + r\cos\theta\, d\theta,\\ \kappa_0 = r^2d\theta,\ \ \kappa_0 = r\big(\,\cos\theta\, dx^2 - \sin\theta\, dx^1\,\big)\end{gathered}$$

Hence

$$\boxed{\kappa_0 = d\theta},\ \ \boxed{\kappa_1 = \cos\theta\, dx^2 - \sin\theta\, dx^1}.$$

B. $N = 3$.

$$\begin{gathered}\big(\, v^1dt + tdv^1 + dx^1\,\big) \wedge \big(\, v^2dt + tdv^2 + dx^2\,\big) \wedge \big(\, v^3dt + tdv^3 + dx^3\,\big)\\ = t^3dt \wedge dx^1 \wedge dx^2 \wedge dx^3\\ +t^2dt \wedge \big(\, v^1dv^2 \wedge dv^3 - v^2dv^1 \wedge dv^3 + v^3dv^1 \wedge dv^2\,\big)\\ +tdt \wedge v^1\big(dv^2 \wedge dx^3 + dx^2 \wedge dv^3\big)\\ -tdt \wedge v^2\big(\, dv^1 \wedge dx^3 + dx^1 \wedge dv^3\,\big)\\ +tdt \wedge v^3\big(dv^1 \wedge dx^2 + dx^1 \wedge dv^2\,\big)\\ +dt \wedge \big(\, v^1dx^2 \wedge dx^3 - v^2dx^1 \wedge dx^3 + v^3dx^1 \wedge dx^2\,\big)\\ = t^3dt\ \wedge dx^1 \wedge dx^2 \wedge dx^3\\ +t^2dt \wedge \underbrace{\big(\, v^1dv^2 \wedge dv^3 - v^2dv^1 \wedge dv^3 + v^3dv^1 \wedge dv^2\,\big)}_{=\kappa_0}\\ +tdt \wedge \underbrace{\Big(\big(- v^3dv^2 + v^2dv^3\big) \wedge dx^1 + \big(- v^1dv^3 + v^3dv^1\big) \wedge dx^2 + \big(- v^2dv^1 + v^1dv^2\big) \wedge dx^3 \Big)}_{=\kappa_1}\\ +dt \wedge \underbrace{\big(\, v^1dx^2 \wedge dx^3 - v^2dx^1 \wedge dx^3 + v^3dx^1 \wedge dx^2\,\big)}_{=\kappa_2}.\end{gathered}$$

The point of the above computations is to illustrate a certain reproducing property of the canonical forms. This is a microlocal manifestation of the reproducing properties of the curvature measure that we will discuss soon.

In general, the angular form κ_0 is independent of the x-coordinates. If we denote by $\vec{\boldsymbol{v}}$ the vector field

$$\vec{\boldsymbol{v}} := \sum_{i=1}^{N} v^i \partial_{v^i},$$

then

$$\kappa_0\big|_{\boldsymbol{S}(\boldsymbol{V})} = \vec{\boldsymbol{v}} \lrcorner \left(dv^1 \wedge \cdots \wedge v^N \right)\big|_{\boldsymbol{S}(\boldsymbol{V})},$$

and we see that the restriction of κ_0 to $\boldsymbol{S}(\boldsymbol{V})$ is the Euclidean area form. □

1.5. **Crofton formulæ.** The curvature measures satisfy certain *reproducing properties* which, among many other things, show that, the curvature measure μ_k of a compact submanifold of $\boldsymbol{V}$ is determined by the Euler characteristics of its submanifolds of codimension k. The Crofton formulæ express this fact in a precise fashion. To describe these formuæ we need to make a brief detour into the foundations of integral geometry.

- We denote by $\mathbf{Gr}_k(\boldsymbol{V})$ and respectively $\mathbf{Gr}^c(\boldsymbol{V})$ the Grassmannian of *vector subspaces* of $\boldsymbol{V}$ of dimension k and respectively codimension c.
- We denote by $\mathbf{Graff}_k(\boldsymbol{V})$ and respectively $\mathbf{Graff}^c(\boldsymbol{V})$ the Grassmannian of *affine subspaces* of $\boldsymbol{V}$ of dimension k and respectively codimension c.

Let $L \in \mathbf{Graff}^c(\boldsymbol{V})$. We define $L_\parallel \in \mathbf{Gr}^c(\boldsymbol{V})$ to be the unique codimension c vector space parallel to L, and we set

$$O(L) := L \cap (L_\parallel)^\perp.$$

Let $\mathcal{T} \to \mathbf{Gr}^c(V)$ be the tautological vector bundle over $\mathbf{Gr}^c(\boldsymbol{V})$. Denote by $\mathcal{T}^\perp$ the orthogonal complement of $\mathcal{T}$ in the trivial bundle over $\mathbf{Gr}^c(\boldsymbol{V})$ with fiber $\boldsymbol{V}$.

We have a natural diffeomorphism between $\mathbf{Graff}^c(\boldsymbol{V})$ and the total space of $\mathcal{T}^\perp$ that associates to each affine subspace L the point

$$O(L) := L \cap (L_\parallel)^\perp$$

situated in the fiber $(L_\parallel)^\perp$ of $\mathcal{T}^\perp$ over $L_\parallel$.

The orthogonal group $O(\boldsymbol{V})$ acts transitively on $\mathbf{Gr}^c(\boldsymbol{V})$ so, up to a multiplicative constant, there exists only one $O(\boldsymbol{V})$-invariant density on $\mathbf{Gr}^c(\boldsymbol{V})$. Following [17] we set

$$\begin{bmatrix} n \\ k \end{bmatrix} := \binom{n}{k} \frac{\boldsymbol{\omega}_n}{\boldsymbol{\omega}_k \boldsymbol{\omega}_{n-k}}. \tag{1.9}$$

where we recall that $\boldsymbol{\omega}_n$ denotes the volume of the unit n-dimensional Euclidean ball. Denote by $|d\nu| = |d\nu_{N,c}|$ the unique $O(\boldsymbol{V})$-invariant volume density on $\mathbf{Gr}^c(\boldsymbol{V})$ such that

$$\int_{\mathbf{Gr}^c(\boldsymbol{V})} |d\nu| = \begin{bmatrix} N \\ c \end{bmatrix} = \begin{bmatrix} N \\ N-c \end{bmatrix}.$$

In [20] we give an explicit, metric description of this density.

The total space of $\mathcal{T}^\perp$ is equipped with a volume density $d\tilde{\nu} = d\tilde{\nu}_{N,c}$ which is locally the product of the density $|d\nu|$ on $\mathbf{Gr}^c(\boldsymbol{V})$ and the metric density on the each fiber of $\mathcal{T}^\perp$. More precisely, $d\tilde{\nu}$ is uniquely determined by the equality for any compactly supported continuous function

$$\int_{\mathbf{Graff}^c(\boldsymbol{V})} f\big(\tilde{L}\big)|d\tilde{\nu}(\tilde{L})| = \int_{\mathbf{Gr}^c(\boldsymbol{V})} \left(\int_{L^\perp} f\big(\boldsymbol{p}+L\big)dV_{L^\perp}(\boldsymbol{p})\right)|d\nu(L)|,$$

for any compactly supported continuous function $f : \mathbf{Graff}^c(\boldsymbol{V}) \to \mathbb{R}$.

Theorem 1.12 (Crofton Formula)**.** *Suppose that X is a compact submanifold of $\boldsymbol{V}$ of dimension m. Then, for every $0 \leq p \leq m-k$ we have*

$$\begin{bmatrix} p+k \\ p \end{bmatrix} \mu_{p+k}(X) = \int_{\mathbf{Graff}^k(\boldsymbol{V})} \mu_p(L\cap X)\,|d\tilde{\nu}|(L). \tag{1.10}$$

In particular, for $p=0$, we deduce

$$\boxed{\mu_k(X) = \int_{\mathbf{Graff}^k(\boldsymbol{V})} \chi(L\cap X)|d\tilde{\nu}|(L)}. \tag{1.11}$$

For a proof we refer to [20, 25]. The equalities (1.10) are sometimes referred to as *reproducing formulaæ*. They are special cases of the so called *kinematic formulæ*, [9, 25].

Example 1.13. Suppose that C is a closed, smoothly embedded curve in $\mathbb{R}^2$. For simplicity, assume that C is contained in a disk of (large) radius R centered at the origin.

Observe that $\mathbf{Graff}^1(\mathbb{R}^2)$, the Grassmanian of affine lines in $\mathbb{R}^2$ can be identified topologically with the Möbius band. An affine line in $\mathbb{R}^2$ is uniquely determined by two parameters $\theta \in \mathbb{R}/\pi\mathbb{Z}$, $t \in \mathbb{R}$, so the line $L_{\theta,t}$ is described by the equation.

$$x\cos\theta + y\sin\theta = t.$$

The density $|d\theta dt|$ is $O(2)$ invariant. The Grassmanian $\mathbf{Gr}^1(\mathbb{R}^2)$ is the submanifold of $\mathbf{Graff}^1(\mathbb{R}^2)$ cut-out by the equation $t=0$. In this case

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\cdot\frac{\pi}{2\cdot 2} = \frac{\pi}{2},$$

The normalized density $d\tilde{\nu}_{2,1}$ is

$$d\tilde{\nu} = \frac{1}{2}|d\theta dt|.$$

For generic θ, t, the line $L_{\theta,t}$ intersectcs the curve C transversally in finitely many points and

$$\chi(L_{\theta,t}\cap C) = \#(L_{\theta,t}\cap C).$$

We deduce that

$$\text{length}\,(C) = \mu_1(C) = \frac{1}{2}\int_0^\pi \int_{\mathbb{R}} \#(L_{\theta,t}\cap C)dtd\theta = \frac{1}{2}\int_0^\pi\int_{-R}^R \#(L_{\theta,t}\cap C)dt\,d\theta. \qquad \square$$

Remark 1.14. We can take (1.11) as our starting point for the definition of $\mu_k(S)$ for any compact subset $S \subset \mathbb{R}^N$ and set

$$\bar{\mu}_k(S) := \int_{\mathbf{Graff}^k(\boldsymbol{V})} \chi(L \cap S)|d\tilde{\nu}|(L).$$

We deduce from the Mayer-Vietoris principle that $\bar{\mu}_k$ satisfies the inclusion exclusion-principle and thus we could take it as definition of curvature measure. H. Federer has shown in [6] that if S is a set with positive reach that the volumes of tubes of small radii around S are expressed by a formula identical to (1.5). □

2. Tame sets and singular Morse theory

To extend the concept of curvature measure to singular subsets of $\boldsymbol{V}$ we need to clearly delineate classes of singular sets that have "reasonable" behavior. Fortunately, the advances in model theory that began in the early 1980s will allow us to single out such classes.

2.1. Tame sets.

An $\mathbb{R}$-*structure* is a family $\mathcal{S} = \{\mathcal{S}^n\}_{n\in\mathbb{N}}$, where each $\mathcal{S}^n$, is a collection of subsets of $\mathbb{R}^n$ satisfying the following conditions.

- $\mathbf{E}_1$**:** For each n, the collection $\mathcal{S}^n$ contains all the real algebraic subsets of $\mathbb{R}^n$, i.e., the subsets described by finitely many polynomial equations.
- $\mathbf{E}_2$**:** For each n, the collection $\mathcal{S}^n$ contains all the closed affine half-spaces of $\mathbb{R}^n$.
- $\mathbf{P}_1$**:** For each n, $\mathcal{S}^n$ is closed under boolean operations, $\cup$, $\cap$ and complement.
- $\mathbf{P}_2$**:** If $A \in \mathcal{S}^m$, and $B \in \mathcal{S}^n$, then $A \times B \in \mathcal{S}^{m+n}$.
- $\mathbf{P}_3$**:** If $A \in \mathcal{S}^m$, and $T : \mathbb{R}^m \to \mathbb{R}^n$ is an affine map, then $T(A) \in \mathcal{S}^n$.

