HODGE NUMBERS OF COMPLETE INTERSECTIONS

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1. Holomorphic Euler characteristics

Suppose X is a compact Kähler manifold of dimension n and E is a holomorphic vector bundle. For every $p \leq \dim_{\mathbb{C}} X$ we have a sheaf $\Omega^p(E)$ whose sections are holomorphic $(p,0)$-forms with coefficients in E. We set

$$H^{p,q}(X,E) := H^q(X, \Omega^p(E)\,), \;\; h^{p,q}(X,E) := \dim_{\mathbb{C}} H^{p,q}(X,E),$$

and we define the holomorphic Euler characteristics

$$\chi^p(X,E) := \sum_{q\geq 0} (-1)^q h^{p,q}(X,E).$$

It is convenient to introduce the generating function of these numbers

$$\chi_y(X,E) := \sum_{p\geq 0} y^p \chi^p(X,E).$$

Observe that

$$\Omega^p(E) \cong \Omega^0(\Lambda^p T^* X^{1,0})$$

so that

$$h^{p,q}(X,E) = h^{0,q}(X, \Lambda^p T^* X^{1,0} \otimes E)$$

and

$$\chi^p(X,E) = \chi^0(X, \Lambda^p T^* X^{1,0} \otimes E).$$

If E is the trivial holomorphic line bundle $\underline{\mathbb{C}}$ then we write $\chi_y(X)$ instead of $\chi_y(X, \underline{\mathbb{C}})$. Observe that

$$\chi_y(X)\,|_{y=-1} = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) = \chi(X),$$

$$\chi^{n-p}(X) = (-1)^n \chi^p(X).$$

Hence for $n = 1$ we have

$$\chi(X) = 2\chi^0(X),$$

while for $n = 2$ we have

$$\chi(X) = 2\chi^0(X) - \chi^1(X).$$

Example 1.1. $X = \mathbb{P}^N$ then

$$h^{p,q}(\mathbb{P}^N) = \begin{cases} 1 & \text{if} \quad 0 \leq p = q \leq N \\ 0 & \text{if} \quad p \neq q \end{cases}.$$

Hence

$$\chi^p(\mathbb{P}^N) = (-1)^p, ;\chi_{-y}(\mathbb{P}^N) = \sum_{p=0}^N y^p = \frac{y^{N+1}-1}{y-1}.$$

□

Date: March, 2005.

2. The Riemann-Roch-Hirzebruch formula

The main tool for computing the holomorphic Euler characteristics $\chi^p(X, E)$ is based on the following fundamental result.

Theorem 2.1 (Riemann-Roch-Hirzebruch)**.**

$$\chi^0(X, E) = \langle \mathbf{td}(X)\, \mathbf{ch}(E), [X] \rangle,$$

where

$$\mathbf{td}(X) = \sum_{k \geq 0} \mathbf{td}_k(X), \quad \mathbf{td}_k(X) \in H^{2k}(X, \mathbb{Q}),$$

denotes the Todd genus of the complex tangent bundle of X,

$$\mathbf{ch}(E) = \sum_{k \geq 0} \mathbf{ch}_k(E), \quad \mathbf{ch}_k(E) \in H^{2k}(X, \mathbb{Q}),$$

denotes the Chern character of E and $\langle \bullet, \bullet \rangle$ denotes the Kronecker pairing between cohomology and homology.

We have the following immediate corollary.

Corollary 2.2.

$$\chi^p(X, E) = \chi^0(X, \Lambda^p T^* X^{1,0} \otimes E) = \langle \mathbf{td}(X)\, \mathbf{ch}(\Lambda^p T^* X^{1,0})\, \mathbf{ch}(E), [X] \rangle.$$

For a vector bundle $V \to X$ we define

$$\mathbf{ch}_y(V) = \sum_{p \geq 0} y^p\, \mathbf{ch}(\Lambda^p V).$$

For simplicity we set $T^*X = T^*X^{1,0}$. The result in the previous corollary can be rewritten as

$$\chi_y(X, E) = \langle \mathbf{td}(X) \cdot \mathbf{ch}_y(T^*X)\, \mathbf{ch}(E), [X] \rangle.$$

To use this equality we need to recall a few basic properties of the Todd genus and the Chern character of a complex vector bundle.

