SCHUBERT CALCULUS ON THE GRASSMANNIAN OF HERMITIAN LAGRAGIAN SPACES

Abstract. We describe a Schubert like stratification on the Grassmannian of hermitian lagrangian spaces in $\mathbb{C}^n \oplus \mathbb{C}^n$ which is a natural compactification of the space of hermitian $n \times n$ matrices. The closures of the strata define integral cycles and we investigate their intersection theoretic properties. The methods employed are Morse theoretic.

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Introduction

A hermitian lagrangian subspace, is a subspace $L$ of a the complex Hermitian vector space $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ satisfying

$$L^\perp = JL,$$

where $J : \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n$ is the unitary operator with the block decomposition

$$J = \begin{bmatrix} 0 & -\mathbb{1}_{\mathbb{C}^n} \\ \mathbb{1}_{\mathbb{C}^n} & 0 \end{bmatrix}.$$

We denote by $\text{Lag}_{h}(n)$ the Grassmannian of such subspaces. This space can be identified with a more familiar space.

Denote by $F^\pm \subset \mathbb{C}^{2n}$ the $\pm i$ eigenspace of $J$,

$$F^\pm = \{(e, \mp ie) : e \in \mathbb{C}^n\}.$$

Arnold has shown in [2] that $L \subset \mathbb{C}^{2n}$ is a hermitian lagrangian subspace if and only if, when viewed as a subspace of $F^+ \oplus F^-$, it is the graph of a unitary operator $F^+ \to F^-$. Thus we have a natural diffeomorphism

$$U(n) \to \text{Lag}_{h}(n).$$

With an eye towards infinite dimensional applications, we describe in this paper a natural, Schubert like stratification of $\text{Lag}_{h}(n)$ and its intersection theoretic properties.

Much like in the case of usual Grassmannians, this stratification has a Morse theoretic description. We denote by \((e_i)\) the canonical unitary basis of \(\mathbb{C}^n\), and we define the Hermitian operator \(A : \mathbb{C}^n \to \mathbb{C}^n\) by setting
\[
Ae_i = \left( i - \frac{1}{2} \right) e_i, \quad \forall i = 1, \ldots, n.
\]
The operator \(A\) defines a function
\[
f = f_A : U(n) \to \mathbb{R}, \quad f(S) = -\text{Re} \text{ tr}(AS) + \frac{n^2}{2}.
\]
This is a Morse function with one critical point \(S_I \in U(n)\) for every subset \(I \subset \{1, \ldots, n\}\). More precisely
\[
S_I e_i = \begin{cases} 
  e_i & i \in I \\
  -e_i & i \notin I.
\end{cases}
\]
Its Morse index is
\[
\text{ind} (S_I) = f(S_I)
\]
so that this function is self-indexing. We denote by \(W_I^{\pm}\) the stable/unstable manifold of \(S_I\).
The unstable manifolds can be identified with the orbits of a real algebraic group acting on \(\text{Lag}_h(n)\) so that, according to [21], the stratification given by these unstable manifolds satisfies the Whitney regularity condition. In particular, this implies that our gradient flow satisfies the Morse-Smale transversality condition. We can thus define the Morse-Floer complex, and it turns out that the boundary operator of this complex is trivial. The ideas outlined so far are classical, going back to the pioneering work of Pontryagin [29], and we recommend [10] for a nice presentation.

Given that the Morse-Floer complex is perfect it is natural to ask if the unstable manifolds \(W_I^{\pm}\) define geometric cycles in any reasonable way, and if so, investigate their intersection theory. M. Goresky [13] has explained how to associate cycles to Whitney stratified objects but this approach seems difficult to use in concrete computations. Instead, we chose to work with the theory of subanalytic cycles developed by R. Hardt [14, 15, 16].

The manifolds \(W_I^{\pm}\) have finite dimensional volume, and carry natural orientations \(or_I\), and thus define integration currents \([W_I, or_I]\). In Proposition 5.5 we show that the closure of \(W_I^{-}\) is a naturally oriented pseudo-manifold, i.e., it admits a stratification by smooth manifolds, with top stratum oriented, while the other strata have (relative) codimension at least 2. Using the fact that the current \([W_I^{-}, or_I]\) is a subanalytic current as defined in [16], it follows that \(\partial[W_I^{-}, or_I] = 0\) in the sense of currents. We thus get cycles \(\alpha_I \in H_*(U(n), \mathbb{Z})\). These cycles are exactly de cycles produced by V.A. Vasil’ev in [32] by a different method.

The currents \([W_I^{-}, or_I]\) define a perfect subcomplex of the complex of integrally flat currents. This subcomplex is isomorphic to the Morse-Floer complex, and via the finite-volume-flow technique of Harvey-Lawson [17] we conclude that the cycles \(\alpha_I\) form an integral basis of \(H_*(U(n), \mathbb{Z})\). The cycle \(\alpha_I\) has codimension
\[
\text{codim} \alpha_I = \sum_{i \in I} (2i - 1).
\]
We denote by \(\alpha_I^\dagger \in H^*(U(n), \mathbb{Z})\) its Poincaré dual. When \(I\) is a singleton, \(I = \{i\}\), we use the simpler notation \(\alpha_i\) and \(\alpha_i^\dagger\) instead of \(\alpha_{\{i\}}\) and respectively \(\alpha_{\{i\}}^\dagger\). We call the cycles \(\alpha_i\) the basic Arnold-Schubert cycles. Let us explain this choice of terminology.
We used Arnold’s name since he was the first to give a clear description in [1] of the (real counterpart of the) cycle $\alpha_1$ which has codimension 1. The intersection of a loop of lagrangians with this cycle computes the Maslov index of that loop.

Schubert’s name appears because the cells $W_I^+$ can be described by a Schubert-like incidence relation: if $k = \# I$, then $S \in W_I^+$ if and only if the subspace $K_S = \ker(1 - S)$ is $k$ dimensional and belongs to a certain Schubert cell in the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$.

We used the attribute basic because in Theorem 6.5 we proved the following intersection formula. If $I = \{ i_1 < \cdots < i_k \}$ then

$$\alpha_I^\dagger = \alpha_{i_1}^\dagger \cup \cdots \cup \alpha_{i_k}^\dagger.$$  \hfill (\ast)

This shows that the basic classes generate the cohomology ring $H^*(U(n), \mathbb{Z})$. In particular, we obtain the well known fact that the integral cohomology ring of $U(n)$ is an exterior algebra with one generator in each of the dimensions $1, 3, \ldots, 2n - 1$. The above intersection formula shows that we can take the basic classes $\alpha_I^\dagger$ as generators of this exterior algebra.

It is well known that the cohomology of $U(n)$ is related via transgression to the cohomology of its classifying space $BU(n)$. We prove that the basic class $\alpha_i$ is obtainable by transgression from the Chern class $c_i$.

More precisely, denote by $E$ the rank $n$ complex vector bundle over $S^1 \times U(n)$ obtained from the trivial vector bundle

$$\mathbb{C}^n \times [0, 1] \times U(n) \to [0, 1] \times U(n)$$

by identifying the point $g \in \mathbb{C}^n$ in the fiber over $(1,g) \in [0, 1] \times U(n)$ with the point $g \bar{z}$ in the fiber over $(0,g) \in [0, 1] \times U(n)$. We denote by $p : S^1 \times U(n) \to U(n)$ the natural projection, and by $p_! : H^*(S^1 \times U(n), \mathbb{Z}) \to H^* U(n), \mathbb{Z})$ the induced Gysin map. We prove a Thom-Porteous type formula (Theorem 5.8) asserting that

$$\alpha_i = p_!(c_i(E)).$$

The reader familiar with the Chern-Weil construction will immediately recognize the transgression construction in the above equality.

We refer to the above identity as a Thom-Porteous formula because it is a consequence of the classical Thom-Porteous formula describing the Poincaré dual of a Chern class as a certain degeneracy cycle.

We have to comment a bit about the flavor of the proofs. The lagrangian grassmannian $\text{Lag}_k(n)$ has double incarnation: as the unitary group, and as a collection of vector subspaces, and each of these points of view has its uses. The unitary group interpretation is very well suited for global problems, while the grassmannian incarnation is ideal for local computations.

In this paper we solve by local means a global problem, the computation of the intersection of two cycles, and not surprisingly, both incarnations of $\text{Lag}_k(n)$ will play a role in the final solution. Switching between the two points of view requires some lengthy but elementary computations.

The intersection theory investigated in this paper is closely related to the traditional Schubert calculus on complex grassmannians, but uses surprisingly little of the traditional technology. The intersection theory on $\text{Lag}_k(n)$ has one added layer of difficulty because the cycles involved could be odd dimensional, and when computing intersection numbers one has to count a signed number of points, not just a number of geometric points. Not surprisingly, the computations of these signs turned out to be a rather tedious job. Moreover, given that the cycles involved are represented by singular real semi-algebraic objects, the general position arguments are a bit more delicate.
Finally, a few words about the organization of the paper. The first two sections survey known material. In Section 1 we describe carefully Arnold’s isomorphism $U(n) \to \text{Lag}_h(n)$, while in Section 2 we describe the most salient facts concerning the Morse function $f_A$.

In Section 3 we give an explicit description of the unstable manifold $W_I^-$ using Arnold’s graph coordinates. In these coordinates, the unstable manifolds become identified with certain vector subspaces of the vector space of $n \times n$ hermitian matrices.

In Section 4 we investigate the tunnellings of the Morse flow, i.e., gradient flow lines connecting two critical points $S_I, S_J$. We introduce a binary relation “≺” on the set of critical points by declaring $S_J \prec S_I$ if and only if there exists a gradient flow line tunnelling from $S_I$ to $S_J$. In Proposition 4.4 we give purely combinatorial description of this relation which implies that “≺” is in fact a partial order. The transitivity of “≺” is usually deduced as a consequence of the Morse-Smale condition. In Proposition 4.6 we give a more geometric explanation of this transitivity by showing that $W_J^- \subset \text{cl}(W_I^-)$. This shows that $≺$ is very similar to the Bruhat order in the classical case of grassmannians, and leads to a natural stratification of the closure of an unstable manifold, where each stratum is itself an unstable manifold.

In Section 5 we introduce the cycles $\alpha_I$, we show that they form an integral basis of the homology, and we prove the odd version of the Thom-Porteous formula. This section is rather long since we had to carefully describe our orientation conventions. Section 6, contains the proof main intersection result, the identity ($\ast$).

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Notations and conventions

- For any finite set $I$, we denote by $\# I$ or $|I|$ its cardinality.
- $i := \sqrt{-1}$.
- $\mathbb{I}_n := \{-n, \ldots, -1, 1, \ldots, n\}, \mathbb{I}_n^+ = \{1, \ldots, n\}$.
- For an oriented manifold $M$ with boundary $\partial M$, the induced orientation on the boundary is obtained using the outer-normal first convention.
- For any subset $S$ of a topological space $X$ we denote by $\text{cl}(S)$ its closure in $X$.
- For any complex vector space $E$ and every nonnegative integer $m \leq \dim E$ we denote by $\text{Gr}_m(E)$ (respectively $\text{Gr}^m(E)$) the Grassmannian of complex subspaces of $E$ of dimension $m$ (respectively codimension $m$).
- Suppose $E$ is a complex Euclidean vector space of dimension $n$ and $F\mathbb{I} := \{F_0 \subset F_1 \subset \cdots \subset F_n\}$ is a complete flag of subspaces of $E$, i.e., $\dim F_i = 0, \forall i = 0, \ldots, n$. 

For the reader’s convenience we included two technical appendices. The first one is about tame (or o-minimal) geometry and tame flows. We need this facts to prove that the gradient flow of $f$ is Morse-Stokes in the sense of [17]. The criteria for recognizing Morse-Stokes flows proved by Harvey-Lawson in [17] do not apply to our gradient flow, and since or flow can be described quite explicitly, the o-minimal technology is ideally suited for this task.

The second appendix is a fast past introduction to R. Hardt’s intersection theory of subanalytic currents. We describe a weaker form of transversality, and explain in Proposition B.4 how to compute intersections under this milder conditions
For every integer $0 \leq m \leq n$, and every partition $\mu = \mu_1 \geq \mu_2 \cdots$ such that $\mu_1 \leq m$ and $\mu_i = 0$, for all $i > n - m$, we define the Schubert cell $\Sigma_\mu(F_\mu)$ to be the subset of $\text{Gr}^m(E)$ consisting of subspaces $V$ satisfying the incidence relations

$$\dim(V \cap F_j) = i,$$

$\forall i = 1, \ldots, m, \forall j, m + i - \mu_i \leq j \leq m + i - \lambda_{i+1}$.

1. **Hermitian Lagrangians**

We would like to collect in this section a few basic facts concerning hermitian lagrangian spaces which we will need in our study. All of the results are due to V.I. Arnold, [2]. In this section all vector spaces will be assumed finite dimensional.

**Definition 1.1.** A hermitian symplectic space is a triplet $(\hat{E}, R, J)$, where $\hat{E}$ is a complex hermitian space, and $R, J : \hat{E} \to \hat{E}$ are complex linear operators such that

$$R^* = R, \ J^* = -J, \ R^2 = 1_{\hat{E}} = -J^2, \ RJ = -JR.$$ 

The map $R$ is called the reflection. An isomorphism of hermitian symplectic spaces $(\hat{E}_i, R_i, J_i)$, $i = 0, 1$, is a metric preserving linear isomorphism $T : \hat{E}_0 \to \hat{E}_1$ such that $TR_0 = R_1T$, and $TJ_0 = J_1T$.

**Remark 1.2.** Note that a symplectic vector space $(\hat{E}, R, J)$ is equipped with a natural structure of $\mathbb{Z}/2$-graded $C_1$-module, where $C_1$ denotes the Clifford algebra with one generator $e_1$ satisfying $e_1^2 = -1$. The $\mathbb{Z}/2$-grading is given by the reflection $R$.

If $(\hat{E}, R, J)$ is a hermitian symplectic space, and $h(\bullet, \bullet)$ is the hermitian metric on $\hat{E}$, then the symplectic hermitian form associated to this space is the form

$$\omega : \hat{E} \times \hat{E} \to \mathbb{C}, \ \omega(u, v) = h(Ju, v).$$

Observe that $\omega$ is linear in the first variable and conjugate linear in the second variable. Moreover

$$\omega(u, v) = -\overline{\omega(v, u)}, \ \forall u, v \in \hat{E}.$$ 

Observe that the $\mathbb{R}$-bilinear map

$$q(u, v) := \text{Re} \ h(iJu, v)$$

is symmetric, nondegenerate and has signature 0. We denote by $\text{Sp}_h(\hat{E}, J)$ the subgroup of $\text{GL}_C(\hat{E})$ consisting of complex linear automorphisms of $\hat{E}$ which preserve $\omega$, i.e.,

$$\omega(Tu, v) = \omega(u, Tu), \ \forall u, v \in \hat{E}.$$ 

Equivalently,

$$\text{Sp}_h(\hat{E}, J) = \{ T \in \text{GL}_C(\hat{E}); \ T^*JTJ \}.$$ 

If $(\hat{E}, R, J)$ is a hermitian symplectic space, then we set $\hat{E}^\pm := \ker(\pm 1 - R)$. The operator $J$ induces a linear isomorphism $\hat{E}^+ \to \hat{E}^-$. Note that we have a natural structure of hermitian symplectic space on $\hat{E}^+ \oplus \hat{E}^+$ with reflection

$$R = \begin{bmatrix} 1_{\hat{E}^+} & 0 \\ 0 & -1_{\hat{E}^+} \end{bmatrix}, \ \text{and} \ J := \begin{bmatrix} 0 & -1_{\hat{E}^+} \\ 1_{\hat{E}^+} & 0 \end{bmatrix}.$$ 

The linear map $T : \hat{E}^+ \oplus \hat{E}^+ \to \hat{E}$, given by $T(x \oplus y) = x + Jy$ induces an isomorphism of complex symplectic spaces $(E^+ \oplus E^+, R, J) \to (\hat{E}, R, J)$. 


For this reason, in the sequel we will assume that our hermitian symplectic spaces have the form

\[ \hat{E} = E \oplus E, \quad R = R = \begin{bmatrix} 1_E & 0 \\ 0 & -1_E \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1_E \\ 1_E & 0 \end{bmatrix}, \]

and we will refer to it as the standard hermitian symplectic structure on \( E \oplus E \).

**Definition 1.3.** Suppose \((\hat{E}, R, J)\) is a hermitian symplectic space. A hermitian lagrangian subspace of \( \hat{E} \) is a complex subspace \( L \subset \hat{E} \) such that \( L^\perp = L \). We will denote by \( \text{Lag}_h(\hat{E}) \) the set of hermitian lagrangian subspaces of \( \hat{E} \).

**Remark 1.4.** If \( \omega \) is the symplectic form associated to \((\hat{E}, R, J)\) then a subspace \( L \) is hermitian lagrangian if and only if

\[ L = \{ u \in \hat{E}; \quad \omega(u, x) = 0, \quad \forall x \in L \}. \]

This shows that the group \( \text{Sp}_h(\hat{E}, J) \) acts on \( \text{Lag}_h(\hat{E}) \), and it is not hard to prove that the action is transitive. \( \square \)

Observe that if \( L \in \text{Lag}_h(\hat{E}) \) then we have a natural isomorphism \( L \oplus JL \to \hat{E} \). It follows that \( \dim \mathbb{C} L = \frac{1}{2} \dim \mathbb{C} \hat{E} \), and if we set \( 2n := \dim \mathbb{C} \hat{E} \) we deduce that \( \text{Lag}_h(\hat{E}) \) is a subset of the Grassmannian \( \text{Gr}_n(\hat{E}) \) of complex \( n \)-dimensional subspaces of \( \hat{E} \). As such, it is equipped with an induced topology.

**Example 1.5.** Suppose \( E \) is a complex hermitian space. To any linear operator \( A : E \to E \) we associate its graph

\[ \Gamma_A = \{ (x, Ax) \in A \oplus E; \quad x \in E \}. \]

Then \( \Gamma_A \) is a complex lagrangian subspace of \( E \oplus E \) if and only if the operator \( A \) is selfadjoint.

More generally, if \( L \) is a lagrangian subspace in a hermitian symplectic vector space \( \hat{E} \), and \( A : L \to L \) is a linear operator, then the graph of \( JA : L \to JL \) viewed as a subspace in \( L \oplus JL = \hat{E} \) is a lagrangian subspace if and only if \( A \) is a Hermitian operator. \( \square \)

**Lemma 1.6.** Suppose \( E \) is a complex hermitian space, and \( S : E \to E \) is a linear operator. Define

\[ \mathcal{L}_S := \{ \begin{bmatrix} (1 + S)x \\ -i(1 - S)x \end{bmatrix}; \quad x \in E \} \subset E \oplus E. \]

Then \( \mathcal{L}_S \in \text{Lag}_h(E \oplus E) \) if and only if \( S \) is a unitary operator.

**Proof.** Observe that \( \mathcal{L}_S \) is the image of the linear map

\[ E \ni E \oplus E, \quad x \mapsto J_S(x) = \begin{bmatrix} (1 + S)x \\ -i(1 - S)x \end{bmatrix}. \]

This map is injective because

\[(1 + S)x = (1 - S)x = 0 \implies x = Sx = -Sx \implies x = 0.\]

Hence \( \dim \mathbb{C} \mathcal{L}_S = \dim \mathbb{C} E = \frac{1}{2} \dim \mathbb{C} E \oplus E = \dim \mathbb{C} \mathcal{L}_S \), and we deduce that \( \mathcal{L}_S \) is a lagrangian subspace if and only if \( JL_S \subset \mathcal{L}_S \).

We denote by \((\cdot, \cdot)\) the Hermitian scalar product in \( E \) and by \((\cdot, \cdot)\) the Hermitian inner product in \( E \oplus E \). If

\[ u = J_S(x) = \begin{bmatrix} (1 + S)x \\ -i(1 - S)x \end{bmatrix} \implies Ju = \begin{bmatrix} i(1 - S)x \\ (1 + S)x \end{bmatrix}. \]
For every \( v = (1 + S)y \oplus -i(1 - S)y = J_S(y) \in \mathcal{L}_S \) we have
\[
\langle Ju, v \rangle = i(x - Sx, y + Sy) + i(x + Sx, y - Sy)
\]
Now observe that
\[
(x - Sx, y + Sy) = (x, y) + (x, Sy) - (Sx, y) - (Sx, Sy) = (x, Sy) - (Sx, y),
\]
and
\[
(x + Sx, y - Sy) = (x, y) - (x, Sy) + (Sx, y) - (Sx, y) = -(x, Sy) + (Sx, y)
\]
so that
\[
\langle J_L S(x), J_S(y) \rangle = 2(x, y) - 2(Sx, Sy), \ \forall x, y \in E.
\]
We deduce that
\[
J_L S \subset L^\perp_S \text{ if and only if } (x, y) = (Sx, Sy), \ \forall x, y \in E \iff \text{the operator } S \text{ is unitary.}
\]

**Proposition 1.7** (Arnold). Suppose \( E \) is a complex hermitian space, and denote by \( U(E) \) the group of unitary operators \( S : E \to E \). Then the map
\[
U(E) \ni S \mapsto L_S \in \text{Lag}_h(E \oplus E)
\]
is a homeomorphism. In particular, we deduce that \( \text{Lag}_h(\hat{E}) \) is a smooth, compact, connected, orientable real manifold of dimension
\[
\dim \text{Lag}_h(\hat{E}) = (\dim \mathbb{C} E)^2.
\]

**Proof.** For simplicity we set \( \hat{E} := E \oplus E \). Set
\[
F^\pm := \ker(\pm i - J) \subset \hat{E}.
\]
Note that
\[
F^\pm := \left\{ \begin{bmatrix} x \\ \mp i x \end{bmatrix} ; x \in E \right\}.
\]
Denote by \( I_{\pm} : E \to F^\pm \) the linear isomorphism
\[
E \ni x \mapsto \begin{bmatrix} x \\ \mp i x \end{bmatrix} \in F^\pm.
\]
The key fact behind Proposition 1.7 is the following.

**Lemma 1.8.** If \( L \in \text{Lag}_h(\hat{E}) \) then \( L \cap F^\pm = \{0\} \).

**Proof.** Suppose \( f \in F^\pm \cap L \). Then \( Jf \in L^\perp \) so that \( \langle Jf, f \rangle = 0 \). On the other hand, \( Jf = \pm if \) so that
\[
0 = \langle Jf, f \rangle = \pm i |f|^2 \implies f = 0.
\]
Using the isomorphism \( \hat{E} = F^+ \oplus F^- \) we deduce from the above lemma that \( L \) can be represented as the graph of a linear isomorphism \( T = T_L : F^+ \to F^- \), i.e.,
\[
L = \{f \oplus T f; \ f \in F^+ \}.
\]
We denote by \( S_L : E \to E \) the linear map given by the commutative diagram
\[
\begin{array}{ccc}
E & \xrightarrow{S_L} & E \\
\downarrow J_+ & & \downarrow J_- \\
F_+ & \xrightarrow{T} & F^- \\
\end{array}
\]
The operator $T$ can be described in terms of the map $S_L$ as $F^+ \ni J_+ x \mapsto J_-(S_L x) \in F^-$, $x \in E$.

Its graph is the subset of $F^+ + F^-$

$$L = \Gamma_T = \{ J_+ x + J_-(S_L x); \ x \in E \} = \{ \begin{bmatrix} x \\ -ix \\ isLx \end{bmatrix} + \begin{bmatrix} S_L x \\ isLx \end{bmatrix}; \ x \in E \} = \mathcal{L}_{S_L}.$$  

From Lemma 1.6 we deduce that the operator $S_L$ is unitary, and that the map

$$\text{Lag}_h(\hat{E}) \ni L \mapsto S_L \in U(E)$$

is the inverse of the map $S \mapsto \mathcal{L}_S$. To prove that $L$ is a homeomorphism we first need to recall the topology on $\text{Lag}_h(\hat{E})$.

For every subspace $L \subset \hat{E}$ we denote by $P_L$ the orthogonal projection onto $L$ and we form the reflection

$$R_L = P_L - P_L^* = 2P_L - \mathbb{1}.$$  

The subspace $L$ is lagrangian if and only if the reflection $R_L$ anticommutes with $J$. This shows that the map $L \mapsto P_L$ embeds $\text{Lag}_h(\hat{E})$ as a compact subset of the space of selfadjoint operators $\hat{E} \to \hat{E}$ equipped with the norm topology.

Since the group $U(E)$ is also compact, it suffices to show that the map $S \mapsto \mathcal{L}_S$ is continuous. To achieve this we will give an explicit description of the projector $P_{\mathcal{L}_S}$ in terms of the unitary operator $S$.