A set X is called $\mathcal{S}$-*definable* if $X \in \mathcal{S}^n$ for some n. A map $f : A \to B$ is called $\mathcal{S}$-*definable* if its graph is such.

The structure $\mathcal{S}$ is called *tame* if it satisfies the *tameness* or *o-minimality* condition singled out by A. Pillay and Ch. Steinhorn in [23].

- **T:** Any set $A \in \mathcal{S}^1$ is a *finite* union of open intervals (a, b), $-\infty \leq a < b \leq \infty$, and singletons $\{r\}$. □

A *tame category* is a category whose objects are the sets of a tame structure $\mathcal{S}$ and whose morphisms are the $\mathcal{S}$-definable continuous maps. We will refer to the sets in a tame structure $\mathcal{S}$ as $\mathcal{S}$-definable or *definable*, if no confusion is possible. Proving that a certain collection of subsets of Euclidean space forms a tame structure requires rather nontrivial techniques from model theory.

Example 2.1. (a) The collection of all semialgebraic sets is a tame structure. We denote it by $\mathcal{S}_{alg}$. Any structure, tame or not, contains $\mathcal{S}_{alg}$.

(b) A *restricted analytic function* is a function $f : \mathbb{R}^n \to \mathbb{R}$ which is identically zero outside an open ball B, while on the ball it coincides with the restriction of a real analytic function defined on an open set containing the ball. We denote by $\mathcal{S}_{an}$ the smallest structure containing $\mathcal{S}_{alg}$ and the graphs of restricted analytic functions. Work of Gabrielov, Hardt and Hironaka shows that $\mathcal{S}_{an}$ is tame. The sets in $\mathcal{S}_{an}$ are called

globally subanalytic. One can show that any globally subanalytic set admits stratifications with real analytic strata. For example, any geometric realization of a finite simplicial complex is a globally subanalytic set.
(c) Denote by $\mathcal{S}_{\exp}$ the smallest structure that contains $\mathcal{S}_{an}$ and the graph of the exponential function e^x. A groundbreaking work of A. Wilkie shows that $\mathcal{S}_{\exp}$ is tame. $\square$

Suppose that $\mathcal{S}$ is a tame category. Then any set that can be described using only the logical operators AND, OR, NOT, the quantifiers $\exists$, $\forall$ and sets known to belong to $\mathcal{S}$ are also sets in $\mathcal{S}$.

Example 2.2. (a). Suppose that $A \subset \mathbb{R}^n$ is an $\mathcal{S}$-definable set and $f : A \to \mathbb{R}^m$ is $\mathcal{S}$ definable . Then, for any definable set $B \subset \mathbb{R}^m$ the preimage $f^{-1}(B)$ is also definable. Indeed if $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denotes the natural projection, then

$$f^{-1}(B) = \pi\big(\Gamma_f \cap A \times B\big).$$

In particular, the diagonal

$$\Delta_A = \big\{ (a_1, a_2) \in A \times A : \; a_1 = a_2 \big\}$$

as the preimage of $\{0\}$ via the definable map $(a_1, a_2) \mapsto a_1 - a_2$.
(b) Suppose that $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, $C \subset \mathbb{R}^\ell$, $f : A \to B$ and $g : B \to C$ are $\mathcal{S}$-definable, then $g \circ f$ is also $\mathcal{S}$-definable. Indeed

$$\Gamma_{g\circ f} = \big\{ (a, c) \in A \times C : \;\; \exists b \in B \text{ such that } (a, b) \in \Gamma_f \text{ and } (b, c) \in \Gamma_g \big\}.$$

Consider the $\mathcal{S}$-definable set

$$X := \Gamma_f \times \Gamma_g \cap A \times \Delta_B \times C \subset A \times B \times B \times C.$$

Then $\Gamma_{g\circ f}$ is the image of X via the natural projection $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^n \times \mathbb{R}^\ell$.
(c) Suppose that $A \subset \mathbb{R}^n$ is $\mathcal{S}$-definable, then its Euclidean closure $\boldsymbol{cl}(A)$ is also $\mathcal{S}$-definable. Indeed,

$$\boldsymbol{cl}(A) = \big\{ x \in \mathbb{R}^n : \;\; \forall \varepsilon > 0 \; \exists a \in A; |a - x|^2 < \varepsilon^2 \big\}.$$

Consider the set

$$X := \big\{ (a, x, \varepsilon) \in A \times \mathbb{R}^n \times (0, \infty); \;\; |a - x|^2 - \varepsilon^2 < 0 \big\}$$

The set X is definable as the preimage of $(-\infty, 0)$ via the polynomial map

$$(a, x, \varepsilon) \mapsto |a - x|^2 - \varepsilon^2.$$

Consider next the set

$$Y := \big\{ (x, \varepsilon) \in \mathbb{R} \times (0, \infty) : \; \exists a \in A : \; (a, x, \varepsilon) \in X \big\}.$$

The set Y is definable as the image of X via the natural projection $(a, x, \varepsilon) \mapsto (x, \varepsilon)$. Moreover

$$\boldsymbol{cl}(A) = \big\{ x \in \mathbb{R}^n : \;\; \forall \varepsilon > 0 \; (x, \varepsilon) \in Y \big\}.$$

Note that the complement of $\boldsymbol{cl}(A)$ is

$$B := \big\{ x \in \mathbb{R}^n : \exists \varepsilon > 0 : \;\; (x, \varepsilon) \notin Y \big\}.$$

The set B is definable as the projection of the complement of Y on $\mathbb{R}^n$ via the natural map$\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. The closure $\boldsymbol{cl}(A)$ is the complement of B and thus definable. □

Example 2.3. The $\mathcal{S}$-definable sets and functions many pleasant properties. Fix $r \in \mathbb{Z}_{>0}$.

(i) Each definable set X is a *finite* disjoint union

$$X = \bigsqcup_k X_k, \tag{2.1}$$

where each stratum X_k is a C^r-submanifold. The dimension of X is defined to be

$$\dim X := \max_k \dim X_k,$$

(ii) The stratifications of X as above can be chosen to satisfy regularity conditions such as theWhitney condition or the Verdier condition.

(iii) Every tame set $X \subset \mathbb{R}^n$ is definably triangulable. More precisely if $\bar{X}$ denotes the closure of X in the one-point compactification of $\mathbb{R}^n$, then there exists a finite affine simplicial complex K and a definable homeomorphism $\Phi : K \to \bar{X}$ such that X is a union of images of open faces. We set

$$\chi_{\mathrm{def}}(X) := \sum_\sigma (-1)^{\dim \sigma},$$

where the summation is carried over all the open faces σ of K such that $\Phi(\sigma) \subset X$. This integer is called the *definable Euler characteristic* of X and it is independent of the choice of triangulation.

(iv) If X is a tame set and $f : X \to \mathbb{R}$ then X admits a C^r-stratification as in (i) such that the restriction of f to each stratum is C^r.

(v) If X, Y are tame sets and $F : X \to Y$ is a definable continuous map, then there exists a triangulation of Y such that, over each open face of the triangulation of Y the map F is a definably trivial fibre bundle. In particular, in the collection of tame sets $F^{-1}(y)$, $y \in Y$ there are only *finitely many* topological types.

(vi) Suppose that X, Λ are definable sets and $S \subset \Lambda \times X$ is also definable. For $\lambda \in \Lambda$ we set

$$S_\lambda := S \cap \{\lambda\} \times X.$$

Then the function $\Lambda \ni \lambda \mapsto \chi_{\mathrm{def}}(S_\lambda) \in \mathbb{Z}$ is definable. In particular, it has finite range.

(vii) Every bounded k-dimensional definable subset has finite k-dimensional Hausdorff measure.

We refer to [30] for details and many more properties of definable sets. □

Remark 2.4. (a) If $X \subset \boldsymbol{V}$ is locally closed[2] in $\boldsymbol{V}$, then $\chi_{\mathrm{def}}(X)$ as defined above coincides with the Euler characteristic of the Borel-Moore homology of X. In particular if X is compact the usual Euler characteristic coincides with the tame one.

[2]A subset of a topological space S is locally closed if it is a closed subset of an open set $U \subset S$

(b) The definable Euler characteristic satisfies the *inclusion-exclusion principle*

$$\chi_{\mathrm{def}}(A \cup B) = \chi_{\mathrm{def}}(A) + \chi_{\mathrm{def}}(B) - \chi_{\mathrm{def}}(A \cap B), \;\; \forall A, B \in \mathcal{S}. \tag{2.2}$$

□

Example 2.5. The definable Euler characteristic of an open n dimensional ball is

$$\chi_{\mathrm{def}}(B^n) = (-1)^n.$$

In particular $\chi_{\mathrm{def}}(\,(0,1)\,) = -1$. Let S be the tame set obtained by removing from the closed triangle

$$T := \bigl\{\,(x,y) \in \mathbb{R}^2;\;\; x, y \geq 0,\; x + y \leq 1\,\bigr\}$$

the open interval $I := (0,1) \times \{0\}$. Then

$$\chi_{\mathrm{def}}(S) = \chi_{\mathrm{def}}(T \setminus I) = \chi_{\mathrm{def}}(T) - \chi_{\mathrm{def}}(I) = 2.$$

On the other hand, the set S is contractible so its usual topological Euler characteristic is 1. □

We should mention a surprising property of tame sets.

Theorem 2.6 (Scissor equivalence)**.** *Suppose that A, B are $\mathcal{S}$-definable sets. Then the following statements are equivalent.*

(i) *There exists a (not necessarily continuous) definable bijection $A \to B$.*
(ii) $\dim A = \dim B$ *and* $\chi_{\mathrm{def}}(A) = \chi_{\mathrm{def}}(B)$.

□

2.2. **Homological Morse theory on tame spaces.** We fix an o-minimal category of sets $\mathcal{S}$. We will refer to the sets and maps in this category as *tame* or *definable*.

Suppose X is a locally closed tame subset of $\boldsymbol{V}$, and S is a closed tame subset of X. We define the local cohomology of X along S (with real coefficients) to be

$$H^\bullet_S(X) := H^\bullet(X, X \setminus S).$$

Observe that $H^\bullet_S(X)$ collects the obstructions to "propagation" to X of the cohomology classes in$X\setminus$. Indeed from the long exact sequence in cohomology

$$\cdots \to H^q(X) \to H^q(X \setminus S) \xrightarrow{\delta} H^{q+1}(X, X \setminus S) = H^{q+1}S(X) \to \cdots$$

we deduce that if $H^{q+1}S(X)$, then any cohomology class in $X \setminus S$ is the restriction of a cohomology class in X.

We can now define the *local cohomology sheaves* $\mathcal{H}^\bullet_S = \mathcal{H}^\bullet_{X/S}$ to be the sheaves on X associated to the presheaves $U \longmapsto H^\bullet_{S\cap U}(U)$.

If $x \in X$ and $U_n(x)$ denotes the open ball of radius $1/n$ centered at x, then for every $m \leq n$ we have morphisms $H^\bullet_{S\cap U_m}(U_m) \to H^\bullet_{S\cap U_n}(U_n)$, and then the stalk of $\mathcal{H}^p_S$ at x is the inductive limit $\mathcal{H}^\bullet_S(x) := \lim_{n\to\infty} H^\bullet_{S\cap U_n}(U_n)$. Observe that since X is locally conical we have

$$\mathcal{H}^\bullet_S(x) = 0 \text{ for every } \; x \in (X \setminus S). \tag{2.3}$$

We set

$$\chi_S(X) := \sum_k (-1)^k \dim H^k_S(X), \;\; \chi_S(x) := \sum_{k\geq 0} (-1)^k \dim \mathcal{H}^k_S(x).$$

We have a Grothendieck spectral sequence converging to $H^\bullet_S(X)$ whose E_2 term is $E_2^{p,q} = H^p(X, \mathcal{H}^q_S)$.

