

Geometric connections and geometric Dirac operators on contact manifolds

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Version 2: January 16, 2001

Introduction

Suppose (M^{2n+1}, η, g, J) is a positively oriented, metric contact manifold. More precisely, this means that η is a 1-form such that $\frac{1}{n!} \eta \wedge (d\eta)^n = dv_g$, J is a skew-symmetric endomorphism of TM such that

$$J^2 X = -X + \eta(X)\xi, \quad d\eta(X, Y) = g(JX, Y), \quad \forall X, Y \in \text{Vect}(M),$$

and ξ is the Reeb vector field determined by $\eta(\xi) = 1$, $\xi \lrcorner d\eta = 0$. Set $V = \ker \eta$.

The operator J induces an almost complex structure on V , and we get decompositions

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}, \quad \Lambda^* V^* \otimes \mathbb{C} = \bigoplus_{0 \leq p+q \leq 2n} \Lambda^{p,q} V^*.$$

We set $\Omega^{p,q}(V^*) := C^\infty(\Lambda^{p,q} V^*)$. The Lie derivative along ξ has the property $L_\xi \Omega^{0,p}(V^*) \subset \Omega^{0,p}(V^*) \oplus \Omega^{1,p-1}(V^*)$, and we define $L_\xi^V : \Omega^{0,p}(V^*) \rightarrow \Omega^{0,p}(V^*)$ by $L_\xi^V \phi = (L_\xi \phi)^{0,p}$. The operator iL_ξ^V is symmetric. There exists a natural operator

$$\bar{\partial}_V : \Omega^{0,*}(V^*) \rightarrow \Omega^{0,*+1}(V^*).$$

We can form a *contact Hodge-Dolbeault* operator

$$\mathcal{H} : \Omega^{0,*}(V^*) \rightarrow \Omega^{0,*}(V^*)$$

which with respect to the decomposition $\Omega^{0,even} \oplus \Omega^{0,odd}(V^*)$ has the block form

$$\mathcal{H} = \begin{bmatrix} -iL_\xi^V & \sqrt{2}(\bar{\partial}_V + \bar{\partial}_V^*) \\ \sqrt{2}(\bar{\partial}_V^* + \bar{\partial}_V) & iL_\xi^V \end{bmatrix}$$

*This work was partially supported by NSF grant DMS-0071820.

This is a symmetric Dirac type operator and it is an example of geometric Dirac operator, i.e. an operator of Dirac type defined entirely in geometric terms with no mention of $spin^c$ structures.

On the other hand, the contact form defines a $spin^c$ structure with determinant line K_M^{-1} , where canonical line bundle of M is defined by

$$K_M := \det V^{0,1} \cong \Lambda^{n,0} V^*.$$

The associated bundle of complex spinors is $\mathbb{S}_c = \Lambda^{0,*} V^*$. The Clifford multiplication by $\mathbf{i}\eta$ is an involution of $\mathbb{S}_c$ and the ± 1 eigenspaces are

$$\mathbb{S}_c^\pm \cong \Lambda^{0,even/odd} V^*.$$

A metric connection ∇ on TM such that $\nabla J = 0$ is called a *contact connection*. If additionally M is a *CR* manifold, i.e. the distribution $V^{1,0}$ is integrable, then we define a *CR* connection to be a contact connection such that its torsion satisfies

$$T(X, Y) = 0 \quad \forall X, Y \in C^\infty(V^{1,0}).$$

A metric connection on TM together with a hermitian A connection on K_M^{-1} canonically define a Dirac operator $\mathfrak{D}(\nabla, A)$ on $\mathbb{S}_c$. The connection ∇ is called *nice* if $\mathfrak{D}(\nabla, A)$ is symmetric for any hermitian connection A on K_M^{-1} . Two metric connections ∇^1 and ∇^2 are called *Dirac equivalent* if there exists a hermitian connection A on K_M^{-1} such that $\mathfrak{D}(\nabla^1, A) = \mathfrak{D}(\nabla^2, A)$.

The first question we address in this paper is the following.

- *Can we find a contact connection ∇ and a hermitian connection A on K_M^{-1} $\mathfrak{D}(\nabla, A) = \mathcal{H}$? (A connection ∇ with this property is said to be *adapted to $\mathcal{H}$* .)*

Suppose additionally that M is also *spin*. We denote by $\mathfrak{D}_0$ the associated *spin* Dirac operator. The second question we ask is the following.

- *Does there exist a contact connection ∇ on TM such that $\mathfrak{D}(\nabla) = \mathfrak{D}_0$? In other words, is there any contact connection Dirac equivalent to the Levi-Civita connection?*

To address these questions we rely on the work P. Gauduchon, (see [5] or §2.1, §2.2), concerning hermitian connections on almost-hermitian manifolds. We can naturally associate two almost hermitian manifolds to M .

- The cylinder $\hat{M} = \mathbb{R} \times M$ with metric $\hat{g} = dt^2 + g$ and almost complex structure $\hat{J}$ defined by $\hat{J}\partial_t = \xi$, $\hat{J}|_V = J$.
- The symplectization $\tilde{M} = \mathbb{R}_+ \times M$ with symplectic form $\omega = \hat{d}(t\eta)$, metric $\tilde{g} = dt^2 + \eta^{\otimes 2} + tg|_V$, and almost complex structure $\tilde{J} = \hat{J}$.

To answer the first question we use the cylinder case and a certain natural perturbation of the first canonical connection on $(T\hat{M}, \hat{g}, \hat{J})$. This new connection on $T\hat{M}$ preserves the splitting $T\hat{M} = \mathbb{R}\partial_t \oplus TM$ and induces a connection on TM with the required properties (see §3.1). Moreover, when M is a *CR* manifold this connection coincides with the Webster connection, [11, 14].

To answer the second question we use the symplectization $\tilde{M}$ and a natural perturbation of the Chern connection on $T\tilde{M}$. We obtain a new connection on $\tilde{M}$ whose restriction to

$\{1\} \times M$ is a contact connection (see §3.3). When M is CR this contact connection is also CR , but it never coincides with the Webster connection. We are not aware whether this contact connection has been studied before.

These two connections are examples of geometric connections. In fact we prove a much stronger result.

Theorem. (a) *On any metric contact manifold there exists a nice contact connection adapted to $\mathcal{H}$ and a nice contact connection Dirac equivalent to the Levi-Civita connection. If the manifold is CR these connections are also CR .*
(b) *On a CR manifold each Dirac equivalence class of connections contains at most one nice CR connection. Moreover, the Webster connection is the unique nice CR connection adapted to $\mathcal{H}$.*

Finally, we present several Weitzenböck formulæ involving the operator $\mathcal{H}$ (see §3.2). We expect these facts will have applications to three dimensional Seiberg-Witten theory.

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1 General properties geometric Dirac operators

§1.1 Dirac operators compatible with a metric connection Suppose (M, g) is an oriented, n -dimensional Riemannian manifold. We will denote a generic local, oriented, synchronous frame of TM by (e_i) . Its dual coframe is denoted by (e^i) . We will denote the natural duality between a vector space and its dual by $\langle \bullet, \bullet \rangle$.

A *metric connection* on TM is a connection ∇ on TM such that

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad \forall X, Y, Z \in \text{Vect}(M).$$

The *torsion* of a metric connection ∇ is the TM -valued 2-form $T = T(\nabla)$ defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

The Levi-Civita connection, denoted by D in the sequel is the metric connection uniquely determined by the condition $T(D) = 0$. Any metric connection ∇ can be uniquely written as $D + A$, where $A \in \Omega^1(\text{End}_-(TM))$, where End_- denotes the space of skew-symmetric endomorphisms. A is called the *potential* of ∇ .

There are natural *isomorphisms*

$$\Omega^2(TM) \longrightarrow \Omega^2(T^*M), \quad T \mapsto T^\dagger$$

$$\Omega^1(\text{End}_-(TM)) \mapsto \Omega^2(T^*M), \quad A \mapsto A^\dagger$$

defined as follows.

$$\Omega^2(TM) \ni T \mapsto T^\dagger, \quad \langle X, T^\dagger(Y, Z) \rangle = g(X, T(Y, Z))$$

and

$$\Omega^1(\text{End}_-(TM)) \ni A \mapsto A^\dagger, \quad \langle X, A^\dagger(Y, Z) \rangle = g(A_X Y, Z) =: A^\dagger(X; Y, Z),$$

$\forall X, Y, Z \in \text{Vect}(M)$. In local coordinates, if

$$T(e_j, e_k) = \sum_i T_{jk}^i e_i, \quad A_{e_i} e_j = \sum_k A_{ij}^k e_k$$

then

$$T^\dagger(e_j, e_k) = \sum_i T_{jk}^i e^i, \quad A^\dagger(e_j, e_k) = \sum_i A_{ij}^k e^i,$$

or equivalently, $T_{ijk}^\dagger = T_{jk}^i$, $A_{ijk}^\dagger = A_{ij}^k$. To simplify the exposition, when working in local coordinates, we will write A_{ijk} instead of $A_{ijk}^\dagger$ etc. Define

$$\text{tr} : \Omega^2(T^*M) \rightarrow \Omega^1(M), \quad \Omega^2(T^*M) \ni (B_{ijk}) \mapsto (\text{tr } B) = \sum_{i,k} B_{ik} e^k$$

and the *Bianchi map*

$$\mathfrak{b} : \Omega^2(T^*M) \rightarrow \Omega^3(M),$$

$$\Omega^2(T^*M) \ni (B_{ijk}) \mapsto \mathfrak{b}B = \sum_{i < j < k} (B_{ijk} + B_{kij} + B_{jki}) e^i \wedge e^j \wedge e^k.$$

Note that if $B \in \Omega^3(M) \subset \Omega^2(T^*M)$ then $B = \frac{1}{3} \mathfrak{b}B$.

For any $A \in \text{End}(TM)$ and $\alpha \in \Omega^1(M)$ we define $A \wedge \alpha \in \Omega^2(T^*M)$ by the equality

$$(A \wedge \alpha)(X; Y, Z) = (AX)_\flat \wedge \alpha(Y, Z), \quad \forall X, Y, Z \in \text{Vect}(M),$$

where $\bullet_\flat$ (resp. $\bullet^\flat$) denotes the g -dual of a vector (resp. covector) $\bullet$. The following elementary result lists some basic properties of the above operation.

Lemma 1.1. *Let $A \in \text{End}(TM)$, $\alpha \in \Omega^1(M)$ and set*

$$A_+ = \frac{1}{2}(A + A^*), \quad A_- = \frac{1}{2}(A - A^*).$$

Then

$$\text{tr}(A \wedge \alpha) = (\text{tr } A)\alpha - A^t \alpha,$$

and

$$\mathfrak{b}(A \wedge \alpha) = 2\omega_{A_-} \wedge \alpha,$$

where

$$\omega_{A_-}(X, Y) = g(A_- X, Y), \quad \forall X, Y \in \text{Vect}(M).$$

Using the above operations we can orthogonally decompose $\Omega^2(T^*M)$ as

$$\Omega^2(T^*M) = \Omega^1(M) \oplus \Omega^3(M) \oplus \Omega_0^2(T^*M)$$

where

$$\Omega_0^2 := \left\{ A \in \Omega^2(T^*M); \quad \mathfrak{b}A = \text{tr } A = 0 \right\},$$

and $\Omega^1(M)$ embeds in $\Omega^2(T^*M)$ via the map

$$\Omega^1(M) \rightarrow \Omega^2(T^*M), \quad \alpha \mapsto \tilde{\alpha} := \frac{1}{n-1} (\mathbf{1}_{TM} \wedge \alpha)$$

Using this orthogonal splitting we can decompose any $A \in \Omega^2(T^*M)$ as

$$A = \widetilde{\text{tr } A} + \frac{1}{3}\mathfrak{b}A + P_0 A, \quad P_0 A := A - \widetilde{\text{tr } A} - \frac{1}{3}\mathfrak{b}A \in \Omega_0^2(T^*M).$$

The next result, whose proof can be found in [5], states that a metric connection is determined by its torsion in a very explicit way.

Proposition 1.2. *Suppose that ∇ is a metric connection with potential A and torsion T . Then*

$$T^\dagger = -A^\dagger + \mathfrak{b}A^\dagger, \tag{1.1}$$

$$A^\dagger = -T^\dagger + \frac{1}{2}\mathfrak{b}T^\dagger. \tag{1.2}$$

In particular

$$\mathfrak{b}A^\dagger = \frac{1}{2}\mathfrak{b}T^\dagger, \quad \text{tr } A^\dagger = -\text{tr } T^\dagger.$$

Since all the computations we are about to perform are local we can assume that M is equipped with a *spin* structure and we denote by $\mathbb{S}$ the associated complex spinor bundle¹. We have a *Clifford multiplication map*

$$\mathbf{c} : \Omega^*(M) \rightarrow \text{End}(\mathbb{S}).$$

A hermitian connection $\tilde{\nabla}$ on $\mathbb{S}$ is said to be compatible with the Clifford multiplication and the metric connection ∇ on TM if

$$\tilde{\nabla}_X (\mathbf{c}(\alpha)\psi) = \mathbf{c}(\nabla_X \alpha)\psi + \mathbf{c}(\alpha)\tilde{\nabla}_X \psi, \quad \forall X \in \text{Vect}(M), \quad \alpha \in \Omega^1(M), \quad \psi \in C^\infty(\mathbb{S}).$$

We denote by $\mathfrak{A}_\nabla = \mathfrak{A}_\nabla(\mathbb{S})$ the space of hermitian connections on $\mathbb{S}$ compatible with the Clifford multiplication and ∇ .

Proposition 1.3. *The space $\mathfrak{A}_\nabla(\mathbb{S})$ is an affine space modelled by the space $\mathfrak{i}\Omega^1(M)$ of imaginary 1-forms on M .*

Proof Suppose $\tilde{\nabla}^0, \tilde{\nabla}^1 \in \mathfrak{A}_\nabla$. Set $C := \tilde{\nabla}^1 - \tilde{\nabla}^0 \in \Omega^1(\text{End}(\mathbb{S}))$. Since both $\tilde{\nabla}^i$, $i = 0, 1$, are compatible with the Clifford multiplication and ∇ we deduce that for every $X \in \text{Vect}(M)$ the endomorphism $C(X) := X \lrcorner C$ commutes with the Clifford multiplication. Since the fibers of $\mathbb{S}$ are irreducible Clifford modules we deduce from Schur's Lemma that $C(X)$ is a constant in each fiber, i.e $C \in \Omega^1(M) \otimes \mathbb{C}$. Since both $\tilde{\nabla}^i$ are hermitian connections we conclude that C must be purely imaginary 1-form. ■

Definition 1.4. *A geometric Dirac operator on $\mathbb{S}$ is a first order partial differeintial operator $\mathfrak{D}$ of the form*

$$\mathfrak{D} = \mathfrak{D}(\tilde{\nabla}) : C^\infty(\mathbb{S}) \xrightarrow{\tilde{\nabla}} C^\infty(T^*M \otimes \mathbb{S}) \xrightarrow{\mathbf{c}} C^\infty(\mathbb{S})$$

where $\tilde{\nabla} \in \mathfrak{A}_\nabla(\mathbb{S})$ for some metric connection ∇ on TM . The geometric Dirac operator is called nice if it is formally self-adjoint.