Let $S \in U(E)$ so that

$$\mathcal{L}_S = \begin{bmatrix} (1+S)x \\ -i(1-S)x \end{bmatrix}; \ x \in E \} = \mathcal{L}_{S_L} = \{ \begin{bmatrix} i(1-S)x \\ (1+S)x \end{bmatrix}; \ x \in E \}.$$  

For simplicity, we set $P_S := P_{\mathcal{L}_S}$. Let $u \oplus v \in \hat{E}$. Then

$$P_S \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (1+S)x \\ -i(1-S)x \end{bmatrix} \iff \exists y \in E: \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (1+S)x \\ -i(1-S)x \end{bmatrix} + \begin{bmatrix} i(1-S)y \\ (1+S)y \end{bmatrix}$$

$$\iff \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (1+S) \\ -i(1-S) \end{bmatrix} \begin{bmatrix} i(1-S) \\ (1+S) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}.$$  

Observe that

$$\begin{bmatrix} (1+S) & \ i(1-S) \\ -i(1-S) & (1+S) \end{bmatrix}^{-1} = \frac{1}{4} S^{-1} = \frac{1}{4} \begin{bmatrix} \ i(1-S) & (1+S) \\ -i(1-S) & (1+S) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \ i(1-S) & (1+S) \\ -i(1-S) & (1+S) \end{bmatrix}.$$  

where we used the equality $S^* = S^{-1}$. Hence

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \ i(1-S) & (1+S) \\ -i(1-S) & (1+S) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$  

We deduce

$$P_S \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (1+S)x \\ -i(1-S)x \end{bmatrix} = \frac{1}{4} \begin{bmatrix} (1+S)(1+S^*) & \ i(1+S)(1-S^*) \\ -i(1-S)(1+S^*) & (1-S)(1-S^*) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$
Hence
\[
\hat{\text{S}} \text{chubert calculus on the Grassmannian of Hermitian Lagrangian Spaces 9}
\]

\[
\begin{bmatrix}
1 + \frac{1}{2}(S + S^*) & \frac{i}{2}(S - S^*) \\
\frac{i}{2}(S - S^*) & 1 - \frac{1}{2}(S + S^*)
\end{bmatrix}
\begin{bmatrix}
u \\
u
\end{bmatrix}.
\]

Hence
\[
P_{L_S} = \frac{1}{2}
\begin{bmatrix}
1 + \frac{1}{2}(S + S^*) & \frac{i}{2}(S - S^*) \\
\frac{i}{2}(S - S^*) & 1 - \frac{1}{2}(S + S^*)
\end{bmatrix}
\]
which shows that the map $S \mapsto \mathcal{L}_S$ is continuous.

Suppose $A : E \to E$ is a selfadjoint operator. Then as we know its graph $\Gamma_A$ is a lagrangian in $\hat{E} = E \oplus E$. Moreover (see e.g. [26, p.222])
\[
P_{\mathcal{L}_A} = \begin{bmatrix}
(1 + A^2)^{-1} & A(1 + A^2)^{-1} \\
A(1 + A^2)^{-1} & A^2(1 + A^2)^{-1}
\end{bmatrix}.
\]

If we denote by $S = S_A$ the unitary operator associated to $\Gamma_A$, $S = S_{\mathcal{L}_A}$ then using (1.2) we deduce
\[
2(1 + A^2)^{-1} = 1 + \frac{1}{2}(S + S^*), \quad 2A(1 + A^2)^{-1} = \frac{i}{2}(S - S^*).
\]

Hence
\[
1 + S = 2(1 + A^2)^{-1} - 2iA(1 + A^2)^{-1} = 2(1 - iA)(1 - iA)^{-1}(1 + iA)^{-1} = 2(1 + iA)^{-1},
\]
so that
\[
S = 2(1 + iA)^{-1} - 1 = (1 + iA)^{-1}(2I - 1 - iA) = (1 - iA)(1 + iA)^{-1}.
\]
The last expression is the so called Cayley transform of $iA$. We will denote it by $\mathcal{C}(iA)$. Thus, for every symmetric operator $A : E \to E$ we have
\[
S_{\mathcal{L}_A} = \mathcal{C}(iA) := (1 - iA)(1 + iA)^{-1}.
\]

Observe that we have a left action “$*$” of $U(E) \times U(E)$ on $U(E)$ given by
\[
(T_+, T_-) * S = T_- ST_+, \quad \forall T_+, T_-, S \in U(E).
\]
This action is transitive, and the stabilizer of any $S \in U(E)$ is conjugate with the diagonal subgroup of $U(E)$.

Through the diffeomorphism $\mathcal{L} : U(E) \to \text{Lag}_h(\hat{E})$ we obtain a left $U(E) \times U(E)$-left action on $\text{Lag}_h(\hat{E})$. More precisely, if $L = \mathcal{L}_S$, $S \in U(E)$ so that
\[
L = \left\{ \begin{bmatrix}
(1 + S)x \\
i(1 - S)x
\end{bmatrix} : x \in E \right\}
\]
Then
\[
\mathcal{L}_{T_- ST_+} = \left\{ \begin{bmatrix}
(1 + T_- ST_+^*)x \\
i(1 - T_- ST_+^*)x
\end{bmatrix} : x \in E \right\}
\]
To obtain a more geometric description of this action we need to consider the symplectic unitary pair
\[
U(\hat{E}, J) := E(\hat{E}) \cap \text{Sp}_h(\hat{E}, J) = \left\{ T \in U(\hat{E}) : TJ = JT \right\}.
\]
The subspaces $F^\pm$ are invariant subspaces of any operator $T \in U(\hat{E}, J)$ so that we have an isomorphism $U(\hat{E}, J) \cong U(F^+) \times U(F^-)$. Now identify $F^\pm$ with $E$ using the isometries
\[
\frac{1}{\sqrt{2}}J_\pm : E \to F^\pm, \quad J : E \oplus E \to F^+ \oplus F^-.
\]
We obtain an isomorphism
\[ U(\hat{E}, J) \ni T \mapsto (T_+, T_-) \in U(E) \times U(E). \]
Moreover, for any lagrangian \( L \in \text{Lag}_h(\hat{E}) \), and \( S \in U(E) \), and any \( T \in U(\hat{E}, J) \) we have
\[ S_{TL} = (T_+, T_-) * S_L, \quad L_{(T_+, T_-) * S} = TL_S. \] (1.6)

2. Morse flows on the Grassmannian of hermitian lagrangians

In this section we would like to describe a few properties of some nice Morse functions on the Grassmannian of complex lagrangian subspaces. The main source for all these facts is the very nice paper by I.A. Dynnikov and A.P. Vesselov, [10].

Suppose \( E \) is complex Hermitian space of complex dimension \( n \). We equip the space \( \hat{E} = E \oplus E \) with the canonical complex symplectic structure. Define \( \hat{E}^+ = E \oplus 0 \), \( \hat{E}^- = 0 \oplus E \).

We say that \( \hat{E}^+ \) (respectively \( \hat{E}^- \)) is the horizontal (respectively vertical) lagrangian subspace of \( \hat{E} \).

For every symmetric operator \( A : \hat{E}^+ \to \hat{E}^+ \) we denote by \( \hat{A} : \hat{E} \to \hat{E} \) the symmetric operator
\[ \hat{A} = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} : \hat{E} \to \hat{E}. \]
Define
\[ f_A : U(\hat{E}^+) \to \mathbb{R}, \quad f_A(S) := \text{Re} \text{tr}(AS), \]
and
\[ \varphi_A : \text{Lag}_h(\hat{E}) \to \mathbb{R}, \quad L \mapsto \text{Re} \text{tr}(\hat{A}PL). \]
Using the equality (1.2) we deduce that
\[ \varphi_A(LS) = f_A(S), \quad \forall S \in U(\hat{E}^+), \]
i.e., the diagram below is commutative
\[ \text{Diagram} \]

**Proposition 2.1.** If \( \ker A = \{0\} \) then an unitary operator \( S \in U(\hat{E}^+) \) is a critical point of \( f_A \) if and only if there exists a unitary basis \( e_1, \ldots, e_n \) of \( E \) consisting of eigenvectors of \( A \) such that
\[ Se_k = \pm e_k, \quad \forall k = 1, \ldots, n. \]

**Proof.** Denote by \( \mathfrak{u}(E) \) the Lie algebra of \( U(\hat{E}^+) \), i.e., the space of skew-hermitian operators \( H : E \to E \). For every \( H \in \mathfrak{u}(E) \) we have
\[ \frac{d}{dt} |_{t=0} f_A(Se^{tH}) = \text{Re} \text{tr}(ASH). \]
Thus, \( S \) is a critical point of \( f_A \) if and only if
\[ \text{Re} \text{tr}(ASH) = 0, \quad \forall H \in \mathfrak{u}(E). \]
Thus, the operator $AS : \mathbb{E} \to \mathbb{E}$ is orthogonal to the subspace $\mathfrak{u}(\mathbb{E}) \subset \text{End}(\mathbb{E})$ with respect to the inner product
\[\langle U, V \rangle := \text{Re} \, \text{tr}(UV^*) .\]
The orthogonal complement of $\mathfrak{u}(\mathbb{E})$ in $\text{End}(\mathbb{E})$ consists of the symmetric operators. Hence $S$ is a critical point of $f_A$ if and only if $AS$ is symmetric, i.e.,
$$AS = (AS)^* = S^* \, A = S^{-1} \, A \iff SAS = A .$$
Fix a unitary basis $e_1, \ldots, e_n$ which diagonalizes $A$, so that $Ae_k = \alpha_k e_k$, $\alpha_k \in \mathbb{R}^*$, $k = 1, 2, \ldots , n$. Then
$$\alpha_i \delta_{ij} = \alpha_i (e_i, e_j) = (SAe_i, e_j) = (S^* A, e_j) = \alpha_i (e_i, Se_j), \forall i, j.$$Hence $Se_j \perp e_i$, $\forall i \neq j$. This means that $Se_j$ is collinear with $e_j$ so that $e_j$ is an eigenvector of $S$. Hence there exist complex numbers $z_j \in \mathbb{C}$, $|z_j| = 1$, such that $Se_j = z_j e_j$.

From the equality $SAS = A$ we deduce
$$z_j^2 \alpha_j = \alpha_j, \forall j = 1, 2, \ldots , n.$$Since $\ker A \neq 0$ we deduce $z_j = \pm 1$.

We can reformulate the above result by saying that when $\ker A \neq 0$, then a unitary operator $S$ is a critical point of $f_A$ if and only if $S$ is an involution and both $\ker(1 - S)$ and $\ker(1 + S)$ are invariant subspaces of $A$. Equivalently this means
$$S = S^*, \quad S^2 = 1_E, \quad SA = AS.$$To obtain more detailed results, we fix an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{E}$. For any $\bar{\alpha} \in \mathbb{R}^n$ such that
$$0 < \alpha_1 < \cdots < \alpha_n , \quad \text{(2.1)}$$we denote by $A = A_{\bar{\alpha}}$ the symmetric operator $E \to E$ defined by $Ae_k = \alpha_k e_k$, $\forall k$, and we set $f_{\bar{\alpha}} := f_{A_{\bar{\alpha}}}$, and by $\text{Cr}_{\bar{\alpha}} \subset U(\tilde{\mathbb{E}}^+)$, the set of critical points of $f_{\bar{\alpha}}$.

For every $\bar{c} \in \{\pm 1\}^n$ we define $S_{\bar{c}} \in U(\tilde{\mathbb{E}}^+)$ by
$$S_{\bar{c}} e_k = e_k e_k, \quad k = 1, 2, \ldots , n.$$Then
$$\text{Cr}_{\bar{\alpha}} = \{ S_{\bar{c}} ; \quad \bar{c} \in \{\pm 1\}^n \}.$$Note that this critical set is independent of the vector $\bar{\alpha}$ satisfying (2.1). For this reason we will use the simpler notation $\text{Cr}_n$ when referring to this critical set.

To compute the index of $f_{\bar{\alpha}}$ at the critical point $S_{\bar{c}}$ we need to compute the Hessian
$$Q_{\bar{c}}(H) := \frac{d^2}{dt^2}_{t=0} \text{Re} \, \text{tr}(AS e^{tH}), \quad H \in \mathfrak{u}(\mathbb{E}).$$We have
$$Q_{\bar{c}}(H) = \text{Re} \, \text{tr} A_{\bar{\alpha}} S_{\bar{c}} H^2 = - \text{Re} \, \text{tr} A_{\bar{\alpha}} S_{\bar{c}} H H^* .$$Using the basis $(e_i)$ we can represent $H \in \mathfrak{u}(\mathbb{E})$ as $H = iZ$, where $Z$ is a hermitian matrix $(z_{ij})_{1 \leq i, j \leq n}$, $z_{jk} = \overline{z_{kj}}$. Note that $z_{ij}$ is a real number, while $z_{ij}$ can be any complex number if $i \neq j$. Then a simple computation shows
$$Q_{\bar{c}}(iZ) = - \sum_{i,j} (\epsilon_i \alpha_i + \epsilon_j \alpha_j)|z_{ij}|^2 = - \sum_i \epsilon_i \alpha_i |z_{ii}|^2 - 2 \sum_{i < j} (\epsilon_i \alpha_i + \epsilon_j \alpha_j)|z_{ij}|^2 . \quad \text{(2.2)}$$
Hence, the index of $f_{\tilde{\alpha}}$ at $S_{\tilde{\epsilon}}$ is
\[ \mu_{\tilde{\alpha}}(\tilde{\epsilon}) := \# \{ i; \ \epsilon_i = 1 \} + 2 \# \{ (i, j); \ i < j, \ \epsilon_i \alpha_i + \epsilon_j \alpha_j > 0 \} \]
Observe that if $i < j$ then $\epsilon_i \alpha_i + \epsilon_j \alpha_j > 0$ if and only if $\epsilon_j = 1$. Hence
\[ \mu_{\tilde{\alpha}}(\tilde{\epsilon}) = \sum_{\epsilon_j = 1} (2j - 1). \]
In particular, we see that the index is independent of the vector $\tilde{\alpha}$ satisfying the conditions (2.1).

It is convenient to introduce another parametrization of the critical set. Recall that $I^+_n := \{ 1, \ldots, n \}$.

For every subset $I \subset I^+_n$ we denote by $S_I \in U(\bar{E}^+)$ the unitary operator defined by
\[ S_I e_j = \begin{cases} e_j & j \in I \\ -e_j & j \not\in I. \end{cases} \]
Then
\[ \text{Cr}_n := \{ S_I; \ I \subset I^+_n \}, \]
and the index of $S_I$ is
\[ \mu_I = \sum_{i \in I} (2i - 1). \quad (2.3) \]
The co-index is
\[ \bar{\mu}_I = \mu_I^c = n^2 - \mu_I, \quad (2.4) \]
where $I^c$ denotes the complement of $I$, $I^c := I^+_n \setminus I$.

**Definition 2.2.** We define the weight of a finite subset $I \subset \mathbb{Z}_{>0}$ to be the integer
\[ w(I) = \begin{cases} 0 & I = \emptyset \\ \sum_{i \in I} (2i - 1) & I \neq \emptyset. \end{cases} \]

Hence $\mu_I = w(I)$. Let us observe a remarkable fact.

**Proposition 2.3.** Let
\[ \tilde{\xi} = \left( \frac{1}{2}, \frac{3}{2}, \cdots, \frac{2n-1}{2} \right) \in \mathbb{Q}^n, \]
and set
\[ f_0 := f_{\tilde{\xi}}, \ \varphi_0 := \varphi_{\tilde{\xi}}. \]
Then for every $I \subset I^+_n$ we have
\[ w(I) = f_0(S_I) + \frac{n^2}{2} = \varphi_0(\Lambda_I) + \frac{n^2}{2}. \]
In other words the gradient flow of $f_{\xi}$ is self-indexing, i.e.,
\[ f_0(S_I) - f_0(S_J) = w(J) - w(I). \]
Proof. We have
\[ f_0(S_I) = \frac{1}{2} (w(I) - w(I^c)). \]
On the other hand, we have
\[ \frac{1}{2} (w(I) + w(I^c)) = \frac{1}{2} w(I^n) = \frac{n^2}{2}. \]
Adding up the above equalities we obtain the desired conclusion.

The (positive) gradient flow of the function \( f_A \) has an explicit description. More precisely, we have the following result, [10, Proposition 2.1].

**Proposition 2.4.** Suppose \( A = A_\alpha \) where \( \alpha \in \mathbb{R}^n \) satisfies (2.1). We equip \( U(\hat{E}^+) \) with the left invariant metric induced from the inclusion in the Euclidean space \( \text{End}_\mathbb{C}(E) \) equipped with the inner product
\[ (X, Y) = \text{Re} \, \text{tr}(XY^*). \]
We denote by \( \nabla f_A \) the gradient of \( f_A : U(\hat{E}^+) \to \mathbb{R} \) with respect to this metric, and we denote by
\[ \Phi_A : \mathbb{R} \times U(\hat{E}^+) \to U(\hat{E}^+), \quad S \mapsto \Phi_A^t(S) \]
the flow defined by \( \nabla f_A \), i.e., the flow associated to the o.d.e. \( \dot{S} = \nabla f_A(S) \). Then
\[ \Phi_A^t(S) = (\sinh(tA) + \cosh(tA)S)(\cosh(tA) + \sinh(tA)S)^{-1}, \quad \forall S \in U(\hat{E}^+), \quad t \in \mathbb{R}. \]

It is convenient to have a lagrangian description of the above results via the diffeomorphism \( L : U(\hat{E}^+) \to \text{Lag}_h(\hat{E}) \). First, we use this isomorphism to transport isometrically the metric on \( U(\hat{E}^+) \). Next, for every \( I \subset I_n^+ \) we set \( \Lambda_I := L_{S_I} \). For every \( i \in I_n^+ \) we define
\[ e_i := e_i \oplus 0 \in E \oplus E, \quad f_i := 0 \oplus e_i \in E \oplus E. \]
Then
\[ \Lambda_I = \ker(1 - S_I) \oplus \ker(1 + S_I) = \text{span} \{ e_i; \ i \in I \} \oplus \text{span} \{ f_j; \ j \in I^c \}. \]
The lagrangians \( \Lambda_I \) are the critical points of the function \( \varphi_A : \text{Lag}_h(\hat{E}) \to \mathbb{R} \).

Using (1.1) and (2.5) we deduce that for every \( S \in U(\hat{E}^+) \) we have
\[ \mathcal{L}_{\Phi_A^t(S)} = e^{tA} \mathcal{L}_S. \]
The above equality describes the (positive) gradient flow of \( \varphi_A \). We denote this flow by \( \Psi_A^t \).

We can use the lagrangians \( \Lambda_I \) to produce the Arnold atlas, [1]. Define
\[ \text{Lag}_h(\hat{E}) |_I := \{ L \in \text{Lag}_h(\hat{E}); \ L \cap \Lambda_I = 0 \}. \]
Then \( \text{Lag}_h(\hat{E}) |_I \) is an open subset of \( \text{Lag}_h(\hat{E}) \) and
\[ \text{Lag}_h(\hat{E}) = \bigcup_I \text{Lag}_h(\hat{E}) |_I. \]
Denote by \( \text{End}_\mathbb{C}^+(\Lambda_I) \) the space of self-adjoint endomorphisms of \( \Lambda_I \). We have a diffeomorphism
\[ \text{End}_\mathbb{C}^+(\Lambda_I) \to \text{Lag}_h(\hat{E}) |_I. \]
Proposition 2.5. Suppose we have thus obtained the following result. The inverse map \( \Gamma_{JT} : \Lambda_I \rightarrow \Lambda_I^\perp \) regarded as a subspace in \( \Lambda_I \oplus \Lambda_I^\perp \cong \hat{E} \). More precisely, if the operator \( T \) is described in the orthonormal basis \( \{ e_i, f_j : \ i \in I, \ j \in I^c \} \) by the Hermitian matrix \( (t_{ij})_{1 \leq i, j \leq n} \), then the graph of \( JT \) is spanned by the vectors

\[
e_i(T) := e_i + \sum_{i' \in I} t_{ii'} f_{i'} - \sum_{j \in I^c} t_{ij} e_j, \quad i \in I,
\]

\[
f_j(T) := f_j + \sum_{i \in I} t_{ij} f_i - \sum_{j' \in I^c} t_{j'j} e_{j'}, \quad j \in I^c.
\]

The inverse map

\[ A_I : \text{Lag}_I(\hat{E})_I \rightarrow \text{End}_C^\perp(\Lambda_I) \]

is known as the Arnold coordinates on \( \text{Lag}_I(\hat{E})_I \).

In applications, a subspace of a Hermitian vector space is described in terms of the orthogonal projection on that subspace. We would like explain how to describe the Arnold coordinates in terms of orthogonal projections.

If \( L \in \text{Lag}_I(\hat{E})_I \), \( T = A_I(L) \), and \( R_L \) is the orthogonal reflection in \( L \), then for every \( u \in \Lambda_I \), the vector \( Tu \in \Lambda_I \) is obtained by solving the equation

\[
u + JT u \in L \iff R_L(u + JT u) = u + JT u
\]

\[
\iff \forall v \in \Lambda_I \langle R_L(u + JT u), Jv \rangle = \langle Tu, v \rangle.
\]

Now observe that

\[
\langle R_LJT u, Jv \rangle = -\langle JR_LT u, v \rangle = \langle J^2R_LT u, v \rangle = -\langle R_LT u, v \rangle
\]

so that

\[
\langle (1 + R_L)Tu, v \rangle = \langle R_Lu, Jv \rangle
\]

We notice that \( (1 + R_L) = 2P_L \) = the orthogonal projection onto \( L \), and we deduce

\[
\langle Tu, P_Lv \rangle = \frac{1}{2} \langle R_LT u, Jv \rangle = -\frac{1}{2} \langle JR_LT u, v \rangle = -\langle JP_Lu, v \rangle.
\]

If we denote by \( P_l \) the orthogonal projection onto \( \Lambda_I \) we deduce

\[
\langle Tu, P_Lv \rangle = \langle P_LT u, v \rangle = \langle P_l P_LT u, v \rangle, \quad \langle JP_Lu, v \rangle = \langle P_l JP_Lu, v \rangle.
\]

so that

\[
P_l P_LT = P_l JP_L.
\]

We denote by \( M_L \) the inverse of the map

\[
P_l P_L : \Lambda_I \rightarrow \Lambda_I.
\]

We have \( M_{\Lambda_I} = 1 \) and

\[
T = M_LP_l JP_L.
\]

We have thus obtained the following result.

**Proposition 2.5.** Suppose \( L \in \text{Lag}_I(\hat{E})_I \). Then the operator \( P_l P_L : \Lambda_I \rightarrow \Lambda_I \) is symmetric and invertible, and if \( M_L \) denotes its inverse then

\[
A_I(L) = M_L P_l JP_L = -P_L JP_L M_L.
\]
If \( t \mapsto L_t \) is a smooth family of lagrangians such that \( L_{t=0} = L_0 \in \text{Lag}_h(\hat{E})_I \), then it defines a tangent vector \( \dot{L}_0 \in T_{L_0} \text{Lag}_h(\hat{E})_I \), and thus can be described by a symmetric operator

\[
\dot{T} : \Lambda_I \to \Lambda_I, \quad \dot{T} = \frac{d}{dt}|_{t=0} A_I(L_t).
\]

To find \( \dot{T} \), we set

\[
P_t := P_{L_t}, \quad T_t = A_I(L_t), \quad M_t := M_{L_t},
\]

\[
\dot{P}_0 = \frac{d}{dt}|_{t=0} P_t : \hat{E} \to \hat{E}, \quad \dot{M}_0 = \frac{d}{dt}|_{t=0} M_t,
\]

and we differentiate the equality

\[
T_t = M_t P_t J P_t
\]

at \( t = 0 \). We deduce

\[
\dot{T} = \dot{M}_0 P_t J P_t + P_t J \dot{P}_0,
\]

and because \( P_t J P_t = 0 \) we deduce

\[
\dot{T} = \frac{d}{dt}|_{t=0} A_I(L_t) = P_t J \dot{P}_0 = -P_t \dot{P}_0 J. \quad (2.8)
\]

Let \( I \subset \mathbb{N}_n \). If \( L \in \text{Lag}_h(\hat{E})_I \) has Arnold coordinates \( A_I(L) = T \), i.e., \( T \) is a symmetric operator

\[
T : \Lambda_I \to \Lambda_I,
\]

and \( L = \Gamma_{JT} \), then \( \Phi^t_{A_I} L = e^{t\hat{A}} \Gamma_{JT} \) is spanned by the vectors

\[
e^{t\alpha_i} e_i + \sum_{i' \in I^c} t_{i'i} e^{-t\alpha_{i'}} f_{i'} - \sum_{j \in I^c} t_{j} e^{t\alpha_{j}} e_j, \quad i \in I,
\]

\[
e^{-t\alpha_j} f_j + \sum_{i \in I^c} t_{ij} e^{-t\alpha_i} f_i - \sum_{j' \in I^c} t_{j'} e^{t\alpha_{j'}} e_{j'}, \quad j \in I^c
\]

or, equivalently, by the vectors

\[
e_i + \sum_{i' \in I^c} t_{i'i} e^{-t(\alpha_{i'} + \alpha_i)} f_{i'} - \sum_{j \in I^c} t_{j} e^{t(\alpha_j - \alpha_i)} e_j, \quad i \in I, \quad (2.9a)
\]

\[
f_j + \sum_{i \in I^c} t_{ij} e^{t(\alpha_i - \alpha_j)} f_i - \sum_{j' \in I^c} t_{j'} e^{-t(\alpha_{j'} + \alpha_j)} e_{j'}, \quad j \in I^c.
\]

(2.9b)

This shows that \( e^{t\hat{A}} \Gamma_{JT} \in \text{Lag}_h(\hat{E})_I \), so that \( \text{Lag}_h(\hat{E})_I \) is invariant under the flow \( \Psi_A \).