If it happens that the local cohomology sheaves are supported by finite sets then

$$H^p(X, \mathcal{H}^q_S) = 0, \;\; \forall p > 0,$$

so that the spectral sequence degenerates at the E_2-terms. In this case we have

$$H^q_S(X) \cong H^0\big(\,X, \mathcal{H}^q_S\,\big) \cong \bigoplus_{x\in X} \mathcal{H}^q_S(x). \tag{2.4}$$

In particular

$$\chi_S(X) = \sum_{x\in X} \chi_S(x). \tag{2.5}$$

Suppose now that X is a compact connected tame subset of $\boldsymbol{V}$ and $f : X \to \mathbb{R}$ is a definable, continuous function. We will write

$$X_{f\geq c} := \big\{\, x \in X;\;\; f(x) \geq c\,\big\}, \;\; X_{f\leq c} := \big\{\, x \in X;\;\; f(x) \leq c\,\big\}, \;\; \text{etc.}$$

Fix a real number c, and consider the sheaves of local cohomology $\mathcal{H}^\bullet_{f\geq c}$ associated to the closed subset set $S = X_{f\geq c} \subset X$.

Definition 2.7. A point $x \in X$ is called a *homological* critical point of f if

$$\mathcal{H}^\bullet_{f\geq c}(x) \neq 0, \;\; \text{where} \;\; c = f(x).$$

We denote by $\mathbf{Cr}_f$ the set of homological critical points of f. We say that f is *nice* if $\mathbf{Cr}_f$ is finite. □

Lemma 2.8 (Kashiwara)**.** [3] *Suppose that X is a compact connected tame subset of $\boldsymbol{V}$ and $f : X \to \mathbb{R}$ is a nice continuous tame function that contains no homological critical points on the level set $\{f = c\}$. Then the inclusion induced morphism*

$$H^\bullet(X_{f<c+\varepsilon}) \to H^\bullet(X_{f<c})$$

is an isomorphism for all $\varepsilon > 0$ sufficiently small. □

Fix a compact tame set $X \subset \boldsymbol{V}$. For $x_0 \in \boldsymbol{V}$ and $r > 0$ we set

$$B^X_r(x_0) = \big\{\, x \in X;\;\; \mathrm{dist}(x, x_0) < r\,\big\}.$$

Given a continuous tame map $f : \boldsymbol{V} \to \mathbb{R}$ we define the *index* of f relative to X at a point $x_0 \in \boldsymbol{V}$ to be the integer

$$\boxed{\boldsymbol{m}_X(f, x_0) := \chi_{f\geq c}(x) = \lim_{r\searrow 0} \chi\Big(\, B^X_r(x_0),\; B^X_r(x_0) \cap \{f < c\}\,\Big)}, \tag{2.6}$$

[3]A much more general result is true where the tameness of X and f are not required. The proof is much more sophisticated and for details we refer to [16].

where $c = f(x_0)$. Since X is closed,

$$\boxed{\boldsymbol{m}_X(f, x_0) = 0, \quad \forall x_0 \in \boldsymbol{V} \setminus X}. \tag{2.7}$$

Due to the local conical structure of X we have

$$\boxed{\boldsymbol{m}_X(f, x_0) = 1 - \lim_{r \searrow 0} \chi\Big(B_r^X(x_0) \cap \big\{ f < f(x_0) \big\} \Big), \quad \forall x_0 \in X}. \tag{2.8}$$

Definition 2.9. A point $x \in X$ is called a *numerically critical point* of the tame continuous function $f : \boldsymbol{V} \to \mathbb{R}$ relative to X if $\boldsymbol{m}_X(f, x) \neq 0$. We denote by $\mathbf{Cr}^{\#}(f, X)$ the set of numerically critical points of f relative to X. □

Example 2.10 (Stratified Morse functions)**.** Any tame set $X \subset \boldsymbol{V}$ is equipped with a (nonunique) Whitney stratification. A tame continuous function $f : \boldsymbol{V} \to \mathbb{R}$ whose restriction to X is stratified Morse function in the sense of Goresky-MacPherson [12] is nice. One can show (see [12, 22]) that there exists a subset $\Delta_X \subset S(\boldsymbol{V}^*)$, such that $\dim \Delta_X < \dim S(\boldsymbol{V}^*) = (n-1)$ and for any $\boldsymbol{\xi} \in S(\boldsymbol{V}^*) \setminus \Delta_X$, the induced function $\xi : X \to \mathbb{R}$ is a stratified Morse function. The integer $\boldsymbol{m}_X(\boldsymbol{\xi}, x_0)$ can be identified with the Euler characteristic of the local Morse data of $\boldsymbol{\xi}$ at $\boldsymbol{x}_0$ as defined in [12]. □

Example 2.11. (a) Note that if X is a compact C^2-submanifold of $\boldsymbol{V}$, f is a Morse function on X, and x is a critical point of f with Morse index λ, then $\boldsymbol{m}(f, x) = (-1)^{\lambda}$.

Indeed, if x is a nondegenerate critical point then, for r sufficiently small, $B_r^X(x) \cap \{f < f(x)\}$ is homotopic to a sphere of dimension $\lambda - 1$ so

$$\chi\Big(B_r^X(x) \cap \{f < f(x)\} \Big) = 1 + (-1)^{\lambda - 1}.$$

In this case $\mathbf{Cr}^{\#}(f, X) = \mathbf{Cr}(f, X)$.

(b) If X is a compact, convex, subanalytic subset of $\boldsymbol{V}$ and $\xi : \boldsymbol{V} \to \mathbb{R}$ is a linear map, then a point $x \in X$ is numerically critical for the restriction of ξ to X if and only if x is a minimum point for ξ. In this case we have $\boldsymbol{m}_X(\xi, x) = 1$.

(c) Suppose that X is a compact domain in $\boldsymbol{V}$ with C^2 boundary. It has a natural Whitney stratification with two strata: the interior and the boundary of X. If $\xi \in \boldsymbol{V}^* \setminus 0$ defines a stratified Morse on X, then and $x \in X$ is a stratified critical point of $\xi\big|_X$ if and only if $x \in \partial X$ the dual vector $\xi_\uparrow \in \boldsymbol{V}$ is perpendicular to $T_x \partial X$. Moreover, the following hold.

- If $\xi_\uparrow$ is outward pointing, then $\boldsymbol{m}_X(\xi, x) = 0$.
- If $\xi_\uparrow$ is inward pointing, then $\boldsymbol{m}_X(\xi, x) = (-1)^{\mu_\xi(x)}$, where $\mu_\xi(x)$ is the Morse index of $\xi\big|_{\partial X}$ at x.

(d) Suppose that X_1, X_2 are compact subanalytic. The equation (2.6) implies the following inclusion-exclusion formula

$$\boldsymbol{m}_{X_1 \cup X_2}(f, x_0) = \boldsymbol{m}_{X_1}(f, x_0) + \boldsymbol{m}_{X_2}(f, x_0) - \boldsymbol{m}_{X_1 \cap X_2}(f, x_0), \quad \forall x_0 \in \boldsymbol{V}. \tag{2.9}$$

□

3. The normal cycle of a tame set

To define the concept of normal cycle of a tame set we need to use the language of currents. The next subsection introduces the basic vocabulary. Our terminology concerning currents closely follows that of Federer [7] (see also the more accessible [18, 19]). However, we changed some notations to better resemble notations used in algebraic topology.

3.1. **Currents.** Suppose that X is a smooth oriented Riemann manifold of dimension n. We denote by $\Omega_k(X)$ the space of k-dimensional currents in X, i.e., the topological dual space of the space $\Omega^k_{cpt}(X)$ of smooth, compactly supported k-forms on X. We will denote by

$$\langle \bullet, \bullet \rangle : \Omega^k_{cpt}(X) \times \Omega_k(X) \to \mathbb{R}$$

the natural pairing. The boundary of a current $T \in \Omega_k(X)$ is the $(k-1)$-current defined via the Stokes formula

$$\langle \alpha, \partial T \rangle := \langle d\alpha, T \rangle, \;\; \forall \alpha \in \Omega^{k-1}_{cpt}(X).$$

We obtain a chain complex

$$\cdots \xrightarrow{\partial} \Omega_{k+1}(X) \xrightarrow{\partial} \Omega_k(X) \xrightarrow{\partial} \cdots .$$

Its homology is the Borel-Moore homology of X with real coefficients.

Similarly we define $\Omega_k^{cpt}(X)$ to be the dual of the space $\Omega^k(X)$ of smooth forms on X, not necessarily with compact support. The currents in $\Omega_k^{cpt}(X)$ are called *compactly supported* currents. We get a chain complex $\big(\Omega^{cpt}_\bullet(X), \partial\big)$ whose homology is the singular homology of X with real coefficients.

For every $\alpha \in \Omega^k(X)$, $T \in \Omega_m(X)$, $k \leq m$ define $\alpha \cap T \in \Omega_{m-k}(X)$ by

$$\langle \beta, \alpha \cap T \rangle = \langle \alpha \wedge \beta, T \rangle, \;\; \forall \beta \in \Omega^{n-m+k}_{cpt}(X).$$

To a pair $(S, \boldsymbol{or}_S)$ consisting of a k-dimensional C^1-submanifold $S \subset X$ and an orientation $\boldsymbol{or}_S$ on S we can associated a k-dimensional current $[S, \boldsymbol{or}_S] \in \Omega_k(X)$ defined by the equality

$$\langle\, \alpha, [S, \boldsymbol{or}_S] \rangle = \int_{(S, \boldsymbol{or}_S)} \alpha, \;\; \forall \alpha \in \Omega^k_{cpt}(X). \tag{3.1}$$

If X, Y are smooth manifolds, then for any C^1-map $F : X \to Y$ and any current $T \in \Omega_k(X)$ such that the restriction of F to $\operatorname{supp} T$ is proper, we can define a *pushforward* current $F_*T \in \Omega_k(Y)$. Moreover

$$\partial F_* T = F_* \partial T.$$

3.2. **Subanalytic currents.** For any globally subanalytic subset $X \subset \mathbb{R}^n$ we denote by $\mathcal{C}_k(X)$ the Abelian subgroup of $\Omega_k(\mathbb{R}^n)$ generated by currents of the form $[S, \boldsymbol{or}_S]$, as above, where $S \subset \mathbb{R}^n$ is an orientable, globally subanalytic analytic C^1-submanifold of $\mathbb{R}^n$ whose closure is compact and contained in X and $\boldsymbol{or}_S$ is an orientation on S. We will refer to the elements of $\mathcal{C}_k(X)$ as *subanalytic (integral) k-chains* in X. One can show that the support of a subanalytic k-chain is a subanalytic set. Moreover. the boundary of a subanalytic chain is a subanalytic chain, i.e.,

$$\partial \mathcal{C}_k(X) \subset \mathcal{C}_{k-1}(X).$$

For any open subanalytic set $U \subset \boldsymbol{V}$ there exists a natural map

$$\mathcal{C}_k(X) \ni T \mapsto T \cap U \in \mathcal{C}_k(X \cap U).$$

If one thinks of $T \in \mathcal{C}_k(X)$ as the current of integration over a "piecewise" smooth k-dimensional variety $\mathcal{T}$, then $T \cap U$ corresponds to the integration over $\mathcal{T} \cap U$.

To describe the intersection theory of subanalytic chains we need to recall a fundamental slicing result of R. Hardt, [14, Theorem 4.3].

Suppose that E_0, E_1 are two oriented real Euclidean spaces of dimensions n_0 and respectively n_1, $f : E_0 \to E_1$ is a real analytic map, and $T \in \mathcal{C}_{n_0-c}(E_0)$ a subanalytic current of codimension c.

If y is a regular value of f, then the fiber $f^{-1}(y)$ is a submanifold equipped with a natural coorientation and thus defines a subanalytic current $[f^{-1}(y)]$ in E_0 of codimension n_1, i.e., $[f^{-1}(y)] \in \mathcal{C}_{d_0-d_1}(E_0)$.