Locally, a geometric Dirac operator has the form

$$\mathfrak{D}(\tilde{\nabla}) = \sum_i \mathbf{c}(e^i) \tilde{\nabla}_{\mathcal{E}_i}.$$

Every metric connection ∇ canonically determines a connection $\hat{\nabla} \in \mathfrak{A}_\nabla(\mathbb{S})$ locally described as follows. If the $\underline{\text{so}}(n)$ -valued 1-form ω associated by the frame (e_i) to the connection ∇ is defined by

$$\nabla e_j = \sum_{i,k} e^k \otimes \omega_{kj}^i e_i, \quad \omega_{kj}^i + \omega_{ki}^j = 0,$$

then the induced connection on $\mathbb{S}$ is given by the $\text{End}_-(\mathbb{S})$ -valued 1-form (see [9])

$$\hat{\omega} = -\frac{1}{4} \sum_{i,j,k} e^k \otimes \omega_{kj}^i \mathbf{c}(e^i) \mathbf{c}(e^j). \quad (1.3)$$

¹ $\mathbb{S}$ is $\mathbb{Z}_2$ -graded if $n = \dim M$ is even and it is ungraded if n is odd.

We set $\mathfrak{D}(\nabla) := \mathfrak{D}(\hat{\nabla})$ and $\mathfrak{D}_0 := \mathfrak{D}(\hat{D})$. $\mathfrak{D}_0$ is the usual *spin* Dirac operator. We see that every geometric operator has the form

$$\mathfrak{D} = \mathfrak{D}(\nabla) + \mathbf{c}(\mathbf{ia})$$

where ∇ is a metric connection on M and $a \in \Omega^1(M)$. The connection ∇ is called *nice* if $\mathfrak{D}(\nabla)$ is nice. We denote by $\mathfrak{A}_{\text{nice}}(M)$ the space of nice connections on M .

Proposition 1.5. *The connection ∇ with torsion T is nice if and only if $\text{tr } T^\dagger = 0$.*

Proof Note that

$$\nabla_i e_j = \nabla_j e_i + T_{ij}, \quad \forall i, j \quad (1.4)$$

and

$$\text{div}_g(e_i) = 0, \quad \forall i. \quad (1.5)$$

We have (at x_0)

$$\mathfrak{D}^* = \sum_k \hat{\nabla}_k^* \mathbf{c}(e^k)^* = \sum_k \tilde{\nabla}_k \mathbf{c}(e^k) = \sum_k \mathbf{c}(\nabla_k e^k) + \sum_k \mathbf{c}(e^k) \hat{\nabla}_k = \mathbf{c} \left(\sum_k \nabla_k e^k \right) + \mathfrak{D}.$$

Thus ∇ is nice if and only if

$$\mathbf{c} \left(\sum_k \nabla_k e^k \right) = 0$$

We compute easily that

$$(\nabla_j e^i)(e_k) = -e^i(\nabla_j e_k) = -g(e_i, \nabla_j e_k) = -g(e_i, \nabla_k e_j + T_{jk})$$

so that

$$\nabla_j e^i = - \sum_k g(e_i, \nabla_k e_j + T_{jk}) e^k. \quad (1.6)$$

Hence

$$\sum_k \nabla_k e^k = - \sum_k \sum_i g(e_k, \nabla_i e_k + T_{ki}) e^i$$

$$(g(e_k, \nabla_i e_k) = 0 \text{ at } x_0)$$

$$= - \sum_i \left(\sum_k g(e_k, T_{ki}) \right) e^i.$$

This concludes the proof of the proposition. ■

Proposition 1.6. *Suppose that $\nabla = D + A$ is a nice connection on TM . Then*

$$\mathfrak{D}(\nabla) = \mathfrak{D}_0 + \frac{1}{2}\mathbf{c}(\mathfrak{b}A^\dagger) = \mathfrak{D}_0 + \frac{1}{4}\mathbf{c}(\mathfrak{b}T^\dagger).$$

Proof Observe that

$$\hat{\nabla} = \hat{D} - \frac{1}{4} \sum_{i,j,k} e^k \otimes A_{kj}^i \mathbf{c}(e^i) \mathbf{c}(e^j) = \hat{D} - \frac{1}{4} \sum_{i,j,k} e^k \otimes A_{kji} \mathbf{c}(e^i) \mathbf{c}(e^j)$$

so that

$$\mathfrak{D}(\nabla) - \mathfrak{D}_0 = -\frac{1}{4} \sum_{i,j,k} A_{kji} \mathbf{c}(e^k) \mathbf{c}(e^i) \mathbf{c}(e^j) = \frac{1}{4} \sum_{i,j,k} A_{kij} \mathbf{c}(e^k) \mathbf{c}(e^i) \mathbf{c}(e^j).$$

Since $\text{tr } A = 0$ we deduce that the contributions corresponding to triplets (i, j, k) where two entries are identical add up to zero. Hence

$$\frac{1}{4} \sum_{i,j,k} A_{kij} \mathbf{c}(e^k) \mathbf{c}(e^i) \mathbf{c}(e^j) = \frac{1}{4} \sum_{i < j < k} \left((\mathfrak{b}A)_{ijk} - (\mathfrak{b}A)_{jik} \right) \mathbf{c}(e^i) \mathbf{c}(e^j) \mathbf{c}(e^k) = \frac{1}{2} \mathbf{c}(\mathfrak{b}A). \quad \blacksquare$$

Corollary 1.7. *Suppose $\mathfrak{D} = \mathfrak{D}_0 + \mathbf{c}(\varpi)$, $\varpi \in \Omega^3(M)$. Then*

$$\mathfrak{D} = \mathfrak{D}(\nabla)$$

where $\nabla = D + A$, $A^\dagger = \frac{2}{3}\varpi$. $\blacksquare$

The above result can also be rephrased in the language of superconnections described e.g. in [1]. Suppose $\varpi \in \Omega^3(M)$. The operator $d + \mathbf{c}(\varpi)$ is a superconnection on the trivial line bundle $\underline{\mathbb{C}}$. Taking the tensor product it with the connection $\hat{D}$ on $\mathbb{S}$ we obtain a superconnection on $\mathbb{S} = \mathbb{C} \otimes \mathbb{S}$

$$\mathbb{A}_\varpi := \varpi \otimes \mathbf{1} + \mathbf{1} \otimes \hat{D} : C^\infty(\mathbb{S}) \rightarrow \Omega^*(\mathbb{S}).$$

The Dirac operator determined by this superconnection is

$$\mathbf{c} \circ \mathbb{A}_\varpi = \mathfrak{D}_0 + \mathbf{c}(\omega).$$

Two connections $\nabla^0, \nabla^1 \in \mathfrak{A}_{\text{nice}}(M)$ will be called *Dirac equivalent* if

$$\mathfrak{D}(\hat{\nabla}^0) = \mathfrak{D}(\hat{\nabla}^1).$$

The above results show that two connections ∇^0 and ∇^1 are Dirac equivalent if and only if

$$\mathbf{c}(\mathfrak{b}T(\nabla^0)^\dagger) = \mathbf{c}(\mathfrak{b}T(\nabla^1)^\dagger) \iff \mathfrak{b}T(\nabla^1)^\dagger = \mathfrak{b}T(\nabla^0)^\dagger.$$

§1.2 Weitzenböck formulæ Suppose (E, h) is a Hermitian vector bundle over M . A *generalized Laplacian* is a *formally self-adjoint*, second order partial differential operator $L : C^\infty(E) \rightarrow C^\infty(E)$ whose principal symbol satisfies

$$\sigma_L(\xi) = -|\xi|_g^2 \mathbf{1}_E.$$

The following classical result is the basis of all the constructions in this section. We include here a proof because of its relevance in the sequel.

Proposition 1.8. ([1, Sec. 2.1], [6, Sec. 4.1.2]) *Suppose L is a generalized Laplacian on E . Then there exists a unique hermitian connection $\tilde{\nabla}$ on E and a unique selfadjoint endomorphism $\mathcal{R}$ of E such that*

$$L = \tilde{\nabla}^* \tilde{\nabla} + \mathcal{R} \quad (1.7)$$

We will refer to this presentation of a generalized Laplacian as the Weitzenböck presentation of L .

Proof Choose an arbitrary hermitian connection ∇ on E . Then $L_0 = \nabla^* \nabla$ is a generalized Laplacian so that $L - L_0$ is a first order operator which can be represented as

$$L - L_0 = A \circ \nabla + B$$

where

$$A : C^\infty(T^*M \otimes E) \rightarrow C^\infty(E)$$

is a bundle morphism and B is an endomorphism of E . We will regard A as an $\text{End}(E)$ -valued 1-form on M . Hence

$$L = \nabla^* \nabla + A \circ \nabla + B. \quad (1.8)$$

The connection ∇ induces a connection on $\text{End}(E)$ which we continue to denote with ∇

$$\nabla : C^\infty(\text{End}(E)) \rightarrow \Omega^1(\text{End}(E)).$$

We define the *divergence* of A by

$$\mathbf{div}_g(A) := -\nabla^* A.$$

If (e_i) is a local synchronous frame at x_0 and, if $A = \sum_i A_i e^i$, then, at x_0 , we have

$$\mathbf{div}_g(A) = \sum_i \nabla_i A_i.$$

Note that since $(L - L_0) = \sum_i A_i \nabla_i + B$ is formally selfadjoint we deduce

$$A_i^* = -A_i, \quad \mathbf{div}_g(A) = B - B^*. \quad (1.9)$$

We seek a hermitian connection $\tilde{\nabla} = \nabla + C$, $C \in \Omega^1(\text{End}(E))$ and an endomorphism $\mathcal{R}$ of E such that

$$\tilde{\nabla}^* \tilde{\nabla} + \mathcal{R} = \nabla^* \nabla + A \circ \nabla + B.$$

We set $C_i := e_i \lrcorner C$ so that we have the local description

$$\tilde{\nabla} = \sum_i e^i \otimes (\nabla_i + C_i), \quad C_i^* = -C_i, \quad \forall i.$$

Then, as in [9], Example 9.1.26, we deduce that, at x_0

$$\tilde{\nabla}^* \tilde{\nabla} = - \sum_i (\nabla_i + C_i)(\nabla_i + C_i)$$

$$(\langle C_i \rangle^2 := C_i C_i^* = -C_i^2)$$

$$= - \sum_i \nabla_i^2 - \sum_i \nabla_i C_i - 2 \sum_i C_i \nabla_i + \sum_i \langle C_i \rangle^2$$

$$(\langle C \rangle^2 = \sum_i \langle C_i \rangle^2)$$

$$= \nabla^* \nabla - 2C \circ \nabla - \mathbf{div}_g(C) + \langle C \rangle^2 = \nabla^* \nabla + A \circ \nabla + B - \mathcal{R}.$$

We deduce immediately that

$$C = -\frac{1}{2}A, \quad \mathcal{R} = B - \frac{1}{2}\mathbf{div}_g(A) - \langle C \rangle^2 \stackrel{(1.9)}{=} \frac{1}{2}(B + B^*) - \frac{1}{4}\langle A \rangle^2. \quad (1.10)$$

The proposition is proved. ■

If $\mathfrak{D}$ is a geometric Dirac operator on $\mathbb{S}$ then both $\mathfrak{D}^* \mathfrak{D}$ and $\mathfrak{D} \mathfrak{D}^*$ are generalized Laplacians. Suppose now that ∇ is a nice connection on our spin manifold (M, g) . It determines a nice Dirac operator $\mathfrak{D}(\nabla)$. We denote by $\nabla^\mathfrak{w}$ and respectively $\mathcal{R}_\nabla$ the Weitzenböck connection and respectively remainder of the generalized Laplacian $\mathfrak{D}(\nabla)^2$. A classical result of Lichnerowicz states that if ∇ is the Levi-Civita connection then $\nabla^\mathfrak{w} = \hat{\nabla}$ and $\mathcal{R} = \frac{s}{4}$, where s is the scalar curvature of the Riemann metric g . When ∇ is not symmetric the situation is more complicated. We will present some general formulæ describing $\nabla^\mathfrak{w}$ and $\mathcal{R}$.