Proposition 2.3. *(a) Suppose $L \to X$ is a complex line bundle and set $x = c_1(L) \in H^2(X, \mathbb{Z})$. Then*

$$\mathbf{td}(L) = \frac{x}{1 - e^{-x}} = e^{x/2} \cdot \frac{x/2}{\sinh(x/2)}, \quad \mathbf{ch}(L) = e^x, \quad \mathbf{ch}_y(L) = 1 + y e^x.$$

(b) If

$$0 \to E_0 \to E \to E_1 \to 0.$$

is a short exact sequence of complex vector bundles then

$$\mathbf{td}(E) = \mathbf{td}(E_0)\, \mathbf{td}(E_1), \quad \mathbf{ch}(E) = \mathbf{ch}(E_0) + \mathbf{ch}(E_1),$$

$$\mathbf{ch}_y(E_0 \oplus E_1) = \mathbf{ch}_y(E_0) \cdot \mathbf{ch}_y(E_1).$$

Moreover

$$\mathbf{ch}(E_0 \otimes E_1) = \mathbf{ch}(E_0)\, \mathbf{ch}(E_1).$$

□

Example 2.4. Suppose $X = \mathbb{P}^N$. We want to compute $\mathbf{td}(X)$ and $\mathbf{ch}_y(T^*X^{1,0})$. Denote by $H \to \mathbb{P}^N$ the hyperplane line bundle. Its sections can be identified with linear maps $\mathbb{C}^{N+1} \to \mathbb{C}$. We denote its first Chern class by h. As is known we have the equality

$$\langle h^N, [\mathbb{P}^N] \rangle = \int_{\mathbb{P}^N} h^N = 1$$

and an isomorphisms of rings

$$H^\bullet(\mathbb{P}^N, \mathbb{Z}) \cong \mathbb{Z}[h]/(h^{N+1}).$$

The dual H^* of H is the tautological line bundle which is a subbundle of the trivial bundle $\underline{\mathbb{C}}^{N+1} \to \mathbb{P}^N$. We denote by Q the quotient $\underline{\mathbb{C}}^{N+1}/H^*$. The tangent bundle of $\mathbb{P}^N$ can be identified with $\mathrm{Hom}(H^*, Q) \cong H \otimes Q$. By tensoring the short exact sequence

$$0 \to H^* \to \underline{\mathbb{C}}^{N+1} \to Q \to 0$$

with H we obtain the short exact sequence

$$0 \to \underline{\mathbb{C}} \to H^{N+1} \to TX \to 0$$

and we deduce

$$\mathbf{td}(H^{N+1}) = \mathbf{td}(\underline{\mathbb{C}})\,\mathbf{td}(TX) \Longrightarrow \mathbf{td}(TX) = \mathbf{td}(H)^{N+1} = \Big(\frac{h}{1-e^{-h}}\Big)^{N+1}. \tag{2.1}$$

Similarly, from the sequence

$$0 \to T^*X^{1,0} \to (H^*)^{N+1} \to \underline{\mathbb{C}} \to 0$$

we deduce

$$(1+y)\cdot \mathbf{ch}_y(T^*X) = \mathbf{ch}_y(H^*)^{N+1} = (1+ye^{-h})^{N+1} \Longrightarrow$$

$$\mathbf{ch}_y(T^*X) = \frac{(1+ye^{-h})^{N+1}}{1+y}. \tag{2.2}$$

3. Hodge numbers of hypersurfaces in $\mathbb{P}^N$.

Consider the line bundle $mH = H^{\otimes m}$ on $X = \mathbb{P}^N$. Its sections can be identified with degree m homogeneous polynomials in the variables $(z_0, \cdots, z_N)$. For a generic section s the zero set is a smooth hypersurface of degree m. We denote it by Z. We want to compute $h^{p,q}(Z)$. We follow closely the approach in [1]. For this we need to use the following fundamental result.