We would like to define the intersection of T and $[f^{-1}(y)]$ as a subanalytic current $f^{-1}(y) \bullet T \in \mathcal{C}_{n_0-c-n_1}(E_0)$. It turns out that this is possibly quite often, even in cases when y is not a regular value.

Theorem 3.1 (Slicing Theorem)**.** *Let E_0, E_1, T and f be as above, denote by dV_{E_1} the Euclidean volume form on E_1, by $\boldsymbol{\omega}_{n_1}$ the volume of the unit ball in E_1.*

Let $\mathcal{R}_f(T)$ be the set of quasi-regular values of f with respect to T, i.e., the points $y \in E_1$ such that

$$\operatorname{codim}\big(\operatorname{supp} T \cap f^{-1}(y)\big) \geq c + n_1,$$

$$\operatorname{codim}\big(\operatorname{supp} \partial T \cap f^{-1}(y)\big) \geq c + n_1 + 1.$$

We denote by $D_f(T)$ the complement of $\mathcal{R}_f(T)$ in E_1, and we will refer to is as the T-discriminant of f.

For every $\varepsilon > 0$ and $y \in E_1$ we define $T \bullet_\varepsilon f^{-1}(y) \in \Omega_{n_0-c-n_1}(E_0)$ by

$$\big\langle\, \alpha, T \bullet_\varepsilon f^{-1}(y) \,\big\rangle := \frac{1}{\boldsymbol{\omega}_{n_1}\varepsilon^{n_1}} \Big\langle \,\big(I_{f^{-1}\big(B_\varepsilon(y)\big)} f^* dV_{E_1} \big) \wedge \alpha,\ T \,\Big\rangle,$$

$\forall \alpha \in \Omega^{n_0-c-n_1}_{cpt}(E_0)$. Then, the following hold.

(i) *The discriminant $D_f(T)$ is a "thin" subanalytic subset of E_1, i.e.,*

$$\operatorname{codim}_{E_1} D_f(E_1) \geq 1.$$

(ii) *For every $y \in \mathcal{R}_f(T)$, the currents $T \bullet_\varepsilon f^{-1}(y)$ converge weakly as $\varepsilon > 0$ to a subanalytic current $T \bullet f^{-1}(y) \in \mathcal{C}_{n_0-c-n_1}(E_0)$ called the f*-slice *of T over y. Moreover, the map $\mathcal{R}_f \ni y \mapsto \langle T, f, y\rangle \in \mathcal{C}_{d_0-c-d_1}(\boldsymbol{V})$ is continuous in the flat topology.*

□

The above slicing theorem generalizes to real analytic manifolds E_0, E_1. In particular consider the natural projection

$$\pi : S(\boldsymbol{V}^*) \times \boldsymbol{V} \to S(\boldsymbol{V}^*).$$

The fiber of π over $\xi \in S(T^*V)$ is $G_\xi = \{\xi\} \times \boldsymbol{V}$.

Given a subanalytic chain $T \in \mathcal{C}_{N-1}\big(S(\boldsymbol{V}^*) \times \boldsymbol{V} \big)$ there exists a subanalytic subset $D_f(T) \subset S(\boldsymbol{V}^*)$ of codimension at least 1 such that, for any $\xi \in S(\boldsymbol{V}^*) \setminus D_f(T)$ the graph $G_{-\xi}$ intersects $\operatorname{supp} T$ along a finite set $\mathcal{F}$, and one can define local intersection multiplicities

$$m_\xi : \mathcal{F} \to \mathbb{Z}$$

so that the intersection chain $G_{-\xi} \bullet T$ is

$$G_{-\xi} \bullet T = \sum_{\hat{\boldsymbol{p}} \in G_{-\xi} \cap \operatorname{supp} T} m_\xi(\hat{\boldsymbol{p}}) \delta_{\hat{\boldsymbol{p}}}.$$

3.3. **The normal cycle.** We have the following crucial uniqueness theorem.

Theorem 3.2 (Uniqueness theorem). *Suppose that $X \subset \boldsymbol{V}$ is a compact subanalytic set. Then there exists at most one tame compactly supported current $\boldsymbol{N} \in \Omega^{cpt}_{n-1}\big(S(\boldsymbol{V}^*) \times \boldsymbol{V} \big)$ satisfying the following conditions.*

(i) *The current $\boldsymbol{N}$ is a cycle, i.e., $\partial \boldsymbol{N} = 0$.*

(ii) *The current $\boldsymbol{N}$ is Legendrian, i.e., $\alpha \cap \boldsymbol{N} = 0$, where α is the canonical contact form on $S(\boldsymbol{V}^*) \times \boldsymbol{V}$.*

(iii) *There exists a tame subset $\Delta \subset S(\boldsymbol{V}^*)$ with the following properties.*
- *The set Δ is "thin", i.e., $\dim \Delta < \dim S(\boldsymbol{V}^*)$.*
- *For any $\xi \in S(\boldsymbol{V}^*) \setminus \Delta$ the function $(-\xi|_X)$ is a stratified Morse function, the intersection $T \bullet G_\xi$ is well defined, and it satisfies*

$$\boldsymbol{N} \bullet G_\xi = \sum_{\boldsymbol{x} \in X} \boldsymbol{m}_X(-\xi, \boldsymbol{x}) \delta_{(\xi, \boldsymbol{x})}.$$

Remark 3.3. The above result has been anticipated by many people. The first to have observed this seems to have been P. Wintgen [33] (1979) when X is a PL subset of $\boldsymbol{V}$. His ideas were further elaborated and developed by J. Cheeger, W. Müller and R. Schrader [4] a few years after Wintgen's pioneering work. The issue was finally settled by Kashiwara and Schapira [16] and J. Fu [10]. Theorem 3.2 is due to J. Fu [10] who proved the uniqueness result without any tameness assumption. Kashiwara and Schapira used a completely different approach, using their microlocal techniques for investigating the derived category of constructible of sheaves.

The dramatic developments in tame geometry, [30] afford a considerablyy simpler approach. The above version of Fu's uniqueness theorem was first stated and proved in [21]. □

Definition 3.4. A compact subanalytic set $X \subset \boldsymbol{V}$ is called *geometric* if there exists tame compactly supported current $T \in \Omega^{cpt}_{n-1}\big(S(\boldsymbol{V}^*) \times \boldsymbol{V} \big)$ satisfying the conditions (i)-(iii) in Theorem 3.2. The cycle T is called the *normal cycle in $\boldsymbol{V}$* of the compact subanalytic subset X of $\boldsymbol{V}$. It is denoted by $\boldsymbol{N}_X^{\boldsymbol{V}}$, or simply $\boldsymbol{N}_X$. □

Example 3.5. (a) Any compact, subanalytic C^2-submanifold X of $\boldsymbol{V}$ is a geometric set. In this case $\boldsymbol{N}_X$ is the current of integration described by the unit conormal bundle $S(T^*_X V)$. This follows from (1.2). The tube formula now reads

$$\operatorname{vol}\bigl(\mathbb{T}_r(X)\bigr)=\sum_{j=0}^{\dim X}\frac{1}{\boldsymbol{\sigma}_{N-1-j}}\bigl\langle\kappa_j,\boldsymbol{N}_X\bigr\rangle\boldsymbol{\omega}_{N-j}r^{N-j},\tag{3.2}$$

where κ_j are the canonical forms; see Definition 1.10.

(b) Suppose that X is the closure of a bounded subanalytic domain with C^2 boundary. Then X is a geometric set and its normal cycle is the current of integration along the graph of the (outer) Gauss map of ∂X.

More precisely, for any $\boldsymbol{x}\in\partial X$ we denote by $\boldsymbol{\nu}(\boldsymbol{p})\in S(\boldsymbol{V}^*)$ the outer conormal to ∂X at $\boldsymbol{x}$. The outer Gauss map is given by the correspondence

$$\partial X\ni\boldsymbol{p}\mapsto\boldsymbol{\nu}(\boldsymbol{p})\in S(\boldsymbol{V}^*)$$

and its graph is

$$G_{\boldsymbol{\nu}}:=\bigl\{\,(\,\boldsymbol{\nu}(\boldsymbol{x}),\boldsymbol{x});\;\;\boldsymbol{x}\in\partial X\,\bigr\}\subset S(\boldsymbol{V}^*)\times\boldsymbol{V}.$$

This follows from Example 2.11(c). Let us observe that the normal cycle, as graph of the Gauss map, capture all the curvature features of ∂X.

If we define the tube of radius r around X to be

$$\mathbb{T}_r(X):=\bigl\{\,\boldsymbol{p}\in\boldsymbol{V};\;\;\operatorname{dist}(\boldsymbol{p},X)\,\bigr\},$$

then, for r sufficiently small, we have a tube formula

$$\operatorname{vol}\bigl(\mathbb{T}_r(X)\bigr)=\operatorname{vol}(X)+\sum_{j=0}^{\dim\boldsymbol{V}-1}\frac{1}{\boldsymbol{\sigma}_{N-1-j}}\bigl\langle\kappa_j,\boldsymbol{N}_X\bigr\rangle\boldsymbol{\omega}_{N-j}r^{N-j},\tag{3.3}$$

Moreover

$$\mu_0(X)=\chi(X).$$

For more details we refer to [20, §9.3.5]. □

Theorem 3.6 (J. Fu)**.** *Any compact subanalytic set $X\subset\boldsymbol{V}$ is geometric.*

Idea of proof. One can show that for any compact subanalytic set X admits an *aura*, i.e., a proper subanalytic C^2-function $f:\boldsymbol{V}\to[0,\infty)$ such that $X=\{f=0\}$. For $\varepsilon>0$, the "tube"

$$X_\varepsilon:=\{f\leq\varepsilon\}$$

is a compact, subanalytic C^2-domain and thus it has a normal cycle $\boldsymbol{N}_{X_\varepsilon}$. One then shows that, as $\varepsilon\searrow 0$, the current $\boldsymbol{N}_{X_\varepsilon}$ converges weakly to a subanalytic current satisfying all the conditions (i)-(ii) of the Uniqueness Theorem 3.2. For details we refer to [21].

□

The convergence $\boldsymbol{N}_{X_\varepsilon}\to\boldsymbol{N}_X$ used in the above proof is a manifestation of the following more general result due to J. Fu [10, Thm. 4.3].

Theorem 3.7 (Approximation theorem)**.** *Let $X \subset K \subset \boldsymbol{V}$ be compact subanalytic sets and $f : K \to [0, \infty)$ a continuous subanalytic function such that $X = f^{-1}(0)$. Set*

$$X_\varepsilon = \{ f \leq \varepsilon \}.$$

Then, as $\varepsilon \to 0$, we have $\boldsymbol{N}_{X_\varepsilon} \to \boldsymbol{N}_X$ weakly. □

3.4. **The characteristic cycle of Kashiwara and Schapira.** To describe this object we need to first describe the coning construction

$$\mathrm{Cone} : \Omega_k^{cpt}\big(S(\boldsymbol{V}^*) \times \boldsymbol{V} \big) \to \Omega_{k+1}(\boldsymbol{V}^* \times \boldsymbol{V}).$$

Let $T \in \Omega_k^{cpt}\big(S(\boldsymbol{V}^*) \times \boldsymbol{V} \big)$. Form the current

$$[0, \infty) \times T \in \Omega_{k+1}\big(\mathbb{R} \times S(\boldsymbol{V}^*) \times \boldsymbol{V} \big).$$

Now observe that we have a radial multiplication map

$$\mu : \mathbb{R} \times S(\boldsymbol{V}^*) \times \boldsymbol{V} \to T^*\boldsymbol{V} = \boldsymbol{V}^* \times \boldsymbol{V}, \ \ (t, \boldsymbol{\xi}, \boldsymbol{v}) \mapsto (t\boldsymbol{\xi}, \boldsymbol{v}).$$

The restriction of this map to $[0, \infty) \times \operatorname{supp} T$ is proper so we have a well defined current

$$\mathrm{Cone}(T) := \mu_*\big([0, \infty) \times T \big) \in \Omega_{k+1}(\boldsymbol{V}^* \times \boldsymbol{V}) = \Omega_{k+1}(T^*\boldsymbol{V}).$$

This current is a *conic current* in the sense that, for any $t > 0$ we have

$$\mu_*^t\big(\mathrm{Cone}(T) \big) = \mathrm{Cone}(T),$$

where $\mu^t : T^*\boldsymbol{V} \to T^*\boldsymbol{V}$ is the multiplication by t in the fiber directions. Let

$$Z^0 : \boldsymbol{V} \to T^*\boldsymbol{V}$$

denote the zero section

If $X \subset \boldsymbol{V}$ is a compact tame subset, we denote by $[X] \in \Omega_N(\boldsymbol{V})$ the current of integration over the interior of X. We set $[X] = 0$ if X has empty interior.