$$\begin{aligned} \mathfrak{D}^2 &= \sum_{i,j} \mathbf{c}(e^i) \hat{\nabla}_i \mathbf{c}(e^j) \hat{\nabla}_j = \sum_{i,j} \mathbf{c}(e^i) \mathbf{c}(e^j) \hat{\nabla}_i \hat{\nabla}_j + \sum_{i,j} \mathbf{c}(e^i) \mathbf{c}(\nabla_i e^j) \hat{\nabla}_j \\ &= - \sum_i \hat{\nabla}_i^2 + \sum_{i < j} \mathbf{c}(e^i) \mathbf{c}(e^j) [\hat{\nabla}_i, \hat{\nabla}_j] + \sum_{i,j} \mathbf{c}(e^i) \mathbf{c}(\nabla_i e^j) \nabla_j \\ &= \hat{\nabla}^* \hat{\nabla} + \sum_{i,j} \mathbf{c}(e^i) \mathbf{c}(\nabla_i e^j) \nabla_j + \sum_{i < j} \hat{R}_{ij} \mathbf{c}(e^i) \mathbf{c}(e^j) \end{aligned}$$

where

$$\hat{R} = \sum_{i < j} e^i \wedge e^j \hat{R}_{ij}$$

denotes the curvature of $\hat{\nabla}$. We need to better understand the quantity A (the coefficient of the first order part of $\mathfrak{D}^2$) which at x_0 is defined as

$$A = \sum_{i,j} e^j \otimes \mathbf{c}(e^i) \mathbf{c}(\nabla_i e^j).$$

Using (1.6) we deduce

$$\begin{aligned} A &= \sum_{i,j} e^j \otimes \mathbf{c}(e^i) \mathbf{c}(-\sum_k \langle e_j, \nabla_k e_i + T_{ik} \rangle e^k) \\ &= -\sum_{i,j,k} e^j \otimes \langle e_j, \nabla_k e_i \rangle \mathbf{c}(e^i) \mathbf{c}(e^k) - \sum_{i,j,k} e^j \otimes \langle e_j, T_{ik} \rangle \mathbf{c}(e^i) \mathbf{c}(e^k) \\ &= \sum_j e^j \sum_k \langle e_j, \nabla_k e_k \rangle - \sum_j e^j \otimes \sum_{i \neq k} \langle e_j, \nabla_k e_i \rangle \mathbf{c}(e^i) \mathbf{c}(e^k) - \sum_{j,i,k} e^j \otimes \langle e_j, T_{ik} \rangle \mathbf{c}(e^i) \mathbf{c}(e^k) \\ (\langle e_j, \nabla_k e_k \rangle &= -\langle \nabla_k e_j, e_k \rangle \text{ at } x_0, \nabla_k e_i - \nabla_i e_k = T_{ki} = -T_{ik}) \\ &= -\sum_j e^j \sum_k \langle \nabla_k e_j, e_k \rangle + \sum_j e^j \otimes \sum_{i < k} \langle e_j, T_{ik} \rangle \mathbf{c}(e^i) \mathbf{c}(e^k) \\ &\quad - 2 \sum_j e^j \otimes \sum_{i < k} \langle e_j, T_{ik} \rangle \mathbf{c}(e^i) \mathbf{c}(e^k) \\ (\text{switch the order of summation in the first term}) \\ &= -\sum_k \left(\sum_j \langle \nabla_k e_j, e_k \rangle e^j \right) - \sum_j e^j \otimes \sum_{i < k} \langle e_j, T_{ik} \rangle \mathbf{c}(e^i) \mathbf{c}(e^k) \\ &= \sum_k \nabla_k e^k - \sum_j e^j \otimes \sum_{i < k} \langle e_j, T_{ik} \rangle \mathbf{c}(e^i) \mathbf{c}(e^k) \\ (\sum_k \nabla_k e^k &= 0) \\ &= -\frac{1}{2} \sum_{i,j,k} e^j \otimes \langle e_j, T_{ik} \rangle \mathbf{c}(e^i) \mathbf{c}(e^k) =: -\alpha(T). \end{aligned}$$

We deduce

$$\mathfrak{D}^2 = \hat{\nabla}^* \hat{\nabla} - \alpha(T) \circ \hat{\nabla} + c(\hat{R}) \quad (1.11)$$

where

$$c(\hat{R}) := \sum_{i < j} \mathbf{c}(e^i) \mathbf{c}(e^j) \hat{R}_{ij}.$$

Using the equalities (1.10) we reach the following conclusion.

Proposition 1.9. *We have the Weitzenböck formula*

$$\mathfrak{D}^2 = (\nabla^{\mathfrak{w}})^* \nabla^{\mathfrak{w}} + \mathcal{R}_{\nabla}$$

where

$$\nabla^{\mathfrak{w}} = \hat{\nabla} + \frac{1}{2}\alpha(T) = \hat{\nabla} + \frac{1}{4} \sum_{i,j,k} e^i \otimes T_{ijk} \mathbf{c}(e^j) \mathbf{c}(e^k). \quad (1.12)$$

$$\mathcal{R}_{\nabla} = \frac{1}{2}(c(\hat{R}) + c(\hat{R})^*) - \frac{1}{4}\langle \alpha(T) \rangle^2, \quad (1.13)$$

where T denotes the torsion of ∇ and $\hat{R}$ the curvature of $\hat{\nabla}$. ■

Remark 1.10. Observe that $\nabla^{\mathfrak{w}}$ is the connection on $\mathbb{S}$ induced by the nice connection $\nabla' = \nabla + A$ where $A^\dagger = T^\dagger$. Using (1.1) we deduce

$$T^\dagger(\nabla') = T(\nabla)^\dagger - A^\dagger + \mathfrak{b}A^\dagger = \mathfrak{b}T(\nabla)^\dagger$$

The Weitzenböck remainder can be given a more explicit description. More precisely we know from Proposition 1.6 that

$$\mathfrak{D}(\nabla) = \mathfrak{D}_0 + \frac{1}{4}\mathbf{c}(\mathfrak{b}T^\dagger).$$

We set $\varpi := \frac{1}{4}\mathfrak{b}T^\dagger$. As explained at the end of §1.1, $\mathfrak{D}(\nabla)$ is the Dirac operator associated to the superconnection $\hat{D} + \varpi$. Using [2, Thm. 1.3] we deduce that the Weitzenböck remainder of $\mathfrak{D}^2$ is

$$\mathcal{R}_{\nabla} = \frac{s}{4} + \mathbf{c}(d\varpi) - 2\|\varpi\|^2 = \frac{s}{4} + \frac{1}{4}\left(\mathbf{c}(d\mathfrak{b}T^\dagger) - \frac{1}{2}\|\mathfrak{b}T^\dagger\|^2\right).$$

where $\|\bullet\|$ denotes the pointwise norm of a differential form and s denotes the scalar curvature of g .

The following result summarizes the main facts we proved so far.

Theorem 1.11. *Denote by $\mathfrak{D}_{spin}$ the spin-Dirac operator induced by the Levi-Civita D , $\mathfrak{D}_{spin} = \mathfrak{D}(\hat{D})$. Any geometric Dirac operator $\mathfrak{D}$ can be written as*

$$\mathfrak{D} = \mathfrak{D}_{spin} + \mathbf{c}(\varpi) + \mathbf{c}(\mathbf{ia}), \quad a \in \Omega^1(M), \quad \varpi \in \Omega^3(M).$$

Additionally, if $\nabla = D + \frac{2}{3}\varpi + U$, where $U \in \Omega^2(T^*M)$ is such that

$$\mathrm{tr} U = 0 = \mathfrak{b}U = 0$$

then

$$\mathfrak{D} = \mathfrak{D}(\hat{\nabla}) + \mathbf{c}(\mathbf{ia})$$

and

$$\mathfrak{D}(\hat{\nabla})^2 = (\nabla^{\mathfrak{w}})^* \nabla^{\mathfrak{w}} + \mathcal{R}_{\nabla} + \mathbf{c}(\mathbf{ida})$$

where

$$\mathcal{R}_{\nabla} = \frac{1}{4}s(g) + (\mathbf{c}(d\varpi) - 2\|\varpi\|^2)$$

The last theorem has an obvious extension where we replace $\mathbb{S}$ by the complex spinor bundle $\mathbb{S}_\sigma$ determined by a $spin^c$ -structure σ on M . This case requires the choice of a hermitian connection on the line bundle $\det \mathbb{S}_\sigma$. In the spin case $\det \mathbb{S} \cong \mathbb{C}$ and the additional hermitian connection on the trivial line bundle is encoded by the imaginary 1-form $\mathbf{i}a$ appearing in the statement of Theorem 1.11.

2 Dirac operators on almost-hermitian manifolds

§2.1 Basic differential geometric objects on an almost-hermitian manifolds In this subsection we survey a few differential geometric facts concerning almost complex manifolds. For more details we refer to [5, 7, 8] which served as sources of inspiration.

Consider an almost-hermitian manifold (M^{2n}, g, J) . Recall that this means that (M, g) is a Riemann manifold and J is a skew-symmetric endomorphism of TM such that $J^2 = -1$. Fix $x_0 \in M$ and $(e_1, f_1, \dots, e_n, f_n)$ a local, oriented orthonormal frame of TM . We also assume it is adapted to J that is

$$f_j = J e_j, \quad \forall j = 1, \dots, n.$$

We denote by $(e^1, f^1, \dots, e^n, f^n)$ the dual coframe. Let $\mathbf{i} := \sqrt{-1}$ and fix one such adapted local frame. We split $TM \otimes \mathbb{C}$ into $\pm \mathbf{i}$ -eigen-subbundles of J , $TM^{1,0}$ and $T^{0,1}$. These are naturally equipped with hermitian metrics induced by g and have natural local unitary frames near p_0

$$TM^{1,0} : \quad \varepsilon_k := \frac{1}{\sqrt{2}}(e_k - \mathbf{i}f_k), \quad k = 1, \dots, n,$$

$$TM^{0,1} : \quad \bar{\varepsilon}_k := \frac{1}{\sqrt{2}}(e_k + \mathbf{i}f_k), \quad k = 1, \dots, n.$$

Form by duality $T^*M^{1,0}$ and $T^*M^{0,1}$ with local unitary frames given by

$$\varepsilon^k := \frac{1}{\sqrt{2}}(e^k + \mathbf{i}f^k), \quad k = 1, \dots, n$$

and respectively,

$$\bar{\varepsilon}^k := \frac{1}{\sqrt{2}}(e^k - \mathbf{i}f^k), \quad k = 1, \dots, n.$$

We have unitary decompositions

$$\Lambda^m T^*M \otimes \mathbb{C} = \bigoplus_{p+q=m} \Lambda^{p,q} T^*M, \quad m = 0, \dots, 2n$$

where

$$\Lambda^{p,q} T^*M := \Lambda^p T^*M^{1,0} \otimes \Lambda^q T^*M^{0,1}.$$

Set $K_M := \Lambda^{n,0} T^*M$. We denote by $P^{p,q}$ the unitary projection onto $\Lambda^{p,q}$ and define

$$\bar{\partial} : \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M), \quad \bar{\partial} := P^{p,q+1} \circ d$$

and

$$\partial : \Omega^{p,q}(M) \rightarrow \Omega^{p+1,q}(M), \quad \partial := P^{p+1,q} \circ d.$$

Define $d^c : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ by

$$d^c \alpha(X_0, X_1, \dots, X_p) = \alpha(-JX_0, -JX_1, \dots, -JX_p).$$

The space $\Omega^3(M) \otimes \mathbb{C}$ splits unitarily as

$$\Omega^3 \otimes \mathbb{C} = \Omega^+ \oplus \Omega^-,$$

where

$$\Omega^+ := \Omega^{2,1} \oplus \Omega^{1,2}, \quad \Omega^- := \Omega^{3,0} \oplus \Omega^{0,3}.$$

Finally, introduce the involution $\mathfrak{M}$ on $\Omega^2(T^*M)$ defined by

$$\mathfrak{M}B(X; Y, Z) = B(X; JY, JZ).$$

Observe that

$$\psi^+ = \mathfrak{b}\mathfrak{M}\psi^+, \quad \forall \psi^+ \in \Omega^+.$$

We denote by $\Omega^{1,1}(T^*M)$ the 1-eigenspace of $\mathfrak{M}$ and by $\Omega_s^{1,1}(T^*M)$ the intersection of $\ker \mathfrak{b}$ to $\Omega^{1,1}(T^*M)$. Thus

$$A \in \Omega_s^{1,1}(T^*M) \iff A = \mathfrak{M}A, \quad \mathfrak{b}A = 0.$$

The Nijenhuis tensor $N \in \Omega^2(TM)$ is defined by

$$N(X, Y) := \frac{1}{4}([JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]), \quad \forall X, Y \in \text{Vect}(M).$$

Notice that $N(JX, Y) = N(X, JY) = -JN(X, Y)$. This implies immediately that $\text{tr } N^\dagger = 0$.

We denote by D the Levi-Civita connection determined by the metric g and by ω the fundamental two form defined by

$$\omega(X, Y) = g(JX, Y), \quad \forall X, Y \in \text{Vect}(M).$$

Locally we have

$$\omega = \mathbf{i} \sum_j \varepsilon^j \wedge \bar{\varepsilon}^j.$$

The *Lee form* θ determined by (g, J) is defined by

$$\theta = \Lambda(d\omega) = -J\Lambda((d^c\omega)^+),$$

where Λ denotes the contraction by ω , $\Lambda = (\omega \wedge)^*$, and J acts on the 1-form α by

$$J\alpha(X) = -\alpha(JX), \quad \forall X \in \text{Vect}(M).$$

We have the following identity

$$g((D_X J)Y, Z) = -\frac{1}{2}d\omega(X, JY, JZ) + \frac{1}{2}d\omega(X, Y, Z) + 2g(N(Y, Z), JX). \quad (2.1)$$

The form ω determines the skew-symmetric part of $N^\dagger$ via the identity

$$\mathfrak{b}N^\dagger = (d^c\omega)^-.$$

The almost complex structure defines a Cauchy-Riemann operator

$$\bar{\partial}_J : C^\infty(TM^{1,0}) \rightarrow \Omega^{0,1}(TM^{1,0})$$

defined by

$$X \lrcorner \bar{\partial}_J Y = [X, Y]^{1,0}, \quad \forall X \in C^\infty(TM^{0,1}), \quad Y \in C^\infty(TM^{1,0}).$$

A *Hermitian connection* on TM is a metric connection ∇ such that $\nabla J = 0$. A Hermitian connection ∇ is completely determined $\psi_+ := \frac{1}{3}(\mathfrak{b}T^\dagger)^+$ and $B := (T^\dagger)_s^{1,1}$ via the equality (see [5, Sec. 2.3])

$$T(\nabla)^\dagger = N^\dagger + \frac{1}{8}(d^c\omega)^+ - \frac{3}{8}\mathfrak{M}(d^c\omega^+) + \frac{9}{8}\psi^+ - \frac{3}{8}\mathfrak{M}\psi^+ + B.$$

We will denote the above connection by $\nabla(\psi^+, B)$. When $B = 0$ we write $\nabla(\psi^+)$ instead of $\nabla(\psi^+, B)$. Observe that if T is the torsion of $\nabla(\psi^+, B)$ then

$$\mathfrak{b}T^\dagger = \mathfrak{b}N^\dagger + 3\psi^+ = (d^c\omega)^- + 3\psi^+ = \mathfrak{b}N^\dagger + 3\psi^+.$$

Using the formulæ [5, (1.3.5), (1.4.9)] and the equality $\psi^+ = \mathfrak{b}\mathfrak{M}\psi^+, \forall \psi^+ \in \Omega^+$ we deduce that

$$\text{tr } \mathfrak{M}\psi^+ = -2J\Lambda\psi^+, \quad \forall \psi^+ \in \Omega^+(M).$$

Since $\text{tr } N^\dagger = 0$ we deduce that the trace of the torsion of $\nabla(\psi^+, B)$

$$\text{tr } T(\nabla(\psi^+, B)) = \text{tr } B + \frac{3}{4}J\Lambda((d^c\omega)^+ + \psi^+) = \text{tr } B - \frac{3}{4}\theta + \frac{3}{4}J\Lambda\psi^+.$$

Example 2.1. The *first canonical connection* (see [5, Sec. 2.5] or [8]) is the Hermitian connection ∇^0 defined by $B = 0$ and

$$\mathfrak{b}T_0^\dagger = (d^c\omega)^- - (d^c\omega^+)$$

so that $\psi^+ = -\frac{1}{3}(d^c\omega)^+$. Its torsion is

$$T_0^\dagger = N^\dagger - \frac{1}{4}((d^c\omega)^+ + \mathfrak{M}(d^c\omega)^+).$$

In general, it is not a nice connection since $\text{tr } T_0^\dagger = -\frac{1}{2}\theta$.