Theorem 3.1 (Lefschetz). *For $k < N-1 = \frac{1}{2}\dim_{\mathbb{R}} Z$ the induced morphism*

$$H^k(\mathbb{P}^N, \mathbb{C}) \to H^k(Z, \mathbb{C})$$

is an isomorphism. Moreover the morphism

$$H^{N-1}(\mathbb{P}^N, \mathbb{Z}) \to H^{N-1}(Z, \mathbb{C})$$

is one-to-one. □

We deduce that for $k < (N-1)$

$$b_k(Z) = b_k(\mathbb{P}^N) = \begin{cases} 1 & \text{if} \quad k \in 2\mathbb{Z} \\ 0 & \text{if} \quad k \in 2\mathbb{Z}+1 \end{cases}$$

Using the equalities

$$b_k(Z) = \sum_{p=0}^{k} h^{p,k-p}(Z)$$

we deduce

$$h^{p,q}(Z) = h^{p,q}(\mathbb{P}^N), \ \ \forall p+q < (N-1).$$

Hence if we set $\nu = (N-1) = \dim_{\mathbb{C}} Z$ we deduce

$$\chi^0(Z) = 1 - (-1)^{\nu} h^{0,\nu}(Z), \ \ \chi^1(Z) = -1 + (-1)^{\nu-1} h^{1,\nu-1}, \ \ \text{etc.}$$

Thus to compute the Hodge numbers it suffices to compute $\chi_y(Z)$.

The first Chern class of mH is mh so that

$$\mathbf{ch}(mH) = e^{mh}.$$

If we denote by $N_Z \to Z$ the normal bundle of the embedding $Z \hookrightarrow X$ then we have the adjunction formula

$$(mH)\mid_Z \cong N_Z$$

and a short exact sequence

$$0 \to TZ \to TX\mid_Z \to (mH)\mid_Z \to 0.$$

Using Proposition 2.3(b), and the identities (2.1) and (2.2) we deduce that

$$\mathbf{td}(Z)\cdot \mathbf{td}((mH)\mid_Z) = \mathbf{td}(X)\mid_Z \Longrightarrow \mathbf{td}(Z) = \Bigl(\frac{h}{1-e^{-h}}\Bigr)^{N+1}\mid_Z \cdot \frac{1-e^{-mh}}{mh}\mid_Z$$

and

$$\mathbf{ch}_y(T^*Z) = \mathbf{ch}_y(T^*X)\mid_Z \cdot \mathbf{ch}_y(-mH)^{-1}\mid_Z = \frac{(1+ye^{-h})^{N+1}}{(1+y)(1+ye^{-mh})}\mid_Z .$$

Hence

$$\chi_y(Z) = \langle \mathbf{td}(Z)\,\mathbf{ch}_y(T^*Z), [Z]\rangle$$

$$= \Bigl\langle \Bigl(\frac{h}{1-e^{-h}}\Bigr)^{N+1}\mid_Z \cdot \frac{1-e^{-mh}}{mh}\mid_Z \frac{(1+ye^{-h})^{N+1}}{(1+y)(1+ye^{-mh})}\mid_Z, [Z]\Bigr\rangle.$$

Since the cohomological class Poincaré dual to $[Z]$ in X is mh we deduce

$$\chi_y(Z) = \Bigl\langle \Bigl(\frac{h}{1-e^{-h}}\Bigr)^{N+1} \cdot \frac{1-e^{-mh}}{mh} \cdot \frac{(1+ye^{-h})^{N+1}}{(1+y)(1+ye^{-mh})}\cdot mh, [X]\Bigr\rangle$$

$$= \Bigl\langle h^{N+1}\Bigl(\frac{1+ye^{-h}}{1-e^{-h}}\Bigr)^{N+1} \cdot \frac{1-e^{-mh}}{(1+y)(1+ye^{-mh})}, [X]\Bigr\rangle$$

We deduce that $\chi_y(Z)$ can be identified with the coefficient of z^{-1} in the Laurent expansion of the function

$$z \longmapsto \Bigl(\frac{1+ye^{-z}}{1-e^{-z}}\Bigr)^{N+1} \cdot \frac{1-e^{-mz}}{(1+y)(1+ye^{-mz})}.$$

This coefficient can be determined using the residue formula so that

$$\chi_y(Z) = \frac{1}{2\pi\mathbf{i}}\int_{|z|=\varepsilon} \Bigl(\frac{1+ye^{-z}}{1-e^{-z}}\Bigr)^{N+1} \cdot \frac{1-e^{-mz}}{(1+y)(1+ye^{-mz})} dz.$$