If $\boldsymbol{N}_X \in \Omega_{N-1}^{cpt}\big(S(\boldsymbol{V}^*) \times \boldsymbol{V} \big)$ denotes the normal current of X, then the *Kashiwara-Schapira charateristic cycle* of X to be

$$\boxed{\mathcal{C}_X := Z_*^0[X] + \mathrm{Cone}\big(\boldsymbol{N}_X \big) \in \Omega_N\big(T^*\boldsymbol{V} \big)}.$$

Theorem 3.2 implies that $\mathcal{C}_X$ is the unique tame integral current satisfying the following conditions.

Theorem 3.8. *The current $\mathcal{C}_X \in \Omega_n(\boldsymbol{V})$ satisfies the following properties.*

(i) $\mathcal{C}_X$ *is Lagrangian, i.e.,* $\omega \cap \mathcal{C}_X = 0$.
(ii) $\mathcal{C}_X$ *is a conic cycle.*
(iii) $\mathcal{C}_X$ *satisfies the intersection property*

$$\mathcal{C}_X \bullet G_\xi = \sum_{\boldsymbol{x} \in X} \boldsymbol{m}_X(-\xi, \boldsymbol{x}) \delta_{(\xi, \boldsymbol{x})}.$$

for generic $\xi \in \boldsymbol{V}^*$.

□

Based on the above uniqueness result one can prove[4] the following *product formula*

Theorem 3.9 (Product formula). *If $\boldsymbol{V}_1, \boldsymbol{V}_2$ are two oriented real Euclidean spaces and $X_i \subset \boldsymbol{V}_i$, $i = 1, 2$, are two compact subanalytic sets, then*

$$\boxed{\mathcal{C}_{X_1 \times X_2} = \mathcal{C}_{X_1} \times \mathcal{C}_{X_2}}. \tag{3.4}$$

□

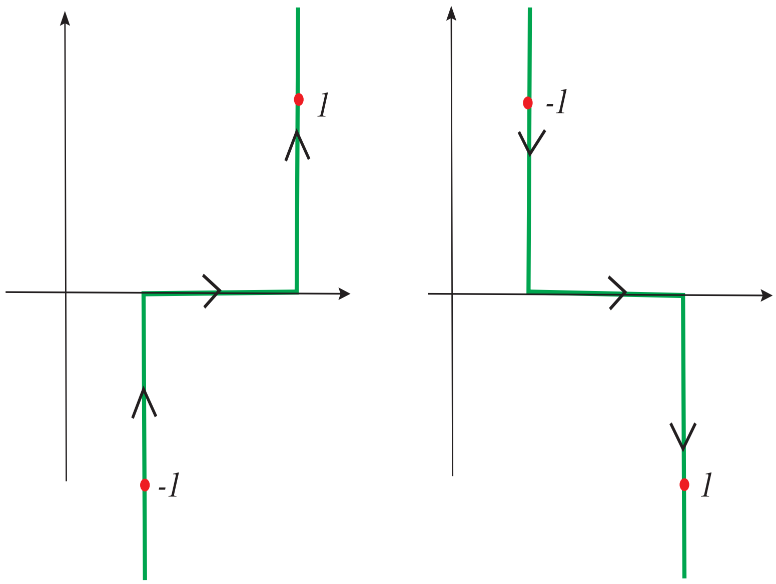

Figure 1. *Above, the Cartesian plane $\mathbb{R}^2$ is identified with $T^*\mathbb{R}$. In the left-hand-side we have depicted (in green) the conormal cycle of a compact interval $[a, b]$ while in the right-hand-side we have depicted the conormal cycle of an open interval (a, b). The normal cycles in each case consist of the union of red points with their corresponding multiplicities.*

4. Curvature measures of tame sets

4.1. **Constructible functions and valuations.** Fix a tame structure $\mathcal{S}$. We will refer to the sets in this category as *definable*.

A *constructible* function on $\boldsymbol{V}$ is a definable function $f : \boldsymbol{V} \to \mathbb{R}$ such that $f(\boldsymbol{V}) \subset \mathbb{Z}$. This means that

- the range of f is *finite*, and
- for any $n \in \mathbb{Z}$, the preimage $f^{-1}(n)$ is a definable subset of $\boldsymbol{V}$.

We denote by $\mathcal{F}(\boldsymbol{V})$ the set of constructible functions and by $\mathcal{F}_0(\boldsymbol{V})$ the space of constructible functions with compact support. Note that $\mathcal{F}(\boldsymbol{V})$ is an Abelian group and $\mathcal{F}_0(\boldsymbol{V})$ is a subgroup.

[4]The proof is not as straightforward as it may seem. It requires nontrivial input from either stratified Morse theory, or from geometric measure theory.

For any definable subset $X \subset \boldsymbol{V}$, the indicator function

$$I_X(\boldsymbol{p}) = \begin{cases} 1, & \boldsymbol{p} \in X \\ 0, & \boldsymbol{p} \in \boldsymbol{V} \setminus X, \end{cases}$$

is constructible. More generally, any constructible function is a linear combination with $\mathbb{Z}$-coefficients of such indicator functions. More precisely,

$$f = \sum_{n \in \mathbb{Z}} n I_{f^{-1}(n)}.$$

The decomposition of f as a linear combination of indication functions *is not unique*. This is due to the *inclusion-exclusion formula*

$$\boxed{I_{X_1 \cup X_2} = I_{X_1} + I_{X_2} - I_{X_1 \cap X_2}}. \tag{4.1}$$

This is in some rather precise sense the only relation satisfied by the elements in the group $\mathcal{F}(\boldsymbol{V})$.

Denote by $\mathcal{S}_b(\boldsymbol{V})$ the collection of *bounded* $\mathcal{S}$-definable subsets of $\boldsymbol{V}$ and by $\mathcal{S}_{cpt}(\boldsymbol{V})$ the collection of *compact* $\mathcal{S}$-definable subsets of $\boldsymbol{V}$. Note that

$$A, B \in \mathcal{S}_b(\boldsymbol{V}) \Rightarrow A \cup B, A \cap B, A \setminus B \in \mathcal{S}_b(\boldsymbol{V}).$$

Clearly the collection of indicator functions

$$\{ I_A; \;\; A \in \mathcal{S}_b(\boldsymbol{V}) \}$$

generates the Abelian group $\mathcal{F}_0(\boldsymbol{V})$. Less obvious is that, the smaller collection

$$\{ I_A; \;\; A \in \mathcal{S}_{cpt}(\boldsymbol{V}) \}$$

generates the Abelian group $\mathcal{F}_b(\boldsymbol{V})$. This follows from the triangulation theorem, [30, §8.2].

More generally, given a definable set $X \subset \boldsymbol{V}$, we denote by $\mathcal{S}(X)$ the collection of definable subsets of X and by $\mathcal{F}(X)$ the subgroup of $\mathcal{F}(\boldsymbol{V})$ consisting of the constructible functions $f : \boldsymbol{V} \to \mathbb{Z}$ that vanish outside X.

Given a definable set X and an Abelian group G we define a *G-valuation* or *G-valued finitely additive measure* on $\mathcal{S}(\boldsymbol{X})$ to be a map $\mu : \mathcal{S}(X) \to G$ satisfying the inclusion-exclusion principle, i.e.,

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B), \;\; \forall A, B \in \mathcal{S}(\boldsymbol{V}).$$

Note that (4.1) implies that the correspondence

$$\mathcal{S}(X) \ni A \mapsto I_A \in \mathcal{F}(X)$$

is a $\mathcal{F}(X)$-valuation. More generally, if G is an Abelian group and $L : \mathcal{F}(X) \to G$ is a morphism of groups, then the correspondence

$$\mathcal{S}(X) \ni A \mapsto L\big(I_A\big) \in G$$

is a G-valuation. It turns out that, if G is torsion free, then all the G-valuations are obtained in this fashion. More precisely, we have the following important result. For a proof we refer to [17].

Theorem 4.1 (Groemer's extension theorem)**.** *Suppose that $X \subset \boldsymbol{V}$ is a definable set, G is a torsion free Abelian group and $\mu : \mathcal{S}(X) \to G$ is a G-valuation. Then, there exists a unique group morphism $L = L_\mu : \mathcal{F}(X) \to G$ that*

$$\mu(A) = L_\mu(I_A), \;\; \forall A \in \mathcal{S}(X).$$

The morphism L_μ is often called the integral with respect to the valuation μ *and, for $\alpha \in \mathcal{F}(X)$ we set*

$$\int \alpha d\mu := L_\mu(\alpha). \qquad \square$$

4.2. **Integration with respect to Euler characteristic and motivic Radon transforms.** We deduce from the inclusion-exclusion principle (2.2) that the Euler characteristic

$$\chi_{\mathrm{def}} : \mathcal{S}(\boldsymbol{V}) \to \mathbb{Z}$$

is a valuation and thus leads to an Euler characteristic integral

$$\mathcal{F}(\boldsymbol{V}) \ni f \mapsto \int f d\chi_{\mathrm{def}}.$$

This was previously constructed by other means by P. Schapira [26] and O. Viro [31].

The correspondence $X \mapsto \mathcal{F}(X)$ has several pleasant functorial properties. Suppose that $\alpha : X \to Y$ is a tame continuous map. For any constructible function $g : Y \to \mathbb{Z}$ the pullback

$$\alpha^*(g) = g \circ \alpha : X \to \mathbb{Z}$$

is also constructible and induces a morphism of Abelian groups

$$\alpha^* : \mathcal{F}(Y) \to \mathcal{F}(X).$$

Less obvious is the fact that α induces a *push-forward* or *integration-along-fibers* morphism

$$\alpha_* : \mathcal{F}(X) \to \mathcal{F}(Y).$$

To construct it, we associate to any definable $f \in \mathcal{F}(X)$ the function

$$\alpha_*(f) : Y \to \mathbb{Z}, \;\; \alpha_*(f)(y) := \int_{\alpha^{-1}(y)} f(x) d\chi_{\mathrm{def}}(x). \tag{4.2}$$

The property (vii) in Example 2.3 shows that $\alpha_*(f)$ is indeed a constructible function. Moreover, the inclusion-exclusion principle (2.2) shows that the correspondence

$$\alpha_* : \mathcal{S}(X) \to \mathcal{F}(Y), \;\; A \mapsto \alpha_*(I_A)$$

is an $\mathcal{F}(Y)$-valuation and thus extends to a group morphism $\alpha_* : \mathcal{F}(X) \to \mathcal{F}(Y)$. We want to remark that when Y consists of a single point, the push-forward α_* coincides with the integral with respect to the Euler characteristic. Often one uses the alternate notation

$$\int_\alpha := \alpha_*.$$

We have a *projection formula* which states that for any $f \in \mathcal{F}(Y)$ we have

$$\alpha_* \circ \alpha^*(f)(y) = \chi_{\mathrm{def}}\big(\,\alpha^{-1}(y)\,\big) f(y). \tag{4.3}$$

Given tame continuous maps $\alpha : X \to Y$ and $\beta : Y \to Z$ we have

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_*. \tag{4.4}$$

This last equality can be viewed as a topological version of Fubini's theorem.