Example 2.2. The *Chern connection* or the *second fundamental connection*, [5, 8], is the *unique* Hermitian connection ∇ on TM such that

$$\nabla^{0,1} = \bar{\partial}_J.$$

We will denote it by ∇^c . Alternatively (see [5, Sec. 2.5]), it is the hermitian connection defined by $B = 0$ and $\flat T^\dagger = (d^c \omega)^- + (d^c \omega)^+$, i.e it is determined by $\psi^+ = \frac{1}{3}(d^c \omega)^+$. Its torsion is given by

$$T_c^\dagger = N^\dagger + \frac{1}{2} \left((d^c \omega)^+ - \mathfrak{M}(d^c \omega)^+ \right).$$

In general, it is not a nice connection since $\text{tr } T_c^\dagger = -\theta$.

§2.2 The Hodge-Dolbeault operator The almost hermitian manifold M is equipped with a canonical *spin^c* structure and the associated complex spinor bundle is

$$\mathbb{S}_c := \Lambda^{0,*} T^* M = \bigoplus_{p \geq 0} \Lambda^{0,p} T^* M.$$

Note that $\det \mathbb{S}_c = K_M^{-1}$. The Chern connection induces a hermitian connection $\det \nabla^c$ on K_M^{-1} and we denote by $\mathfrak{D}_c$ the geometric Dirac operator induced by the Levi-Civita connection D and the connection $\det \nabla^c$.

If M is spinnable, then a choice of spin structure is equivalent to a choice of a square root of K_M and in this case $\mathbb{S}_c := \mathbb{S} \otimes K_M^{-1/2}$.

The bundle $\mathbb{S}_c$ has a natural Dirac type operator, the Hodge-Dolbeault operator

$$\mathcal{H}_J := \sqrt{2}(\bar{\partial} + \bar{\partial}^*) : C^\infty(\mathbb{S}_J) \rightarrow C^\infty(\mathbb{S}_J).$$

We have the following result [2, Thm.2.2] and [5, Sec.3.6].

$$\mathcal{H}_J = \mathfrak{D}_c - \frac{1}{4} \left\{ \mathbf{c}((d^c \omega)^+) - \mathbf{c}((d^c \omega)^-) \right\}.$$

Using Theorem 1.11 we deduce that $\mathcal{H}_J$ is a geometric Dirac operator, more precisely $\mathfrak{H}_J$ is induced by $\widehat{\nabla} \otimes \mathbf{1} + \mathbf{1} \otimes \det \nabla^c$, where ∇ is the connection

$$\nabla = D - \frac{1}{6}((d^c \omega)^+ - (d^c \omega)^-)$$

with torsion

$$T^\dagger = \frac{1}{3}(d^c(\omega)^- - (d^c \omega)^+).$$

A stronger result is true. Using the results in the previous subsection we deduce the following result.

Theorem 2.3. *For every $B \in \Omega_s^{1,1}(T^*M)$ such that $\text{tr } B = \frac{1}{2}\theta$ there exists a **Hermitian** connection $\nabla^b = \nabla^b(B)$ uniquely determined by the following conditions.*

- (i) ∇^b is nice.
- (ii) ∇^b is Dirac equivalent to ∇^0 .

Proof Since $\nabla^b = \nabla(\psi^+, B)$ is strongly Dirac equivalent to ∇ we deduce that its torsion satisfies

$$\flat T_b = (d^c \omega)^- - (d^c \omega)^+.$$

Thus we need to choose $\psi^+ = -\frac{1}{3}(d^c \omega)^+$. Now observe that

$$0 = \text{tr } T_b^\dagger = \text{tr } B - \frac{1}{2}\theta = 0, \quad \blacksquare$$

Definition 2.4. We will refer to any of the connections ∇^b constructed in Theorem 2.3 as a **basic connection** determined by an almost Hermitian structure.

The torsion of a basic connection $\nabla^b(B)$ is

$$T_b^\dagger = N^\dagger - \frac{1}{4} \left((d^c \omega)^+ + \mathfrak{M}(d^c \omega)^+ \right) + B.$$

Observe also that the first and second fundamental connection coincide of an almost Kähler structure coincide and they are both basic. They are precisely the connections used by Taubes, [13], to analyze the Seiberg-Witten monopoles on a symplectic manifold.

For any basic connection ∇^b we have the following identities ([5, Sec. 3.5])

$$(\bar{\partial}\phi)(Z_0, Z_1, \dots, Z_p) = \sum_{j=0}^p (-1)^j \nabla_{Z_j}^b \phi(Z_0, \dots, \hat{Z}_j, \dots, Z_p), \quad (2.2a)$$

$$\bar{\partial}^* \phi(Z_1, \dots, Z_{p-1}) = - \sum_{i=1}^n \left(e_i \lrcorner \nabla_{e_i}^b \phi + f_j \lrcorner \nabla_{f_i}^b \phi \right) (Z_1, \dots, Z_{p-1}), \quad (2.2b)$$

$$\forall Z_0, \dots, Z_p \in C^\infty(T^{0,1}M), \quad \phi \in \Omega^{0,p}(M).$$

3 Dirac operators on contact 3-manifolds

§3.1 Differential objects on metric contact manifolds We review a few basic geometric facts concerning metric contact manifolds. For more details we refer to [3, 12].

A *metric contact manifold* (m.c. manifold for brevity) is an oriented manifold of odd dimension $2n + 1$ equipped with a Riemann metric g and a 1-form η such that

- $|\eta(x)|_g = 1, \forall x \in M$. Denote by $\xi \in \text{Vect}(M)$ the metric dual of η and set $V := \ker \eta \subset TM$. V is a hyperplane sub-bundle of TM and we denote by P_V the orthogonal projection onto V .

- There exists $J : TM \rightarrow TM$ such that

$$d\eta(X, Y) = g(JX, Y), \quad \forall X, Y \in \text{Vect}(M).$$

and

$$J^2 X = -X + \eta(X)\xi, \quad \forall X \in \text{Vect}(M).$$

Definition 3.1. A **contact metric connection** on (M^{2n+1}, η, J, g) is a metric connection such that $\nabla J = 0 = \nabla \xi$.

The manifold M is called *positively oriented* if the orientation induced by the nowhere vanishing $(2n+1)$ -form $\eta \wedge (d\eta)^n$ coincides with the given orientation of M . In this case

$$dv_g = \frac{1}{n!} \eta \wedge (d\eta)^n$$

Set $\omega := d\eta$. The metric g is completely determined by η and J via the equality

$$g(X, Y) = \eta(X)\eta(Y) + d\eta(X, JY) = \eta(X)\eta(Y) + \omega(X, JY).$$

We have decompositions

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}, \quad V^* \otimes \mathbb{C} = (V^*)^{1,0} \oplus (V^*)^{0,1}$$

and we set

$$K_M := \det(V^*)^{1,0}.$$

Set $\Phi := L_\xi J$. The operator Φ is a traceless, symmetric endomorphism of V (see [3]). Since $L_\xi(J^2) = 0$ we deduce

$$J\Phi + \Phi J = 0 \implies (J\Phi)^* = (J\Phi) \quad (3.1)$$

Define the Nijenhuis tensor $N \in \Omega^2(TM)$ by

$$N(X, Y) = \frac{1}{2} \left\{ J^2[X, Y] + [JX, JY] - J[X, JY] - J[JX, Y] \right\}.$$

Notice that

$$N(\xi, X) = -\frac{1}{2} J\Phi X, \quad \forall X \in \text{Vect}(M).$$

(M, g, η) is a *Cauchy-Riemann manifold* (*CR* for brevity) if and only if $JN(X, Y) = 0$, $\forall X, Y \in C^\infty(V)$. Equivalently, this means, and

$$N(X, Y) + \omega(X, Y)\xi = -J^2 N(X, Y) = 0, \quad \forall X, Y \in C^\infty(V).$$

In this case, the Nijenhuis tensor can be given the more compact description

$$N^\dagger = \frac{1}{2} J\Phi \wedge \eta - \eta \otimes d\eta.$$

In particular, M is a *CR* manifold when $\dim M = 3$. Arguing exactly as in [3, p.53] we obtain the following result.

Proposition 3.2. If D denotes the Levi-Civita connection of (M, g) then

$$g((D_X J)Y, Z) = g(JX, N(Y, Z)) + \frac{1}{2}(\eta \wedge d\eta)(JX, Y, Z).$$

$\forall X, Y, Z \in \text{Vect}(M)$.

To each metric contact manifold M we can associate an almost Hermitian manifold $(\hat{M}, \hat{g}, \hat{J})$ defined as follows.

$$\hat{M} = \mathbb{R} \times M, \quad \hat{g} = dt^2 + g, \quad \hat{J}\partial_t = \xi,$$

We will denote by $\hat{d}$ the exterior differentiation on $\hat{M}$. If we set

$$\hat{\omega}(X, Y) = \hat{g}(\hat{J}X, Y), \quad \forall X, Y \in \text{Vect}(\hat{M})$$

then $\hat{\omega} = dt \wedge \eta + \omega$ and $\hat{d}\hat{\omega} = -dt \wedge \omega$. We deduce that the Lee form $\theta = \Lambda(-dt \wedge d\eta)$ is $-ndt$. We will work with local, oriented orthonormal frames $(e_0, f_0, e_1, \dots, e_n, f_n)$ adapted to $\hat{J}$ such that

$$e_0 = \partial_t, \quad f_0 = \xi, \quad e^0 = dt, \quad f^0 = \eta$$

$$\hat{\omega} = \mathbf{i}\varepsilon^0 \wedge \bar{\varepsilon}^0 + \mathbf{i} \sum_{k=1}^n \varepsilon^k \wedge \bar{\varepsilon}^k, \quad \hat{d}\hat{\omega} = -\frac{\mathbf{i}}{\sqrt{2}}(\varepsilon^0 + \bar{\varepsilon}^0) \wedge \sum_{k=1}^n \varepsilon^k \wedge \bar{\varepsilon}^k.$$

Hence

$$\hat{d}^c \hat{\omega} = -\frac{1}{\sqrt{2}}(\varepsilon^0 - \bar{\varepsilon}^0) \wedge \sum_{k=1}^n \varepsilon^k \wedge \bar{\varepsilon}^k = -\eta \wedge d\eta$$

so that $(\mathfrak{b}\hat{N}^\dagger) = (\hat{d}^c \hat{\omega})^- = 0$. We have the following result, [3].

Proposition 3.3.

$$\hat{N}(X, Y) = \frac{1}{2}N(X, Y) + \frac{1}{2}\omega(X, Y)\xi, \quad \forall X, Y \in \text{Vect}(M),$$

$$\hat{N}(\partial_t, X) = \frac{1}{4}\Phi X, \quad \forall X \in \text{Vect}(M).$$

Observe that $\hat{N}^\dagger|_M = \frac{1}{2}N^\dagger + \frac{1}{2}\eta \otimes d\eta$ so that

$$0 = \mathfrak{b}\hat{N}^\dagger|_M = \frac{1}{2}\mathfrak{b}N^\dagger + \frac{1}{2}\mathfrak{b}(\eta \otimes d\eta) = \frac{1}{2}\mathfrak{b}N^\dagger + \frac{1}{2}\eta \wedge d\eta.$$

Hence

$$\mathfrak{b}N^\dagger = -\eta \wedge d\eta.$$

We want to find $B \in \Omega_s^{1,1}(T^*\hat{M})$ such that $\text{tr} B = -\frac{n}{2}dt$ and the basic connection it induces on $T^*\hat{M}$ is compatible with the splitting $\partial_t \oplus TM$. The torsion of such a connection is

$$\begin{aligned} \hat{T}_b^\dagger &= \hat{N}^\dagger - \frac{1}{4} \left((\hat{d}^c \hat{\omega})^+ + \mathfrak{M}(\hat{d}^c \hat{\omega})^+ \right) + B \\ &= \hat{N}^\dagger + \frac{1}{4} (\eta \wedge \omega + \mathfrak{M}(\eta \wedge \omega)) + B. \end{aligned}$$

Thus $\flat T_b^\dagger = \eta \wedge d\eta$. Using Proposition 1.2 we deduce that $\nabla^b = D + A$ where

$$A_b^\dagger = \frac{1}{2}\flat T_b^\dagger - T_b^\dagger = \frac{1}{4}\left(\eta \wedge d\eta - \mathfrak{M}(\eta \wedge d\eta)\right) - \hat{N}^\dagger - B.$$

Thus, for all $X, Y \in \text{Vect}(M)$ which are t -independent we have

$$\hat{g}(\nabla_t^b X, Y) = A_b^\dagger(\partial_t; X, Y)$$

Since

$$B(\partial_t; \bullet, \bullet) = 0 \quad \text{and} \quad \hat{g}(\hat{N}(X, Y), \partial_t) = 0, \quad \forall X, Y \in \text{Vect}(M).$$

we deduce

$$\hat{g}(\nabla_t^b X, Y) = -\frac{1}{4}\mathfrak{M}(\eta \wedge d\eta)(\partial_t; X, Y) = 0.$$

Similarly, we deduce

$$\hat{g}(\nabla_t^b X, \partial_t) = A_b^\dagger(\partial_t; X, \partial_t) = 0.$$

Thus

$$\nabla_t^b Z = 0, \quad \forall Z \in \text{Vect}(M).$$

Since ∇^b is a metric connection we deduce

$$\hat{g}(\nabla_\bullet^b \partial_t, \partial_t) = 0.$$

On the other hand, $\forall X, Y \in \text{Vect}(M)$ we have

$$\begin{aligned} \hat{g}(\nabla_X^b \partial_t, Y) &= A_b^\dagger(X; \partial_t, Y) \\ &= -\frac{1}{4}\mathfrak{M}\eta \wedge d\eta(X, \partial_t, Y) - \hat{g}(\hat{N}(\partial_t, Y), X) - B(X; \partial_t, Y) \\ &= \frac{1}{4}g(X_V, Y_V) - \frac{1}{4}g(\Phi Y, X) - B(X; \partial_t, Y), \end{aligned}$$

where $X_V = P_V X$, $Y = P_V Y$. Next, $\forall X, Y \in \text{Vect}(M)$, we have

$$\begin{aligned} \hat{g}(\nabla_X^b Y, \partial_t) &= A_b^\dagger(X; Y, \partial_t) = -\frac{1}{4}\mathfrak{M}\eta \wedge d\eta(X; Y, \partial_t) - \hat{g}(\hat{N}(Y, \partial_t), X) - B(X; Y, \partial_t) \\ &= -\frac{1}{4}g(X_V, Y_V) + \frac{1}{4}g(\Phi Y, X) - B(X; Y, \partial_t). \end{aligned}$$