If we denote by \( A_I \) the restriction of \( \hat{A} \) to \( \Lambda_I \), and we regard \( A_I \) as a symmetric operator \( \Lambda_I \to \Lambda_I \), then we deduce from the above equalities that

\[
e^{t\hat{A}} \Gamma_{JT} = \Gamma_{J e^{tA_I} T e^{tA_I}}.
\]

We can rewrite the above equality in terms of Arnold coordinates as

\[
A_I(\Psi^t L) = e^{tA_I} A_I(L) e^{tA_I}, \quad \forall L \in \text{Lag}_h(\hat{E})_I. \quad (2.10)
\]
3. Unstable Manifolds

The unstable manifolds of the positive gradient flow of \( \varphi_A \) have many similarities with the Schubert cells of complex Grassmannians, and we would like to investigate these similarities in great detail.

The stable/unstable variety of \( \Lambda_I \) with respect to the positive gradient flow \( \Psi^I_A \) is defined by

\[
W^\pm_I := \{ L \in \text{Lag}_h(\hat{E}) : \lim_{t \to \pm \infty} e^{\pm tA}L = \Lambda_I \}.
\]

Observe that if \( L \in W^-_I \), then \( L \in \text{Lag}_h(\hat{E})_I \), and if \( T = \Lambda_I(L) \), then \( e^{tA}L \) is spanned by the vectors (2.9a) and (2.9b).

This implies that the unstable variety \( W^-_I \) is described in the Arnold coordinates by the linear equations

\[
t_{ij} = 0 \quad \text{if} \quad i, j \in I, \quad \text{or} \quad j \in I^c, \quad i \in I \quad \text{and} \quad j < i.
\]

We can rewrite the last system of equalities in the more compact form

\[
W^-_I = \{ T \in \text{End}^+(\Lambda_I) : \ t_{ji} = 0, \ \forall 1 \leq j \leq i, \ i \in I \}.
\] (3.1)

This shows that \( W^-_I \) has real codimension \( \sum_{i \in I} (2i - 1) \). This agrees with our previous computation (2.4) of the index of \( \Lambda_I \).

Thus

\[
\text{codim}_\mathbb{R} W^-_I = w(I), \quad \dim_\mathbb{R} W^-_I = n^2 - w(I) = \frac{1}{2} \dim_\mathbb{R} \text{Lag}_h(\hat{E}) - \varphi_0(\Lambda_I).
\]

It is convenient to express \( W^-_I \) in terms of orthogonal projections. Thus if \( L \in \text{Lag}_h(\hat{E})_I \), then using Proposition 2.5, we deduce that \( L \in W^-_I \) if and only if

\[
\langle M_L P_i J P'_i e_{i'}, e_i \rangle = 0 = \langle M_L P_i P'_i f_j, e_i \rangle, \quad \forall i, i' \in I, \ j \in I^c, \ j < i.
\] (3.2)

For any \( L \in \text{Lag}_h(\hat{E}) \) we set

\[
L^\pm := L \cap \hat{E}^\pm.
\]

The dimension of \( L^+ \) is called the depth of \( L \), and will be denoted by \( \delta(L) \).

From the description (3.1) of the unstable variety \( W^-_I \), \#I = k we deduce the following result.\(^1\)

**Proposition 3.1.** Let \( L \in \text{Lag}_h(\hat{E})_I, I \subset \{1, \cdots, n\}, k = \#I \). We denote by \( S \in U(\hat{E}^+) \) the unitary operator corresponding to \( L \). The following statements are equivalent.

(a) \( L \in W^-_I \)

(b) \( L \in \text{Lag}_h(\hat{E})_I \) and \( \lim_{t \to \infty} e^{-tA}L^+ = \Lambda^+_I \).

(c) \( \dim L^+ = k \) and \( \lim_{t \to \infty} e^{-tA}L^+_I = \Lambda^+_I \).

(d) \( \dim \ker(1 - S) = k \) and \( \lim_{t \to \infty} e^{-tA}\ker(1 - S) = \Lambda^+_I \).

**Proof.** The description (3.1) shows that (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c). Suppose that \( L \) satisfies (c) and let \( \Lambda_J = \lim_{t \to \infty} e^{-tA}L \), i.e., \( L \in W^-_J \). Then using the implication (a) \( \Rightarrow \) (b) for the unstable manifold \( W^-_J \) we deduce

\[
\lim_{t \to \infty} e^{tA}L^+ = \Lambda^+_J.
\]

\(^1\) The characterization in Proposition 3.1 depends essentially on the fact that the eigenvalues of \( A \) satisfy the inequalities \( 0 < \alpha_1 < \cdots < \alpha_n \).
On the other hand, since \( L \) satisfies (c) we have
\[
\lim_{t \to \infty} e^{tA}L^+ = \Lambda^+_I.
\]
This implies \( I = J \) which proves the implication (c) \( \Rightarrow \) (a). Finally, observe that (d) is a reformulation of (c) via the diffeomorphism \( S : \text{Lag}(E) \to U(\hat{E}^+) \).

The condition \( \lim_{t \to \infty} e^{-tA}L^+ = \Lambda^+_I \) can be rephrased as an incidence condition. We write
\[
I = \{ \nu_1 > \cdots > \nu_k \}.
\]
We have \( \lim_{t \to \infty} e^{-tA}L^+ \to \Lambda^+_I \) if and only if \( L \) is the graph of a linear map
\[
A : \Lambda^+_I \to \Lambda^+_I, \quad Ae_i = \sum_{j \in I^c} a^j_i e_i, \quad \forall i \in I,
\]
such that,
\[
a^j_i = 0, \quad \forall i \in I, \quad j \in I^c, \quad j < i.
\]
We consider the complete flag \( \mathcal{F} \mathcal{I}_* = \{ \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n \} \) of subspaces of \( \hat{E}^+ \),
\[
f_j = \text{span}_C \{ e_i; \ i \leq j \},
\]
and we form de dual flag \( \mathcal{F} \mathcal{I}^* = \{ \mathcal{F}^0 \subset \cdots \subset \mathcal{F}^n \} \),
\[
\mathcal{F}^j := \mathcal{F}_{n-j}^\perp = \text{span}_C \{ e_i; \ i > n - j \}.
\]
Then \( \lim_{t \to \infty} e^{-tA}L^+ \to \Lambda^+_I \) if and only if
\[
\forall i = 0, 1, \ldots, k : \ \text{dim}_C(\mathcal{L}^+ \cap \mathcal{F}_{\nu}^j) = i, \quad \forall \nu, \ n + 1 - \nu_1 \leq \nu \leq n - \nu_{i+1}, \ \nu_0 = n + 1.
\]
or, equivalently
\[
\forall i = 0, 1, \ldots, k : \ \text{dim}_C(\mathcal{L}^+ \cap \mathcal{F}^\nu) = i, \quad \forall \nu, \ n + 1 - \nu_1 \leq \nu \leq n - \nu_{i+1}, \ \nu_0 = n + 1.
\]
We define \( \mu_i \) so that
\[
n - k + i - \mu_i = n + 1 - \nu_i \iff \mu_i = \nu_i - (k + 1 - i),
\]
and we obtain a partition \( \mu_I = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0) \). We deduce that \( L^+ \in \Sigma_I(\mathcal{F} \mathcal{I}^*) \), where \( \Sigma_I(\mathcal{F} \mathcal{I}^*) \) denotes the Schubert cell associated to the partition \( \mu \), and the flag \( \mathcal{F} \mathcal{I}^* \).

Remark 3.2. The partition \( (\mu_1, \ldots, \mu_k) \) can be given a very simple intuitive interpretation. We describe the set \( I \) by placing \( \bullet \)'s on the positions \( i \in I \), and \( \circ \)'s on the positions \( j \in I^c \). If \( I = \{ \nu_1 > \cdots > \nu_k \} \), then \( \mu_i \) is equal to the number of \( \circ \)'s situated to the left of the \( \bullet \) located on the position \( \nu_i \). Thus
\[
\mu_k = (k - 1, 0, \ldots, 0), \quad \mu_{\{1,\ldots,k-1,k+1,\ldots,n\}} = 1_{n-k} = (1, \ldots, 1).
\]

A critical lagrangian \( \Lambda_I \) is completely characterized by its depth \( k = \delta(\Lambda_I) = \# I \), and the associated partition \( \mu \). More precisely,
\[
I = \{ \mu_1 + k > \mu_2 + k - 1 > \cdots > \mu_k + 1 \}.
\]
(3.3)
The Ferres diagram of the partition \( \mu_I \) fits inside a \( k \times (n - k) \) rectangle.

We denote by \( \mathcal{E}_n \) the set
\[
\mathcal{E}_n = \{ (m, \mu); \ k \in \{0, \ldots, n\}, \ \mu \in \mathcal{P}_{m,n-m} \}.
\]
Corollary 3.4. Let $L \in \text{Lag}_h(\hat{E})$ and set $S := S(L) \in U(E)$. Then the following hold.

(a) $L \in W^-_{(m, \mu)}$ if and only if 
\[ \dim \ker (1 - S) = m \quad \text{and} \quad \ker (1 - S) \in \Sigma_{\mu}(FL^*) \subset \text{Gr}_m(E). \]

(b) $L \in W^+_{(k, \lambda)}$ if and only if 
\[ \dim \ker (1 + S) = n - k \quad \text{and} \quad \ker (1 + S) \in \Sigma_{\lambda^*}(FL^*) \subset \text{Gr}_k(E). \]

Finally, we can give an invariant theoretic description of the unstable manifolds $W^-_I$. 

Remark 3.3. There is a remarkable involution in this story. More precisely, the operator $J : \hat{E} \to \hat{E}$ defines a diffeomorphism $J : \text{Lag}_h(\hat{E}) \to \text{Lag}_h(\hat{E})$, $L \mapsto JL$. 

We list some of the properties of this involution.

- $f_{A}(JL) = -f_{A}(L)$, $\forall L \in \text{Lag}_h(\hat{E})$, because $P_{JL} = 1_{\hat{E}} - P_L$ and $\hat{A}$ and $\text{tr} \hat{A} = 0$.
- $e^{t\hat{A}}J = J e^{-t\hat{A}}$ because $J \hat{A} = -\hat{A}J$.
- $J\lambda^* = (J\lambda)^*$, $\forall L \in \text{Lag}_h(\hat{E})$.
- $J\Lambda_I = \Lambda_{I^c}$, $\forall I \subset \Pi^+_n$.
- $JW^+_{(k, \lambda)} = W^+_{(k^*, \lambda^*)}$, $\forall I \subset \{1, \ldots, n\}$.

The involution is transported by the diffeomorphism $S : \text{Lag}_h(\hat{E}) \to U(E)$ to the involution $S \mapsto -S$ on $U(E)$. 

Proposition 3.1 can be rephrased as follows.

Corollary 3.4. Let $L \in \text{Lag}_h(\hat{E})$ and set $S := S(L) \in U(E)$. Then the following hold.

(a) $L \in W^-_{(m, \mu)}$ if and only if 
\[ \dim \ker (1 - S) = m \quad \text{and} \quad \ker (1 - S) \in \Sigma_{\mu}(FL^*) \subset \text{Gr}_m(E). \]

(b) $L \in W^+_{(k, \lambda)}$ if and only if 
\[ \dim \ker (1 + S) = n - k \quad \text{and} \quad \ker (1 + S) \in \Sigma_{\lambda^*}(FL^*) \subset \text{Gr}_k(E). \]
**Definition 3.5.** (a) We define the *symplectic annihilator* of a subspace \( U \subset \hat{E} \) to be the \[ U^\dagger := JU^\perp, \]
where \( U^\perp \) denotes the orthogonal complement.

(b) A subspace \( U \subset \hat{E} \) is called *isotropic* (respectively *coisotropic*) if \( U \subset U^\dagger \) (respectively \( U^\dagger \subset U \)). (Observe that a lagrangian subspace is a maximal isotropic space.)

(c) An *isotropic flag* of \( \hat{E} \) is a collection of isotropic subspaces \( 0 = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \cdots \subset \mathcal{I}_n, \dim \mathcal{I}_k = k, \ n = \frac{1}{2} \dim_C \hat{E}. \)

The top space \( \mathcal{I}_n \) is called the *lagrangian subspace associated to* \( \mathcal{I}_\bullet \). \( \Box \)

Consider the isotropic flag \( \mathcal{I}_\bullet \) given by
\[ \mathcal{I}_\ell = \text{span}\{ e_i; \ i > n - \ell \}. \]
If \( I = \{ \nu_0 < \cdots < \nu_1 \} \), then \( L \in W^-_I \) if and only if
\[ \forall i = 0, 1, \ldots, k: \dim_C (L \cap \mathcal{I}_\nu) = i, \ \forall \nu, n + 1 - \nu \leq \nu \leq n - \nu_1, \ \nu_0 = n + 1. \]

Define the (real) Borel group
\[ \mathcal{B} \coloneqq \{ T \in \text{Sp}_h(\hat{E}, J); \ T\mathcal{I}_\ell \subset \mathcal{I}_\ell, \ \forall \ell \in \mathbb{I}_n^+ \}. \]

Observe that the unstable manifolds are \( \mathcal{B} \)-invariant subsets, and in fact, \( W^-_I \) can be identified with the \( \mathcal{B} \)-orbit of \( \Lambda_I \). Using [21, Theorem 3.3] and [26, Theorem 8.1] we deduce the following result.

**Proposition 3.6.** *The collection of unstable manifolds* \( (W^-_I)_{I \subset \mathbb{I}_n^+} \) *defines a Whitney regular stratification of* \( \text{Lag}_h(\hat{E}) \). *In particular, the gradient flow* \( e^{t\hat{A}} \) *on* \( \text{Lag}_h(\hat{E}) \) *satisfies the Morse-Smale transversality condition. \( \Box \)

4. **Tunnellings**

The main problem we want to investigate in this section is the structure of *tunnellings*. Given \( M, K \subset \mathbb{I}_n^+ \), then a *tunnelling* from \( \Lambda_M \) to \( \Lambda_K \) is a gradient trajectory
\[ t \mapsto \Psi^t L = e^{t\hat{A}} L, \ L \in \text{Lag}_h(\hat{E}) \]
such that
\[ \lim_{t \to \infty} \Psi^{-t}_A L = \Lambda_M, \ \lim_{t \to \infty} \Psi^t_A L = \Lambda_K. \]
We denote by \( \mathcal{T}(M, K) \) the set of tunnellings from \( \Lambda_M \) to \( \Lambda_K \), and we say that \( M \) *covers* \( K \), and write this \( K \prec M \), if \( \mathcal{T}(M, K) \neq \emptyset \). Equivalently, \( K \prec M \) if and only if \( W^-_M \cap W^+_K \neq \emptyset \).

Observe that
\[ L \in W^+_K \iff JL \in W^-_K. \]

Hence
\[ W^-_M \cap W^+_K = W^-_M \cap JW^-_K, \]
so that
\[ K \prec M \iff W^-_M \cap JW^-_K \neq \emptyset. \]
Let us observe that, although the flow $\Psi^t_A$ depends on the choice of the hermitian operator $A : \hat{E}^+ \to \hat{E}^+$, the equality (3.1) shows that the unstable manifolds $W^+_J$ are independent of the choice of $A$. Thus, we can choose $A$ such that

$$Ae_i = \frac{2i-1}{2}e_i, \quad \forall i = 1, \ldots, n.$$ 

Using Proposition 2.3 on self-indexing we obtain the following result.

**Proposition 4.1.** If $J \prec I$, then $w(J) > w(I)$ so that $\dim W^-_J < \dim W^+_I$. \hfill $\square$

**Definition 4.2.** (a) For any nonempty set $K \subset \mathbb{I}^+_n$ of cardinality $k$ we denote by $\nu_K$ the unique strictly decreasing function $\nu_K : \{1, \ldots, k\} \to \mathbb{I}^+_n$ whose range is $K$, i.e.,

$$K = \{ \nu_K(k) < \cdots < \nu_K(1) \}.$$ 

(b) We define a partial order $\prec$ on the collection of subsets of $\mathbb{I}^+_n$ by declaring $J \prec I$ if either $I = \emptyset$, or $#J \geq #I$, and for every $1 \leq \ell \leq #I$ we have $\nu_I(\ell) \leq \nu_J(\ell)$. \hfill $\square$

We have the following elementary fact whose proof is left to the reader.

**Lemma 4.3.** Let $K, M \subset \mathbb{I}^+_n$. Then the following statements are equivalent.

(a) $K \prec M$

(b) For every $\ell \in \mathbb{I}^+_n$ we have $\#(K \cap [\ell, n)) \geq \#(M \cap [\ell, n))$.

(c) $M^c \prec K^c$. \hfill $\square$

**Proposition 4.4.** Suppose $K, M \subset \mathbb{I}^+_n$. Then $K \prec M$ if and only if $K \subset M$.

**Proof.** Suppose $L \in W^-_M$. Then $(JL)^+ = JL^-$, and we deduce that

$$L \in W^-_M \cap W^+_K \iff \lim_{t \to \infty} e^{-tA}L^+ = \Lambda^+_M \quad \text{and} \quad \lim_{t \to \infty} e^{-tA}JL^- = \Lambda^+_K.$$ 

In other words,

$$L \in W^-_M \cap W^+_K \iff L^+ \in \Sigma_M(Fl^*), \quad JL^- \in \Sigma_{K^c}(Fl^*).$$

We denote by $U^+ = U^+_L$ the orthogonal complement of $L^+$ in $\hat{E}^+$, and by $T = (t_{ij})_{1 \leq i,j}$ the Arnold coordinates of $L$ in the chart $\text{Lag}_h(\hat{E})_M$.

Observe that $U^+$ contains $JL^-$, the subspace $L^+$ is spanned by the vectors

$$v_i = e_i - \sum_{j \in M^c, j \geq i} t_{ij}e_j, \quad i \in M,$$

and $U^+$ is spanned by the vectors

$$u_j = e_j + \sum_{i \in M^c, i \geq j} t_{ij}e_i = e_j + \sum_{i \in M^c, i < j} t_{ij}e_i, \quad j \in M^c.$$ 

If we write

$$M^c = \{j_{n-m} < \cdots < j_1\}, \quad K^c = \{\ell_{n-k} < \cdots < \ell_1\},$$

then the condition $JL^- \in \Sigma_{K^c}$ is equivalent with the existence of linearly independent vectors of the form

$$w_k = e_k + \sum_{s \geq k} a_{sk}e_s, \quad k \in K^c,$$ 

which span $JL^-$. Arguing exactly as in [22, §3.2.2] we deduce that the inclusion

$$JL^- = \text{span} \{ w_k; \quad k \in K^c \} \subset U^+ = \text{span} \{ u_j; \quad j \in M^c \}$$
Proposition 4.6. Suppose $M, K \subset \mathbb{P}^n$. The following statements are equivalent.

(a) $K \prec M$.
(b) $W_K^- \subset \text{cl}(W_M^-)$.

Proof. The implication (b) $\Rightarrow$ (a) follows from the above remark. Conversely assume

$$K = \{i_k < \cdots < i_1\} \prec \{j_m < \cdots < j_1\} = M,$$

i.e., $j_1 \leq i_1, \ldots, j_m \leq i_m$.

To prove $W_K^- \subset \text{cl}(W_M^-)$ we find it more convenient to use the identification of Lag$_n(\hat{E})$ with $U(\hat{E}^+)$. Suppose $S$ is a unitary operator such that $\mathcal{L} \in W_K^-$. Then there exist scalars $s_{pq} \in \mathbb{C}, q = 1, \ldots, k, p > i_q$ such that

$$\ker(\mathbb{I} - S) = \text{span}\{v_q = e_{i_q} + \sum_{p > i_q} s_{pq} e_p; \ q = 1, \ldots, k\}.$$  

We need to produce a family of unitary operators $(S_\varepsilon)_{\varepsilon > 0}$ such that $\mathcal{L}_{S_\varepsilon} \in W^{-} \text{ and}$

$$\lim_{\varepsilon \to 0} S_\varepsilon = S.$$  

Define

$$V_{\varepsilon} := \text{span}\{v_r(\varepsilon) = \varepsilon e_{j_r} + v_r; \ r = 1, \ldots, m \leq k\}.$$  

Observe that

$$V_0 := \lim_{\varepsilon \to 0} V_\varepsilon \subset \ker(\mathbb{I} - S),$$  

and any unitary operator such that $\ker(\mathbb{I} - T) = V_\varepsilon$ satisfies $\mathcal{L}_T \in W_M^-$. Choose a unitary basis $u_1, \ldots, u_m$ of $V_0$ and complete it to a unitary basis $u_1, \ldots, u_n$ of $\hat{E}^+$ consisting of eigenvector of $S$ such that

$$Su_q = e^{i\varepsilon} u_q, \ \forall q = 1, \ldots, n, \ \ker(\mathbb{I} - S) = \text{span}\{u_1, \ldots, u_k\}.$$  

Next, choose a unitary basis $u_1(\varepsilon), \ldots, u_m(\varepsilon)$ of $V_\varepsilon$ such that

$$\lim_{\varepsilon \to 0} u_r(\varepsilon) = u_r, \ \forall r = 1, \ldots, m,$$  

Conversely, if (4.2) holds then, arguing as in [22, §3.2.2] we can find vectors $w_k$ as in (4.1) and complex numbers $\tau_{ij}$ $i \in M, j \in M^c, i < j$, such that

$$\text{span}\{w_k: k \in M^c\} \subset \text{span}\{e_j + \sum_{i \in M, 1 < j} \tau_{ij} e_i; j \in M^c\}.$$  

Next complete the collection $(\tau_{ij})$ to a collection $(t_{ij})_{1 \leq i, j \leq n}$ such that $t_{ij} = \bar{\tau}_{ji}$, $\forall i, j$, and $t_{ij} = 0$ if $i \in M$ and $j < i$. The collection $(t_{ij})$ can be viewed as the Arnold coordinates in the chart Lag$_n(\hat{E})$ of a Lagrangian $L \in W_M^- \cap W_K^+$. \hfill $\square$

Remark 4.5. Proposition 4.4 implies that if $K \prec M$ and $M \prec N$ then $K \prec N$, so that $\prec$ is a partial order relation. This fact has an interesting consequence.

If $K_0, K_1, \cdots, K_\nu \subset \mathbb{P}^n$ are such that for every $i = 1, \ldots, \nu$ there exists tunnelling from $\Lambda_{K_{i-1}}$ to $\Lambda_{K_i}$ then there must exist tunnelling from $\Lambda_{K_0}$ to $\Lambda_{K_\nu}$. \hfill $\square$

Proof. The implication (b) $\Rightarrow$ (a) follows from the above remark. Conversely assume

$$K = \{i_k < \cdots < i_1\} \prec \{j_m < \cdots < j_1\} = M,$$

i.e., $j_1 \leq i_1, \ldots, j_m \leq i_m$.