Given two continuous tame maps

$$X_0 \xrightarrow{f_0} S, \;\; X_1 \xrightarrow{f_1} S$$

we define their fiber product

$$X_0 \times_S X_1 := \Big\{ (x_0, x_1) \in X_0 \times X_1; \;\; f_0(x_0) = f_1(x_1) \Big\}$$

Observe that

$$X_0 \times_S X_1 = (f_0 \times f_1)^{-1}(\Delta_S),$$

where Δ_S denotes the diagonal of $S \times S$. $X_0 \times_S X_1$ is equipped with natural maps

$$X_0 \xleftarrow{p_0} X_0 \times_S X_1 \xrightarrow{p_1} X_1.$$

We obtain in this fashion a *Cartesian square*

$$\begin{tikzcd} X_0 \times_S X_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] & X_1 \arrow[d, "f_1"] \\ X_0 \arrow[r, "f_0"'] & S \end{tikzcd} \tag{†}$$

We have the *base change formula* stating that

$$(f_0)^*(f_1)_* = (p_0)_*(p_1)^*. \tag{4.5}$$

The above functoriality concepts have found many remarkable applications; see e.g.[5, 27, 28]. We describe one such simple application.

Given definable sets X, Y, any a definable subset $S \subset X \times Y$ defines an integral transform

$$T_S : \mathcal{F}(X) \to \mathcal{F}(Y), \;\; T_S(f)(y) = \int I_S(x,y) f(x) d\chi_{\mathrm{def}}(x) = (\pi_Y)_*\big(I_S \pi_X^*(f) \big)(y).$$

Given a definable set

$$R \subset Y \times X$$

we form the Cartesian square

$$\begin{tikzcd} S \times_Y R \arrow[rr, "\pi_R"] \arrow[dr, dashed, "p"] \arrow[dd, "\pi_S"'] & & R \arrow[dd, "\pi_Y"] \\ & X \times X & \\ S \arrow[rr, "\pi_Y"'] & & Y \end{tikzcd}, \;\; p(s,r) = (\pi_X(s), \pi_X(r)). \tag{4.6}$$

Assume R satisfies the following condition

$$\exists \mu \neq \nu \in \mathbb{Z} : \;\; \chi_{\mathrm{def}}(p^{-1}(x,x')) = \begin{cases} \mu, & \text{if} \;\; x \neq x' \\ \nu, & \text{if} \;\; x = x' \end{cases} . \tag{4.7}$$

Note that condition (4.7) is equivalent to

$$p_*\Big(I_{S\times_Y R} \Big) = \nu I_{\Delta_X} + \mu I_{X\times X\setminus \Delta_X} = (\nu-\mu) I_{\Delta_X} + I_{X\times X}. \tag{4.8}$$

It is convenient to think of S as the graph of a multivalued map $F_S : X \dashrightarrow Y$,

$$(x,y) \in S \Longleftrightarrow x \xrightarrow{F} y.$$

Then, roughly speaking

$$T_S(f) = (F_S)_*(f).$$

Similarly, we can think of R as the graph of a multivalued map $F_R : Y \dashrightarrow X$. Then the graph of $F_R \circ F_S : X \to X$ is the image of $S \times_Y R$ via the map p. In other words

$$x \xrightarrow{F_R\circ F_S} x' \Longleftrightarrow p^{-1}(x,x') \neq \emptyset.$$

Also we can identify the fiber $p^{-1}(x,x')$ with the set of paths

$$x \xrightarrow{F_S} y \xrightarrow{F_R} x'$$

We can loosely reformulate (4.8) as follows.

- The "number" of paths $x \xrightarrow{F_S} y \xrightarrow{F_R} x$ is independent of x and it is the integer ν.
- The "number" of paths $x \xrightarrow{F_S} y \xrightarrow{F_R} x'$ is independent of $x \neq x'$ and it is the integer μ.

Theorem 4.2 (The Inversion Formula, P. Schapira [28])**.** *Suppose S and R satisfy the condition (4.7). Then for every $\varphi \in \mathcal{F}(X)$ we have*

$$T_R \circ T_S(\varphi) = (\nu-\mu)\varphi + \mu\left(\int_X \varphi \, d\chi_{\mathrm{def}} \right).$$

Proof. Consider the following diagram

$$\begin{tikzcd}
 & & S\times_Y R \arrow[dl, "\pi_S"'] \arrow[d, "p"] \arrow[dr, "\pi_R"] & & \\
 & S \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & X\times X & R \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_X"] & \\
X & & Y & & X
\end{tikzcd}$$

We deduce from the base change formula (4.5)that

$$T_R\circ T_S(\varphi) = (\pi_X)_* \circ \; \pi_Y^* \circ (\pi_Y)_* \circ \pi_X^*(\varphi) = (\pi_X)_* \circ (\pi_R)_* \circ \pi_S^* \circ \pi_X^*(\varphi)$$
$$\stackrel{(4.4)}{=} (\pi_X\circ\pi_R)_* \circ (\pi_X\circ \pi_S)^*(\varphi).$$

Now look at the commutative diagram

$$\begin{tikzcd} & S\times_Y R \arrow[dl, "\pi_X\circ\pi_S"'] \arrow[d, "p"] \arrow[dr, "\pi_X\circ\pi_R"] & \\ X & X\times X \arrow[l, "p_1"] \arrow[r, "p_2"'] & X \end{tikzcd} \iff \pi_X\circ\pi_S = p_1\circ p,\quad \pi_X\circ\pi_R = p_2\circ p.$$

We deduce

$$T_R\circ T_S(\varphi) = (p_2\circ p)_*\circ(p_1\circ p)^*(\varphi) = (p_2)_*\circ p_*\circ p^*(p_1^*(\varphi)).$$

At this point observe that for every $\psi\in\mathcal{F}(X\times X)$ we have according to the projection formula (4.3)

$$p_*\circ p^*\psi = \underbrace{\big(p_*p^*I_{X\times X}\big)}_{=:K(x,x')}\psi$$

Hence

$$T_R\circ T_S(\varphi)(x') = \int_{p_2} K(x,x')\varphi(x)d\chi_{\mathrm{def}}(x)$$

More precisely, this means that

$$T_R\circ T_S(\varphi)(x') = \int_X K(-,x')\varphi(-)d\chi_{\mathrm{def}}.$$

Using the condition (4.8) we deduce

$$K(x,x') = \chi_{\mathrm{def}}\big(p^{-1}(x,x')\big) = (\nu-\mu)I_{\Delta_X} + \mu I_{X\times X}.$$

Hence

$$\int_X K(x,x')\varphi(x)d\chi_{\mathrm{def}}(x) = (\nu-\mu)\phi(x') + \mu\Big(\int_X\varphi\Big).$$

This concludes the proof of the Inversion Formula. □

Denote by $\mathbf{Graff}^1(\boldsymbol{V})$ the Grassmannian of affine hyperplanes of $\boldsymbol{V}$. This is a definable subset of a suitable Euclidean space since it is semi-algebraic. Consider the incidence relations

$$\mathcal{I} := \big\{\,(\boldsymbol{p},H)\in\boldsymbol{V}\times\mathbf{Graff}^1(\boldsymbol{V});\;\;\boldsymbol{p}\in H\,\big\},$$
$$\mathcal{I}^{\#} := \big\{\,(H,\boldsymbol{p})\in\mathbf{Graff}^1(\boldsymbol{V})\times\boldsymbol{V};\;\;\boldsymbol{p}\in H\,\big\}.$$

This defines a tautological "roofs" or double-fibrations

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \mathcal{I}^{\#} \arrow[dl, "\lambda"'] \arrow[dr, "\rho"] & \\ \boldsymbol{V} & & \mathbf{Graff}^1(\boldsymbol{V}) & & \boldsymbol{V} \end{tikzcd}$$

$$\alpha(\boldsymbol{p},H) = \boldsymbol{p} = \rho(H,\boldsymbol{p}),\quad \beta(\boldsymbol{p},H) = H = \lambda(H,\mathbb{P}).$$

We define the *topological Radon transform* to be the morphism of groups

$$\mathcal{R} : \mathcal{F}(\boldsymbol{V}) \to \mathcal{F}\big(\,\mathbf{Graff}^1(\boldsymbol{V})\,\big),\ \ \mathcal{R}(f) = T_{\mathcal{J}}(f) = \beta_*\alpha^*(f).$$

For example, for $S \in \mathcal{S}(\boldsymbol{V})$ we have

$$\mathcal{R}(I_S)(H) = \chi(S \cap H).$$

The dual Radon transform is then defined to be

$$\mathcal{R}_\# : \mathcal{F}\big(\,\mathbf{Graff}^1(\boldsymbol{V})\,\big) \to \mathcal{F}(\boldsymbol{V}),\ \ \mathcal{R}_\#(\varphi) = T_{\mathcal{J}\#}(\varphi) = \lambda_*\rho^*(\varphi).$$

Let us show that the condition (4.7) is satisfied. In this case

$$p^{-1}(x, x') = \big\{\,(x, H, x');\ \ x, x' \in H \in \mathbf{Graff}^1(\boldsymbol{V})\,\big\}.$$

Set $N := \dim \boldsymbol{V}$. We deduce

$$p^{-1}(x, x) \cong \mathbb{RP}^{N-1},\ \ p^{-1}(x, x') \cong \mathbb{RP}^{N-2},\ \ \forall x \neq x',$$

so

$$\mu = \begin{cases} 0, & N\text{-even}, \\ 1, & N\text{-odd}, \end{cases} \qquad \nu = \begin{cases} 1, & N\text{-even}, \\ 0, & N\text{-odd}. \end{cases}$$

The inversion formula implies that

$$\mathcal{R}_\#\mathcal{R}(f) = (-1)^{N+1}\varphi + \frac{1}{2}\Big(\,1 + (-1)^N\,\Big)\left(\int \varphi d\chi\right) I_{\boldsymbol{V}}.$$

Note that if N is odd, the above equality shows that

$$\mathcal{R}_\#\mathcal{R}(f) = f.$$

This has a surprising consequence when applied to the special case $f := I_X$, $X \in \mathcal{S}(\boldsymbol{V})$. It shows that a tame set X in an odd dimensional vector space is completely determined by the Euler characteristics of its intersections with all the affine hyperplanes in $\boldsymbol{V}$. The same is true in even dimensions as well.

4.3. **Curvature measures.** It tuns out that the correspondence $X \mapsto \boldsymbol{N}_X$ is also valuation.

Proposition 4.3. *The correspondence*

$$\mathcal{S}_{cpt}(\boldsymbol{V}) \ni X \mapsto \boldsymbol{N}_X \in \Omega^{N-1}\big(\,S(T^*\boldsymbol{V})\,\big)$$

is a valuation.

Proof. We have to show that, given compact subanalytic sets X_1, X_1, then

$$\boxed{\boldsymbol{N}_{X_1\cup X_2} = \boldsymbol{N}_{X_1} + \boldsymbol{N}_{X_2} - \boldsymbol{N}_{X_1\cap X_2}}. \tag{4.9}$$

To see why this happens note that the current in the right-hand-side is a Legendrian cycle, so it satisfies the conditions (i) and (ii) of the Uniqueness Theorem 3.2. The condition (iii) follows from the local inclusion-exclusion formula (2.9). □

The last result, coupled with Groemer's Extension Theorem shows that we can associate a compactly supported Legendrian cycle $\boldsymbol{N}_f$ to any constructible function $f \in \mathcal{C}_0(\boldsymbol{V})$ so that the correspondence

$$\mathcal{C}_b(\boldsymbol{V}) \ni f \mapsto \boldsymbol{N}_f \in \Omega^{n-1}\big(S(\boldsymbol{V}^*) \times \boldsymbol{V} \big)$$

is a morphism of Abelian groups.