Lemma 3.4. *There exists $B_0 \in \Omega_s^{1,1}(T^*\hat{M})$ such that $\text{tr } B = -\frac{n}{2}dt$ and*

$$B(\partial_t; \bullet, \bullet) = 0. \quad (3.2a)$$

$$B(X; Y, \partial_t) = \frac{1}{4}g(X, \Phi Y) - \frac{1}{4}g(X_V, Y_V), \quad \forall X, Y \in \text{Vect}(V). \quad (3.2b)$$

Proof Define

$$B = \frac{1}{4}(\Phi \wedge dt + J\Phi \wedge \eta) - \frac{1}{4}(P_V \wedge dt + JP_V \wedge \eta) + \frac{1}{2}\eta \otimes d\eta$$

and we set

$$B_0 = \frac{1}{4}(\Phi \wedge dt + J\Phi \wedge \eta), \quad B_1 = -\frac{1}{4}(P_V \wedge dt + JP_V \wedge \eta).$$

We need to show that this definition is correct, i.e. the above B satisfies all the required conditions (3.2a), (3.2b) and

$$\text{tr } B = -\frac{n}{2}dt, \quad \flat B = 0$$

$$B \in \Omega^{1,1}(T^*M).$$

Here the elementary properties in Lemma 1.1 will come in handy. Since Φ and $J\Phi$ are symmetric and traceless we deduce that

$$\text{tr } B_0 = 0, \quad \flat B_0 = 0.$$

The condition $B_0 \in \Omega^{1,1}$ follows from the identity $\phi J = -J\Phi$. Now observe that $B_1 \in \Omega^{1,1}$ and

$$\flat B_1 = -\frac{1}{2}\eta \wedge d\eta, \quad \text{tr } B_1 = -\frac{n}{2}dt.$$

Finally $\eta \otimes d\eta \in \Omega^{1,1}$, it is traceless and

$$\flat(\eta \otimes d\eta) = \eta \wedge d\eta.$$

The condition (3.2b) follows by direct computation. The Lemma follows putting together the above facts. ■

If we choose B as in Lemma 3.4 we deduce

$$\hat{g}(\nabla_{\bullet}^b X, \partial_t) = 0, \quad \forall X \in \text{Vect}(M).$$

The above computations show that the basic connection ∇^b of $(\hat{M}, \hat{g}, \hat{J})$ determined by B_0 preserves the orthogonal splitting $T\hat{M} = \langle \partial_t \rangle \oplus TM$ and thus induces a *nice* contact

metric connection ∇^w on TM . We will call ∇^w the *generalized Webster connection* of M for reasons which will be explained below. To compute its torsion observe that

$$\hat{N}^\dagger|_M = \frac{1}{2} \left\{ N^\dagger + \eta \otimes d\eta \right\},$$

and $\mathfrak{M}(\eta \wedge d\eta)|_M = \eta \otimes d\eta$. Finally

$$B|_M = \frac{1}{4}(J\Phi) \wedge \eta - \frac{1}{4}JP_V \wedge \eta + \frac{1}{2}\eta \otimes d\eta.$$

Since on M we have the equality $JP_V = J$, the torsion T_w of ∇^w given by

$$T_w^\dagger = \frac{1}{2}N^\dagger + \frac{5}{4}\eta \otimes d\eta + \frac{1}{4}\eta \wedge d\eta + \frac{1}{4}(J\Phi - J) \wedge \eta \quad (3.3)$$

Moreover, $\flat T_w = \eta \wedge d\eta$.

Suppose now that M is a *CR*-manifold. Then

$$N^\dagger = \frac{1}{2}J\Phi \wedge \eta - \eta \otimes d\eta$$

and thus

$$T_w^\dagger = \frac{3}{4}\eta \otimes d\eta + \frac{1}{4}\eta \wedge d\eta - \frac{1}{4}(J \wedge \eta) + \frac{1}{2}J\Phi \wedge \eta.$$

We deduce

$$T_w(X, Y) = d\eta(X, Y)\xi, \quad \forall X, Y \in \text{Vect}(V).$$

In particular, because the distribution $V^{1,0}$ is integrable we deduce

$$T_w(X, Y) = 0, \quad \forall X, Y \in C^\infty(V^{1,0}).$$

A contact metric connection with the above property will be called a *CR* metric connection. Next observe that for $X, Y \in C^\infty(V)$ we have

$$g(X, T_w(\xi, Y)) = T^\dagger(X, \xi, Y) = -\frac{1}{4}d\eta(X, Y) + \frac{1}{4}g(JX, Y) + \frac{1}{2}g(J\Phi X, Y).$$

Hence

$$T_w(\xi, Y) = \frac{1}{2}J\Phi Y.$$

Since $\Phi J = -J\Phi$ we deduce

$$JT_w(\xi, X) = -T_w(\xi, JX)$$

Using [12, Prop. 3.1], we deduce that when M is a Cauchy-Riemann manifold, the connection ∇^w on (V, J) is the Tannaka-Webster connection determined by the *CR* structure (see [4, 11, 12, 14] for more details). The generalized Webster connection we have constructed does not agree with the generalized Tannaka connection constructed by S.Tanno in [12] because that connection is not compatible with J if M is not a *CR*-manifold.

Finally let us point out that when M is a CR manifold then

$$g(\nabla_\xi^w X, Y) = g(D_\xi X, Y) + \frac{1}{2} \mathfrak{b} T_w^\dagger(\xi, X, Y) - T_w^\dagger(\xi; X, Y) = g(D_\xi X - \frac{1}{2} JX, Y)$$

so that

$$\nabla_\xi^w = D_\xi^V := P_V D_\xi - \frac{1}{2} J.$$

Example 3.5. We consider in great detail the special case of a metric, contact, *spin* 3-manifold M . M is automatically a CR -manifold so that the torsion of the (geberalized) Webster connection satisfies

$$T_w(X, Y) = \frac{1}{2} d\eta(X, Y)\xi, \quad T_w(\xi, X) = \frac{1}{2} J\Phi X, \quad \forall X, Y \in C^\infty(V)$$

$$\mathfrak{b} T_w^\dagger = \eta \wedge d\eta.$$

The *spin* Dirac operator $\mathfrak{D}_0$ on M is related to the Dirac operator $\mathfrak{D}(\nabla^w)$ by the equality

$$\mathfrak{D}(\nabla^w) = \mathfrak{D}_0 + \frac{1}{4} \mathfrak{c}(\mathfrak{b} T^\dagger) = \mathfrak{D}_0 + \frac{1}{4} \mathfrak{c}(\eta \wedge d\eta) = \mathfrak{D}_0 - \frac{1}{4}.$$

When M is Sasakian, i.e. $\Phi = 0$, the above equality shows that $\mathfrak{D}(\nabla^w)$ coincides with the adiabatic Dirac operator introduced in [10] (see in particular [10, Eq.(2.20)] with $\lambda = \frac{1}{2}$, $\delta = 1$).

Later on we will need to compare the connections $\det \nabla^c$ and $\det \nabla^b$ induced by the Chern connection ∇^c and respectively ∇^b on $K_{\hat{M}}^{-1}$.

Proposition 3.6.

$$\det \nabla^c = \det \nabla^b + \frac{n\mathbf{i}}{2} \eta.$$

Proof Denote by ∇^0 the first fundamental connection of $(\hat{M}, \hat{J})$. We have

$$\nabla^b = \nabla^0 - B,$$

where B is described in Lemma 3.4. Set $\delta := \varepsilon_0 \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$. Then for every vector field X on $\hat{M}$ we have

$$\det \nabla_X^b \delta = \det \nabla_X^0 \delta - B_X \delta$$

Observe that

$$B_X \varepsilon^k = \sum_{j=0}^n C_k^j \varepsilon_j$$

so that $B_X \delta = (\sum_{j=0}^n C_k^j) \delta$. On the other hand, $C_k^k = g_c(B_X \varepsilon_k, \bar{\varepsilon}_k)$ where g_c denotes the complex bilinear extension of g .

$$C_k^k = \frac{1}{2} g_c(B_X(e_k - \mathbf{i} f_k), e_k + \mathbf{i} f_k) = \mathbf{i} g(B_X e_k, J e_k) + \mathbf{i} g(B_X f_k, J f_k)$$

Thus

$$\sum_k C_k^k = -\mathbf{i} \sum_{k=0}^n \left(g(J B_X e_k, e_k) + g(J B_X f_k, f_k) \right) = -\mathbf{i} \operatorname{tr} J B_X. \quad (3.4)$$

The equality

$$B = \frac{1}{4} \left\{ (\Phi + P_V) \wedge dt - (J\Phi + J P_V) \wedge \eta \right\} + \frac{1}{2} \eta \otimes d\eta$$

so that

$$\begin{aligned} \hat{g}(B_X Y, JY) &= \frac{1}{4} \left\{ \hat{g}(\Phi X, Y) dt(JY) - \hat{g}(J\Phi X, Y) \eta(JY) \right\} \\ &+ \frac{1}{4} \left\{ \hat{g}(P_V X, Y) dt(JY) - \hat{g}(J P_V X, Y) \eta(JY) \right\} + \frac{1}{2} \eta(X) d\eta(Y, JY). \end{aligned}$$

We see that $\operatorname{tr} J B_X \neq 0$ only if $X = \xi$ in which case shows that the sum (3.4) is n . Hence

$$\nabla^b \delta = \nabla^0 \delta - \mathbf{i} n \eta.$$

On the other hand we have the identity, [5, Eq. (2.7.6)],

$$\det \nabla^c = \det \nabla^0 + \frac{\mathbf{i}}{2} J \theta = \det \nabla^0 - \frac{n \mathbf{i}}{2} J dt = \det \nabla^b + \frac{n \mathbf{i}}{2} \eta. \quad \blacksquare$$

Corollary 3.7.

$$F(\det \nabla^c) = F(\det \nabla^b) + \frac{n \mathbf{i}}{2} d\eta. \quad \blacksquare$$

§3.2 Geometric Dirac operators on contact manifolds Consider the Hodge-Dolbeault operator $\hat{\mathcal{H}}$ on $\hat{M}$

$$\hat{\mathcal{H}} = \sqrt{2}(\bar{\partial} + \bar{\partial}^*) : \Omega^{0,*}(\hat{M}) \rightarrow \Omega^{0,*}(\hat{M}).$$

It is a geometric Dirac operator and it is

$$\hat{\mathcal{H}} = \sqrt{2} \sum_{k=0}^n (\hat{\mathbf{c}}(\varepsilon^k) \hat{\nabla}_{\varepsilon_k} + \hat{\mathbf{c}}(\bar{\varepsilon}) \hat{\nabla}_{\bar{\varepsilon}_k})$$

where $\hat{\mathbf{c}}$ denotes the Clifford multiplication on $\hat{\mathbb{S}}_c \cong \Lambda^{0,*} T^* \hat{M}$, $\hat{\nabla} = \hat{\nabla}^b \otimes \mathbf{1} + \mathbf{1} \otimes \det \nabla^c$, and $\det \nabla^c$ denotes the Hermitian connection on $K_{\hat{M}}^{-1}$ induced by the Chern connection on $T\hat{M}$. More precisely

$$\hat{\mathbf{c}}(\bar{\varepsilon}^k) = \sqrt{2} \bar{\varepsilon}^k \wedge \bullet, \quad \hat{\mathbf{c}}(\varepsilon^k) = -\sqrt{2} \varepsilon^k \lrcorner \bullet.$$

Above, $\varepsilon^k \lrcorner \bullet$ denotes the odd derivation of $\Omega^{0,*}(\hat{M})$ uniquely determined by the requirements

$$\varepsilon^k \lrcorner \varepsilon^j = \delta_{kj}, \quad \forall j, k = 0, \dots, n.$$

We want to point out that

$$(\varepsilon^k \wedge)^* = \varepsilon^k \lrcorner.$$

We set

$$\mathcal{J} := \hat{\mathbf{c}}(dt) = \frac{1}{\sqrt{2}} \hat{\mathbf{c}}(\varepsilon^0) + \hat{\mathbf{c}}(\bar{\varepsilon}^0), \quad \mathbb{S}_c := \hat{\mathbb{S}}_c^+|_{0 \times M}.$$

Note that

$$\hat{\mathbb{S}}_c|_M \cong \mathbb{S}_c \oplus \mathcal{J}\mathbb{S}_c.$$

The metric contact structure on M produces a $U(n)$ -reduction of the tangent bundle TM which in general has only a $SO(2n+1)$ -structure. This $U(n)$ -reduction induces a $spin^c$ structure on M and $\mathbb{S}_c$ is the associated bundle of complex spinors and

$$\det \mathbb{S}_c \cong K_M^{-1}.$$

The Clifford multiplication on $\mathbb{S}_c$ is defined by the equality

$$\mathbf{c}(\alpha) = \mathcal{J} \hat{\mathbf{c}}(\alpha), \quad \forall \alpha \in \Omega^1(M).$$

Along M we can identify $\hat{\mathbb{S}}_c^-$ with $\mathcal{J}\mathbb{S}_c^+$ and as such $\mathcal{J}$ we can write.

$$\mathcal{J} = \begin{bmatrix} 0 & -G^* \\ G & 0 \end{bmatrix}, \quad GG^* = G^*G = \mathbf{1}_{\mathbb{S}_c}.$$

We can view the Hodge-Dolbeault operator as an operator on $\mathbb{S}_c \oplus \mathbb{S}_c$

$$\hat{\mathcal{H}} = \mathcal{J} \left(\hat{\nabla}_t^b - \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -G\mathcal{H}G^* \end{bmatrix} \right), \quad \mathcal{H}^* = \mathcal{H}.$$

$\mathcal{H}$ is the geometric Dirac operator induced by $\hat{\nabla}^w \otimes \mathbf{1} + \mathbf{1} \otimes \det \nabla^c$. We want to provide a more explicit description of the operator $\mathcal{H}$. Observe that

$$C^\infty(\hat{\mathbb{S}}_c^+) = \Omega^{0,even}(\hat{M}) = \Omega^{0,even}(V^*) \oplus \varepsilon^0 \wedge \Omega^{0,odd}(V^*)$$

where

$$\Omega^{0,p}(V^*) := C^\infty(\Lambda^p(V^*)^{0,1}).$$

We can represent $\psi \in C^\infty(\hat{\mathbb{S}}_c^+)$ as a sum

$$\psi = \psi_+ \oplus \varepsilon^0 \wedge \psi_-, \quad \psi_+ \in \Omega^{0,even}(V^*), \quad \psi_- \in \Omega^{0,odd}(V^*).$$