If we make the change in variables

$$\zeta = 1-e^{-z} \Longleftrightarrow e^{-z} = 1-\zeta, \ \ e^{-mz} = (1-\zeta)^m,$$

$$e^{-z}dz = -d\zeta \Longrightarrow dz = -\frac{1}{1-\zeta} d\zeta$$

we deduce that $\chi_y(Z)$ is given by

$$\chi_y(Z) = \frac{1}{2\pi\mathbf{i}}\int_{C_\varepsilon}\Big(\frac{1+y(1-\zeta)}{\zeta}\Big)^{N+1}\cdot\frac{1-(1-\zeta)^m}{(1+y)(1-\zeta)(1+y(1-\zeta)^m)}d\zeta,$$

where C_ε is a small closed path with winding number around 0 equal to 1. This residue is equal to the coefficient of ζ^N in the ζ-Taylor expansion of the function

$$f_N(y,\zeta) = \frac{\big(\,1+y(1-\zeta)\,\big)^{N+1}}{(1-\zeta)(1+y)}\frac{1-(1-\zeta)^m}{1+y(1-\zeta)^m}$$

For our purposes it is perhaps more productive to understand the y-expansion on $f_N(y,\zeta)$

$$f_N(y,\zeta) = \sum_{p\geq 0} f_{N,p}(\zeta)y^p,$$

and then compute the coefficient of ζ^N in $f_{N,p}(\zeta)$. Set $u = (1-\zeta)$. We have

$$\frac{1-(1-\zeta)^m}{(1+y)(1+y(1-\zeta)^m)} = \frac{1-u^m}{(1+y)(1+u^my)} = (1-u^m)\Big(\sum_{i\geq 0}(-1)^iy^i\Big)\Big(\sum_{j\geq 0}(-1)^ju^{mj}y^j\Big)$$

$$= (1-u^m)\sum_{k\geq 0}(-1)^k\Big(\sum_{j=0}^{k}u^{mj}\Big)y^k = \sum_{k\geq 0}(-1)^k(1-u^{m(k+1)})y^k.$$

$$(1+uy)^{N+1} = \sum_{j=0}^{N+1}\binom{N+1}{j}u^jy^j.$$

From the equalities

$$\sum_{p\geq 0} f_{N,p}(\zeta)y^p = f_y(\zeta) = \frac{1}{u}\Big(\sum_{k\geq 0}(-1)^k(1-u^{m(k+1)})y^k\Big)\Big(\sum_{j=0}^{N+1}\binom{N+1}{j}u^jy^j\Big)$$

we deduce

$$f_{N,p}(\zeta) = \frac{1}{u}\sum_{k=0}^{p}(-1)^k\binom{N+1}{p-k}(1-u^{m(k+1)})u^{p-k}.$$

Note that

$$f_{N,0}(\zeta) = \frac{(1-u^m)}{u} = \frac{1-(1-\zeta)^m}{(1-\zeta)} = \frac{1}{1-\zeta} - (1-\zeta)^{m-1}.$$

$$f_{N,1}(\zeta) = (N+1)(1-u^m) - \frac{(1-u^{2m})}{u} = (N+1)\big(\,1-(1-\zeta)^m\,\big) - (1-\zeta)^{-1} + (1-\zeta)^{2m-1}.$$

The holomorphic Euler characteristic $\chi^p(Z)$ is the coefficient of ζ^N in $f_{N,p}(\zeta)$. For any power series $a(x) = \sum_{n\geq 0} a_nx^n$ we set

$$T_n(a(x)) := a_n.$$

We want to discuss a few special cases.

- $N = 2$. In this case Z is a plane curve and we have $\chi^0(Z) = 1 - h^{0,1}(Z) = \frac{1}{2}\chi(Z)$. Then

$$T_2(f_{2,0}) = \chi^0(Z) = T_2(1-\zeta)^{-1} - T_2(1-\zeta)^{m-1} = 1 - \frac{(m-1)(m-2)}{2}$$

We deduce

$$h^{0,1}(Z) = h^{1,0}(Z) = \frac{(m-1)(m-2)}{2} = \frac{1}{2}b_1(Z)$$

so that Z is a Riemann surface of genus $\frac{(m-1)(m-2)}{2}$. Note that if $m = N+1 = 3$ we have

$$h^{N-1,0} = h^{1,0} = 1,$$

so that Z is an elliptic curve. Z is a 1-dimensional Calabi-Yau manifold.