To prove $W_K^- \subset \text{cl}(W_M^-)$ we find it more convenient to use the identification of Lag$_n(\hat{E})$ with $U(\hat{E}^+)$. Suppose $S$ is a unitary operator such that $\mathcal{L} \in W_K^-$. Then there exist scalars $s_{pq} \in \mathbb{C}, q = 1, \ldots, k, p > i_q$ such that

$$\ker(\mathbb{I} - S) = \text{span}\{v_q = e_{i_q} + \sum_{p > i_q} s_{pq} e_p; \ q = 1, \ldots, k\}.$$  

We need to produce a family of unitary operators $(S_\varepsilon)_{\varepsilon > 0}$ such that $\mathcal{L}_{S_\varepsilon} \in W^{-} \text{ and}$

$$\lim_{\varepsilon \to 0} S_\varepsilon = S.$$  

Define

$$V_{\varepsilon} := \text{span}\{v_r(\varepsilon) = \varepsilon e_{j_r} + v_r; \ r = 1, \ldots, m \leq k\}.$$  

Observe that

$$V_0 := \lim_{\varepsilon \to 0} V_\varepsilon \subset \ker(\mathbb{I} - S),$$  

and any unitary operator such that $\ker(\mathbb{I} - T) = V_\varepsilon$ satisfies $\mathcal{L}_T \in W_M^-$. Choose a unitary basis $u_1, \ldots, u_m$ of $V_0$ and complete it to a unitary basis $u_1, \ldots, u_n$ of $\hat{E}^+$ consisting of eigenvector of $S$ such that

$$Su_q = e^{i\varepsilon} u_q, \ \forall q = 1, \ldots, n, \ \ker(\mathbb{I} - S) = \text{span}\{u_1, \ldots, u_k\}.$$  

Next, choose a unitary basis $u_1(\varepsilon), \ldots, u_m(\varepsilon)$ of $V_\varepsilon$ such that

$$\lim_{\varepsilon \to 0} u_r(\varepsilon) = u_r, \ \forall r = 1, \ldots, m,$$  

Conversely, if (4.2) holds then, arguing as in [22, §3.2.2] we can find vectors $w_k$ as in (4.1) and complex numbers $\tau_{ij}$ $i \in M, j \in M^c, i < j$, such that

$$\text{span}\{w_k: k \in M^c\} \subset \text{span}\{e_j + \sum_{i \in M, 1 < j} \tau_{ij} e_i; j \in M^c\}.$$  

Next complete the collection $(\tau_{ij})$ to a collection $(t_{ij})_{1 \leq i, j \leq n}$ such that $t_{ij} = \bar{\tau}_{ji}$, $\forall i, j$, and $t_{ij} = 0$ if $i \in M$ and $j < i$. The collection $(t_{ij})$ can be viewed as the Arnold coordinates in the chart Lag$_n(\hat{E})$ of a Lagrangian $L \in W_M^- \cap W_K^+$. \hfill $\square$
and a unitary basis $u_{m+1}(\varepsilon), \ldots, u_n(\varepsilon)$ of $V_\varepsilon^1$ such that
\[
\lim_{\varepsilon \to 0} u_r(\varepsilon) = u_r, \quad \forall r > m.
\]
Now define the unitary operators $S_\varepsilon$ such that
\[
S_\varepsilon u_r(\varepsilon) = \begin{cases} u_r(\varepsilon) & r \leq m \\ e^{i(t_r + \varepsilon)}u_r(\varepsilon) & r > m. \end{cases}
\]
Observe that $L_{S_\varepsilon} \subset W^-_M$ and $S_\varepsilon \to S$. □

**Corollary 4.7.** For any $M \subset \mathbb{I}_n^+$ we have
\[
\text{cl}(W^-_M) = \bigcup_{K \leq M} W^-_K.
\]

**Corollary 4.8.** Let $K, M \subset \mathbb{I}_n^+$, and set $k = \#K$, $m = \#M$, $\lambda = \mu_K$, $\mu = \mu_M$. The following statements are equivalent.
- $W^-_K \subset \text{cl}(W^-_M)$ and $\dim W^-_K = \dim W^-_M - 1$.
- $\{1\} \in K$ and $M = K \setminus \{1\}$.
- $k = m + 1$, $\mu_m = 1$ and $\lambda = \mathbb{T}_{k+1} \mu$.

□

5. **Arnold-Schubert cells, varieties and cycles**

We want to use the results we have proved so far to describe a very useful collection of subsets of $\text{Lag}_h(\hat{E})$. We begin by describing this collection using the identification $\text{Lag}_h(\hat{E}) \cong U(\hat{E}^+)$. For every complete flag $Fl_\bullet = \{ 0 = F_0 \subset F_1 \subset \cdots \subset F_n = E \}$ of $\hat{E}^+$, and for every subset $I = \{ \nu_k < \cdots < \nu_1 \} \subset \mathbb{I}_n^+$, we denote $Fl^\bullet_I$ the dual flag $F^\dagger_j := F^1_{n-j}$, we set $\nu_0 = n+1$, $\nu_{k+1} = 0$, and we denote by $W^-_I(Fl^\bullet_I)$ the set
\[
\{ S \in U(\hat{E}^+); \dim \mathbb{C} F^-_j \cap \ker(1 - S) = j, \quad \nu_j + 1 \leq n - \nu < \nu_j, \quad j = 0, \ldots, k \} = \{ S \in U(\hat{E}^+); \quad \dim \mathbb{C} F^1_{n-j} \cap \ker(1 - S) = j, \quad \nu_j + 1 \leq \ell < \nu_j, \quad j = 0, \ldots, k \}.
\]
We say that $W^-_I(Fl^\bullet_I)$ is the Arnold-Schubert (AS) cell of type $I$ associated to the flag $Fl^\bullet_I$. Its closure, denoted by $X_I(Fl^\bullet_I)$ is called the AS variety of type $I$, associated to the flag $Fl^\bullet_I$. We want to point out that
\[
S \in W^-_I(Fl^\bullet_I) \implies \dim \mathbb{C} \ker(1 - S) = \#I.
\]
If we fix a unitary basis $\underline{e} = \{ e_1, \ldots, e_n \}$ of $\hat{E}^+$ we obtain a flag
\[
Fl^\bullet_I(\underline{e}), \quad Fl^\bullet_I(\underline{e}) := \text{span}_\mathbb{C} \{ e_j; \quad j \leq \nu \}.
\]
We set
\[
W^+_I(\underline{e}) := W^+_I(Fl^\bullet_I(\underline{e})).
\]
As we know, the unitary symplectic group $U(\hat{E}, J) \cong U(\hat{E}^+) \times U(\hat{E}^+)$ acts on $U(\hat{E}^+)$, by
\[
(U_+, U_-) \ast S = U_-SU_+^*,
\]
and we set
\[
W^+_I(Fl^\bullet_I(\underline{e}), U_+, U_-) := (U_+, U_-) \ast W^+_I(Fl^\bullet_I(\underline{e})).
\]
We denote by $X_I(F\ell_\bullet, U_+, U_-)$ the closure of $W_I^-(F\ell_\bullet, U_+, U_-)$, and we will refer to When $I$ is a singleton, $I = \{\nu\}$, we will use the simpler notation $W_\nu^-$ and $X_\nu$ instead of $W_{\{\nu\}}^-$ and $X_{\{\nu\}}$.

Moreover, for every unit complex number $\nu$ we set

$$W_I^-(F\ell_\bullet, \nu) = W_I^-(F\ell_\bullet, \nu, 1) = W_I^-(F\ell_\bullet, 1, \nu)$$

$$= \{ S \in U(\hat{E}^+) : \dim_C F^I_\ell \cap \ker(\nu - S) = j, \ \nu_{j+1} \leq \nu_j, \ j = 0, \ldots, n \}.$$  

When $F\ell_\bullet = F\ell_\bullet(\nu)$ we will use the alternative notation

$$W_I^-(F\ell_\bullet(\nu)) = W_I^-(F\ell_\bullet(\nu, \nu)),$$

and we will refer to

$$X_I(F\ell_\bullet(\nu, \nu)) = X_I(F\ell_\bullet(\nu, \nu)). \quad (5.1)$$

**Example 5.1.** Suppose $e_1, \ldots, e_n$ is an orthonormal basis of $\hat{E}^+$. We form the flag $F\ell_\bullet$ given by

$$F\ell_j := \text{span}_C \{ e_i ; \ i \leq j \}.$$  

For every $\nu \in \mathbb{N}_n^+$ and every unit complex number $\rho$ we have

$$W_\nu^-(F\ell_\bullet, \rho) = \left\{ S \in U(\hat{E}^+) : \exists z_{\nu+1}, \ldots, z_n \in \mathbb{C} : \ker(\rho - S) = \text{span}\{ e_\nu + \sum_{j > \nu} z_je_j \} \right\}.$$  

Moreover

$$X_\nu(F\ell_\bullet, \rho) = \left\{ S \in U(\hat{E}^+) : \ker(\rho - S) \cap \text{span}\{ e_\nu, \ldots, e_n \} \neq 0 \right\}. \quad \Box$$

**Definition 5.2.** Let $I \subset \mathbb{N}_n^+$. We say that a subset $\Sigma \subset \text{Lag}_h(\hat{E}) = U(E)$ is an Arnold-Schubert (AS) cell, respectively variety, of type $I$ if there exists a flag $F\ell_\bullet$ of $E$, and $U_\pm \in U(E)$ such that such that $\Sigma = W_I^-(F\ell_\bullet, U_+ , U_-)$, respectively $\Sigma = X_I(F\ell_\bullet, U_+ , U_-)$. We will refer to $X_\nu$ as the basic AS varieties.

Note that an AS cell of type $I$ is a non-closed, smooth, semi-algebraic submanifold of $\text{Lag}_h(\hat{E})$, semi-algebraically diffeomorphic to $\mathbb{R}^{n^2 - w(I)}$. The AS cells can be given a description as incidence loci of lagrangian subspaces of $\hat{E}$.

We denote by $\text{FLAG}_{\text{iso}}(\hat{E})$ the collection of isotropic flags of $\hat{E}$. The unitary symplectic group

$$U(\hat{E}, J) = \{ T \in U(\hat{E}) : \ TJ = JT \},$$

maps isotropic subspaces to isotropic subspaces and thus acts on $\text{FLAG}_{\text{iso}}$. It is easily seen that this action is transitive.

For any flag $J_\bullet \in \text{FLAG}_{\text{iso}}$, and any subset

$$I = \{ \nu_1 > \cdots > \nu_k \} \subset \mathbb{N}_n^+$$

we set $\nu_0 = n + 1$, $\nu_{k+1} = 0$, and we define

$$W_I^-(J_\bullet) := \{ L \in \text{Lag}_h(\hat{E}) : \dim(J_\nu \cap L) = k, \ \dim L \cap J_\nu = i, \ \forall n - \nu_{i+1} \leq \nu \leq n + 1 - \nu \}.$$  

If we choose a complete flag $F\ell_\bullet$ of $\hat{E}^+$, then the dual flag $F\ell^\dagger_\bullet$ is an isotropic flag, and we observe that the diffeomorphism $L : U(\hat{E}^+) \rightarrow \text{Lag}_h(\hat{E})$ sends $W_I^-(F\ell_\bullet)$ to $W_I^-(F\ell^\dagger_\bullet)$. If $e$ is a unitar basis, then we ill write

$$W_I^-(e) := W_I^-(F\ell_\bullet(e) \dagger)$$
As we explained earlier, the unitary symplectic group $U(\mathcal{E}, J)$ is isomorphic to $U(\mathcal{E}^+) \times U(\mathcal{E}^+)$, so that every $T \in U(\mathcal{E}, J)$ can be identified with a pair $(T_+, T_-) \in U(\mathcal{E}^+) \times U(\mathcal{E}^+)$, such that for every $S \in U(\mathcal{E}^+)$ we have

$$TL_S = L_{T_{\cdot}ST\cdot}.$$

We deduce that $W_1(FI_\bullet, T_+, T_-) \to TW_1(FI^\bullet)$.

Since $U(\mathcal{E}, J)$ acts transitively on $\text{FLAG}_{\text{iso}}$, we conclude that any $AS$ cell is of the form $W_1(\mathcal{J}_\bullet)$ for some flag $\mathcal{J}_\bullet \in \text{FLAG}_{\text{iso}}$.

In the sequel we will use the notation $W_1^-$ when referring to $AS$ cells viewed as subsets of the unitary group $U(\mathcal{E}^+)$, and the notation $W_1^+$ when referring to $AS$ cells viewed as subsets of the Grassmannian Lag$_h(\mathcal{E})$.

We would like to associate cycles to the $AS$ cells, and to do this we must first fix some orientation conventions. First we need to fix an orientation on Lag$_h(\mathcal{E})$ which is orientable because it is diffeomorphic to the connected Lie group $U(\mathcal{E}^+)$. To fix an orientation on $U(\mathcal{E})$ it suffices to pick an orientation on the Lie algebra $\mathfrak{u}(\mathcal{E}^+) = T_1U(\mathcal{E}^+)$. This induces an orientation on each tangent space $T_SU(\mathcal{E})$ via the left translation isomorphism

$$T_1U(\mathcal{E}^+) \to T_SU(\mathcal{E}^+), \quad T_1U(\mathcal{E}^+) \ni X \mapsto SX \in T_SU(\mathcal{E}^+).$$

To produce such an orientation we first choose a unitary basis of $\mathcal{E}^+$,

$$\mathfrak{e} = \{ e_1, \ldots, e_n \}.$$

We can then describe any $X \in \mathfrak{u}(\mathcal{E}^+)$ as a skew-hermitian matrix

$$X = (x_{ij})_{1 \leq i,j \leq n}$$

We identify $\mathfrak{u}(\mathcal{E}^+)$ with the space of Hermitian operators $\mathcal{E}^+ \to \mathcal{E}^+$, by associating to the skew-hermitian operator $X$ the hermitian operator $Z = -iX$. Hence $X = iZ$, and we write

$$z_{ij}(X) := -i x_{ij}.$$

Note that $z_{ii} \in \mathbb{R}, \forall i$, but $z_{ij}$ may not be real if $i \neq j$. The functions $(z_{ij})_{1 \leq i,j \leq n}$ define linear coordinates on $\mathfrak{u}(\mathcal{E}^+)$ which via the exponential map define coordinate in an open neighborhood of 1 in $U(\mathcal{E}^+)$. More precisely, to any sufficiently small Hermitian matrix $Z = (z_{ij})_{1 \leq i,j \leq n}$ one associates the unitary operator $e^{-iZ}$.

Using the above linear coordinates we obtain a decomposition of $\mathfrak{u}(\mathcal{E}^+)$, as a direct sum between the real vector space with coordinates $z_{ii}$, and a complex vector space with complex coordinates $z_{ij}, i < j$. The complex summand has a canonical orientation, and we orient the real summand using the ordered basis

$$\partial_{z_{11}}, \ldots, \partial_{z_{nn}}.$$

Equivalently, if we set $\theta_{ij} = \text{Re} z_{ij}, \varphi_{ij} = \text{Im} z_{ij}$, $\theta^i = z_{ii}$, then the functions $\theta^i, \theta_{ij}, \varphi_{ij} : \mathfrak{u}(\mathcal{E}^+) \to \mathbb{R}$ form a basis of the real dual of $\mathfrak{u}(\mathcal{E}^+)$. The function $z_{ij} : \mathfrak{u}(\mathcal{E}^+) \to \mathbb{C}$ are $\mathbb{R}$-linear and we have

$$\theta_{ij} \wedge \varphi_{ij} = \frac{1}{2i} z_{ij} \wedge \bar{z}_{ij} = \frac{1}{2i} \bar{z}_{ij} \wedge z_{ji}.$$

Equivalently, if we set $\theta_{ij} = \text{Re} z_{ij}, \varphi_{ij} = \text{Im} z_{ij}, \theta^i = z_{ii}$, then the functions $\theta^i, \theta_{ij}, \varphi_{ij} : \mathfrak{u}(\mathcal{E}^+) \to \mathbb{R}$ form a basis of the real dual of $\mathfrak{u}(\mathcal{E}^+)$. The function $z_{ij} : \mathfrak{u}(\mathcal{E}^+) \to \mathbb{C}$ are $\mathbb{R}$-linear and we have

$$\theta_{ij} \wedge \varphi_{ij} = \frac{1}{2i} z_{ij} \wedge \bar{z}_{ij} = \frac{1}{2i} \bar{z}_{ij} \wedge z_{ji}.$$
The above orientation of $\omega(\hat{E}^+)$ is described by the volume form

$$\Omega_n = \left( \bigwedge_{i=1}^{n} \theta^i \right) \wedge \left( \bigwedge_{1 \leq i < j \leq n} \varphi^{ij} \right).$$

The volume form $\Omega_n$ on $\omega(\hat{E}^+)$ is uniquely determined by the unitary basis $e_i$, and depends continuously on $e$. Since the set of unitary bases is connected, we deduce that the orientation determined by $\omega$ is independent of the choice of the unitary basis $e$. We will refer to this as the canonical orientation on the group $U(1) \cong S^1$. Note that when $\dim_{\mathbb{C}} \hat{E}^+ = 1$, the canonical orientation of $U(1) \cong S^1$ coincides with the counterclockwise orientation on the unit circle in the plane.

We will need to have a description of this orientation in terms of Arnold coordinates. For a lagrangian $\Lambda \in \text{Lag}_h(\hat{E})$ we denote by $\text{Lag}_h(\hat{E})_{\Lambda}$ the Arnold chart

$$\text{Lag}_h(\hat{E})_{\Lambda} = \{ L \in \text{Lag}_h(\hat{E}) : L \cap \Lambda^\perp \}. $$

The Arnold coordinates identify this open set with the space $\text{End}^+_{\mathbb{C}}(\Lambda)$ of hermitian operators $\Lambda \rightarrow \Lambda$. By choosing a unitary basis of $\Lambda$ we can identify such an operator $A$ with a Hermitian matrix $(a_{ij})_{1 \leq i, j \leq n}$, and we can coordinatize $\text{End}^+_{\mathbb{C}}(\Lambda)$ using the functions $(a_{ij})_{1 \leq i, j \leq n}$. We orient $\text{Lag}_h(\hat{E})_{\Lambda}$ using the form

$$(-1)^n \left( \bigwedge_{i=1}^{n} da_{ii} \right) \wedge \left( \bigwedge_{1 \leq i < j \leq n} \frac{1}{2i} da_{ij} \wedge da_{ji} \right)$$

We will refer to this as the canonical orientation on the chart $\text{Lag}_h(\hat{E})_{\Lambda}$. We want to show that this orientation convention agrees with the canonical orientation on the group $U(\hat{E})$.

The relationship between the Arnold coordinates on the chart $\text{Lag}_h(\hat{E})_{\Lambda} = \text{Lag}_h(\hat{E})_{\Lambda}^+$ and the above coordinates on $U(\hat{E}^+)$ is given by the Cayley transform. More precisely, if $S = e^{iZ}$, $Z$ Hermitian matrix, and $A$ are the Arnold coordinates of the associated lagrangian $\mathcal{L}_S$, the according to (1.4) we have

$$1 + S = 2(1 + iA)^{-1} \iff iA = 2(1 + S)^{-1} - 1.$$

To see whether this correspondence is orientation preserving we compute its differential at $S = 1$, i.e., $A = 0$. We set

$$S_t := e^{itZ}, \quad iA_t = 2(1 + S_t)^{-1} - 1$$

we deduce upon differentiation at $t = 0$ that $\dot{A}_0 = -2Z$.

Thus, the differential at $1$ of the Cayley transform is represented by a negative multiple of identity matrix in our choice of coordinates. This shows that the canonical orientation on the chart $\text{Lag}_h(\hat{E})_{\Lambda}$ agrees with the canonical orientation on the group $U(\hat{E}^+)$.

To show that this happens for any chart $\text{Lag}_h(\hat{E})_{\Lambda}$ we choose $T = (T_+, T_-) \in U(\hat{E}, J)$ such that $T\hat{E} = \Lambda$. Then

$$\text{Lag}_h(\hat{E})_{\Lambda} = T \text{Lag}_h(\hat{E})_{\Lambda}^+,$$

We fix a unitary basis $\{e_1, \ldots, e_n\}$ of $\hat{E}^+$ and we obtain unitary basis $e'_i = Te_i$ of $\Lambda$. Using these bases we obtain Arnold coordinates

$$A : \text{Lag}_h(\hat{E})_{\Lambda}^+ \to \text{End}^+_{\mathbb{C}}(\mathbb{C}^n), \quad A' : \text{Lag}_h(\hat{E})_{\Lambda} \to \text{End}^+_{\mathbb{C}}(\mathbb{C}^n).$$
Let $L \in \text{Lag}_h(\hat{E})_{E^+} \cap \text{Lag}_h(\hat{E})_{E^3}$, the Arnold coordinates of $L$ in the chart $\text{Lag}_h(\hat{E})_{E^3}$ are equal to the Arnold coordinates of $L' = T^{-1}L$ in the chart $\text{Lag}_h(\hat{E})_{E^+}$, i.e.,

$$A'(L) = A(T^{-1}L).$$

Using (1.6) we deduce

$$S_{T^{-1}L} = T^*S_LT_+ = T^{-1}S_L.$$  

Form (1.4) and (1.5) we deduce

$$S_L = \mathcal{E}_i(A(L)) := (1 - iA(L))(1 + iA(L))^{-1},$$

$$iA'(L) = iA(T^{-1}L) = 2(1 + T^*S_LT_+)^{-1} - 1 = \mathcal{E}_i^{-1}(T^*S_LT_+).$$

We seen that the transition map

$$\text{End}_\mathbb{C}(\mathbb{C}^n) \ni A(L) \mapsto A'(L) \in \text{End}_\mathbb{C}(\mathbb{C}^n)$$

is the composition of the maps

$$\text{End}_\mathbb{C}(\mathbb{C}^n) \xrightarrow{\mathcal{E}_i} U(n) \xrightarrow{T^*} U(n) \xrightarrow{\mathcal{E}_i^{-1}} \text{End}_\mathbb{C}(\mathbb{C}^n).$$

This composition is orientation preserving if and only if the map $S \mapsto T^*S$ is such. Now we remark that the map $S \mapsto T^*S$ is indeed orientation preserving because it is homotopic to the identity map since $U(\hat{E}^+)$ is connected.

Fix $I = \{\nu_1 < \cdots < \nu_k\} \subset \mathbb{N}^r$, and a unitary basis of $\hat{E}^+$,

$$g = \{e_1, \ldots, e_n\}.$$

We want to describe a canonical orientation on the $\text{AS}$ cell $W^{-}_{I} = W^{-}_I(g)$. We will achieve this by describing a canonical co-orientation.

The cell $W^{-}_{I}$ is contained in the Arnold chart $\text{Lag}_h(\hat{E})_{I}$, and it is described in the Arnold coordinates $(t_{pq})_{1 \leq p \leq q \leq n}$ on this chart by the system of linearly independent equations

$$t_{ji} = 0, \quad i \in I, \quad j \leq i.$$

We set

$$u_{pq} = \text{Re} t_{pq}, \quad v_{pq} = \text{Im} t_{pq}, \quad \forall 1 \leq p < q.$$

The conormal bundle $T^*_{W^{-}_{I}} \text{Lag}_h(\hat{E})$ of $W^{-}_{I} \subset \text{Lag}_h(\hat{E})$ is the kernel of the natural restriction map $T^* \text{Lag}_h(\hat{E})|_{W^{-}_{I}} \rightarrow T^*W^{-}_{I}$. This bundle morphism is surjective and thus we have a short exact sequence of bundles over $W^{-}_{I}$,

$$0 \rightarrow T^*_{W^{-}_{I}} \text{Lag}_h(\hat{E}) \rightarrow T^* \text{Lag}_h(\hat{E})|_{W^{-}_{I}} \rightarrow T^*W^{-}_{I} \rightarrow 0. \quad (5.2)$$

The 1-forms $du_{ji}, dv_{ji}, dt_{ji}, i \in I, \quad j < i$, trivialize the conormal bundle. We can orient the conormal bundle $T^*_{W^{-}_{I}} \text{Lag}_h(\hat{E})$ using the form

$$\omega_I = (-1)^{w(I)} dt_I \wedge \left( \bigwedge_{j < i, \quad i \in I} du_{ji} \wedge dv_{ji} \right), \quad (5.3)$$

where $dt_I$ denotes the wedge product of the 1-forms $dt_{ii}, i \in I$, written in increasing order,

$$dt_I = dt_{\nu_k \nu_k} \wedge \cdots \wedge dt_{\nu_1 \nu_1}.$$  

We denote by $\text{or}^{-I}$ this co-orientation, and we will refer to it as the canonical co-orientation. As explained in Appendix B this co-orientation induces a canonical orientation $\text{or}^{-I}$ on $W^{-}_{I}$. We denote by $|W^{-}_{I}, \text{or}^{-I}|$ the current of integration thus defined.
To understand how to detect this co-orientation in the unitary picture we need to give a
unitary description of the Arnold coordinates on $W_I(\mathcal{E}) \subset U(\hat{E}^+)$. 