The above decomposition can be alternatively described as follows. The operator

$$\mathbf{c}(\eta) = \mathcal{J}\hat{\mathbf{c}}(\eta) : C^\infty(\hat{\mathbb{S}}_c^+) \rightarrow C^\infty(\hat{\mathbb{S}}_c^+)$$

satisfies $\mathbf{c}(\eta)^2 = -1$ and thus $\mathbf{c}(\mathbf{i}\eta)$ is an involution of $C^\infty(\hat{\mathbb{S}}_c^+)$. More explicitly

$$\mathbf{c}(\eta) = \frac{\mathbf{i}}{2}(\hat{\mathbf{c}}(\bar{\varepsilon}^0) + \hat{\mathbf{c}}(\varepsilon^0))(\hat{\mathbf{c}}(\bar{\varepsilon}^0) - \hat{\mathbf{c}}(\varepsilon^0)) = \mathbf{i}(\bar{\varepsilon}^0 \wedge - \varepsilon^0 \lrcorner)(\bar{\varepsilon}^0 \wedge + \varepsilon^0 \lrcorner).$$

Thus, for every $\phi \in \Omega^{0,*}(V^*)$ we have

$$\mathbf{c}(\mathbf{i}\eta)(\bar{\varepsilon}^0 \wedge \phi) = -\bar{\varepsilon}^0 \wedge \phi, \quad \mathbf{c}(-\mathbf{i}\eta)\phi = \phi$$

This shows that the above decomposition is defined by the ± 1 eigenspaces of the involution $\mathbf{c}(\eta)$. The restriction of the operator $\bar{\partial} : \Omega^{0,*}(\hat{M}) \rightarrow \Omega^{0,p}(\hat{M})$ to $\Omega^{0,*}(V^*)$ decomposes into two parts. More precisely, if $\phi \in \Omega^{0,*}(V^*)$ then

$$\bar{\partial}\phi = \bar{\varepsilon}^0 \wedge \bar{\partial}_0\phi + \bar{\partial}_V\phi := \frac{1}{2}(1 + \mathbf{c}(\mathbf{i}\eta))\bar{\partial} + \frac{1}{2}(1 - \mathbf{c}(\mathbf{i}\eta))\bar{\partial}.$$

Note that

$$\bar{\partial}_0\phi := \varepsilon^0 \lrcorner \bar{\partial}\phi \in \Omega^{0,p}(V^*), \quad \bar{\partial}_V \in \Omega^{0,p+1}(V^*).$$

We will regard $\bar{\partial}_0$ and $\bar{\partial}_V$ as operators

$$\bar{\partial}_0 : \Omega^{0,*}(V^*) \rightarrow \Omega^{0,*}(V^*), \quad \bar{\partial}_V : \Omega^{0,*}(V^*) \rightarrow \Omega^{0,*+1}(V^*).$$

Pick a t -independent section $\psi = C^\infty(\hat{\mathbb{S}}_c^+)$. It decomposes as

$$\psi = \psi_+ + \bar{\varepsilon}^0 \wedge \psi_-, \quad \psi_\pm \in \Omega^{0,even/odd}(V^*).$$

We have the equality

$$\hat{\mathcal{H}} \begin{bmatrix} \psi \\ 0 \end{bmatrix} = - \begin{bmatrix} 0 & -G^* \\ G & 0 \end{bmatrix} \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -G\mathcal{H}G^* \end{bmatrix} \begin{bmatrix} \psi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{H}G^* \\ G\mathcal{H} & 0 \end{bmatrix} \begin{bmatrix} \psi \\ 0 \end{bmatrix}$$

Thus

$$\sqrt{2}(\bar{\partial} + \bar{\partial}^*)\psi = G\mathcal{H}\psi = \hat{\mathbf{c}}(dt)\mathcal{H}\psi \implies \mathcal{H}\psi = -\sqrt{2}\mathcal{J}(\bar{\partial} + \bar{\partial}^*)\psi.$$

We compute

$$\begin{aligned} (\bar{\partial} + \bar{\partial}^*)(\psi_+ + \bar{\varepsilon}^0 \wedge \psi_-) &= \bar{\partial}\psi_+ + (\bar{\partial}\bar{\varepsilon}^0) \wedge \psi_- - \bar{\varepsilon}^0 \wedge \bar{\partial}\psi_- + \bar{\partial}^*\psi_+ + \bar{\partial}^*(\bar{\varepsilon}^0 \wedge \psi_-) \\ (\bar{\partial}\bar{\varepsilon}^0 &= 0) \\ &= \bar{\varepsilon}^0 \wedge \bar{\partial}_0\psi_+ + \bar{\partial}_V\psi_+ - \bar{\varepsilon}^0 \wedge \bar{\partial}_V\psi_- + (\bar{\varepsilon}^0 \wedge \bar{\partial}_0 + \bar{\partial}_V)^*\psi_+ + \bar{\partial}^*(\bar{\varepsilon}^0 \wedge \psi_-) \\ &= \bar{\varepsilon}^0 \wedge (\bar{\partial}_0\psi_+ - \bar{\partial}_V\psi_-) + \bar{\partial}_V\psi_+ + \bar{\partial}_V^*\psi_+ + \bar{\partial}_0^*(\bar{\varepsilon}^0 \lrcorner \psi_+) + \bar{\partial}^*(\bar{\varepsilon}^0 \wedge \psi_-) \end{aligned}$$

$$= \bar{\varepsilon}^0 \wedge (\bar{\partial}_0 \psi_+ - \bar{\partial}_V \psi_-) + \bar{\partial}_V \psi_+ + \bar{\partial}_V^* \psi_+ + \bar{\partial}^* (\bar{\varepsilon}^0 \wedge \psi_-).$$

To proceed further we need to provide a more explicit description for $\bar{\partial}^* (\bar{\varepsilon}^0 \lrcorner) \psi_-$. We denote by $\langle \bullet, \bullet \rangle_M$ the L^2 -inner product on M . For every t -independent compactly supported $\alpha \in \Omega^{0, \text{odd}}(M)$ we have $\alpha = \alpha_- + \bar{\varepsilon}^0 \wedge \alpha_+$, $\alpha_{\pm} \in \Omega^{0, \text{odd/even}}(V^*)$, and

$$\langle \alpha, \bar{\partial}^* (\bar{\varepsilon}^0 \wedge \phi_-) \rangle_M = \langle \bar{\partial} \alpha, \bar{\varepsilon}^0 \wedge \phi_- \rangle_M = \langle \bar{\varepsilon}^0 \wedge \bar{\partial}_0 \alpha_-, \bar{\varepsilon}^0 \wedge \phi_- \rangle_M - \langle \bar{\varepsilon}^0 \wedge \bar{\partial}_V \alpha_+, \bar{\varepsilon}^0 \wedge \phi_- \rangle_M$$

$$= \langle \bar{\partial}_0 \alpha_-, \phi_- \rangle_M - \langle \bar{\partial}_V \alpha_+, \phi_- \rangle_M = \langle \alpha_-, \bar{\partial}_0^* \phi_- \rangle_M - \langle \alpha_+, \bar{\partial}_V^* \phi_- \rangle_M$$

We conclude

$$\bar{\partial}^* (\bar{\varepsilon}^0 \wedge \phi_-) = \bar{\partial}_0^* \phi_- - \bar{\varepsilon}^0 \wedge \bar{\partial}_V^* \phi_-,$$

and

$$(\bar{\partial} + \bar{\partial}^*)(\psi_+ + \bar{\varepsilon}^0 \wedge \psi_-) = \bar{\varepsilon}^0 \wedge (\bar{\partial}_0 \psi_+ - \bar{\partial}_V \psi_- - \bar{\partial}_V^* \phi_-) + \bar{\partial}_V \psi_+ + \bar{\partial}_V^* \psi_+ + \bar{\partial}_0^* \phi_-.$$

Now observe that

$$\hat{\mathbf{c}}(dt)\bullet = \frac{1}{\sqrt{2}}(\hat{\mathbf{c}}(\bar{\varepsilon}^0) + \hat{\mathbf{c}}(\varepsilon^0))\bullet = (\bar{\varepsilon}^0 \wedge \bullet - \varepsilon^0 \lrcorner \bullet)$$

so that

$$\begin{aligned} \mathcal{H}\psi &= -\sqrt{2}(\varepsilon^0 \lrcorner - \bar{\varepsilon}^0 \wedge) \left\{ \bar{\varepsilon}^0 \wedge (\bar{\partial}_0 \psi_+ - \bar{\partial}_V \psi_- - \bar{\partial}_V^* \phi_-) + \bar{\partial}_V^* \psi_+ + \bar{\partial}_V \psi_+ + \bar{\partial}_0^* \psi_- \right\} \\ &= -\sqrt{2} \left\{ (\bar{\partial}_0 \psi_+ - \bar{\partial}_V \psi_- - \bar{\partial}_V^* \phi_-) - \bar{\varepsilon}^0 \wedge (\bar{\partial}_V^* \psi_+ + \bar{\partial}_V \psi_+ + \bar{\partial}_0^* \psi_-) \right\}. \end{aligned}$$

In block form

$$\mathcal{H} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \sqrt{2} \begin{bmatrix} -\bar{\partial}_0 & (\bar{\partial}_V^* + \bar{\partial}_V) \\ (\bar{\partial}_V^* + \bar{\partial}_V) & \bar{\partial}_0^* \end{bmatrix} \cdot \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$$

The above equality can be further simplified as follows. If $\phi \in \Omega^{0,p}(V^*) \subset \Omega^*(M) \otimes \mathbb{C}$ then

$$d\phi \in \eta \wedge \left(\Omega^{0,p}(V^*) + \Omega^{1,p-1}(V^*) \right) \oplus \Omega^{0,p+1}(V^*) \oplus \Omega^{1,p}(V^*) \oplus \Omega^{2,p-1}(V^*).$$

and

$$-\sqrt{2}\bar{\partial}_0 \phi = -\mathbf{i}(\xi \lrcorner d\phi)^{0,p} =: -\mathbf{i}L_{\xi}^V \phi.$$

On the other hand, the identity (2.2a) implies

$$\bar{\partial}_0 \phi = \nabla_{\varepsilon_0}^b \phi = \frac{\mathbf{i}}{\sqrt{2}} \nabla_{\xi}^w \phi.$$

Since $\mathbf{div}_g \xi = 0$ the operator $\mathbf{i}\nabla_\xi^w$ is symmetric and so must by $\mathbf{i}L_\xi^V$. Hence $\bar{\partial}_0^* \phi = \mathbf{i}L_\xi^V$ and

$$\mathcal{H} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \begin{bmatrix} -\mathbf{i}L_\xi^V & \sqrt{2}(\bar{\partial}_V^* + \bar{\partial}_V) \\ \sqrt{2}(\bar{\partial}_V + \bar{\partial}_V^*) & \mathbf{i}L_\xi^V \end{bmatrix} \cdot \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$$

or equivalently,

$$\mathcal{H} = \mathbf{c}(\mathbf{i}\eta)L_\xi^V + \begin{bmatrix} 0 & \sqrt{2}(\bar{\partial}_V + \bar{\partial}_V^*) \\ \sqrt{2}(\bar{\partial}_V^* + \bar{\partial}_V) & 0 \end{bmatrix}. \quad (3.5)$$

We will refer to $\mathcal{H}$ as the *contact Hodge-Dolbeault operator*. The next result summarizes the results we have proved so far.

Theorem 3.8. *Suppose (M^{2n+1}, g, η) is a metric contact manifold, $V := \ker \eta$. Denote by $\mathbb{S}_c$ the bundle of complex spinors associated to the spin^c structure determined by the contact structure. Denote the corresponding Clifford multiplication by $\mathbf{c}$.*

(i) $\mathbb{S}_c \cong \Lambda^{0,*}V^*$, $\mathbf{c}(\mathbf{i}\eta)\phi = (-1)^p\phi$, $\forall \phi \in \Omega^{0,p}(V^*)$. We decompose

$$\mathbb{S}_c = \mathbb{S}_c^+ \oplus \mathbb{S}_c^-, \quad \mathbb{S}_c^\pm = \Lambda^{0, \text{even/odd}}(V^*).$$

(ii) The operator $\mathcal{H} : C^\infty(\mathbb{S}_c) \rightarrow C^\infty(\mathbb{S}_c)$ defined by (3.5) is a geometric Dirac operator induced by the connection ∇^w on TM and $\det \nabla^c$ on $\det \mathbb{S}_c$.