Suppose $X \subset \boldsymbol{V}$. Using (1.8) as inspiration we define the curvature measures of X to be

$$\mu_j(X) := \frac{1}{\boldsymbol{\sigma}_{N-1-j}} \big\langle \kappa_j, \boldsymbol{N}_X \big\rangle, \quad j = 0, 1, \dots, N-1.$$

where $\kappa_0, \dots, \kappa_{N-1}$ are the canonical forms; see Definition 1.10. The resulting correspondences

$$\mathcal{S}_{cpt}(\boldsymbol{V}) \ni X \mapsto \mu_j(X) \in \mathbb{R}$$

are valuations. They define "integrals"

$$\mathcal{C}_b(\boldsymbol{V}) \ni \alpha \mapsto \int \alpha d\mu_j := \frac{1}{\boldsymbol{\sigma}_{N-1-j}} \big\langle \kappa_j, \boldsymbol{N}_\alpha \big\rangle \in \mathbb{R}.$$

Proposition 4.4.

$$\mu_0(X) = \chi(X), \quad \forall X \in \mathcal{S}_{cpt}(\boldsymbol{V}). \tag{4.10}$$

Idea of proof. Choose proper C^3, definable functions $f : \boldsymbol{V} \to [0, \infty)$ such that $f^{-1}(0) = X$. Form the "tubes"

$$X_\varepsilon := \big\{ f \leq \varepsilon \big\}.$$

Then $\boldsymbol{N}_{X_\varepsilon} \to \boldsymbol{N}_X$ weakly as $\varepsilon \searrow 0$ and we deduce

$$\mu_0(X_\varepsilon) = \big\langle \kappa_0, \boldsymbol{N}_{X_\varepsilon} \big\rangle \to \big\langle \kappa_0, \boldsymbol{N}_X \big\rangle = \mu_0(X).$$

Since X_ε is the closure of a bounded domain with C^2-boundary we deduce from Example 3.5(b) that

$$\mu_0(X_\varepsilon) = \chi(X_\varepsilon), \quad \forall \varepsilon \ll 1.$$

On the other hand, for $\varepsilon > 0$ sufficiently small we have $\chi(X_\varepsilon) = \chi(X)$. □

For any compact tame set $X \subset \boldsymbol{V}$ we denote by $\mu_N(X)$ its N-dimensional Lebesgue measure, $N = \dim \boldsymbol{V}$. Clearly $\mu_N(X)$ is also a valuation of $\mathcal{S}_{cpt}(\boldsymbol{V})$. The valuations $\mu_0(X), \mu_1(X), \dots, \mu_N(X)$ of a compact tame set $X \subset \boldsymbol{V}$ were defined extrinsically, by relying the fact that X lives inside the Euclidean space $\boldsymbol{V}$, so a more appropriate notation would be $\mu_k^{\boldsymbol{V}}(X)$.

Suppose now that $\boldsymbol{V}_1$ is another finite dimensional oriented Euclidean space of dimension $N_1 > N$ containing $\boldsymbol{V}$ as a subspace. The tame subset $X \subset \boldsymbol{V}$ can now be viewed as a subset of $\boldsymbol{V}_1$ and, as such, its normal cycle $\boldsymbol{N}_X^{\boldsymbol{V}_1}$ is an $(N_1 - 1)$ dimensional cycle. Conceivably, the resulting curvature measures

$$\mu_0^{\boldsymbol{V}_1}(X), \dots, \mu_N^{\boldsymbol{V}_1}(X)$$

could be different from $\mu_0^{\boldsymbol{V}}(X), \mu_1^{\boldsymbol{V}}(X), \dots, \mu_N^{\boldsymbol{V}}(X)$. This is not the case.

Theorem 4.5 (J.Fu, [10])**.** *If X is a compact subanalytic subset of the oriented Euclidean space $\boldsymbol{V}$, and $\boldsymbol{V}_1$ is another finite dimensional oriented Euclidean space of dimension $N_1 > N$ containing $\boldsymbol{V}$ as a subspace, then*

$$\mu_k^{\boldsymbol{V}_1}(X) = \mu_k^{\boldsymbol{V}}(X),\ \ \forall k = 0, 1, \dots, \dim \boldsymbol{V}. \qquad \square$$

Thus, in the sequel we will not need indicate the ambient space when referring to the valuations μ_k.

Example 4.6 (The curvature measures of balls)**.** Denote by $B_R(\boldsymbol{p})$ the closed ball of radius R centered at $\boldsymbol{p} \in \boldsymbol{V}$. Note that in this case the tube $\mathbb{T}_r\big(\,B_R(\boldsymbol{p})\,\big)$ is the ball $B_{R+r}(\boldsymbol{p})$. Using (3.3) we deduce

$$\boldsymbol{\omega}_N(R+r)^N = \operatorname{vol}\big(\,\mathbb{T}_r\big(\,B_R(\boldsymbol{p})\,\big)\,\big) = \operatorname{vol}\big(\,B_R(\boldsymbol{p})\,\big) + \sum_{j=0}^{N-1} \omega_{N-j} r^{N-j} \mu_j\big(\,B_R(\boldsymbol{p})\,\big).$$

Expanding in powers of r the left-hand-side of the above equality we deduce.

$$\boxed{\mu_j\big(\,B_r(\boldsymbol{p})\,\big) = \frac{\boldsymbol{\omega}_N}{\boldsymbol{\omega}_{N-j}}\binom{N}{j} r^j = \left[\begin{matrix} N \\ j \end{matrix}\right] \boldsymbol{\omega}_j r^j}. \tag{4.11}$$

$\square$

The tube formula is a special case of Fu's kinematic formula, [10], generalizing an earlier version of Federer, [6].

Theorem 4.7 (Baby kinematic formula)**.** *Suppose that $X \in \mathcal{S}_{cpt}(\boldsymbol{V})$, $N = \dim \boldsymbol{V}$. Then, for any $r > 0$, we have*

$$\boxed{\int_{\boldsymbol{V}} \chi\big(\,X \cap B_r(\boldsymbol{p})\,\big)\,|d\boldsymbol{p}| = \sum_{j=0}^{N} \mu_{N-j}(X) \boldsymbol{\omega}_j r^j}. \qquad \square$$

The characteristic polynomial of a compact subanalytic subset $X \subset \boldsymbol{V}$ is

$$M_X(t) := \sum_{j=0}^{\dim \boldsymbol{V}} \mu_j(X) t^j.$$

The correspondence $X \mapsto M_X(t)$ is an $\mathbb{R}[t]$-valuation on $\mathcal{S}_{cpt}(\boldsymbol{V})$. The product formula (3.4) implies a product formula for characteristic polynomials, [10].

Theorem 4.8 (Product formula)**.** *If $\boldsymbol{V}_1, \boldsymbol{V}_2$ are two oriented real Euclidean spaces and $X_i \subset \boldsymbol{V}_i$, $i = 1, 2$, are two compact subanalytic sets, then*

$$\boxed{M_{X_1 \times X_2}(t) = M_{X_1}(t) M_{X_2}(t)}. \tag{4.12}$$

$\square$

We have the following fundamental result generalizing the Crofton formulæ (1.11).

Theorem 4.9 (Fu)**.** *Suppose that $X \in \mathcal{S}_{cpt}(\boldsymbol{V})$. Then for any $0 < k, p < N$, $k+p < N$, we have*

$$\begin{bmatrix} p+k \\ p \end{bmatrix} \mu_{p+k}(X) = \int_{\mathbf{Graff}^k(X)} \mu_p(L \cap X)\, |d\tilde{\nu}|(L). \tag{4.13}$$

In particular, for $p = 0$, we have

$$\mu_k(X) = \int_{\mathbf{Graff}^k(\boldsymbol{V})} \mu_0(X \cap L)\, |d\tilde{\nu}|(L) = \int_{\mathbf{Graff}^k(\boldsymbol{V})} \chi(X \cap L)\, |d\tilde{\nu}|(L). \tag{4.14}$$

□

The following is a trivial consequence of the above result.

Corollary 4.10. *Suppose that $X \in \mathcal{S}_{cpt}(\boldsymbol{V})$. Then*

$$\mu_j(X) = 0, \;\; \forall \dim X < j < N = \dim \boldsymbol{V}.$$

Proof. Note that a generic affine plane $L \subset \boldsymbol{V}$ of codimension $j > \dim X$ will not intersect X so the integrands in (4.14) are a.e. zero.

□

4.4. **Examples.** Let us describe how to compute the normal cycle and the curvature measures of a few simple examples. At various places we will use the natural isometry $\boldsymbol{V}^* \to \boldsymbol{V}$ to identify the normal current with a current on $S(\boldsymbol{V}) \times \boldsymbol{V}$.

Example 4.11 (Convex polyhedra)**.** Suppose that $X \subset \boldsymbol{V}$ is a compact, convex polyhedron, i.e., it is a *compact* set that can be described as the intersection of *finitely many* closed half-spaces of $\boldsymbol{V}$.

We have a natural projection $\mathrm{Proj}_X : \boldsymbol{V} \to X$ that associates to a point $\boldsymbol{p} \in \boldsymbol{V}$ *the* point in X closest to $\boldsymbol{p}$. In other words, Proj_X is characterized by the equality

$$\big\|\boldsymbol{p} - \mathrm{Proj}_X(\boldsymbol{p})\big\| = \mathrm{dist}(\boldsymbol{p}, X).$$

Consider the Lipschitz, piecewise hypersurface

$$X_1 := \big\{ \; \mathrm{dist}(\boldsymbol{p}, X) = 1 \; \big\}.$$

This carries a natural orientation as boundary of the domain

$$\big\{ \; \mathrm{dist}(\boldsymbol{p}, X) \leq 1 \; \big\}.$$

Then $\boldsymbol{N}_X$ is the current of integration over the $(n-1)$-dimensional Lipschitz submanifold of $S(\boldsymbol{V}) \times \boldsymbol{V}$ defined as the image of X_1 via the Lipschitz homeomorphism

$$X_1 \ni \boldsymbol{p} \mapsto \big(\boldsymbol{p} - \mathrm{Proj}_X(\boldsymbol{p}), \; \mathrm{Proj}_X(\boldsymbol{p}) \big) \in S(\boldsymbol{V}) \times \boldsymbol{V}.$$

□

Example 4.12 (PL sets)**.** Suppose that X is a compact PL subset of $\boldsymbol{V}$. Fix a triangulation $\mathcal{T}$ of X. Thus $\mathcal{T}$ is a collection of *closed* affine simplices whose union is X and such that any two simplices are either disjoint, or their intersection is a face of both of them. For any simplex $\sigma \in \mathcal{T}$ we denote by b_σ its barycenter. The face σ spans an affine subspace $\mathbf{Aff}(\sigma)$. We denote by $\mathbf{Aff}(\sigma)^{\perp}$ the affine orthogonal complement that passes through b_σ, and by $S^{\perp}_{\varepsilon}(\sigma)$ of radius $\varepsilon > 0$ in $\mathbf{Aff}(\sigma)^{\perp}$ centered at b_σ. Thus

$$\dim S^{\perp}_{\varepsilon}(\sigma) = \dim \boldsymbol{V} - \dim \sigma - 1.$$

The ε-normal link to σ in X is the compact set

$$\mathbf{Lk}^{\perp}_{\varepsilon}(\sigma, X) = X \cap S^{\perp}_{\varepsilon}(\sigma).$$

The *local normal Euler characteristic* of σ in X is then the integer

$$\chi^{\perp}(\sigma, X) := \lim_{\varepsilon \searrow 0} \chi\big(\, \mathbf{Lk}^{\perp}_{\varepsilon}(\sigma, X)\,\big).$$

We then have the following abstract Möbius inversion formula of Gian-Carlo Rota, [29, Chap. 3]

$$I_X = \sum_{\sigma \in \mathfrak{T}} \big(\, 1 - \chi^{\perp}(\sigma, X)\,\big) I_\sigma \tag{4.15}$$

In particular, this implies

$$\boldsymbol{N}_X = \sum_{\sigma \in \mathfrak{T}} \big(\, 1 - \chi^{\perp}(\sigma, X)\,\big) \boldsymbol{N}_\sigma. \tag{4.16}$$

The normal cycles $\boldsymbol{N}_\sigma$ can be determined as in Example 4.11.