**Definition 5.3.** Fix a unitary basis $\mathcal{E}$ of $\hat{E}^+$. For every subset $I \subset \mathcal{I}_n^+$ we define $\mathcal{U}_I \in U(\hat{E}, \mathcal{J})$ the symplectic unitary operator defined by

$$
\mathcal{U}_I(e_k) = \begin{cases} 
  e_k & k \in I \\
  Je_k & k \notin I,
\end{cases} \quad \mathcal{U}_I(f_k) = \begin{cases} 
  f_k & k \in I \\
  Jf_k & k \notin I.
\end{cases}
$$

Via the diffeomorphism

$$
U(\hat{E}, \mathcal{J}) \ni T \mapsto (T_+, T_-) \in U(\hat{E}^+) \times U(\hat{E}),
$$
the operator $\mathcal{U}_I$ corresponds to the pair of unitary operators

$$
\mathcal{U}_I^+ = \mathcal{I}_I, \quad \mathcal{U}_I^- = \mathcal{I}_I^*,
$$
where

$$
\mathcal{I}_I(e_k) = \begin{cases} 
  e_k & k \in I \\
  ie_k & k \notin I.
\end{cases}
$$

Observe that

$$
\mathcal{U}_I \hat{E}^+ = \Lambda_I,
$$
and the Arnold coordinates $A_I$ on $\text{Lag}_h(\hat{E})$ are related to the Arnold coordinates on $\text{Lag}_h(\hat{E})$ via the equality

$$
A_I = \Lambda \circ \Lambda_I^{-1}.
$$

We deduce that if $S \in U(\hat{E}^+)$ is such that $\mathcal{L}_S \in \text{Lag}_h(\hat{E})$ then

$$
A_I(\mathcal{L}_S) = \mathcal{E}_I(\mathcal{U}_I^{-1} \ast S) = \mathcal{E}_I(\mathcal{I}_I \mathcal{S}_I).
$$

**Example 5.4.** Let us describe the orientation of $W_I^- \subset U(\hat{E}^+)$. To any map $\bar{\rho} : \mathcal{I} \rightarrow S^1 \setminus \{1\}$, $j \mapsto \rho_j$, we associate the diagonal unitary operator $D = D_{\bar{\rho}} \in W_I^-$ defined by

$$
D e_j = \begin{cases} 
  1 & j \in I \\
  \rho_j & j \in \mathcal{I}^c.
\end{cases}
$$

Every tangent vector $\dot{S} \in T_D U(\hat{E}^+)$ can be written as $\dot{S} = iDZ$, $Z$ hermitian matrix, so that

$$
Z = -iD^{-1}\dot{S}.
$$

The cotangent space $T_D^* U(\hat{E}^+)$ has a natural basis given by the $\mathbb{R}$-linear forms

$$
\theta^p, \theta^{pq}, \varphi^{pq} : T_S U(\hat{E}) \rightarrow \mathbb{R}, \quad \theta^p(Z) = (Ze_p, e_p),
$$

$$
\theta^{pq}(Z) = \text{Re}(Ze_q, e_p), \quad \varphi^{pq} = \text{Im}(Ze_q, e_p).
$$

To describe the orientation of the conormal bundle $T_S^* U(\hat{E}^+)$ we use the above prescription. The Arnold coordinates on $W_I^-$ are given by

$$
W_I \ni S \mapsto A_I(S) := \mathcal{E}_I(\mathcal{I}_I \mathcal{S}_I) = -i(1 - \mathcal{I}_I \mathcal{S}_I)(1 + \mathcal{I}_I \mathcal{S}_I)^{-1} \in \text{End}^+(\hat{E}^+).
$$

Using the equality

$$
\mathcal{E}_I(\mathcal{I}_I \mathcal{S}_I) = -2i(1 + \mathcal{I}_I \mathcal{S}_I)^{-1} + i1
$$

we deduce

$$
\frac{d}{dt}|_{t=0} A_I(De^{itZ}) = -2i \frac{d}{dt}|_{t=0} (1 + \mathcal{I}_I De^{itZ})^{-1} = -2i \frac{d}{dt}|_{t=0} (1 + \mathcal{I}_I D(1 + itZ) \mathcal{I}_I)^{-1}
$$
\[
= -2i(1 + T_1D)T_1^{-1} \frac{d}{dt} |_{t=0}(1 + tT_1DZT_1(1 + T_1D)T_1^{-1})^{-1}
= -2(1 + T_1D)T_1^{-1}T_1DZT_1(1 + T_1D)T_1^{-1} = \dot{A}.
\]

Hence
\[
Z = -1/2 D^*T_1^*(1 + T_1D)\dot{A}(1 + T_1D)T_1 = -1/2 D^*T_1^*(1 + T_1^2D)\dot{A}(1 + T_1^2D).
\]

Note that
\[
\mathcal{T}_1^2e_j = \begin{cases} e_j & j \in I \\ -e_j & j \in I^c \end{cases}
\]

If \( i \in I \), then for every \( j \leq i \) we have
\[
\langle \dot{Z}e_j, e_i \rangle = -1/2 (\dot{A}(1 + \mathcal{T}_1^2D)e_j, T_1D(1 + \mathcal{T}_1^2D)e_i)
\]
\[
- (\dot{A}(1 + \mathcal{T}_1^2D)e_j, e_i) = \begin{cases} -(\dot{A}e_j, e_i) & j \in I \\ 1/2(\rho_j - 1)(\dot{A}e_j, e_i) & j \in I^c. \end{cases}
\]

If for \( i \in I \) we set
\[
u^i(\dot{A}) = (\dot{A}e_i, e_i), \quad \nu^{ij}(\dot{A}) = \text{Re}(\dot{A}e_j, e_i), \quad \nu^{ij}(\dot{A}) = \text{Im}(\dot{A}e_j, e_i)
\]
then we deduce
\[
u^i = -\varphi^i
\]
\[
u^{ij} \wedge \nu^{ij} = k_j \varphi^{ij}
\]
where \( k_j \) is the positive constant
\[
k_j = \begin{cases} 1 & j \in I \\ 1/4|\rho_j - 1|^2 & j \in I^c. \end{cases}
\]

Using (5.3) we conclude that the conormal bundle to \( W_I^- \) is oriented at \( D \) by the exterior monomial
\[
\theta^I \wedge \left( \bigwedge_{j<i, i \in I} \theta^{ij} \wedge \varphi^{ij} \right),
\]
where \( \theta^I \) denotes the wedge product of \( \{\theta^i\}_{i \in I} \) written in increasing order.

In particular, if \( I = \{\nu\} \) and \( D = S_\nu \), i.e., \( \rho_j = -1, \forall j \in I^c \), then the conormal orientation of \( W_\nu^- \) is given at \( S_\nu = S_{\{\nu\}} \) by the exterior monomial
\[
\omega^\perp = \theta^\nu \wedge (\theta^{1\nu} \wedge \varphi^{1\nu}) \wedge \cdots \wedge (\theta^\nu-1^{\nu} \wedge \varphi^{\nu-1,\nu}).
\]

The tangent space \( T_{S_{\{\nu\}}} W^-_{\{\nu\}} \) is oriented by the exterior monomial
\[
\omega_T = (-1)^{\nu-1} \theta^1 \wedge \cdots \theta^{\nu-1} \wedge \theta^{\nu+1} \wedge \cdots \wedge \theta^n \wedge \left( \bigwedge_{j<k, k \neq \nu} \theta^{jk} \wedge \varphi^{jk} \right). \tag{5.5}
\]

because
\[
\omega^\perp \wedge \omega_T = \Omega_n = \left( \bigwedge_{i=1}^n \theta^i \right) \wedge \left( \bigwedge_{1 \leq i < j \leq n} \theta^{ij} \wedge \varphi^{ij} \right). \tag*{\Box}
\]
Proposition 5.5. We have an equality of currents
\[ \partial [W_I^-, or_I] = 0. \]
In other words, using the terminology of Definition B.3, the pair \((W_I^-, or_I)\) is an elementary cycle.

Proof. The proof relies on the theory of subanalytic currents developed by R. Hardt [16]. For the reader’s convenience we have gathered in Appendix B the basic properties of such currents.

Here is our strategy. We will prove that there exists an oriented, smooth, subanalytic submanifold \(Y_I\) of \(\text{Lag}_h(\hat{E})\) with the following properties.

(a) \(W_I^- \subset Y_I \subset \text{cl}(W_I^-) = X_I\).
(b) \(\dim(X_I \setminus Y_I) < \dim W_I^- - 1\).
(c) The orientation on \(Y_I\) restricts to the orientation \(or_I\) on \(W_I^-\).

Assuming the existence of such a \(Y_I\) we observe first that, \(\dim Y_I = \dim W_I^-\), and that we have an equality of currents
\[ [W_I^-, or_I] = [Y_I, or_I]. \]
Moreover
\[ \text{supp} \partial [Y_I, or_I] \subset \text{cl}(Y_I) \setminus Y_I = X_I \setminus Y_I \]
so that
\[ \dim \text{supp} \partial [Y_I, or_I] < \dim Y_I - 1. \]
This proves that
\[ \partial [W_I^-, or_I] = \partial [Y_I, or_I] = 0. \]
To prove the existence of an \(Y_I\) with the above properties we recall that we have a stratification of \(X_I\), (see Corollary 4.7)
\[ X_I = \bigcup_{J < I} W_J^-, \quad (5.6) \]
where
\[ \dim W_J^- = \dim W_I^- + w(I) - w(J). \]
We distinguish two cases.

**A.** \(1 \in I\). In this case, using Corollary 4.8 we deduce that all the lower strata in the above stratification have codimension at least 2. Thus, we can choose \(Y_I = W_I^-\), and the properties (a)-(c) above are trivially satisfied.

**B.** \(1 \notin I\). In this case, Corollary 4.8 implies that the stratification (5.6) had a unique codimension 1-stratum, \(W_I^*\), where \(I_* := \{1\} \cup I\). We set
\[ Y_I := W_I^- \cup W_{I_*}^-. \]
We have to prove that this \(Y_I\) has all the desired properties. Clearly (a) and (b) are trivially satisfied. The rest of the properties follow from our next result.

**Lemma 5.6.** The set \(Y_I\) is a smooth, subanalytic, orientable manifold.

Proof. Consider the Arnold chart \(\text{Lag}_h(\hat{E})_{I^*}\). For any \(L \in \text{Lag}_h(\hat{E})_{I^*}\) we denote by \(t_{ij}(L)\) its Arnold coordinates. This means that \(t_{ij} = t_{ji}\) and that \(L\) is spanned by the vectors
\[ e_i(L) = e_i + \sum_{i' \in I_*} t_{i'i} f_{i'} - \sum_{j \in I^c} t_{ji} e_{ji}, \quad i \in I_*, \]
We will prove that
\[ f_j(L) = f_j + \sum_{i \in I_\alpha} t_{ji} f_i - \sum_{j' \in I_\alpha} t_{j'j} e_{j'}, \quad j \in I_\alpha^c. \]

The AS cell \( W_I^- \) is described by the equations
\[ t_{ji} = 0, \quad \forall i \in I_\alpha, \quad j \leq i. \]

We will prove that
\[ W_I^- \cap \text{Lag}_{\alpha}(\tilde{E})_{I_\alpha} = \Omega := \{ L \in \text{Lag}_{\alpha}(\tilde{E})_{I_\alpha}; \quad t_{ji}(L) = 0, \quad \forall i \in I, \quad j \leq i, \quad t_{11}(L) \neq 0 \}. \quad (5.7) \]

Denote by \( A \in \text{End}_{\mathbb{C}}^+(\tilde{E}^+) \) the hermitian operator defined by
\[ A e_i = \alpha_i, \quad i \in \mathbb{I}_n^+, \]
where \( \alpha_i \) satisfy
\[ 0 < \alpha_1 < \cdots < \alpha_n. \]

Extend \( A \) to \( \hat{A} : \tilde{E} \to \tilde{E} \) by setting \( \hat{A} f_i = -\alpha_i e_i \). Note that \( L \in W_I^- \) if and only if
\[ \lim_{t \to \infty} e^{-t\hat{A}} L = \Lambda_I. \]

Clearly, if \( L \in \Omega \), then \( L_t = e^{-t\hat{A}} L \) is spanned by the vectors
\[ e_1(L_t) = e_1 + t_{1i} e^{2\alpha_i} f_1 + \sum_{j \in I_\alpha} t_{ji} e^{(\alpha_j - \alpha_i)} f_j, \]
\[ e_i(L_t) = e_i - \sum_{j \in I_\alpha, j \neq i} t_{ji} e^{(\alpha_i - \alpha_j)} e_i, \quad i \in I \]
\[ f_j(L_t) = f_j + \sum_{i \in I_\alpha, i < j} t_{ij} e^{(\alpha_j - \alpha_i)} f_i - \sum_{j' \in I_\alpha^c} t_{j'j} e^{-(\alpha_j + \alpha_{j'})} e_{j'}, \quad j \in I_\alpha^c. \]

We note that as \( t \to \infty \) we have
\[ \text{span}\{e_1(L_t)\} \to \text{span}\{f_1\}, \quad \text{span}\{e_i(L_t)\} \to \text{span}\{e_i\}, \quad \forall i \in I, \]
\[ \text{span}\{f_j(L_t)\} \to \text{span}\{f_j\}, \quad \forall j \in I_\alpha^c. \]

This proves \( L_t \to \Lambda_I \) so that \( \Omega \subset W_I^- \cap \text{Lag}_{\alpha}(\tilde{E})_{I_\alpha} \).

Conversely, let \( L \in W_I^- \cap \text{Lag}_{\alpha}(\tilde{E})_{I_\alpha}, \) i.e., \( L \in W_I^+ \) and
\[ \lim_{t \to \infty} L_t = \Lambda_I, \quad \text{where} \quad L_t = e^{-t\hat{A}} L. \]

Observe that \( L_t \) is spanned by the vectors
\[ e_1(L_t) = e_1 + t_{1i} e^{2\alpha_i} f_1 + \sum_{i \in I} t_{i1} e^{(\alpha_i + \alpha_1)} f_i - \sum_{j \in I_\alpha^c} t_{ji} e^{(\alpha_j - \alpha_i)} e_j, \]
\[ e_i(L_t) = e_i + t_{ii} e^{(\alpha_i + \alpha_1)} f_i + \sum_{i' \in I} t_{i'i} e^{(\alpha_i + \alpha_{i'})} f_{i'} - \sum_{j \in I_\alpha^c} t_{ji} e^{(\alpha_j - \alpha_i)} e_j, \quad i \in I, \]
\[ f_j(L_t) = f_j + t_{ij} e^{(\alpha_j - \alpha_i)} f_i + \sum_{i \in I} t_{ij} e^{(\alpha_i - \alpha_j)} f_i - \sum_{j' \in I_\alpha^c} t_{j'j} e^{-(\alpha_j + \alpha_{j'})} e_{j'}, \quad j \in I_\alpha^c. \]

Observe that
\[ e_i(L_t), f_j(L_t) \perp \text{span}\{e_i; \quad i \in I \} \subset \Lambda_I, \quad \forall j \in I_\alpha^c \]
\[ \text{span}\{e_i(L_t), f_j(L_t); \quad j \in I_\alpha^c \} \to \text{span}\{f_j; \quad j \in I_\alpha^c \} \subset \Lambda_I. \]
On the other hand the line spanned by \( e_1(\mathcal{L}_t) \) converges as \( t \to \infty \) to either the line spanned by \( e_1 \), or to the line spanned by \( f_i, i \in I \). Since the line spanned by \( e_1 \), and the line spanned \( f_i, i \in I \) are orthogonal to \( \Lambda_I \) we deduce

\[
\text{span}\{ e_1(\mathcal{L}_t) \} \to \text{span}\{ f_i \},
\]

which implies

\[
t_{i1} \neq 0, \quad t_{i1} = 0, \quad \forall i \in I.
\]

Hence

\[
e_i(\mathcal{L}_t) = e_i + \sum_{t_i' \in I} t_{i'i} e^{(\alpha_i + \alpha_{i'})} f_{i'} - \sum_{j \in I_i^c} t_{ji} e^{(\alpha_i - \alpha_{j})} e_j, \quad \forall i \in I.
\]

Now observe that

\[
e_i(\mathcal{L}_t) \perp f_j, \quad \forall j \in I^c,
\]

and we conclude that

\[
\text{span}\{ e_i(\mathcal{L}_t); \ i \in I \} \to \text{span}\{ e_i; \ i \in I \}.
\]

Since \((e_i(\mathcal{L}_t) - e_i) \perp e_{i'}, \forall i, i' \in I, \ i \neq i'\), we deduce

\[
\text{span}\{ e_i(\mathcal{L}_t) \} \to \text{span}\{ e_i \}
\]

which implies

\[
t_{i'i'} = 0, \quad t_{ji} = 0, \quad \forall i, i' \in I, \quad j \in I, \quad j < i.
\]

This proves that \( \mathcal{Y}_I^- \cap \text{Lag}_h(\hat{E})_I^- \subset \Omega \) and thus, also the equality (5.7). In particular, this implies that \( \mathcal{Y}_I = W_I^- \cup W_I^- \) is smooth, because in the Arnold chart \( \text{Lag}_h(\hat{E})_I \), which contains the stratum \( W_I^- \) is described by the linear equations

\[
t_{ji} = 0, \quad i \in I, \quad j < i.
\]

(5.8)

To prove that \( \mathcal{Y}_I \) is orientable, we will construct an orientation \( \text{or}_I \) on \( \mathcal{Y}_I \cap \text{Lag}_h(\hat{E})_I^* \) with the property that its restriction to

\[
\mathcal{Y}_I \cap \text{Lag}_h(\hat{E})_I \cap \text{Lag}_h(\hat{E})_I^* \subset W_I^-.
\]

We define an orientation \( \text{or}_I \) on \( \mathcal{Y}_I \cap \text{Lag}_h(\hat{E})_I \) by orienting the conormal bundle of this submanifold using the conormal volume form

\[
\omega_{\mathcal{Y}_I} = (-1)^{w(\mathcal{Y})} dt_I \wedge \left( \bigwedge_{i \in I, \ j \leq i} \frac{1}{2i} dt_{ji} \wedge dt_{ij} \right).
\]

Let

\[
L \in \text{Lag}_h(\hat{E})_I \cap \text{Lag}_h(\hat{E})_I^* \subset W_I^-.
\]

We denote by \( t_{ij}(L) \) its coordinates in the chart \( \text{Lag}_h(\hat{E})_I \). Then \( L \) is spanned by the vectors

\[
e_1(L) = e_1 + t_{11} f_1 + \sum_{i \in I} t_{i1} f_i - \sum_{j \in I^c_i} t_{ji} e_j,
\]

\[
e_i(L) = e_i + t_{1i} f_1 + \sum_{i' \in I} t_{i'i} f_{i'} - \sum_{j \in I^c_i} t_{ji} e_j, \quad i \in I,
\]

\[
f_j(L) = f_j + t_{1j} f_1 + \sum_{i \in I} t_{ij} f_i - \sum_{j' \in I^c_j} t_{j'j} e_{j'}, \quad j \in I^c.
\]
The space $L$ belongs to $\text{Lag}_h(\hat{E})_I$ if and only if
$$ L \cap \Lambda_f^1 = L \cap \text{span}\{ e_j, f_i : i \in I, j \in I^c \} = 0. $$

This is possible if and only if $t_{11} \neq 0$.

We set
$$ f'_1(L) = \frac{1}{t_{11}} e_1(L) = f_1 + \frac{1}{t_{11}} e_1 + \sum_{i' \notin I} \frac{t_{i'i}}{t_{11}} f_{i'} - \sum_{j \notin I^c} \frac{t_{j1}}{t_{11}} e_j $$
$$ e'_i(L) = e_i(L) - t_{11} f'_1(L) $$
$$ = e_i + \sum_{i' \notin I} \left(t_{i'i} - \frac{t_{i'i1}}{t_{11}}\right) f_{i'} - \frac{t_{ii}}{t_{11}} e_1 - \sum_{j \notin I^c} \left(t_{ji} - \frac{t_{ji1}}{t_{11}}\right) e_j, \quad i \in I $$
$$ f'_j(L) = f_j(L) - t_{11} f'_1(L) $$
$$ = f_j + \sum_{i \in I} \left(t_{ij} - \frac{t_{ij1}}{t_{11}}\right) f_i - \frac{t_{jj}}{t_{11}} e_1 - \sum_{j' \notin I^c} \left(t_{jj'} - \frac{t_{jj'1}}{t_{11}}\right) e_{j'} $$

The space $L$ is thus spanned by the vectors $e'_i(L), i \in I$ and $f'_j(L), j \in I^c$, where we recall that $1 \in I^c$. Also, since $t_{11} = t_{11}$ we deduce that
$$ x_{pq} = \bar{x}_{qp}, \quad \forall 1 \leq p, q \leq n. $$

This implies that $x_{pq}$ must be the Arnold coordinates of $L$ in the chart $\text{Lag}_h(\hat{E})_I$.

In these coordinates the canonical orientation $\omega_I$ of $W^-_I$ is obtained from the orientation of the conormal bundle given by the form
$$ \omega_I = (-1)^{e(i)} dx_I \wedge \left( \bigwedge_{j < i, i \in I} \frac{1}{2t} dx_{ij} \wedge dx_{ij} \right), $$
where $dx_I$ denotes the wedge product of the forms $dx_{ii}, i \in I$, in increasing order with respect to $i$.

Observe that
$$ x_{ii} = t_{ii} - \frac{t_{i1t_{11}}}{t_{11}}, \quad \forall i \in I, $$
$$ x_{i1} = \frac{t_{i1}}{t_{11}}, \quad \forall i \in I, $$
$$ x_{ij} = t_{ij} - \frac{t_{ijt_{11}}}{t_{11}}, \quad \forall i, j \in I^c \setminus \{1\}. $$

Observe that along $W^-_I$ we have
$$ t_{11} = t_{ij} = 0, \quad \forall i \in I, j \in I^c, j < i. $$

We will denote by $O(1)$ any differential form on $\text{Lag}_h(\hat{E})_I \cap \text{Lag}_h(\hat{E})_I$ which is a linear combination of differential forms of the type
$$ f(t_{pq}) dt_{pq1} \wedge \cdots \wedge dt_{pnqm}, \quad f|_{W^-_I} = 0. $$

Then
$$ dx_{ii} = dt_{ii} + O(1), \quad \forall i \in I, $$
$$ dx_{ij} = dt_{ij} - \frac{t_{ijt_{11}}}{t_{11}} dt_{i1} + O(1), \quad \forall i \in I, j \in I^c, j \neq 1, $$
\[ dx_{11} = \frac{1}{t_{11}} dt_{11} + O(1). \]

We deduce that
\[ \omega_I = \frac{1}{t_{11}^{2\pi i}} \omega_{I*} + O(1). \]

The last equality shows that the orientations \( \mathbf{o}r_I \) and \( \mathbf{o}r_I^* \) coincide on the overlap \( W_I^- \cap \text{Lag}_h(\hat{E})_I^* \). This concludes the proofs of both Lemma 5.6 and of the Proposition 5.5. \( \square \)

We see that any \( AS \) cell \( W_I(\mathbf{Fl}_*, U_+, U_-) \) defines a subanalytic cycle in \( U(\hat{E}^+) \). For fixed \( I \), any two such cycle are homologous since any one of them is the image of \( [W_I^-, \mathbf{o}r_I] \) via a real analytic map, real analytically homotopic to the identity. Thus they all determine the same homology class
\[ \alpha_I \in H_n^{\text{as}}(\text{Lag}_h(\hat{E}), \mathbb{Z}), \]

called the \textit{AS cycle of type} \( I \subset \Pi_n^+ \). By Poincaré duality we obtain cocycles
\[ \alpha_I^* \in H^{w(I)}(\text{Lag}_h(\hat{E}), \mathbb{Z}). \]

We will refer to these as \textit{AS cocycles of type} \( I \). When \( I = \{ \nu \} \), \( \nu \in \Pi_n^+ \) we will use the simpler notations \( \alpha_\nu \) and \( \alpha_\nu^* \) to denote the \( AS \) cycles and cocycles of type \( \{ \nu \} \). We will refer to these cycles as the \textit{basic AS} (co)cycles.

\textbf{Example 5.7.} Observe that the \( AS \) cycle \( \alpha_0 \) is the orientation cycle of \( \text{Lag}_h(\hat{E}) \).

The codimension 1 basic cycle \( \alpha_1 \) is the so called \textit{Maslov cycle}. It defined by the same incidence relation as the Maslov cycle in the case of real lagrangians, [1].