• $N = 3$ so that Z is an algebraic surface. We have

$$\chi^0(Z) = 1 + h^{0,2}(Z),\ \ \chi^1(Z) = -h^{1,1}(Z),\ \ \chi^2(Z) = \chi^0(Z) = h^{2,0}(Z) + 1.$$

We have

$$\begin{aligned}\chi^0(Z) = T_3(f_{3,0}) = T_3(1-\zeta)^{-1} - T_3(1-\zeta)^{m-1} &= 1 + \binom{m-1}{3}\\ &= 1 + \frac{(m-1)(m-2)(m-3)}{6}\\ \chi^1(Z) = T_3(f_{3,1}) = -4T_3(1-\zeta)^m - T_3(1-\zeta)^{-1} &+ T_3(1-\zeta)^{2m-1}\\ = 4\binom{m}{3} - 1 - \binom{2m-1}{3}&\end{aligned}$$

$$= -1 + 4\frac{m(m-1)(m-2)}{6} - \frac{(2m-1)(2m-2)(2m-3)}{6} = -\frac{4m^3 - 12m + 14m}{6}.$$

Hence

$$h^{2,0}(Z) = h^{0,2}(Z) = \frac{(m-1)(m-2)(m-3)}{6},$$

$$h^{1,1}(Z) = \frac{4m^3 - 12m + 14m}{6},$$

$$b_2(Z) = 2h^{0,2} + h^{1,1}(Z) = 1 + 2\binom{m-1}{3} - 4\binom{m}{3} + \binom{2m-1}{3} = m^3 - 4m^2 + 6m - 2.$$

Note that if $m = (N+1) = 4$ we have

$$h^{N-1,0} = h^{2,0} = 1,\ \ h^{1,1} = 20.$$

In this case we say that X is a $K3$ surface. It is a 2-dimensional Calabi-Yau manifold.

4. Hodge numbers of complete intersection curves in $\mathbb{P}^3$

Consider two generic hypersurfaces $Z_m, Z_n \subset \mathbb{P}^3$ of degree m and respectively n. Their intersection is a curve $C = C_{m,n}$. We want to compute its genus. Denote by N_C the normal bundle of the embedding $C_{m,n} \hookrightarrow \mathbb{P}^3$. $N_C \to C$ is a rank 2 vector bundle over C and we have the adjunction isomorphism

$$N_C \cong (mH \oplus nH)\mid_C .$$

Hence we obtain a short exact sequence

$$0 \to TC \to (T\mathbb{P}^3)\mid_C \to (mH \oplus nH)\mid_C \to 0$$

so that

$$1 + c_1(TC) = \mathbf{ch}(TC) = \mathbf{ch}(T\mathbb{P}^3)\mid_C - \mathbf{ch}(mH)\mid_C - \mathbf{ch}(nH)\mid_C .$$

From the exact sequence

$$0 \to \underline{\mathbb{C}} \to H^{\oplus 4} \to T\mathbb{P}^3 \to 0$$

we deduce

$$\mathbf{ch}(T\mathbb{P}^3) = 4e^h - 1$$

so that

$$1 + c_1(TC) = (4e^h - 1 - e^{mh} - enh)\mid_C$$

so that

$$e(C) = c_1(TC) = (4 - m - n)h\,|_C\,.$$

We deduce that

$$\chi(C) = \langle e(C), [C]\rangle = \langle (4 - m - n)h\,|_C, [C]\rangle.$$

Since $[C]$ is the homology class Poincaré dual to mnh^2 we deduce

$$\chi(C_{m,n}) = \langle (4 - m - n)mnh^3, [\mathbb{P}^3]\rangle = -(m + n - 4)mn$$

so that

$$g(C_{m,n}) = h^{1,0}(C_{m,n}) = 1 + \frac{mn(m + n - 4)}{2}.$$

Remark 4.1. If $X = X^n_{m_1,\cdots,m_k} \subset \mathbb{P}^{n+k}$ is a generic intersection of k hypersurfaces of degrees $m_1, \cdots, m_k$ (so $\dim_{\mathbb{C}} X = n$) then

$$\sum_{n\geq 0} \chi_y(X^n_{m_1,\cdots,m_k})z^{n+k} = \frac{1}{(1 + zy)(1 - z)} \prod_{j=1}^{k} \frac{(1 + zy)^{m_i} - (1 - z)^{m_i}}{(1 + zy)^{m_i} + y(1 - z)^{m_i}}.$$

For a proof, we refer to [1, Appendix I]. □

References

[1] F. Hirzebruch: *Topological Methods in Algebraic Geometry*, Springer Verlag, New York, 1966.

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