(iii) If we denote by $\mathfrak{D}_c$ the Dirac operator on $\mathbb{S}_c$ induced by the Levi-Civita connection on TM and $\det \nabla^c$ on $\det \mathbb{S}_c$ then

$$\mathcal{H} = \mathfrak{D}_c + \frac{1}{4}\mathbf{c}(\eta \wedge d\eta).$$

(iv) Using the identity $F(\det \nabla^c) = F(\det \nabla^w) + \frac{n\mathbf{i}}{2}d\eta$, we deduce that $\mathcal{H}$ satisfies a Weitzenböck formula

$$\mathcal{H}^2 = (\nabla^w)^*(\nabla^w) + \frac{s(g)}{4} + \frac{1}{16}(4\mathbf{c}(d\eta \wedge d\eta) - 2n) + \frac{1}{2}\mathbf{c}(F(\det \nabla^w)) + \frac{n\mathbf{i}}{4}\mathbf{c}(\omega).$$

In particular, if $\dim M = 3$ (so that $n = 1$ and $\mathbf{c}(\eta \wedge d\eta) = -1$) we have

$$\mathfrak{D}_c = \mathcal{H} + \frac{1}{4},$$

$$\mathcal{H}^2 = (\nabla^w)^*(\nabla^w) + \frac{s}{4} - \frac{1}{8} + \frac{1}{2}\mathbf{c}(F(\det \nabla^w)) + \frac{\mathbf{i}}{4}\mathbf{c}(d\eta).$$

We want to discuss in more detail the case $\dim M = 3$. In this case $\Lambda^{0, \text{even}}V^* \cong \underline{\mathbb{C}}$ and $\Lambda^{0, \text{odd}}(V^*) \cong K_M^{-1}$. The above geometric Dirac operator has the simpler form

$$\mathcal{H}^2 = \mathbf{c}(\mathbf{i}\eta)L_\xi + \begin{bmatrix} 0 & \sqrt{2}\bar{\partial}_V^* \\ \sqrt{2}\bar{\partial}_V & 0 \end{bmatrix} = \sqrt{2} \begin{bmatrix} -\bar{\partial}_0 & \bar{\partial}_V^* \\ \bar{\partial}_V & \bar{\partial}_0^* \end{bmatrix}$$

$$= \sqrt{2} \begin{bmatrix} -\bar{\partial}_0 & 0 \\ 0 & \bar{\partial}_0^* \end{bmatrix} + \sqrt{2} \begin{bmatrix} 0 & \bar{\partial}_V^* \\ \bar{\partial}_V & 0 \end{bmatrix} =: Z + T.$$

Note that along M we have $\bar{\partial}_0 = \frac{\mathbf{i}}{\sqrt{2}}\partial_\xi$. We have $\mathcal{H}^2 = Z^2 + T^2 + \{Z, T\}$, where $\{\bullet, \bullet\}$ denotes the anti-commutator of two operators. In this case

$$\{Z, T\} = 2 \begin{bmatrix} 0 & [\bar{\partial}_0, \bar{\partial}_V]^* \\ [\bar{\partial}_0, \bar{\partial}_V] & 0 \end{bmatrix}.$$

The above commutators can be further simplified using the identity (2.2a) of §2.1. In this case the Lee 1-form on $\hat{M}$ is dt . The equality (2.2a) implies that for every t -independent $\phi \in \Omega^{0,*}(V^*) \subset \Omega^{0,*}(\hat{M})$ we have

$$\bar{\partial}\phi(\bar{\varepsilon}_{k_0}, \dots, \bar{\varepsilon}_{k_p}) = \sum_{j=0}^p (-1)^j \nabla_{\bar{\varepsilon}_{k_j}}^b \phi(\bar{\varepsilon}_{k_0}, \dots, \hat{\bar{\varepsilon}}_{k_j}, \dots, \bar{\varepsilon}_{k_p}) = \left(\sum_{k=0}^p \bar{\varepsilon}^k \wedge \nabla_{\bar{\varepsilon}_k}^b \right) \phi.$$

Thus

$$\bar{\partial}_0 \phi = \nabla_{\bar{\varepsilon}_0}^b \phi, \quad \bar{\partial}_V \phi = \left(\sum_{k=1}^p \bar{\varepsilon}^k \wedge \nabla_{\bar{\varepsilon}_k}^b \right) \phi.$$

When $\dim M = 3$ and $\phi = u \in \Omega^{0,0}(V^*) = C^\infty(M) \otimes \mathbb{C}$ we have

$$\begin{aligned} [\bar{\partial}_0, \bar{\partial}_V]u &= \nabla_{\bar{\varepsilon}_0}^b (\bar{\varepsilon}^1 \wedge \nabla_{\bar{\varepsilon}_1}^b u) - \bar{\varepsilon}^1 \wedge \nabla_{\bar{\varepsilon}_1}^b \nabla_{\bar{\varepsilon}_0}^b u \\ &= (\nabla_{\bar{\varepsilon}_0}^b \bar{\varepsilon}^1) \wedge \nabla_{\bar{\varepsilon}_1}^b u + \bar{\varepsilon}^1 \wedge [\nabla_{\bar{\varepsilon}_0}^b, \nabla_{\bar{\varepsilon}_1}^b]u = (\nabla_{\bar{\varepsilon}_0}^b \bar{\varepsilon}^1) \wedge \nabla_{\bar{\varepsilon}_1}^b u + \bar{\varepsilon}^1 \wedge \nabla_{[\bar{\varepsilon}_0, \bar{\varepsilon}_1]}^b u + \bar{\varepsilon}_1 \wedge F_b(\bar{\varepsilon}_0, \bar{\varepsilon}_1)u, \end{aligned}$$

where F_b denotes the curvature of the ∇^b . Denote by T_b the torsion of ∇^b . Observe that

$$\nabla_{\bar{\varepsilon}_0}^b \bar{\varepsilon}^1(\bar{\varepsilon}_1) = -\bar{\varepsilon}^1(\nabla_{\bar{\varepsilon}_0}^b \bar{\varepsilon}_1)$$

so that

$$\begin{aligned} &(\nabla_{\bar{\varepsilon}_0}^b \bar{\varepsilon}^1) \wedge \nabla_{\bar{\varepsilon}_1}^b u + \bar{\varepsilon}^1 \wedge \nabla_{[\bar{\varepsilon}_0, \bar{\varepsilon}_1]}^b u = -\bar{\varepsilon}^1 \wedge \nabla_{\nabla_{\bar{\varepsilon}_0}^b \bar{\varepsilon}_1}^b u + \bar{\varepsilon}^1 \wedge \nabla_{[\bar{\varepsilon}_0, \bar{\varepsilon}_1]}^b u \\ &(\nabla^b \bar{\varepsilon}_0 = 0) \\ &= -\bar{\varepsilon}^1 \wedge (\nabla_{\nabla_{\bar{\varepsilon}_0}^b \bar{\varepsilon}_1}^b - \nabla_{\nabla_{\bar{\varepsilon}_1}^b \bar{\varepsilon}_0}^b - \nabla_{[\bar{\varepsilon}_0, \bar{\varepsilon}_1]}^b)u = \bar{\varepsilon}^1 \wedge \nabla_{T_w(\bar{\varepsilon}_1, \bar{\varepsilon}_0)}^b u = \frac{\mathbf{i}}{\sqrt{2}} \bar{\varepsilon}^1 \wedge \nabla_{T_w(\bar{\varepsilon}_1, \bar{\varepsilon}_0)}^w u \\ &= -\frac{\mathbf{i}}{2\sqrt{2}} \bar{\varepsilon}^1 \wedge \nabla_{J\Phi \bar{\varepsilon}_1}^b u = \frac{\mathbf{i}}{2\sqrt{2}} \bar{\varepsilon}^1 \wedge \nabla_{\Phi J \bar{\varepsilon}_1}^b u \\ &(J\bar{\varepsilon} - 1 = -\mathbf{i}\bar{\varepsilon}_1) \\ &= \frac{1}{2\sqrt{2}} \bar{\varepsilon}^1 \wedge \nabla_{\Phi \bar{\varepsilon}_1}^b u = \frac{1}{2\sqrt{2}} \Phi_c(\partial_V u) =: \mathfrak{I}u. \end{aligned}$$

where Φ_c is the complexification² of Φ . The differential operator by $\mathfrak{T}$ is trivial when $\Phi = 0$, which in the 3-dimensional case is equivalent to M being Sasakian or to $\hat{J}$ being integrable.

Putting together all the above facts we obtain

$$[\bar{\partial}_0, \bar{\partial}_V] = \bar{\varepsilon}^1 \wedge F_b(\bar{\varepsilon}_0, \bar{\varepsilon}_1) + \mathfrak{T}.$$

We conclude

$$\{Z, T\} = 2 \begin{bmatrix} 0 & \overline{F_b(\bar{\varepsilon}_0, \bar{\varepsilon}_1)} \varepsilon^1 \lrcorner \\ \bar{\varepsilon}^1 \wedge F_b(\bar{\varepsilon}_0, \bar{\varepsilon}_1) & 0 \end{bmatrix} + 2\tilde{\mathfrak{T}}, \quad \tilde{\mathfrak{T}} := \begin{bmatrix} 0 & \mathfrak{T}^* \\ \mathfrak{T} & 0 \end{bmatrix}.$$

The zero order operator above can be further simplified by observing that

$$\mathbf{c}(\bar{\varepsilon}^1) = \sqrt{2} \begin{bmatrix} 0 & 0 \\ \bar{\varepsilon}^1 \wedge & 0 \end{bmatrix}, \quad \mathbf{c}(\varepsilon^1) = \sqrt{2} \begin{bmatrix} 0 & -\varepsilon^1 \lrcorner \\ 0 & 0 \end{bmatrix}$$

so that

$$\begin{aligned} \begin{bmatrix} 0 & \overline{F_b(\bar{\varepsilon}_0, \bar{\varepsilon}_1)} \varepsilon^1 \lrcorner \\ \bar{\varepsilon}^1 \wedge F_b(\bar{\varepsilon}_0, \bar{\varepsilon}_1) & 0 \end{bmatrix} &= \frac{1}{\sqrt{2}} \mathbf{c} \left(F_b(\bar{\varepsilon}_0, \bar{\varepsilon}^1) \bar{\varepsilon}^1 - \overline{F_b(\bar{\varepsilon}_0, \bar{\varepsilon}_1)} \varepsilon^1 \right) \\ &= \frac{\mathbf{i}}{2} \mathbf{c} \left(F_w(\xi, \bar{\varepsilon}_1) \bar{\varepsilon}^1 + \overline{F_w(\xi, \bar{\varepsilon}_1)} \varepsilon^1 \right) \end{aligned}$$

Above we denoted by F_w the curvature of ∇^w as a connection on the hermitian line bundle $(V, J) \cong K_M^{-1}$. To get a more suggestive description we write

$$\xi \lrcorner F_w = \mathbf{i}(ae^1 + bf^1),$$

where a, b are locally defined *real valued functions*. Then

$$F_w(\xi, \bar{\varepsilon}_1) \bar{\varepsilon}^1 = \frac{\mathbf{i}}{2}(a + \mathbf{i}b)(e^1 - \mathbf{i}f^1), \quad \overline{F_w(\xi, \bar{\varepsilon}_1)} \varepsilon^1 = -\frac{\mathbf{i}}{2}(a - \mathbf{i}b)(e^1 + \mathbf{i}f^1).$$

Thus

$$F_w(\xi, \bar{\varepsilon}_1) \bar{\varepsilon}^1 + \overline{F_w(\xi, \bar{\varepsilon}_1)} \varepsilon^1 = (-be^1 + af^1) = -\mathbf{i}(*F_w - \eta \wedge (\xi \lrcorner *F_w)).$$

The last term can also be described as $-\mathbf{i}P_V(*F_w)$, where P_V denotes the orthogonal projection $T^M \rightarrow V^*$, and $*$ denotes the *complex linear* extension of the Hodge operator. The above facts now yield the following commutator identities.

$$\{Z, T\} = \mathbf{c}(P_V * F_w) + 2\tilde{\mathfrak{T}}, \tag{3.6a}$$

$$\mathcal{H}^2 = Z^2 + T^2 + \mathbf{c}(P_V * F_w) + 2\tilde{\mathfrak{T}}. \tag{3.6b}$$

² Φ_c is complex linear but it *anticommutes* with J .

Remark 3.9. (a) If we twist the Dirac operator $\mathfrak{D}(\nabla^w)$ by a hermitian connection on the trivial line bundle $\underline{\mathbb{C}}$ we obtain a new Dirac operator $\mathcal{H}_A$ satisfying

$$\mathcal{H}_A = \begin{bmatrix} -i\nabla_\xi^A & \sqrt{2}(\bar{\partial}_V^A)^* \\ \sqrt{2}\bar{\partial}_V^A & i\nabla_\xi^A \end{bmatrix} =: Z_A + T_A.$$

The operators Z_A and T_A satisfy the anticommutation rule

$$\{Z_A, T_A\} = Z_A^2 + T_A^2 + \mathbf{c}(P_V * F_w) + \mathbf{c}(P_V * F_A) + 2\tilde{\mathfrak{T}}_A \quad (3.7)$$

where $\tilde{\mathfrak{T}}_A$ is defined as $\tilde{\mathfrak{T}}$ using instead the operator $\mathfrak{T}_A := \frac{1}{2\sqrt{2}}\Phi_c\partial_V^A$.

(b) The curvature F_w has the local description

$$F_w = -i\rho d\eta + \eta \wedge (\xi \lrcorner F_w).$$

Up to a positive multiplicative constant (depending on various normalization conventions) the scalar ρ is known as the *Webster scalar curvature*. We refer to [4] for more details.

§3.3 Connections induced by symplectizations The symplectization of the positively oriented metric contact manifold (M^{2n+1}, η, g, J) is the manifold $\tilde{M} = \mathbb{R}_+ \times M$ equipped with the symplectic form

$$\tilde{\omega} = dt \wedge \eta + t d\eta = dt \wedge \eta + t\omega.$$

If we denote by $\tilde{d}$ the exterior derivative on $\tilde{M}$ then we can write

$$\tilde{\omega} = \tilde{d}(t\eta).$$

$\tilde{M}$ is equipped with a compatible metric

$$\tilde{g} = dt^2 + \eta^2 + t\omega(\bullet, J\bullet).$$

We denote by $\tilde{J}$ the associated almost complex structure. We will identify M with the slice $\{1\} \times M$ of $\tilde{M}$.

If we fix as before a local, oriented, orthonormal frame $\xi, e_1, f_1, \dots, e_n, f_n$ compatible with the metric contact structure on M then we get a symplectic frame

$$\tilde{e}_0 = \partial_t, \quad \tilde{f}_0 = \xi, \quad \tilde{e}_k = t^{-1/2}e_k, \quad \tilde{f}_k = t^{-1/2}f_k, \quad k = 1, \dots, n.$$

The dual coframe is

$$\tilde{e}^0 = dt, \quad \tilde{f}^0 = \eta, \quad \tilde{e}^k = t^{1/2}e^k, \quad \tilde{f}^k = t^{1/2}f^k, \quad k = 1, \dots, n.$$

We denote by $\tilde{N}$ the Nijenhuis tensor of $\tilde{J}$ and by $\hat{N}$ the Nijenhuis tensor of the almost complex manifold $(\hat{M}, \hat{J})$ used in §3.1. The Chern connection $\tilde{\nabla}^c$ of $(\tilde{M}, \tilde{g}, \tilde{J})$ is the metric connection with torsion $\tilde{T} = \tilde{N}$. In this case

$$\theta = 0, \quad \mathfrak{b}\tilde{T} = 0.$$

Observe that $\tilde{J} = \hat{J}$. We deduce that for $j, k = 1, \dots, n$ we have

$$\begin{aligned}\tilde{N}(\tilde{e}_j, \tilde{e}_k) &= \frac{1}{t} \hat{N}(e_j, e_k), \quad \tilde{N}(\tilde{e}_j, \tilde{f}_k) = \frac{1}{t} \hat{N}(e_j, f_k), \quad \tilde{N}(\tilde{f}_j, \tilde{f}_k) = \frac{1}{t} \hat{N}(f_j, f_k), \\ \tilde{N}(\partial_t, \tilde{e}_j) &= \frac{1}{\sqrt{t}} \hat{N}(\partial_t, e_j), \quad \tilde{N}(\partial_t, \tilde{f}_k) = \frac{1}{\sqrt{t}} \hat{N}(\partial_t, f_k), \\ \tilde{N}(\xi, \tilde{e}_j) &= \frac{1}{\sqrt{t}} \hat{N}(\partial_t, e_j), \quad \tilde{N}(\xi_t, \tilde{f}_k) = \frac{1}{\sqrt{t}} \hat{N}(\partial_t, f_k).\end{aligned}$$

Denote by $\tilde{D}$ the Levi-Civita connection determined by $\tilde{g}$. It determined by (see [9])

$$\begin{aligned}2\tilde{g}(\tilde{D}_X Y, Z) &= X\tilde{g}(Y, Z) + Y\tilde{g}(X, Z) - Z\tilde{g}(X, Y) \\ &\quad + \tilde{g}([X, Y], Z) + \tilde{g}([Z, X], Y) + \tilde{g}(X, [Z, Y]).\end{aligned}$$

We deduce from the above identity that if X, Y are t -independent vectors tangent along M

$$2\tilde{g}(\tilde{D}_t X, Y) = g(X_V, Y_V) = \omega(X, JY),$$

where $X_V := P_V X$.

$$2\tilde{g}(\tilde{D}_X Y, \partial_t) = -\partial_t \tilde{g}(X, Y) = -g(X_V, Y_V) = \omega(JX, Y).$$

As in §3.1 we want to alter $\tilde{\nabla}^c$ by $B \in \Omega_s^{1,1}(T^* \tilde{M})$ such that $\text{tr } B = 0$ so that the new basic hermitian connection $\tilde{\nabla}^b$ with torsion $\tilde{T}_b^\dagger := \tilde{N}^\dagger + B$ satisfies

$$\tilde{\nabla}_X^b \xi = 0, \quad \tilde{g}(\tilde{\nabla}_X^b Y, \partial_t) = 0, \tag{3.8}$$

for all t -independent tangent vectors X, Y along M .