The top codimension basic cycle \( \alpha_n \) can be identified with the integration cycle defined by the embedding
\[ U(n-1) \hookrightarrow U(n), \ U(n-1) \ni T \mapsto T \oplus 1 \in U(n). \quad \square \]

The basic cycles have a remarkable property. To formulate it we need to introduce some fundamental concepts. We denote by \( \mathcal{E} \) the rank \( n = \dim_{\mathbb{C}} \hat{E}^+ \) complex vector bundle over \( S^1 \times \hat{E} \) obtained from the trivial vector bundle
\[ \hat{E}^+ \times [-\pi, \pi] \times U(\hat{E})^+ \to [-\pi, \pi] \times U(\hat{E}^+), \]

by identifying the point \( u \in \hat{E}^+ \) in the fiber over \((\pi, g) \in [-\pi, \pi] \times U(\hat{E})^+ \) with the point \( v = g u \in \hat{E}^+ \) in the fiber over \((\pi, g) \in [-\pi, \pi] \times U(\hat{E})^+ \). Equivalently, consider the \( \mathbb{Z} \)-equivariant bundle
\[ \tilde{\mathcal{E}} = \hat{E}^+ \times \mathbb{R} \times U(\hat{E}) \to \mathbb{R} \times U(\hat{E}), \]

where the \( \mathbb{Z} \)-action is given by
\[ \mathbb{Z} \times ( \hat{E}^+ \times \mathbb{R} \times U(\hat{E})) \ni (n; u, \theta, S) \mapsto (S^n u, \theta + 2n \pi, S) \in \hat{E}^+ \times \mathbb{R} \times U(\hat{E}),. \]

Then \( \mathcal{E} \) is the bundle
\[ \text{Z}\backslash \tilde{\mathcal{E}} \to \text{Z}\backslash (\mathbb{R} \times U(\hat{E})). \]

The sections of this bundle can be identified with maps \( u : \mathbb{R} \times U(\hat{E}^+) \to \hat{E}^+ \) satisfying the equivariance condition
\[ u(\theta + 2\pi, S) = Su(\theta, S), \ \forall (t, S) \in \mathbb{R} \times U(\hat{E}^+). \]

Denote by
\[ \pi_1 : H^*(S^1 \times U(\hat{E}^+), \mathbb{Z}) \to H^{*-1}(U(\hat{E}^+), \mathbb{Z}) \]
the Gysin map determined by the natural projection
\[ \pi : S^1 \times U(\hat{E}^+) \to U(\hat{E}^+). \]
For every \( \nu = \{1, \ldots, n\} \) we define \( \gamma_\nu \in H^{2\nu-1}(U(\hat{E}^+), \mathbb{Z}) \) by setting
\[ \gamma_\nu := \pi_! c_\nu(\mathcal{E}), \]
where \( c_\nu(\mathcal{E}) \in H^{2\nu}(\hat{E}, \mathbb{Z}) \) denotes the \( \nu \)-th Chern class of \( \mathcal{E} \).

**Theorem 5.8** (Thom-Porteous: the odd version). For every \( \nu = \{1, \ldots, n\} \) we have the equality
\[ \alpha^1_\nu = \gamma_\nu. \]

**Proof.** Here is briefly the strategy. Fix a unitary basis \( \mathfrak{e} = \{e_1, \ldots, e_n\} \), and consider the AS variety
\[ \mathcal{X}_\nu(-1) := \{ S \in U(\hat{E}^+) ; \ker(1 + S) \cap \text{span}_C\{e_{\nu_1}, \ldots, e_{\nu_n}\} \geq 1 \}. \]
As we have seen it defines a subanalytic cycle \([\mathcal{X}_\nu(-1), \mathfrak{or}_\nu]\). We will prove that there exists a subanalytic cycle \( c \in S^1 \times U(\hat{E}) \) such that the following happen.
- The (integral) homology class determined by \( c \) is Poincaré dual to \( c_\nu(\mathcal{E}) \).
- We have an equality of subanalytic currents \( \pi_! c = [\mathcal{X}_\nu(-1), \mathfrak{or}_\nu] \).

To construct this analytic cycle we will use the interpretation of \( c_\nu \) as the Poincaré dual of a degeneracy cycle \([18, 22]\).

We set \( V := \text{span}_C\{e_{\nu_1}, \ldots, e_{\nu_n}\} \), and we denote by \( V \) the trivial vector bundle with fiber \( V \) over \( S^1 \times U(\hat{E}) \). Denote by \( \mathbb{P}(V) \) the projective space of lines in \( V \), and by \( p \) the natural projection
\[ p : \mathbb{P}(V) \times S^1 \times (\hat{E}^+) \to S^1 \times (\hat{E}^+). \]
We have a tautological line bundle \( \mathcal{L} \to \mathbb{P}(V) \times S^1 \times (\hat{E}^+) \) defined as the pullback to \( \mathbb{P}(V) \times S^1 \times (\hat{E}^+) \) of the tautological line bundle over \( \mathbb{P}(V) \).

To any bundle morphism \( T : V \to \mathcal{E} \) we can associate in a canonical fashion a bundle morphism \( \hat{T} : \mathcal{L} \to p^* \mathcal{E} \). We regard \( \hat{T} \) as a section of the bundle \( \mathcal{L}^* \otimes p^* \mathcal{E} \). If \( T \) is a \( C^2 \), subanalytic section such that the associated section \( \hat{T} \) vanishes transversally, then the zero set \( Z(\hat{T}) \) is a \( C^1 \) subanalytic manifold equipped with a natural orientation and defines a subanalytic current \([Z(\hat{T})]\). Moreover (see \([18, VI.1]\), the subanalytic current \( p_* [Z(\hat{T})] \) is Poincaré dual to \( c_\nu(\mathcal{E}) \). We will produce a \( C^4 \), subanalytic bundle morphism \( T \) satisfying the above transversality condition, and satisfying the additional equality of currents
\[ \pi_* p_* [Z(\hat{T})] = [\mathcal{X}_\nu(-1), \mathfrak{or}_\nu]. \]
To construct such a morphism \( T \) we first choose a polynomial \( \eta \in \mathbb{R}[\theta] \) satisfying the following conditions
\[ \eta'(\theta) \geq 0, \quad \forall \theta \in [-\pi, \pi], \]
\[ \eta(-\pi) = 0, \quad \eta(\pi) = 1, \quad \eta(0) = \frac{1}{2}, \]
\[ \eta'(0) = \frac{1}{4}, \quad \eta'(-\pi) = \eta'(-\pi) = 0. \]
Note that a bundle morphism \( T : V \to \mathcal{E} \) is uniquely determined by the sections \( Te_j, \nu \leq j \leq n, \) of \( \mathcal{E} \). Now define a vector bundle morphism
\[ T : V \times \{ [-\pi, \pi] \times U(\hat{E}^+) \} \to \hat{E}^+ \times \{ [-\pi, \pi] \times U(\hat{E}^+) \} \]
Proof. Observe that \( \ell \) is a skew-hermitian operator of the unitary operator \( S \). Clearly when \( \eta(t) = 0 \), this is not possible. Hence \( \eta(t) \neq 0,1 \) and thus \( -1 - \eta(t) \eta(t) \) must be an eigenvalue of the unitary operator \( S \). Since \( \eta(t) \in (0,1) \), and the eigenvalues of \( S \) are complex numbers of norm 1, we deduce that \( -1 - \eta(t) \eta(t) \) can be an eigenvalue of \( S \) if and only if \( -1 - \eta(t) \eta(t) = -1 \), so that \( \eta(t) = \frac{1}{2} \). From the properties of \( \eta \) we conclude that this happens if and only if \( \theta = 0 \). Thus

\[
\mathcal{Z}(\hat{T}) = \{ (\ell, \theta, S) \in \mathbb{P}(V) \times S^1 \times U(\hat{E}^+) ; \quad \theta = 0, \quad \ell \in \ker (I + S) \}.
\]

Lemma 5.9. The section \( \hat{T} \) constructed above vanishes transversally.

Proof. Let \((\ell_0, 0, S_0) \in \mathcal{Z}(\hat{T}) \). Fix \( \nu_0 \in V \) spanning \( \ell_0 \). Then we can identify an open neighborhood of \( \ell_0 \) in \( \mathbb{P}(V) \) with an open neighborhood of 0 in the hyperplane \( \ell_0^\perp \cap V \): to any \( u \in \ell_0^\perp \cap V \) we associate the line \( \ell_u \) spanned by \( \nu_0 + u \). We obtain in this fashion a map

\[
\ell_0^\perp \cap V \times (-\pi, \pi) \times U(\hat{E}^+) \ni (u, \theta, S) \xrightarrow{F} (\hat{1} + \eta(\theta)(S - 1))(\nu_0 + u) \in \hat{E}^+,
\]

and we have to prove that the point \((0, 0, S_0) \in \ell_0^\perp \cap V \times (-\pi, \pi) \times U(\hat{E}^+) \) is a regular point of this map.

Choose a smooth path \((-\varepsilon, \varepsilon) \ni t \mapsto (u_t, \theta_t, S_t) \in \ell_0^\perp \cap V \times (-\pi, \pi) \times U(\hat{E}^+) \) such that

\[
u_0 = 0, \quad \theta = 0, \quad S_{t=0} = S_0.
\]

We set

\[
\dot{u} := \frac{d}{dt}|_{t=0} u_t, \quad \dot{\theta} := \frac{d}{dt}|_{t=0} \theta_t, \quad \dot{S}_0 := \frac{d}{dt}|_{t=0} S_t
\]

and

\[
X := S_0^{-1} \dot{S}_0 = S_0^* \dot{S}_0, \quad \text{i.e.,} \quad \dot{S}_0 = S_0 X.
\]

Observe that \( X \) is a skew-hermitian operator \( \hat{E}^+ \to \hat{E}^+ \), and we can identify the tangent space to \( \ell_0^\perp \cap V \times (-\pi, \pi) \times U(\hat{E}^+) \) at \((0, 0, S_0) \) with the space of vectors

\[
(\dot{u}, \dot{\theta}, X) \in \ell_0^\perp \cap V \times \mathbb{R} \times \mathfrak{u}(\hat{E}^+).
\]

Then

\[
\frac{d}{dt}|_{t=0} F(u_t, \theta_t, S_t) = \frac{1}{2} (I + S) \dot{u}_0 + \eta'(0) \dot{\theta}_0 (S_0 - 1)(\nu_0) + \eta(0) \dot{S}_0 \nu_0,
\]
The current $\mathbb{P}\times u(\hat{E}^+)$ is surjective for any nonzero vector $u$. The surjectivity of the differential of $F$ at $(0,0,S_0)$ follows from the fact that the $\mathbb{R}$-linear map 

$$\mathbb{R} \times u(\hat{E}^+) \ni (\hat{\theta}_0, X) \longmapsto (\hat{\theta}_0 + X) v_0 \in \hat{E}^+$$

is surjective for any nonzero vector $v_0 \in \hat{E}^+$. \qed

The above lemma proves that $Z(\tilde{T})$ is a $C^1$ submanifold of $\mathbb{P}(V) \times S^1 \times U(\hat{E}^+)$. It carries a natural orientation which we will describe a bit later. It thus defines a subanalytic current $[Z(\tilde{T})]$. Observe that 

$$Z(\tilde{T}) \subset \mathbb{P}(V) \times \{\theta = 0\} \times U(\hat{E}^+) \subset \mathbb{P}(V) \times S^1 \times U(\hat{E}^+).$$

The current $\pi_* [Z(\tilde{T})]$ is the integration current defined by $Z(\tilde{T})$ regarded as submanifold of $\mathbb{P}(V) \times U(\hat{E}^+)$. As such, it has the description 

$$Z(\tilde{T}) = \left\{ (\ell, S) \in \mathbb{P}(V) \times U(\hat{E}); \ (1 + S)|_{\ell} = 0 \right\}.$$ 

We set 

$$Z(\tilde{T})^* := \left\{ (\ell, S) \in Z(\tilde{T}); \ \ell = \ker(1 + S), \ e_\nu \not\in \ell \right\}.$$ 

Note that the projection 

$$\pi := \mathbb{P}(V) \times U(\hat{E}^+) \rightarrow U(\hat{E}), \ (\ell, S) \mapsto S,$$

maps $Z(\tilde{T})$ surjectively onto $X_\nu(\hat{-1})$. Moreover, $Z(\tilde{T})^*$ is the preimage under $\pi$ of the top stratum $W^r_\nu(-1)$ of $X_\nu(-1)$, 

$$Z(\tilde{T})^* = \pi^{-1}(W^r_\nu(-1)),$$

and the restriction of $\pi$ to $Z(\tilde{T})^*$ is a bijection with inverse 

$$W^r_\nu(-1) \ni S \mapsto (\ker(1 + S), S) \in Z(\tilde{T})^*.$$ 

**Lemma 5.10.** The map $\pi : Z(\tilde{T})^* \rightarrow W^r_\nu$ is a diffeomorphism.

**Proof.** It suffices to show that the differential of $\pi$ is everywhere injective. Let $\zeta_0 = (\ell_0, S_0) \in Z(\tilde{T})^*$. Suppose $\ell_0 = \text{span}\{v_0\}$. We have to prove that if 

$$(-\varepsilon, \varepsilon) \ni (\ell_t, S_t) \in$$

a is a smooth path $Z(\tilde{T})^*$ passing through $\zeta_0$ at $t = 0$ and $\dot{S}_0 := \frac{d}{dt}|_{t=0} \ell_t = 0$, then $\frac{d}{dt}|_{t=0} \ell_t = 0$. 

We write $\ell_t = \text{span}\{v_0 + u_t\}$, where $t \mapsto u_t \in \ell_t^\bot \cap V$ is a $C^1$-path such that $u_0 = 0$. Then 

$$S_t(v_0 + u_t) = -v_0 - u_t, \ \forall t,$$

and differentiating with respect to $t$ at $t = 0$ we get 

$$-\dot{u}_0 = S_0 \dot{u}_0 + \dot{S}_0(v_0) = S_0 \dot{u}_0.$$ 

Hence $\dot{u}_0 \in \ker(1 + S_0)$. We conclude that $\dot{u}_0 = 0$ because $\ker(1 + S)$ is the line spanned by $v_0$, and $u_0 \perp v_0$. \qed

Lemma 5.10 implies that we have an equality of currents 

$$\pi_* p_* [Z(\tilde{T})] = \pm [X_\nu(-1)].$$

To eliminate the sign ambiguity we need to understand the orientation of $Z(\tilde{T})$. 

\[ (-I) v_0 = S_0 v_0, \ n'(0) = \frac{1}{2} \]

$$= \frac{1}{2} (1 + S_0) \dot{u}_0 + \frac{1}{2} \dot{\theta}_0 S_0 v_0 + \frac{1}{2} S_0 X v_0 = \frac{1}{2} (1 + S_0) \dot{u}_0 + \frac{1}{2} S_0 (\dot{\theta}_0 1 + X) v_0.$$
We begin by describing the conormal orientation of \( \mathcal{Z}(\hat{T}) \) at a special point \( \xi_0 = (\ell_0, 0, S_0) \), where
\[
\ell_0 = \text{span}_C\{e_\nu\}, \quad \text{and} \quad S_0 e_i = \begin{cases} -1 & i = \nu \\ 1 & i \neq \nu. \end{cases}
\]
Observe that \( S_0 \) is selfadjoint and belongs to the top dimensional stratum \( W_\nu^{-}(-1) \) of \( X_\nu(-1) \). Denote by \( F \) the differential at \( \xi_0 \) of the map \( F \) described in (5.9).

The fiber at \( \xi_0 \) of the conormal bundle to \( \mathcal{Z}(T) \) is the image of the real adjoint of \( F \),
\[
F^\dagger: T_0^*\hat{E}^+ \to T_{\xi_0}^*(\mathbb{P}(V) \times S^1 \times U(\hat{E}^+)).
\]
Since \( F \) is surjective, its real dual \( F^\dagger \) is injective. The fiber at \( \xi_0 \) of the conormal bundle is the image of \( F^\dagger \), and we have an orientation on this fiber induced via \( F^\dagger \) by the canonical orientation of \( \hat{E}^+ \) as a complex vector space.

The canonical orientation of the real cotangent space \( T_{\xi_0}^*\hat{E}^+ \) is described by the top degree exterior monomial
\[
\alpha^1 \wedge \beta^1 \wedge \cdots \wedge \alpha^n \wedge \beta^n,
\]
where \( \alpha^k, \beta^k \in \text{Hom}_\mathbb{R}(\hat{E}^+, \mathbb{R}) \) are defined by
\[
\alpha^k(x) = \text{Re}(x, e_k), \quad \beta^k(x) = \text{Im}(x, e_k), \quad \forall x \in \hat{E}^+, \quad k = 1, \ldots, n.
\]
For every \( \dot{u}_0 \in V, \dot{u}_0 \perp e_\nu, \dot{\theta}_0 \in \mathbb{R} \) and \( iZ \in \omega(\hat{E}^+) \) we have
\[
F^\dagger \alpha^k(\dot{u}_0, \dot{\theta}_0, iZ) = \text{Re}( F(\dot{u}_0, \dot{\theta}_0, iZ), e_k ) = \frac{1}{2} \text{Re}( (1 + S_0)\dot{u}_0, e_k ) + \frac{1}{2} \text{Re}( S_0(\dot{\theta}_0 + iZ)e_\nu, e_k )
\]
\[
= \frac{1}{2} \text{Re}( \dot{u}_0, (1 + S_0)e_k ) + \frac{1}{2} \text{Re}( (\dot{\theta}_0 + iZ)e_\nu, S_0 e_k ).
\]
\[
F^\dagger \beta^k(\dot{u}_0, \dot{\theta}_0, iZ) = \frac{1}{2} \text{Im}( \dot{u}_0, (1 + S_0)e_k ) + \frac{1}{2} \text{Im}( (\dot{\theta}_0 + iZ)e_\nu, S_0 e_k )
\]
To simplify the final result observe that the restrictions to \( \ell_0^+ \cap V \) of the \( \mathbb{R} \)-linear functions \( \alpha^k, \beta^k, k \leq \nu \) determine a basis of \( \text{Hom}(\ell_0^+ \cap V, \mathbb{R}) \) which we will continue by the same symbols. We denote by \( dt \) the tautological linear map \( T_0^*\mathbb{R} \to \mathbb{R} \).

Recall (see Example 5.4) that the real dual of \( \omega(\hat{E}^+) \) admits a natural basis given by the \( \mathbb{R} \)-linear forms
\[
\theta^i(Z) = (Ze_j, e_j), \quad \theta^i(Z) = \text{Re}(Ze_j, e_i), \quad \varphi^i = \text{Im}(Ze_j, e_i), \quad i < j \in \mathbb{N}^+_n, \quad iZ \in \omega(\hat{E}^+).
\]
Observe that
\[
\theta^i = \theta^j, \quad \varphi^i = -\varphi^j, \quad \forall i \neq j.
\]
For every \( \dot{u}_0 \in \ell_0^+ \cap V, \dot{\theta}_0 \in T_0^*\mathbb{R}, iZ \in \omega(\hat{E}^+) \) we have
\[
\text{Re}( \dot{u}_0, (1 + S_0)e_k ) = \begin{cases} 2\alpha^k(\dot{u}_0) & k > \nu \\ 0 & k \leq \nu, \end{cases}
\]
\[
\text{Im}( \dot{u}_0, (1 + S_0)e_k ) = \begin{cases} 2\beta^k(\dot{u}_0) & k > \nu \\ 0 & k \leq \nu, \end{cases}
\]
\[
\text{Re}( (\dot{\theta}_0 + iZ)e_\nu, S_0 e_k ) = \delta_{\nu k}dt(\dot{\theta}_0) - \begin{cases} 0 & k = \nu \\ \varphi^{k\nu}(Z) & k \neq \nu, \end{cases}
\]
we see that its conormal bundle has a natural orientation given by the exterior form
\[ \text{Im}( (\theta_0 + iZ)e_\nu, S_0 e_k) = \begin{cases} 
\theta^\nu(Z) & k = \nu \\
\theta^k\nu(Z) & k \neq \nu,
\end{cases} \]
We deduce the following.

- If \( k < \nu \) then
  \[ F^1 \alpha^k = -\frac{1}{2} \varphi^{kj}, \quad F^1 \beta^k = \frac{1}{2} \theta^{kj}. \]
- If \( k = \nu \) then
  \[ F^1 \alpha^\nu = \frac{1}{2} dt, \quad F^1 \beta^\nu = \frac{1}{2} d\theta^\nu. \]
- If \( j > \nu \) then
  \[ F^1 \alpha^j = \alpha^j + \frac{1}{2} \varphi^{\nu j}, \quad F^1 \beta^k = \beta^j + \frac{1}{2} \theta^{\nu j}. \]

Thus, the conormal space of \( \mathcal{Z}(\tilde{T}) \hookrightarrow \mathbb{P}(V) \times S^1 \times U(\hat{E}) \) at \( \xi_0 \) has an orientation given by the oriented basis
\[-\varphi^{1\nu}, \theta^{1\nu}, \ldots, -\varphi^{\nu-1,\nu}, \theta^{\nu,1\nu}, dt, d\theta^\nu, \alpha^{\nu+1} + \frac{1}{2} \varphi^{\nu,\nu+1}, \theta^{\nu+1} + \frac{1}{2} \theta^{\nu,\nu+1}, \ldots, \alpha^n + \frac{1}{2} \varphi^{\nu,n}, \beta^n + \frac{1}{2} \theta^{\nu,n} \]
which is equivalent with the orientation given by the oriented basis
\[ \theta^{1\nu}, \varphi^{1\nu}, \ldots, \varphi^{\nu-1,\nu}, dt, d\theta^\nu, \alpha^{\nu+1} + \frac{1}{2} \varphi^{\nu,\nu+1}, \beta^{\nu+1} + \frac{1}{2} \theta^{\nu,\nu+1}, \ldots, \alpha^n + \frac{1}{2} \varphi^{\nu,n}, \beta^n + \frac{1}{2} \theta^{\nu,n} \]
We will represent this oriented basis by the exterior polynomial
\[ \omega^{\text{norm}} \in \Lambda^{2n} T^*_{\xi_0} (\mathbb{P}(V) \times S^1 \times U(\hat{E}^+)) , \]
\[ \omega^{\text{norm}} := \left( \bigwedge_{k < \nu} \theta^{kj} \wedge \varphi^{kj} \right) \wedge dt \wedge d\theta^\nu \wedge \left( \bigwedge_{j = \nu+1} (\alpha^j + \frac{1}{2} \varphi^{\nu j}) \wedge (\beta^j + \frac{1}{2} \theta^{\nu j}) \right) . \]
The zero set \( \mathcal{Z}(\tilde{T}) \) is a smooth manifold of dimension
\[ \dim_{\mathbb{R}} \mathcal{Z}(\tilde{T}) = \dim_{\mathbb{R}} (\mathbb{P}(V) \times S^1 \times U(\hat{E}^+)) - \dim_{\mathbb{R}} \hat{E}^+ \]
\[ = (2n - 2\nu) + 1 + n^2 - 2n = n^2 - (2\nu - 1) = n^2 - w(\nu). \]
The orientation of \( T^*_{\xi_0} (\mathbb{P}(V) \times S^1 \times U(\hat{E}^+)) \) is described by the exterior monomial
\[ \Omega = dt \wedge \left( \bigwedge_{j = \nu+1} \alpha^j \wedge \beta^j \right) \wedge \left( \bigwedge_{i=1}^{n} \theta^i \right) \wedge \left( \bigwedge_{j < i} \beta^j \wedge \varphi^{ji} \right) . \]
The orientation of \( T^*_{\xi_0} \mathcal{Z}(\tilde{T}) \) is given by any \( \omega \in \Lambda^{n^2 - w(\nu)} T^*_{\xi_0} (\mathbb{P}(V) \times S^1 \times U(\hat{E}^+)) \) such that
\[ \hat{\Omega} = \omega^{\text{norm}} \wedge \omega. \]
We can take \( \omega \) to be
\[ \omega = \omega^{\text{tan}} := (-1)^{\nu-1} \theta^1 \wedge \ldots \theta^{\nu-1} \wedge \theta^{\nu+1} \wedge \ldots \wedge \theta^n \wedge \left( \bigwedge_{j < k, k \neq \nu} \theta^j \wedge \varphi^{jk} \right) . \]
(5.10)
If we now think of \( \mathcal{Z}(\tilde{T}) \) as an oriented submanifold of \( \mathbb{P}(V) \times U(\hat{E}) \subset \mathbb{P}(V) \times S^1 \times U(\hat{E}^+) \), we see that its conormal bundle has a natural orientation given by the exterior form
\[ \omega^{\text{norm}}_0 = \left( \bigwedge_{k < \nu} \theta^{kj} \wedge \varphi^{kj} \right) \wedge d\theta^\nu \wedge \left( \bigwedge_{j = \nu+1} (\alpha^j + \frac{1}{2} \varphi^{\nu j}) \wedge (\beta^j + \frac{1}{2} \theta^{\nu j}) \right) . \]
The discussion in Example 5.4 shows that the tangent space \( T_{S_1}W^\nu_{\nu}(-1) \subset T_{S_0}U(\hat{E}^+) \) is also oriented by \( \omega_{\tan} \) which proves that the differential

\[
D\pi : T_{(\ell_0,S_0)}Z(\hat{T}) \to T_{S_0}W^\nu_{\nu}(-1).
\]

This concludes the proof of the Theorem 5.8.

\[ \square \]

**Remark 5.11.** (a) The proof of Theorem 5.8 shows that we have a resolution \( \tilde{\mathcal{X}}_{\nu} \xrightarrow{\pi} \mathcal{X}_{\nu} \) of \( \mathcal{X}_{\nu} \), where

\[
\tilde{\mathcal{X}}_{\nu} = \{(\ell,S) \in \mathbb{P}(V_{\nu}) \times U(\hat{E})^+; \ (1 - S) |_{e_{\ell}} = 0 \},
\]

and \( \pi \) is induced by the natural projection \( \mathbb{P}(V_{\nu}) \times U(\hat{E}) \to U(\hat{E}^+) \). Here \( V_{\nu} := \text{span}_C \{e_\nu, \ldots, e_n\} \).