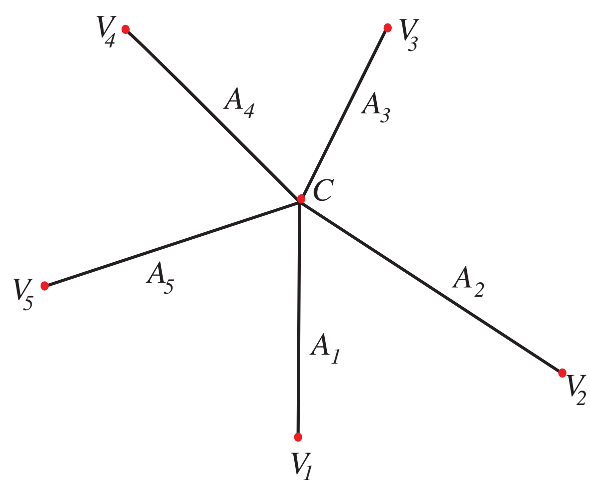

FIGURE 2. *A* 5*-arm star in* $\mathbb{R}^2$.

Let us see how this works in a simple case. Consider the 5-arm star X in Figure 2. It has an obvious triangulation consisting of the 0-dimensional simplices $C, V_1, \dots, V_5$ and the 1-dimensional simplices $A_1, \dots, A_5$. Using (4.16) we deduce

$$I_X = -4 I_C + \sum_{k=1}^{5} I_{A_k}$$

so

$$\boldsymbol{N}_X = -4 \boldsymbol{N}_C + \sum_{k=1}^{5} \boldsymbol{N}_{A_k}. \qquad \square$$

Example 4.13. Given $2N$ real numbers

$$a_1 < b_1,\ a_2 < b_2, \dots, a_N < b_N,$$

we have

$$M_{[a_1,b_1]\times\cdots\times[a_N,b_N]}(t) = \prod_{j=1}^{N}\bigl(1+\lambda_j t\bigr),\ \ \lambda_j = b_j - a_j. \qquad \square$$

Example 4.14 (Curvature measures of C^2-domains)**.** Fix an oriented Euclidean space $\boldsymbol{V}$. Set $N := \dim \boldsymbol{V}$ and denote by $\boldsymbol{D}$ the Levi-Civita connection on $T\boldsymbol{V}$. For simplicity we will denote by $\bullet$ the inner product on $\boldsymbol{V}$. Fix a relatively compact open set $D \subset \boldsymbol{V}$ with C^2-boundary $M := \partial D$.

Definition 4.15. We define the *co-oriented second fundamental form* of D to be the symmetric bilinear map

$$S_M : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to C^2(M),$$

$$S_M(X,Y) = (\boldsymbol{D}_X Y) \bullet \boldsymbol{n},\ \ X, Y \in \mathrm{Vect}(M),$$

where $\boldsymbol{n} : M \to \boldsymbol{V}$ denotes the *outer* unit normal vector field along $M = \partial D$. $\square$

For every symmetric bilinear form B on an Euclidean space we define $\mathrm{tr}_j(B)$ the j-th elementary symmetric polynomial in the eigenvalues of B, i.e.,

$$\sum_{j\geq 0} z^j \,\mathrm{tr}_j(B) = \det(\mathbb{1}_V + zB).$$

Equivalently,

$$\mathrm{tr}_j\, B = \mathrm{tr}\left(\Lambda^j B : \Lambda^j V \to \Lambda^j V\right).$$

Then

$$\mu_k(D) := \frac{1}{\boldsymbol{\sigma}_{N-k-1}}\left(\int_M \mathrm{tr}_{N-k-1}(-S_M)dV_M\right),\ \ 0 \leq k \leq N-1,$$

and

$$\mu_N(D) := \mathrm{vol}\,(D).$$

Note that

$$\mu_{N-1}(D) = \frac{1}{2}\mathrm{vol}\,(\partial D).$$

For $k = 0$ we obtain

$$\chi(D) = \mu_0(D) = \frac{1}{\boldsymbol{\sigma}_{N-1}}\left(\int_M \mathrm{tr}_{N-1}(-S_M)dV_M\right). \qquad \square$$

Example 4.16 (C^2-submanifolds with boundary)**.** Suppose $M \subset \mathbb{R}^N$ is a tame m-dimensional C^2-submanifold with boundary. Let $\nu \in C^2(TM|_{\partial M}$ denote by $\boldsymbol{n}$-the unit outer normal vector field along the boundary ∂M. For $\boldsymbol{p} \in \partial M$ we consider the $(N-m)$-dimensional hemisphere

$$\boldsymbol{S}^+_{\boldsymbol{p}} := \bigl\{\, \boldsymbol{\nu} \in T_{\boldsymbol{p}}M :\ \ \boldsymbol{\nu} \perp T_{\boldsymbol{p}}\partial M,\ \ (\boldsymbol{n}, \boldsymbol{n}(\boldsymbol{p})\,) \geq 0 \,\bigr\}.$$

For $\boldsymbol{\nu} \in \boldsymbol{S}_{\boldsymbol{p}}$ we denote by $S^{\boldsymbol{\nu}}_{\partial M} : \mathrm{Vect}(\partial M) \times \mathrm{Vect}(\partial M) \to C^2(\partial M)$ the second fundamental form,

$$S_{\partial M}(X, Y) = (\boldsymbol{D}_X Y) \bullet \boldsymbol{n}, \quad X, Y \in \mathrm{Vect}(\partial M).$$

Then

$$\mu_k(M) := \frac{1}{\boldsymbol{\sigma}_{N-k-1}} \left(\int_{\partial M} \left(\int_{\boldsymbol{S}^+_{\boldsymbol{p}}} \mathrm{tr}_{m-k-1}(-S^{\boldsymbol{\nu}}_{\partial M}) dV_{\boldsymbol{S}^+_{\boldsymbol{p}}}(\boldsymbol{\nu}) \right) dV_{\partial M}(\boldsymbol{p}) \right) + \int_M \rho_k(\boldsymbol{q}) dV_m(\boldsymbol{q}), \quad k = 0, 1, 2, \dots, m,$$

where

$$\rho_k = \begin{cases} 0, & m-k \text{ is odd}, \\ P_h(-R), & m-k = 2h, \end{cases}$$

and $P_h(R)$ is defined by (1.6). Note that for $k = m-1$ we have

$$\mu_{m-1}(M) = \frac{1}{2} \mathrm{vol}_{m-1}(\partial M).$$

For $k = 0$ we obtain the Gauss-Bonnet formula for manifolds with boundary. □

4.5. **Subanalytic isometries.** As we have seen in Subsection 1.4, the curvature measures of a compact submanifold $M \subset \mathbb{R}^N$ are intrinsic quantities, although they were defined extrinsically. This is a manifestation of a more general phenomenon.

Definition 4.17. Let $K_0 \subset \mathbb{R}^{N_0}$ and $K_1 \subset \mathbb{R}^{N_1}$ be two compact subanalytic sets. A homeomorphism $F : K_0 \to K_1$ is called a *subanalytic* isometry if it is subanalytic and, for any continuous subanalytic path $\gamma : [0, 1] \to K_0$, we have

$$\mathrm{length}\,(\gamma) = \mathrm{length}\,(F \circ \gamma).$$ □

Example 4.18. (a) Suppose that K_0 is a 2-dimesional affine simplicial complex. Assume that its 2-dimensional faces are made of a rigid material and the edges contain hinges that allow the 2-dimensional faces to rotate. Deform (if possible) the simplicial complex K_0 by allowing the movable faces to rotate. The resulting simplicial complex is subanalytically isometric to K_0.

(b) Suppose that K_0 is a flat sheet of paper. When we crumple it, without tearing, we obtain a shape that is subanalytically isometric to K_0.

(c) Suppose that there exists a definable stratification of K_0 by C^k-strata ($k \geq 1$) such that the restriction of F to each stratum is a definable isometric C^k-immersion. Then F is a subanalytic isometry. It turns out that all analytic isometries are of this type. □

Theorem 4.19. *If the compact subanalytic sets $K_0 \subset \boldsymbol{V}_0$ and $K_1 \subset \boldsymbol{V}_1$ are subanalytically isometric, then*

$$\mu_i(K_0) = \mu_i(K_1), \quad \forall i.$$

Proof. We present a proof that we learned from J. Fu, [11]. We argue by induction on the common dimension of K_i. The result is trivially true when $\dim K_i = 0$. Suppose the result is true for compact subanalytic sets of dimension$\leq m-1$. We prove that it is true for compact subanalytic sets of dimension m.

Let $F : K_0 \to K_1$ be a subanalytic isometry. Fix a stratification $\mathcal{S}$ of K_0 with C^3-strata such that the restriction of F to each stratum is a C^3 isometric immersion. Denote by X_0 the union of strata of dimension $\leq m-1$. Set

$$Z_0 = K_0 \setminus X_0.$$

As in the proof of Theorem 3.6 we can find a proper C^2 subanalytic function $f : \boldsymbol{V}_0 \to [0, \infty)$ such that

$$X_0 = f^{-1}(0).$$

There exists $r_0 > 0$ such that any $\varepsilon \in (0, r_0)$ is a regular value of f. Set:

$$X_\varepsilon := \{f \leq \varepsilon\}, \;\; Z_\varepsilon = \{f \geq \varepsilon\}, \;\; Y_\varepsilon = \{f = \varepsilon\}.$$

Then X_ε, and Z_ε are m-dimensional compact subanalytic sets, $X_\varepsilon \cap Z_\varepsilon = Y_\varepsilon$. Additionally Z_ε is a compact C^2-submanifold with boundary Y_ε. Since F is C^3 on $Z_0 \supset Z_\varepsilon$ we defice that $F(Z_\varepsilon)$ is a also a C^2-manifold with boundary. Note that for any $\varepsilon \in (0, r_0)$ we have

$$\begin{aligned} \mu_i(K_0) &= \mu_i\bigl(X_\varepsilon\bigr) + \mu_i\bigl(Z_\varepsilon\bigr) - \mu_i\bigl(Y_\varepsilon\bigr), \\ \mu_i(K_1) &= \mu_i\bigl(\,F(X_\varepsilon)\,\bigr) + \mu_i\bigl(\,F(Z_\varepsilon)\,\bigr) - \mu_i\bigl(\,F(Y_\varepsilon)\,\bigr). \end{aligned} \tag{4.17}$$

Since Y_ε and $F(Y_\varepsilon)$ are C^2 closed submanifolds we deduce.

$$\mu_i\bigl(Y_\varepsilon\bigr) = \mu_i\bigl(\,F(Y_\varepsilon)\,\bigr)$$

Since Z_ε and $F(Z_\varepsilon)$ are C^2-isometric submanifolds with boundary we deduce from the computations in Example 4.16 that

$$\mu_i\bigl(Z_\varepsilon\bigr) = \mu_i\bigl(\,F(Z_\varepsilon)\,\bigr).$$

The induction assumption implies that

$$\mu_i\bigl(X_0\bigr) = \mu_i\bigl(\,F(X_0)\,\bigr).$$

Finally, the Approximation Theorem 3.7 implies that

$$\lim_{\varepsilon \searrow 0} \mu_i\bigl(X_\varepsilon\bigr) = \mu_i\bigl(X_0\bigr) = \mu_i\bigl(\,F(X_0)\,\bigr) = \lim_{\varepsilon \searrow 0} \mu_i\bigl(\,F(X_\varepsilon)\,\bigr).$$

If we let $\varepsilon > 0$ in (4.17) we deduce $\mu_i(K_0) = \mu_i(K_1)$. □

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