We have $\tilde{\nabla} = \tilde{D} + A$, where $A^\dagger = -\tilde{T}_b^\dagger$. Thus we need

$$\begin{aligned}0 &= \tilde{g}(\tilde{\nabla}_X Y, \partial_t) = \tilde{g}(\tilde{D}_X Y, \partial_t) - \tilde{g}(X, \tilde{N}(Y, \partial_t) - B(X; Y, \partial_t)) \\ &= -\frac{1}{2} \omega(JX, Y) + \tilde{g}(X, \tilde{N}(\partial_t, Y)) - B(X; Y, \partial_t)\end{aligned}$$

If $Y = \xi$ we deduce

$$B(X; \xi, \partial_t) = 0.$$

If $Y \in C^\infty(V)$ then we deduce

$$\begin{aligned}0 &= -\frac{1}{2} \omega(JX, Y) + \frac{1}{\sqrt{t}} \tilde{g}(X, \hat{N}(\partial_t, Y)) + B(X; \partial_t, Y) \\ &= \frac{1}{2} g(X, Y) + \frac{1}{4\sqrt{t}} \tilde{g}(X, \Phi Y) + B(X; \partial_t, Y) = \frac{1}{2} g(X, Y) + \frac{\sqrt{t}}{4} g(X, \Phi Y) + B(X; \partial_t, Y)\end{aligned}$$

We conclude that B must satisfy the additional conditions

$$B(\xi; \partial_t, Y) = 0, \quad Y \in C^\infty(V)$$

$$B(X; \partial_t, Y) = -\frac{1}{2\sqrt{t}} \left(\frac{1}{\sqrt{t}} \tilde{g}(X, Y) + \frac{1}{2} \tilde{g}(X, \Phi Y) \right)$$

We write $B = B_0 + B_1$ where B_0 is defined as in Lemma 3.4 by the equality

$$B_0 = \frac{1}{4\sqrt{t}} \left\{ \Phi \wedge dt + J\Phi \wedge \eta \right\}$$

B_1 must satisfy the equalities $\text{tr } B_1 = 0$,

$$B_1(X; \partial_t, Y) = -\frac{1}{2t} \tilde{g}(X, Y), \quad \forall X, Y \in C^\infty(V) \quad (3.9a)$$

$$B_1(X; \xi, \partial_t) = B_1(\xi; \partial_t, Y) = 0, \quad \forall X \in \text{Vect}(M), Y \in C^\infty(V) \quad (3.9b)$$

We try B_1 of the form

$$B_1 = xdt \otimes dt \wedge \eta + y\eta \otimes d\eta + U + V$$

where

$$U = \frac{1}{2t} P_V \wedge dt, \quad V = \frac{1}{2t} J P_v \wedge \eta$$

Clearly $B_1 \in \Omega^{1,1}(T^* \tilde{M})$. Next observe that

$$\flat B_1 = y\eta \wedge d\eta + \flat V = \left(y + \frac{1}{t}\right) \eta \wedge d\eta,$$

$$\text{tr } B_1 = \left(x + \frac{n}{t}\right) dt,$$

Thus, set $x = -\frac{n}{t}$, $y = \frac{1}{t}$. These choices guarantee that $B_1 \in \Omega_s^{1,1}(T^* \tilde{M})$ and $\text{tr } B_1 = 0$. The conditions (3.9a) and (3.9b) can now be verified by direct computation. We can now conclude that if

$$B = \frac{1}{4\sqrt{t}} \left(\Phi \wedge dt + J\Phi \wedge \eta \right) - \frac{n}{t} dt \otimes dt \wedge \eta - \frac{1}{t} \eta \otimes d\eta + \frac{1}{2t} \left(P_V \wedge dt + J P_V \wedge \eta \right)$$

then the connection $\tilde{\nabla}^b$ with torsion $\tilde{N}^\dagger + B$ satisfies the conditions (3.8). These conditions show that $\tilde{\nabla}^b$ induces, by restriction to each slice $\{t\} \times M$, a connection ∇^t on TM . The torsion of $\nabla^1 = \nabla^{t=1}$ is given by

$$\begin{aligned} (T_1)^\dagger &= \tilde{N}^\dagger|_{t=1} + B|_{t=1} = \hat{N}^\dagger|_M + \frac{1}{4} (J\Phi \wedge \eta) - \eta \otimes d\eta + \frac{1}{2} (J P_V \wedge \eta) \\ &= \frac{1}{2} N^\dagger - \frac{1}{2} \eta \otimes d\eta + \frac{1}{2} (J P_V \wedge \eta) + \frac{1}{4} (J\Phi \wedge \eta). \end{aligned}$$

When M is a CR manifold we deduce

$$T_1^\dagger = -\eta \otimes d\eta + \frac{1}{2}J \wedge \eta + \frac{1}{2}J\Phi \wedge \eta.$$

In particular

$$T^1(X, Y) = -\xi d\eta(X, Y), \quad \forall X, Y \in C^\infty(V).$$

This connection *never* coincides with generalized Webster connection constructed in §3.1, because in this case we have $\flat T_1^\dagger = 0$. This shows ∇^1 is Dirac equivalent to the Levi-Civita connection. We have thus proved the following result.

Theorem 3.10. *On every metric contact manifold (M, g, J) there exists a canonical nice contact metric connection $= \nabla^1$ induced by a basic Hermitian connection on the symplectization of M . This contact connection is Dirac equivalent to the Levi-Civita connection and its torsion is given by*

$$T_1^\dagger = \frac{1}{2}N^\dagger - \frac{1}{2}\eta \otimes d\eta + \frac{1}{2}(JP_V \wedge \eta) + \frac{1}{4}(J\Phi \wedge \eta). \quad \blacksquare$$

Let us observe that if M is CR then for every $X, Y \in C^\infty(V)$ we have

$$g(\nabla_\xi^1 X, Y) = -g(\xi, T(X, Y)) = +\omega(X, Y)$$

so that

$$\nabla_\xi^1 = D_\xi^V - J = P_V D_\xi + J = \nabla_\xi^w + J.$$

Proposition 3.11. *Suppose M is a CR manifold. Then*

$$\det \nabla^1 = \det \nabla^w + 3ni\eta = \det \nabla^c + \frac{5ni}{2}\eta.$$

Proof

$$\Delta := T_1^\dagger - T_w^\dagger = \frac{1}{4}(-7\eta \otimes d\eta - \eta \wedge d\eta + 3J \wedge \eta)$$

so that $\nabla^1 = \nabla^w + A$ where

$$\begin{aligned} A^\dagger &= \frac{1}{2}\flat \Delta - \Delta = -\frac{1}{2}\eta \wedge d\eta + \frac{1}{4}(7\eta \otimes d\eta + \eta \wedge d\eta - 3J \wedge \eta) \\ &= \frac{1}{4}(7\eta \otimes d\eta - \eta \wedge d\eta - 3J \wedge \eta). \end{aligned}$$

Set

$$\delta = \varepsilon_1 \wedge \cdots \wedge \varepsilon_n.$$

As in the proof of Proposition 3.6 we have

$$\det \nabla_X^1 \delta = \det \nabla_X^w + \mathbf{i} \left(\sum_{k=1}^n C_k^k \right) \delta$$

where

$$C_k^k = g(A_X e_k, J f_k) + g(A_X f_k, J f_k).$$

The above sum is nontrivial only for $X = \xi$ in which case it is equal to $3n$. We conclude that

$$\det \nabla^1 = \det \nabla^w + 3n\mathbf{i}\eta. \blacksquare$$

Remark 3.12. Let us point out a difference between contact and Hermitian connections. We have shown that there always exist contact connections with torsion T satisfying $\mathfrak{b}T^\dagger = 0$.

On the other hand, if ∇ is a Hermitian connection on an almost complex Hermitian manifold (M, g, J) with Nijenhuis tensor N then its torsion satisfies (see [5])

$$(\mathfrak{b}T)^- = (\mathfrak{b}N^\dagger) = (d^c \omega)^-.$$

If $\dim M = 4$ then always $(d^c \omega)^- = 0$ and in this case it is possible to find Hermitian connections Dirac equivalent to the Levi-Civita connection. However, in higher dimensions this is possible if and only if $(d^c \omega)^- = 0$.

§3.4 Uniqueness results The constructions we performed in the previous subsection may seem a bit ad-hoc but as we will show in this section they produce, at least for CR manifolds, connections uniquely determined by a few natural requirements.

Proposition 3.13. *Suppose (M, η, g, J) is CR connection. Then each Dirac equivalence class of connections contains at most one nice CR connection.*

Proof Suppose ∇ is a nice CR connection with torsion T . Set $\Omega := \mathfrak{b}T$. We get a hermitian connection $\hat{\nabla} = dt \wedge \partial_t + \nabla$ on $(T\hat{M}, \hat{J})$ with the property

$$\mathfrak{b}T(\hat{\nabla})^\dagger = \Omega, \quad \text{tr } T(\hat{\nabla})^\dagger = 0.$$

Denote by ∇^b the basic hermitian connection on $(T\hat{M}, \hat{J})$ we have constructed in §3.1. The results in §2.1 imply that

$$T(\hat{\nabla})^\dagger = T_b^\dagger + \frac{9}{8}\psi^+ - \frac{3}{8}\mathfrak{M}\psi^+ + B =: T_b^\dagger + S,$$

where

$$\psi^+ \in \Omega^{3,+}(\hat{M}), \quad B \in \Omega_s^{1,1}(T^*\hat{M}),$$

$$\Omega = \mathfrak{b}T_b^\dagger + 3\psi^+ = 3\psi^+ + \eta \wedge d\eta,$$

$$B(\partial_t; \bullet, \bullet) = 0 = B(\bullet; \bullet, \partial_t) = 0, \quad \text{tr } B = 0. \quad (*)$$

Thus ψ^+ is uniquely determined. Moreover, since ∇ is a CR connection we deduce that

$$g(X, T(Y, Z)) = 0, \quad \forall X, Y, Z \in C^\infty(V).$$

Since the restriction of ∇^b to M is also a CR connection we deduce

$$S(X; Y, Z) = 0, \quad \forall X, Y, Z \in C^\infty(V).$$

Thus the restriction of B to V is uniquely determined. The condition $B \in \Omega_s^{1,1}(T^*\hat{M})$ coupled with $(*)$ show that the restriction of B to $\mathbb{R}\partial_t \oplus \mathbb{R}\xi \subset T\hat{M}$ is also uniquely determined. This concludes the proof of Proposition 3.13. ■

Remark 3.14. We can use Gauduchon's description of the hermitian connections on $T\hat{M}$ to completely characterize which Dirac equivalence classes of connections on TM contain nice CR connections.

Corollary 3.15. *The Webster connection on a CR manifold is the unique CR connection adapted to $\mathcal{H}$. Moreover, the connection ∇^1 of §3.3 is the unique nice CR connection with torsion satisfying $\mathfrak{b}T^\dagger = 0$.*

References

- [1] N. Berline, E. Getzler, M. Vergne: *Heat Kernels and Dirac Operators*, Springer Verlag, 1992.
- [2] J.-M. Bismut: *A local index theorem for non-Kähler manifolds*, Math. Ann., **284**(1989), 681-699.
- [3] D.E. Blair: *Contact Manifolds in Riemann Geometry*, Lect. Notes in Math., vol. **509**, Springer Verlag 1976.
- [4] S.S. Chern, R.S. Hamilton: *On Riemannian metrics adapted to three dimensional contact manifolds*, in the volume “*Arbeitstagung Bonn 1984*”, Lect. Notes in Math., vol. **1111**, 279-305, Springer Verlag 1985.
- [5] P. Gauduchon: *Hermitian connections and Dirac operators*, Boll. U.M.I. **11-B**(1997), Suppl.fasc.2, 257-288.
- [6] P.B. Gilkey: *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, 2nd Edition, CRC Press, 1995.
- [7] S. Kobayashi, K. Nomizu: *Foundations of Differential Geometry*, Interscience Publishers, New York, 1963.
- [8] A. Lichnerowicz: *Global Theory of Connections and Holonomy Groups*, Nordhoff International Pub., Leyden, 1976.
- [9] L.I. Nicolaescu: *Lectures on the Geometry of Manifolds*, World Scientific, 1996
- [10] L.I. Nicolaescu: *Adiabatic limits of the Seiberg-Witten equations on Seifert manifolds*, Comm. Anal. and Geom., **6**(1998), 301-362.
- [11] C.M. Stanton: *Intrinsic connections for Levi metrics*, Manuscripta Math., **75**(1992), 349-364.
- [12] S. Tanno: *Variational problems on contact Riemannian manifolds*, Trans. Amer. Math. Soc., **314**(1989), 349-379.
- [13] C.H. Taubes: *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Letters, **1**(1994), 809-822.
- [14] S.M. Webster: *Pseudo-hermitian structures on a real hypersurface*, J. Diff. Geom., **13**(1978), 25-41.