We call the map \( \pi \) a resolution, because it is semi-algebraic, proper, and it is a diffeomorphism over the top dimensional stratum \( W^-_{\nu} \) of \( \mathcal{X}_{\nu} \). The map \( \pi \) is also a Bott-Samelson cycle (see [8, 28] for a definition) for the Morse function

\[
U(\hat{E}^+) \ni S \mapsto - \text{Re} \text{tr} AS \in \mathbb{R}
\]

and its critical point \( S_\nu \in U(\hat{E}^+) \) given by

\[
S_\nu e_k = \begin{cases} e_\nu & k = \nu \\ -e_k & k \neq \nu. \end{cases}
\]

All the \( AS \)-varieties \( \mathcal{X}_I \) admit such Bott-Samelson resolutions (see [28]), \( \rho_I : \tilde{\mathcal{X}}_I \to \mathcal{X}_I \), and it is essentially trough these resolutions that the cycles \( \alpha_I \) were defined by Vasil’ev [32].

(b) We have a natural group morphism

\[
\pi_{2n-1}(U(n)) \to \mathbb{Z}, \ (S^{2n-1} \xrightarrow{f} \mathbb{Z}) \mapsto \int_{S^{2n-1}} f^* \alpha_n^* = f_*[S^{2n-1}] \cdot \alpha_n.
\]

Using Bott divisibility theorem, [3], [19, Thm. 24.5.2] and the Theorem 5.8 we deduce that this is an injective morphism whose range is the subgroup \((n - 1)!\mathbb{Z} \subset \mathbb{Z}\).

(c) The rank \( n \) complex vector bundle \( E \to S^1 \times U(n) \) has an interesting homotopic theoretic significance.\(^2\) Its classifying map

\[
F : S^1 \times U(n) \to BU(n)
\]

can be viewed by adjunction as a map

\[
F_1 : U(n) \to \text{Map} (S^1, BU(n)).
\]

The space \( \text{Map} (S^1, BU(n)) \) is the total space of a fibration

\[
\Omega BU(n) \hookrightarrow \text{Map} (S^1, BU(n)) \to BU(n).
\]

The fiber of this fibration is homotopic to \( U(n) \). The map \( F_1 \) will sent \( U(n) \) to a fiber of this fibration, and will be a homotopy equivalence \( U(n) \to \Omega BU(n) \). To understand this map we use the adjunction

\[
\text{Map} (U(n), \Omega BU(n)) \cong \text{Map} (\Sigma G, BU(n))
\]

so the map \( F_1 \) corresponds to a map \( F_2 : \Sigma G \to BU(n) \) which classifies a rank \( n \) complex vector bundle \( E_2 \) over the suspension \( \Sigma U(n) \). This bundle is obtained via the clutching construction, where the clutching over the “Equator” \( U(n) \hookrightarrow \Sigma U(n) \) is given by the identity map \( U(n) \to U(n) \). Equivalently, the map \( \Sigma U(n) \to BU(n) \) is a special case of the inclusion (see [31]) \( \Sigma G \hookrightarrow BG \) for any compact Lie group.

\(^2\)I want to thank my colleague Bruce Williams for explaining to me these homotopic theoretic facts.
6. The Morse-Floer complex and intersection theory

It is well known that the integral cohomology ring of $U(n)$ is an exterior algebra freely generated by elements $x_i \in H^{2i-1}(U(n), \mathbb{Z})$, $i = 1, \ldots, n$. The odd Thom-Porteous theorem implies that as generators $x_i$ of this ring we can take the AS cocycles $\alpha_I^\dagger$. In this section we would like to prove this by direct geometric considerations, and then investigate the cup product of two arbitrary AS cocycles.

**Proposition 6.1.** The AS-cycles $\alpha_I$, $I \subset \mathbb{N}_n^+$, form a $\mathbb{Z}$-basis of $H_\ast(\text{Lag}_h(\hat{E}), \mathbb{Z})$.

**Proof.** We will use a Morse theoretic approach. Consider again the Morse flow $\Psi^t = e^{t\hat{A}}$ on $\text{Lag}_h(\hat{E})$.

**Lemma 6.2.** The flow $\Psi^t$ is a Morse-Stokes flow, i.e., the following hold.

(a) The flow $\Phi^t$ is a finite volume flow, i.e., the $(n^2 + 1)$-dimensional manifold

$$\left\{ \left( t, \Psi^t(L), L \right); \ t \in (0, 1], \ L \in \text{Lag}_h(\hat{E}) \right\} \subset (0, 1] \times \text{Lag}_h(\hat{E}) \times \text{Lag}_h(\hat{E}),$$

has finite volume.

(b) The stable and unstable manifold $W_I^\pm$ have finite volume.

(c) If there exists a tunnelling from $\Lambda_I$ to $\Lambda_J$ then $\dim W_J^- < \dim W_I^-$. 

**Proof.** From Theorem A.6 we deduce that $\Psi^t$ is a tame flow. Proposition A.5 now implies that the flow satisfies (a) and (b). Property (c) follows from Proposition 4.6. \(\square\)

As in Harvey-Lawson [17], we consider the subcomplex $C_\ast(\Psi^t)$ of the complex $C_\ast(\text{Lag}_h(\hat{E}))$ of subanalytic chains generated by the analytic chains $[W_I^-, \text{or}_I]$, and their boundaries. According to [17, Thm. 4.1], the inclusion

$$C_\ast(\Psi^t) \hookrightarrow C_\ast$$

induces an isomorphism in homology.

Proposition 5.5 implies that the complex $C_\ast(\Psi^t)$ is perfect. Hence the AS cycles, which form an integral basis of the complex $C_\ast(\Psi^t)$, also form an integral basis of the integral homology of $\text{Lag}_h(\hat{E})$. \(\square\)

**Remark 6.3.** (a) The complex $C_\ast(\Psi^t)$ is isomorphic to the Morse-Floer complex of the flow $\Phi^t$ (see [26, §2.5] for a definition of the Morse-Floer complex. Although our flow does not satisfy the Morse-Smale transversality condition, it satisfies the Morse-Smale condition in codimensions $\leq 1$ which is all we need to define the Morse-Floer complex.

(b) The fact that the flow $\Phi^t$ does not satisfy the Morse-Smale condition implies that the stratification of $\text{Lag}_h(\hat{E})$,

$$\text{Lag}_h(\hat{E}) = \bigsqcup_{I \in \mathbb{N}_n^+} W_I^-$$

violates the Whitney regularity conditions (see [27, §8]). That is why, instead of working with the stratified chains of M. Goresky [13] we chose to work with the subanalytic chains of R. Hardt [16]. \(\square\)
Using the Poincaré duality on $U(\hat{E}^+)$ we obtain intersection products
\[ \bullet : H_{n^2-p}(U(\hat{E}^+), \mathbb{Z}) \times H_{n^2-q}(U(\hat{E}^+), \mathbb{Z}) \to H_{n^2-p-q}(U(\hat{E}^+), \mathbb{Z}). \]
For every pair of nonempty, disjoint subsets $I, J \subset \mathbb{Z}_+^n$ such that
\[ I = \{ i_1 < \cdots < i_p \}, \quad J = \{ j_1 < \cdots < j_q \}, \]
we define $\epsilon(I, J) = \pm 1$ to be the signature of the permutation $(i_1, \ldots, i_p; j_1, \ldots, j_q)$ of $I \cup J$.

**Proposition 6.4.** Let $I, J \subset \mathbb{Z}_+^n$ such that $w(I) + w(J) = w(\mathbb{I}_n^+) = n^2$. then
\[ \alpha_I \bullet \alpha_J = \begin{cases} 0 & I \cap J \neq \emptyset \\ \epsilon(I, J) & I = J^c. \end{cases} \]

**Proof.** Fix unitary basis $\{e_1, \ldots, e_n\}$ of $\hat{E}^+$, and consider the symmetric operator $A_0 : \hat{E}^+ \to \hat{E}^+$ given by
\[ A_0 e_i = \frac{2i - 1}{2} e_i. \]
We form as usual the associated symmetric operator $\hat{A}_0 : \hat{E} \to \hat{E}$, and the positive gradient flow $e^{\lambda_0}$ on $\text{Lag}_h(\hat{E})$ associated to the Morse function $\varphi_0 : \text{Lag}_h(\hat{E}) \to \mathbb{R}$, $L \mapsto \text{Re} \text{tr}(\hat{A}_0 L)$.

For every critical point $\Lambda_K$ of $\varphi_0$ we have
\[ \dim W_K^- = w(K) = \varphi_0(\Lambda_K) + \frac{n^2}{2}. \]
For every $M \subset \mathbb{Z}_+^n$ we denote by $W_M^+$ the stable manifold at $\Lambda_M^+$.

Let $w(I) + w(J) = w(\mathbb{I}_n^+)$. Using the equality
\[ W_{J^c}^- = JW_J^- \]
we deduce that $W_{J^c}^+$ is also an AS cell of type $I$, so that that we can represent the homology class $\alpha_J$ by the subanalytic cycle given as the integration over the stable manifold $W_{J^c}^+$ equipped with a natural orientation. We denote by $X_{J^c}^+$ its closure.

We have $\varphi_0(\Lambda_{J^c}) = -\varphi_0(\Lambda_J)$ and the equality $w(I) + w(J) = n^2$ translates into the equality
\[ \varphi_0(\Lambda_{J^c}) = \varphi_0(\Lambda_I) =: \kappa. \]
Observe that,
\[ X_{J^c}^+ \setminus \{ \Lambda_{J^c} \} \subset \{ \varphi_0 > \kappa \}, \]
\[ X_I^- \setminus \{ \Lambda_I \} \subset \{ \varphi_0 < \kappa \}. \]
This shows that if $I^c \neq J$ and $w(I^c) = w(J)$ the supports of the subanalytic currents $[X_{J^c}^+]$ and $[X_I^-]$ are disjoint, so that, in this case,
\[ \alpha_I \bullet \alpha_J = 0. \]

When $J = I^c$ we see that the supports of the above subanalytic cycles intersect only at $\Lambda_I$. In fact, only the top dimensional strata of their supports intersect, and they do so transversally. Hence the intersection of the analytic cycles $[X_{J^c}^+]$ and $[X_I^-]$ is well defined, and from Proposition B.4 we deduce
\[ [X_{J^c}^+] \bullet [X_I^-] = \pm [\Lambda_I], \]
where \([\Lambda_I]\) denotes the Dirac 0 dimensional current supported at \(\Lambda_I\). The fact that the correct choice of signs is \(\epsilon(I, I^c)\) follows from our orientation conventions.

Form the above result we deduce that for every cycle \(c \in H_k(U(n), \mathbb{Z})\) we have a decomposition

\[
c = \sum_{w(I) = k} \epsilon(I, I^c)(c \bullet \alpha_{I^c}) \alpha_I.
\] (6.1)

**Theorem 6.5** (Schubert calculus: the odd version). If \(I = \{i_k < \cdots < i_1\} \subset \mathbb{I}_n^+\) then

\[
\alpha_I = \alpha_{i_k} \cdots \alpha_{i_1},
\] (6.2)

or equivalently,

\[
\alpha_I^\dagger = \alpha_{i_k}^ \dagger \wedge \cdots \wedge \alpha_{i_1}^ \dagger.
\] (6.3)

**Proof.** Let us first describe our strategy. Fix a unitary basis \(e\) of \(\hat{E}^+\), an injection \(\rho : I \rightarrow S^1 \setminus \{\pm 1\}, i \mapsto \rho_i\) and we consider the AS varieties \(X_i(\rho_i) = X_i(\rho_i, e), i \in I,\) defined in (5.1). We denote by \([X_i(\rho_i)]\) the associated subanalytic cycles. We will prove the following facts.

**A.** The varieties \(X_i\) intersect quasi-transversally, i.e., for any subset \(J \subset I\) we have

\[
\text{codim} \bigcap_{j \in J} X_j(\rho_j) \geq w(J).
\]

**B.** There exists a continuous semialgebraic map \(\Xi : U(\hat{E}^+) \rightarrow U(\hat{E}^+),\) semialgebraically homotopic to the identity such that

\[
\Xi \bigg( \bigcap_{i \in I} X_i(\rho_i) \bigg) \subset X_I = X_I(1).
\]

**C.** The intersection current \([X_{i_k}(\rho_{i_k})]\) \(\cdots\) \([X_{i_1}(\rho_{i_1})]\) \(\bullet\) \([X_{I^c}(1)]\) is a well defined zero dimensional subanalytic current consisting of a single point with multiplicity \(\epsilon(I, I^c)\).

We claim that the above facts imply (6.2). To see this, note first that **A** implies that according to [14] (see also Appendix B) we can form the intersection current

\[
\eta = [X_{i_k}(\rho_{i_k})]\) \(\cdots\) \([X_{i_1}(\rho_{i_1})].
\]

The current \(\eta\) is a subanalytic current whose homology class is \(\alpha_{i_k} \cdots \alpha_{i_1},\) and its support is

\[
\text{supp}(\eta) = \bigg( \bigcap_{i \in I} X_i(\rho_i) \bigg).
\]

The pushforward \(\Xi_*(\eta)\) is also a subanalytic current and it represents the same homology class since \(\Xi\) is homotopic to the identity. Moreover, property **B** shows that

\[
\text{supp} \Xi_*(\eta) \subset X_I(1).
\]

Consider again the dual AS varieties \(X_I^+, w(J) = w(I).\) In the proof of Proposition 6.4 we have seen that

\[
X_I \cap X_J^+ = \emptyset, \text{ if } J \neq I.
\]

Hence, the equality (6.1) implies that there exists an integer \(k = k(I)\) such that

\[
\alpha_{i_k} \cdots \alpha_{i_1} = k_I \alpha_I,
\]

where

\[
k_I = \epsilon(I, I^c)(\alpha_{i_k} \cdots \alpha_{i_1} \bullet \alpha_{I^c}.
\]
The equality $k_I = 1$ now follows from C.

**Proof of A.** Since the set of unitary operators with simple eigenvalues is open and dense, we deduce that the set

$$Y_J := \bigcap_{j \in J} X_j(\rho_j)$$

contains a dense open subset $O_J$ consisting of operators $S$ such that

$$\dim \mathbb{C} \ker(\rho_j - S) = 1, \quad \forall j \in J.$$ 

For $\nu \in \mathbb{N}_1$ we set $F_\nu := \text{span}\{e_i, \ i \leq \nu\}$.

Observe that if $S \in O_J$, then for every $j \in J$ we have $\ker(\rho_j - S) \subset F_{\nu}^\perp$. Suppose that $J = \{j_m < \cdots < j_1\}$, and define

$$\Phi: O_J \to \mathbb{P}(F_{j_m}^\perp) \times \cdots \times \mathbb{P}(F_{j_1}^\perp), \ S \mapsto (\ker(\rho_{j_m} - S), \ldots, \ker(\rho_{j_1} - S)).$$

The image of $\Phi$ is

$$\Phi(O_J) = \{((\ell_{m}, \ldots, \ell_1) \in \mathbb{P}(F_{j_m}^\perp) \times \cdots \times \mathbb{P}(F_{j_1}^\perp); \ \ell_i \perp \ell_i', \ \forall i \neq i'\}.$$

The resulting map $O_J \to \Phi(O_J)$ is a fibration with fiber over $(\ell_m, \ldots, \ell_1)$ the manifold $\mathcal{F}$ consisting of unitary operators on the subspace $(\ell_m \oplus \cdots \oplus \ell_1)^\perp \subset \hat{E}^+$ which do not have the numbers $\rho_j, j \in J$, in their spectra. The manifold $\mathcal{F}$ is open in this group of unitary operators. Now observe that

$$\dim \mathbb{R} \Phi(O_J) = 2(n - j_m) + 2(n - 1 - j_{m-1}) + \cdots + 2(n - (m - 1) - j_1) = 2nm - m(m - 1) - 2 \sum_{j \in J} j.$$ 

The fiber $\mathcal{F}$ has dimension $(n - m)^2$ so that

$$\dim \mathbb{R} O_J = (n - m)^2 + 2nm - m(m - 1) - 2 \sum_{j \in J} j = n^2 - \sum_{j \in J} (2j - 1).$$

Hence

$$\text{codim} Y_J = \text{codim} O_J = w(J).$$

**Proof of B.** In the proof we will need the following technical result.

**Lemma 6.6.** There exists a continuous semialgebraic map $\xi: S^1 \to S^1$ semialgebraically homotopic to the identity such that

$$\xi^{-1}(1) = \{\rho_i; \ i \in I\}.$$ 

**Proof.** We write $\rho_i = e^{Bi}$, $t_i \in (-\pi, \pi)$ and we consider a semialgebraic map

$$f: [-\pi, \pi] \to [-\pi, \pi]$$

satisfying the following conditions (see Figure 1, where $k = \# I$ )

- $f(\pm \pi) = \pm \pi$.
- $f^{-1}(0) = \{t_i; \ i \in I\}$. 

The image of $\Phi$ is

$$\Phi(O_J) = \{((\ell_{m}, \ldots, \ell_1) \in \mathbb{P}(F_{j_m}^\perp) \times \cdots \times \mathbb{P}(F_{j_1}^\perp); \ \ell_i \perp \ell_i', \ \forall i \neq i'\}.$$
Now define $\xi : S^1 \to S^1$ by setting
$$\xi(e^{it}) = e^{if(t)}, \quad t \in [-\pi, \pi].$$
We have a semialgebraic homotopy between $\xi$ and the identity map given by
$$\xi_s(e^{it}) = e^{i((1-s)f(t)+st)}, \quad s \in [0, 1], \quad t \in [-\pi, \pi].$$

Using the map $\xi$ in Lemma 6.6 we define $\Xi : U(\widehat{E}^+_{I}) \to U(\widehat{E}^+_{I}), S \mapsto \Xi(S)$.

The map $\Xi$ is semialgebraic because its graph $\Gamma_\Xi \subset U(\widehat{E}^+_{I}) \times U(\widehat{E}^+_{I})$ can be given the description
$$\Gamma_\Xi = \{(S, S'); \exists A \in U(\widehat{E}^+_{I}), \ AS^*A^* = \text{Diag}(\lambda_1, \ldots, \lambda_n), \ AS'\text{ }A^* = \text{Diag}(\xi(\lambda_1), \ldots, \xi(\lambda_n)) \}.$$
The continuity of $\Xi$ is classical; see [9, Theorem X.7.2].

If we consider the set $O_I$ defined in the proof of $A$ then we notice that
$$\Xi(O_I) \subset W^I_{-, 1}(\rho),$$
and thus
$$\Xi(\bigcap_{i \in I} X_i(\rho_i)) = \Xi(\text{cl}(O_I)) \subset \text{cl}(W^I_{-, 1}) = X_I(1).$$
This proves $B$.

**Proof of C.** For $\nu \in \mathbb{Z}^+_+ \setminus 1$ and $\rho \neq -1$ we set
$$^*\mathcal{W}^-_{\nu}(\rho) := \{ S \in \mathcal{W}^-_{\nu}; \quad \dim_C \ker(\rho - S) = 1, \quad \ker(1 + S) = 0 \}.$$ 
Note that $^*\mathcal{W}^-_{\nu}(\rho)$ is an open and dense subset of $\mathcal{W}^-_{\nu}(\rho)$. We first want to produce a natural trivializing frame of the conormal bundle of $^*\mathcal{W}^-_{\nu}(\rho)$. Set $\lambda := -i^{1-\rho}$.

The Cayley transform
$$S \mapsto -i(1 - S)(1 + S)^{-1}$$
maps $^*\mathcal{W}^-_{\nu}(\rho)$ onto the subset $\mathcal{R}^*_{\nu}$ of the space of Hermitian operators $A : \widehat{E}^+ \to \widehat{E}^+$ such that
- $\dim \ker(\lambda - A) = 1$.
- $e_j \perp \ker(\lambda - A), \quad \forall j < \nu$.
- $(e_{\nu}, u) \neq 0, \forall u \in \ker(\lambda - A), \ u \neq 0$.

---

I want to thank my colleague Sergei Starchenko for this simple argument.
Note that for any $A \in \mathbb{R}_v^*$ there exists a unique vector $u = u_A \in \ker(\lambda - A)$ such that $(u, e_\nu) = 1$. For $A \in \mathbb{R}_v^*$ we denote by $(\lambda - A)^{-1}$ the unique Hermitian operator $\hat{E}^+ \to \hat{E}^+$ such that

$$(\lambda - A)^{-1} u_A = 0, \quad (\lambda - A)^{-1} (\lambda - A) v = v, \quad \forall v \perp u_A.$$ 

If $(-\varepsilon, \varepsilon) \ni t \mapsto A_t \in \mathbb{R}_v^*$ is a smooth path, and $u_t := u_{A_t}$, then differentiating the equality $A_t u_t = \lambda u_t$ at $t = 0$ we deduce

$$\dot{A}_0 u_0 = (\lambda - A_0) \dot{u}_0.$$ 

Taking the inner product with $u_0$ we deduce

$$(\dot{A}_0 u_0, u_0) = 0.$$ 

We write $\dot{u}_0 = cu_0 + \dot{v}_0$, where $(v_0, u_0) = (v_0, e_j) = 0, \forall j < \nu$. We deduce

$$v_0 = (\lambda - A_0)^{-1} \dot{A}_0 u_0,$$

so that

$$\langle \dot{A}_0 u_0, (\lambda - A_0)^{-1} e_j \rangle = (v_0, e_j) = 0, \quad \forall j < \nu.$$ 

This shows that the fiber at $A_0$ of the conormal bundle of $\mathbb{R}_v^*$ contains the $\mathbb{R}$-linear forms

$$\dot{A}_0 \mapsto \nu^\nu(\dot{A}_0) = u_{\dot{A}_0}^\nu(\dot{A}_0) = (\dot{A}_0 u_0, u_0),$$

$$\dot{A}_0 \mapsto \nu^{ij\nu}(\dot{A}_0) = u_{\dot{A}_0}^{ij\nu}(\dot{A}_0) = \Re(\dot{A}_0 u_0, (\lambda - A_0)^{-1} e_j),$$

$$\dot{A}_0 \mapsto \nu^{ij\nu}(\dot{A}_0) = u_{\dot{A}_0}^{ij\nu}(\dot{A}_0) = \Im(\dot{A}_0 u_0, (\lambda - A_0)^{-1} e_j).$$

Since the vectors $e_j, \, j < \nu$ lie in the orthogonal complement of $\ker(\lambda - A_0)$ we deduce that the vectors $(\lambda - A_0)^{-1} e_j, \, j < \nu$ are linearly independent over $\mathbb{C}$. A dimension count now implies that the above linear forms form a basis of the fiber at $A_0$ of the conormal bundle of $\mathbb{R}_v^*$. Since the forms $u_\nu^\nu, \, u^{ij\nu}_\nu, \, v^{ij\nu}_\nu$ depend smoothly on $A \in \mathbb{R}_v^*$, we deduce that they define a smooth frame of the conormal bundle. Moreover, the canonical orientation of $\mathbb{R}_v^*$ is given by

$$(-1)^{w(\nu)} u_\nu \wedge \bigwedge_{j < \nu} u^{ij\nu} \wedge v^{ij\nu}.$$ 

In particular, we deduce that if $S \in {}^*W^-_v(\rho)$ is a unitary operator such that the vectors $e_i$ are eigenvectors of $S$ then the canonical orientation of the fiber of $S$ of the conormal bundle of ${}^*W^-_v(\rho)$ is given by

$$\theta^\nu \wedge \bigwedge_{j < \nu} \theta^{ij\nu} \wedge \varphi^{ij\nu}$$

where, for any $\hat{S} \in T_S U(\hat{E}^+)$ we have

$$\theta^\nu(\hat{S}) = (-iS^{-1}\hat{S} e_\nu, e_\nu),$$

$$\theta^{ij\nu}(\hat{S}) = \Re(-iS^{-1}\hat{S} e_\nu, e_j),$$

$$\varphi^{ij\nu}(\hat{S}) = \Im(-iS^{-1}\hat{S} e_\nu, e_j).$$

We deduce that if $\rho : I \to S^1 \setminus \{\pm 1\}$ is an injective map then the manifolds $^*W^-_i(\rho_i), \, i \in I$ intersect transversally.

Now observe that the manifolds $^*W^-_i(-\rho_i), \, i \in I$, and $W^-_i(1)$ intersect at a unique point $S_0 \in U(\hat{E}^+)$, where

$$S_0 e_j = \begin{cases} \rho_j e_j & j \in I, \\ e_j & j \in I^c. \end{cases}$$
The computations in Example 5.4 show that this intersection is transversal, and moreover, at $S_0$ we have

$$or_{i_k} \wedge \cdots \wedge or_{i_1} \wedge or_f = \epsilon(I, I') or(T_{S_0}^* U(\hat{E}^+))$$

The equality $k_I = 1$ now follows by invoking Proposition B.4.

**Remark 6.7.** Observe that on the collection of subsets of $\mathbb{I}_n^+$ we have two partial order relations.

$$K \prec M \iff JW_K^{-} \cap W_M^{-} \neq \emptyset \iff W_K^{-} \subset cl(W_M^{-}).$$

and

$$K \supset M \iff \alpha_K^{-} \bullet \alpha_M^{-} \neq 0.$$

Note that $K \supset M \implies K \prec M$, but the converse is not true. Following the analogy with the complex Grassmannian, we could refer to the partial order $\prec$ as the Bruhat order on the set of parts of $\mathbb{I}_n^+$. It would be very interesting to investigate the combinatorial properties of the poset $(\mathbb{I}_n^+, \prec)$. What is its Möbius function? Is this a Cohen-Macaulay poset? □

**APPENDIX A. TAME GEOMETRY**

Since the subject of tame geometry is not very familiar to many geometers we devote this section to a brief introduction to this topic. Unavoidably, we will have to omit many interesting details and contributions, but we refer to [5, 6, 7] for more systematic presentations.

For every set $X$ we will denote by $\mathcal{P}(X)$ the collection of all subsets of $X$.

An $\mathbb{R}$-structure$^4$ is a collection $\mathcal{S} = \{ S^n \}_{n \geq 1}$, $S^n \subset \mathcal{P}(\mathbb{R}^n)$, with the following properties.

**E$_1$:** $S^n$ contains all the real algebraic subvarieties of $\mathbb{R}^n$, i.e., the zero sets of finite collections of polynomial in $n$ real variables.

**E$_2$:** For every linear map $L: \mathbb{R}^m \rightarrow \mathbb{R}$, the half-plane $\{ \vec{x} \in \mathbb{R}^n; \ L(x) \geq 0 \}$ belongs to $S^n$.

**P$_1$:** For every $n \geq 1$, the family $S^n$ is closed under boolean operations, $\cup$, $\cap$, and complement.

**P$_2$:** If $A \in S^n$, and $B \in S^n$, then $A \times B \in S^{n+n}$.

**P$_3$:** If $A \in S^n$, and $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is an affine map, then $T(A) \in S^n$.

**Example A.1** (Semialgebraic sets). Denote by $S_{alg}$ the collection of real semialgebraic sets. Thus, $A \in S_{alg}$ if and only if $A$ is a finite union of sets, each of which is described by finitely many polynomial equalities and inequalities. The celebrated Tarski-Seidenberg theorem states that $S_{alg}$ is a structure.

Given a structure $\mathcal{S}$, then an $\mathcal{S}$-definable set is a set that belongs to one of the $S^n$-s. If $A, B$ are $\mathcal{S}$-definable, then a function $f: A \rightarrow B$ is called $\mathcal{S}$-definable if its graph

$$\Gamma_f := \{(a, b) \in A \times B; \ b = f(a)\}$$

is $\mathcal{S}$-definable. The reason these sets are called definable has to do with mathematical logic.

Given an $\mathbb{R}$-structure $\mathcal{S}$, and a collection $A = \{A_n\}_{n \geq 1}$, $A_n \subset \mathcal{P}(\mathbb{R}^n)$, we can form a new structure $\mathcal{S}(A)$, which is the smallest structure containing $\mathcal{S}$ and the sets in $A_n$. We say that $\mathcal{S}(A)$ is obtained from $\mathcal{S}$ by adjoining the collection $A$.

---

$^4$This is a highly condensed and special version of the traditional definition of structure. The model theoretic definition allows for ordered fields, other than $\mathbb{R}$, such as extensions of $\mathbb{R}$ by “infinitesimals”. This can come in handy even if one is interested only in the field $\mathbb{R}$. 

---
Definition A.2. An $\mathbb{R}$-structure is called \textit{o-minimal} (order minimal) or \textit{tame} if it satisfies the property

\[ T: \text{Any set } A \in S^1 \text{ is a finite union of open intervals } (a, b), \; -\infty \leq a < b \leq \infty, \text{ and}
\]
\[ \text{singletons } \{ r \}. \]

Example A.3. (a) The collection $S_{\text{alg}}$ of real semialgebraic sets is a tame structure.

(b) (Gabrielov-Hironaka-Hardt) A \textit{restricted} real analytic function is a function $f : \mathbb{R}^n \to \mathbb{R}$ with the property that there exists a real analytic function $\tilde{f}$ defined in an open neighborhood $U$ of the cube $C_n := [-1,1]^n$ such that

\[ f(x) = \begin{cases} 
\tilde{f}(x) & x \in C_n \\
0 & x \in \mathbb{R}^n \setminus C_n. 
\end{cases} \]

we denote by $S_{\text{an}}$ the structure obtained from $S_{\text{alg}}$ by adjoining the graphs of all the restricted real analytic functions. Then $S_{\text{an}}$ is a tame structure, and the $S_{\text{an}}$-definable sets are called \textit{globally subanalytic sets}.

(c) (Wilkie, van den Dries, Macintyre, Marker) The structure obtained by adjoining to $S_{\text{an}}$ the graph of the exponential function $\mathbb{R} \to \mathbb{R}$, $t \mapsto e^t$, is a tame structure.

(d) (Khovanski-Speissegger) There exists a tame structure $\hat{S}_{\text{an}}$ with the following properties

\[ \text{(d1)} \; S_{\text{an}} \subset \hat{S}_{\text{an}} \]
\[ \text{(d2)} \; \text{If } U \subset \mathbb{R}^n \text{ is open, connected and } \hat{S}_{\text{an}}\text{-definable, } F_1, \ldots, F_n : U \times \mathbb{R} \to \mathbb{R} \text{ are } \hat{S}_{\text{an}}\text{-definable and } C^1, \text{ and } f : U \to \mathbb{R} \text{ is a } C^1 \text{ function satisfying}
\]
\[ \frac{\partial f}{\partial x_i} = F_i(x, f(x)), \; \forall x \in \mathbb{R}, \; i = 1, \ldots, n, \tag{A.1} \]
\[ \text{then } f \text{ is } \hat{S}_{\text{an}}\text{-definable.} \]

\[ \text{(d3)} \; \text{The structure } \hat{S}_{\text{an}} \text{ is the minimal structure satisfying (d1) and (d2).} \]

The structure $\hat{S}_{\text{an}}$ is called the \textit{pfaffian closure} of $S_{\text{an}}$.

Observe that if $f : (a, b) \to \mathbb{R}$ is $C^1$, $\hat{S}_{\text{an}}$-definable, and $x_0 \in (a, b)$ then the antiderivative $F : (a, b) \to \mathbb{R}$

\[ F(x) = \int_{x_0}^x f(t)dt, \; x \in (a, b), \]

is also $\hat{S}_{\text{an}}$-definable. \hfill \square

The definable sets and function of a tame structure have rather remarkable \textit{tame} behavior which prohibits many pathologies. It is perhaps instructive to give an example of function which is not definable in any tame structure. For example, the function $x \mapsto \sin x$ is not definable in a tame structure because the intersection of its graph with the horizontal axis is the countable set $\pi \mathbb{Z}$ which violates the $o$-minimality condition \textbf{O}.

We will list below some of the nice properties of the sets and function definable in a tame structure $S$. Their proofs can be found in [5, 6].

- \textit{(Curve selection.)} If $A$ is an $S$-definable set, and $x \in \text{cl}(A) \setminus A$, then there exists an $S$ definable continuous map $\gamma : (0,1) \to A$ such that $x = \lim_{t \to 0} \gamma(t)$. 

\footnote{Our definition of pfaffian closure is more restrictive than the original one in [20, 30], but it suffices for the geometrical applications we have in mind.}
• (Closed graph theorem.) Suppose $X$ is a tame set and $f : X \to \mathbb{R}^n$ is a tame bounded function. Then $f$ is continuous if and only if its graph is closed in $X \times \mathbb{R}^n$.

• (Piecewise smoothness of tame functions.) Suppose $A$ is an $S$-definable set, $p$ is a positive integer, and $f : A \to \mathbb{R}$ is a definable function. Then $A$ can be partitioned into finitely many $S$-definable sets $S_1, \ldots, S_k$, such that each $S_i$ is a $C^p$-manifold, and each of the restrictions $f|_{S_i}$ is a $C^p$-function.

• (Triangulability.) For every compact definable set $A$, and any finite collection of definable subsets $\{S_1, \ldots, S_k\}$, there exists a compact simplicial complex $K$, and a definable homeomorphism

$$\Phi : |K| \to A$$

such that all the sets $\Phi^{-1}(S_i)$ are unions of relative interiors of faces of $K$.

• (Dimension.) The dimension of an $S$-definable set $A \subset \mathbb{R}^n$ is the supremum over all the nonnegative integers $d$ such that there exists a $C^1$ submanifold of $\mathbb{R}^n$ of dimension $d$ contained in $A$. Then $\dim A < \infty$, and

$$\dim(cL(A) \setminus A) < \dim A.$$  

• (Crofton formula, [4], [12, Thm. 2.10.15, 3.2.26].) Suppose $E$ is an Euclidean space, and denote by $\text{Graff}^k(E)$ the Grassmannian of affine subspaces of codimension $k$ in $E$. Fix an invariant measure $\mu$ on $\text{Graff}^k(E)$. Denote by $\mathcal{H}^k$ the $k$-dimensional Hausdorff measure. Then there exists a constant $C > 0$, depending only on $\mu$, such that for every compact, $k$-dimensional tame subset $S \subset E$ we have

$$\mathcal{H}^k(S) = C \int_{\text{Graff}^k(E)} \chi(L \cap S) d\mu(L).$$

• (Finite volume.) Any compact $k$-dimensional tame set has finite $k$-dimensional Hausdorff measure.

In the remainder of this section, by a tame set we will understand a $\hat{S}_{an}$-definable set.

**Definition A.4.** A tame flow on a tame set $X$ is a topological flow $\Phi : \mathbb{R} \times X \to X$, $(t, x) \mapsto \Phi_t(x)$, such that the map $\Phi$ is $\hat{S}_{an}$-definable. □

We list below a few properties of tame flows. For proofs we refer to [27].

**Proposition A.5.** Suppose $\Phi$ is a tame flow on a compact tame set $X$. Then the following hold.

(a) The flow $\Phi$ is a finite volume flow in the sense of [17].

(b) For every $x \in X$ the limits $\lim_{t \to \pm \infty} \Phi_t(x)$ exist and are stationary points of $\Phi$. We denote them by $\Phi_{\pm \infty}(x)$.

(c) The maps $x \mapsto \Phi_{\pm \infty}(x)$ are definable.

(d) For any stationary point $y$ of $\Phi$, the unstable variety $W^-_y = \Phi_{-\infty}^{-1}(y)$ is a definable subset of $X$. In particular, if $k = \dim W^-_y$, then $W^-_y$ has finite $k$-th dimensional Hausdorff measure. □

**Theorem A.6** (Theorem 4.3, [27]). Suppose $M$ is a compact, connected, real analytic, $m$-dimensional manifold, $f : M \to \mathbb{R}$ is a real analytic Morse function, and $g$ is a real analytic Morse function.
metric on \( M \) such that in the neighborhood of each critical point \( p \) there exists real analytic coordinates \( (x^i)_{1 \leq i \leq m} \) and nonzero real numbers \((\lambda_i)_{1 \leq i \leq m}\) such that,

\[
\nabla^g f = \sum_{i=1}^{m} \lambda_i \partial_{x^i}, \text{ near } p.
\]

Then the flow generated by the gradient \( \nabla^g f \) is a tame flow.

\[\square\]

**Appendix B. Subanalytic currents**

In this appendix we gather without proofs a few facts about the subanalytic currents introduced by R. Hardt in [16]. Our terminology concerning currents closely follows that of Federer [12] (see also the more accessible [24]). However, we changed some notations to better resemble notations used in algebraic topology.

Suppose \( X \) is a \( C^2 \), oriented Riemann manifold of dimension \( n \). We denote by \( \Omega_k(X) \) the space of \( k \)-dimensional currents in \( X \), i.e., the topological dual space of the space \( \Omega_{cpt}^k(X) \) of smooth, compactly supported \( k \)-forms on \( M \). We will denote by

\[
\langle \bullet, \bullet \rangle : \Omega_{cpt}^k(X) \times \Omega_k(X) \to \mathbb{R}
\]

the natural pairing. The boundary of a current \( T \in \Omega_k(X) \) is the \((k-1)\)-current defined via the Stokes formula

\[
\langle \alpha, \partial T \rangle := \langle d\alpha, T \rangle, \ \forall \alpha \in \Omega_{cpt}^{k-1}(X).
\]

For every \( \alpha \in \Omega^k(M) \), \( T \in \Omega_m(X) \), \( k \leq m \) define \( \alpha \cap T \in \Omega_{m-k}(X) \) by

\[
\langle \beta, \alpha \cap T \rangle = \langle \alpha \wedge \beta, T \rangle, \ \forall \beta \in \Omega_{cpt}^{n-m+k}(X).
\]

We have

\[
\langle \beta, \partial(\alpha \cap T) \rangle = \langle d\beta, (\alpha \cap T) \rangle = \langle \alpha \wedge d\beta, T \rangle
\]

\[
= (-1)^k \langle d(\alpha \wedge \beta) - d\alpha \wedge \beta, T \rangle = (-1)^k \langle \beta, \alpha \cap \partial T \rangle + (-1)^{k+1} \langle \beta, d\alpha \cap T \rangle
\]

which yields the homotopy formula

\[
\partial(\alpha \cap T) = (-1)^{\deg \alpha} \left( \alpha \cap \partial T - (d\alpha) \cap T \right). \tag{B.1}
\]

We say that a set \( S \subset \mathbb{R}^n \) is **locally subanalytic** if for any \( p \in D \) we can find an open ball \( B \) centered at \( p \) such that \( B \cap S \) is globally subanalytic.

**Remark B.1.** There is a rather subtle distinction between globally subanalytic and locally subanalytic sets. For example, the graph of the function \( y = \sin(x) \) is a locally subanalytic subset of \( \mathbb{R}^2 \), but it is not a globally subanalytic set. Note that a compact, locally subanalytic set is globally subanalytic.

If \( S \subset \mathbb{R}^n \) is an orientable, locally subanalytic, \( C^1 \) submanifold of \( \mathbb{R}^n \) of dimension \( k \), then any orientation \( \mathbf{or}_S \) on \( S \) determines a \( k \)-dimensional current \([S, \mathbf{or}_S]\) via the equality

\[
\langle \alpha, [S, \mathbf{or}_S] \rangle := \int_S \alpha, \ \forall \alpha \in \Omega^k_{cpt}(\mathbb{R}^n).
\]

The integral in the right-hand side is well defined because any bounded, \( k \)-dimensional globally subanalytic set has finite \( k \)-dimensional Hausdorff measure. For any open, locally subanalytic subset \( U \subset \mathbb{R}^n \) we denote by \([S, \mathbf{or}_S] \cap U\) the current \([S \cap U, \mathbf{or}_S]\).
We will refer to the elements of \( \mathcal{C}_k(X) \) as subanalytic (integral) \( k \)-chains in \( X \).

Given compact subanalytic sets \( A \subset X \subset \mathbb{R}^n \) we set
\[
\mathcal{Z}_k(X, A) = \{ T \in \mathcal{C}_k(\mathbb{R}^n); \ \text{supp} T \subset A, \ \text{supp} \partial T \subset A \},
\]
and
\[
\mathcal{B}_k(X, A) = \{ \partial T + S; \ T \in \mathcal{Z}_{k+1}(X, A), \ S \in \mathcal{Z}_k(A) \}.
\]
We set
\[
\mathcal{H}_k(X, A) := \mathcal{Z}_k(X, A)/\mathcal{B}_k(X, A).
\]
R. Hardt has proved in [15, 16] that the assignment
\[
(X, A) \mapsto \mathcal{H}_\bullet(X, A)
\]
satisfies the Eilenberg-Steenrod homology axioms with \( \mathbb{Z} \)-coefficients from which we conclude that \( \mathcal{H}_\bullet(X, A) \) is naturally isomorphic with the integral homology of the pair. In fact, we can be much more precise.

If \( X \) is a compact subanalytic we can form the chain complex
\[
\cdots \to \mathcal{C}_{k}(X) \xrightarrow{\partial} \mathcal{C}_{k-1}(X) \xrightarrow{\partial} \cdots
\]
whose homology is \( \mathcal{H}_\bullet(X) \).

If we choose a subanalytic triangulation \( \Phi : [K] \to X \), and we linearly orient the vertex set \( V = V(K) \), then for any \( k \)-simplex \( \sigma \subset K \) we get a subanalytic map from the standard affine \( k \)-simplex \( \Delta_k \) to \( X \)
\[
\Phi^\sigma : \Delta_k \to X.
\]
This defines a current \( [\sigma] = \Phi^\sigma_\ast([\Delta_k]) \in \mathcal{C}_k(X) \). By linearity we obtain a morphism from the group of simplicial chains \( C_\bullet(K) \) to \( \mathcal{C}_\bullet(X) \) which commutes with the respective boundary operators. In other words, we obtain a morphism of chain complexes
\[
C_\bullet(K) \to \mathcal{C}_\bullet(\Phi[K]).
\]
The arguments in [11, Chap.III] imply that this induces an isomorphism in homology.

To describe the intersection theory of subanalytic chains we need to recall a fundamental result of R. Hardt, [14, Theorem 4.3]. Suppose \( E_0, E_1 \) are two oriented real Euclidean spaces of dimensions \( n_0 \) and respectively \( n_1 \), \( f : E_0 \to E_1 \) is a real analytic map, and \( T \in \mathcal{C}_{n_0-c}(E_0) \) a subanalytic current of codimension \( c \). If \( y \) is a regular value of \( f \), then the fiber \( f^{-1}(y) \) is a submanifold equipped with a natural coorientation and thus defines a subanalytic current \( [f^{-1}(y)] \) in \( E_0 \) of codimension \( n_1 \), i.e., \( [f^{-1}(y)] \in \mathcal{C}_{d_0-d_1}(E_0) \). We would like to define the intersection of \( T \) and \( [f^{-1}(y)] \) as a subanalytic current \( T \bullet [f^{-1}(y)] \in \mathcal{C}_{n_0-c-n_1}(E_0) \). It turns out that this is possibly quite often, even in cases when \( y \) is not a regular value.

**Theorem B.2** (Slicing Theorem). Let \( E_0, E_1, T \) and \( f \) be as above, denote by \( dV_1 \) the Euclidean volume form on \( E_1 \), by \( \omega_{n_1} \), the volume of the unit ball in \( E_1 \), and set
\[
\mathcal{R}_f(T) := \{ y \in E_1; \ \text{codim}(\text{supp} T) \cap f^{-1}(y) \geq c+n_1, \ \text{codim}(\text{supp} \partial T) \cap f^{-1}(y) \geq c+n_1+1 \}.
\]
For every \( \varepsilon > 0 \) and \( y \in E_1 \) we define \( T \bullet f^{-1}(y) \in \Omega_{n_0-c-n_1}(E_0) \) by
\[
\langle \alpha, T \bullet f^{-1}(y) \rangle = \frac{1}{\omega_{n_1}^{\varepsilon n_1}} \langle (f^*dV_1) \wedge \alpha, T \cap \{ f^{-1}(B_\varepsilon(y)) \} \rangle, \ \forall \alpha \in \Omega_{cpt-c-n_1}(E_0).
\]
Then for every \( y \in \mathcal{R}_f(T) \), the currents \( T \cdot f^{-1}(y) \) converge weakly as \( \varepsilon \to 0 \) to a subanalytic current \( T \cdot f^{-1}(y) \in \mathcal{C}_{n_0-c-n_1}(E_0) \) called the \( f \)-slice of \( T \) over \( y \), i.e.,

\[
\langle \alpha, T \cdot f^{-1}(y) \rangle = \lim_{\varepsilon \to 0} \frac{1}{\omega_{n_1}^\varepsilon} \langle (f^\ast dV_1) \wedge \alpha, T \cap \{ f^{-1}(B_\varepsilon(y)) \} \rangle, \quad \forall \alpha \in \Omega^{n_0-c-n_1}_{cpt}(E_0).
\]

Moreover, the map

\[
\mathcal{R}_f \ni y \mapsto T \cdot f^{-1}(y) \in \mathcal{C}_{d_0-c-d_1}(\mathbb{R}^n)
\]

is continuous in the locally flat topology.

We will refer to the points \( y \in \mathcal{R}_f(T) \) as the \textit{quasi-regular values of} \( f \) \textit{relative to} \( T \).

Consider an oriented real analytic manifold \( M \) of dimension \( m \), and \( T_i \in \mathcal{C}_{m-c_i}(M) \), \( i = 0, 1 \). We would like to define an intersection current \( T_0 \cdot T_1 \in \mathcal{C}_{m-c_0-c_1}(M) \). This will require some very mild transversality conditions.

The slicing theorem describes this intersection current when \( T_1 \) is the integration current defined by the fiber of a real analytic map. We want to reduce the general situation to this case. We will achieve this in two steps:

- Reduction to the diagonal.
- Localization.

To understand the reduction to the diagonal let us observe that if \( T_0, T_1 \) were homology classes then their intersection \( T_0 \cdot T_1 \) satisfies the identity

\[
j_\ast(T_0 \cdot T_1) = (-1)^{c_0-c_1}(T_0 \times T_1) \cdot \Delta_M,
\]

where \( \Delta_M \) denotes the diagonal class in \( M \times M \), and \( j : M \to M \times M \) denotes the diagonal embedding.

We use this fact to define the intersection current in the special case when \( M \) is an open subset of \( \mathbb{R}^m \). In this case the diagonal \( \Delta_M \) is the fiber over 0 of the difference map

\[
\delta : M \times M \to \mathbb{R}^m, \quad \delta(m_0, m_1) = m_0 - m_1.
\]

If the currents \( T_0, T_1 \) are \textit{quasi-transversal}, i.e.,

\[
\text{codim}(\text{supp } T_0) \cap (\text{supp } T_1) \geq c_0 + c_1,
\]

\[
\text{codim}( (\text{supp } T_0 \cap \text{supp } \partial T_1) \cup (\text{supp } \partial T_0 \cap \text{supp } T_1) ) \geq c_0 + c_1 + 1,
\]

then \( 0 \in \mathbb{R}^m \) is a \( T_0 \times T_1 \)-quasiregular value of \( \delta \) so that the intersection

\[
(T_0 \times T_1) \cdot \delta^{-1}(0) = (T_0 \times T_1) \cdot \Delta_M
\]

is well defined.

The intersection current \( T_0 \cdot T_1 \) is then the unique current in \( M \) such that

\[
j_\ast(T_0 \cdot T_1) = (-1)^{c_0-c_1}(T_0 \times T_1) \cdot \delta^{-1}(0).
\]

If \( M \) is an arbitrary real analytic manifold and the subanalytic currents are quasi-transversal then we define \( T_0 \cdot T_1 \) to be the unique subanalytic current such that for any open subset \( U \) of \( M \) real analytically diffeomorphic to an open ball in \( \mathbb{R}^m \) we have

\[
(T_0 \cdot T_1) \cap U = (T_0 \cap U) \cdot (T_1 \cap U).
\]

One can prove that

\[
\partial(T_0 \cdot T_1) = (-1)^{c_0+c_1}(\partial T_0) \cdot \partial T_1 + T_0 \cdot (\partial T_1),
\]

whenever the various pairs of chains in the above formula are quasi-transversal.
One of the key results in [15, 16] states that this intersection of quasi-transversal chains induces a well defined intersection pairing

\[ i^* : T^* M |_S \to T^* S, \]

where \( i : S \hookrightarrow M \) is the inclusion map. A co-orientation of \( S \) is then an orientation of the conormal bundle. This induces an orientation on the cotangent bundle of \( S \) as follows.

- Fix \( s_0 \in S \), and a positively basis \( b_0 = \{ e^1, \ldots, e^n \} \) of the fiber of \( T^* S \) over \( s_0 \).
- Extend the basis \( b_0 \) to a positively oriented basis \( b = \{ e_0^1, \ldots, e_0^n \} \) of \( T^* _{s_0} M \).
- Orient \( T^* S \) using the ordered basis \( \{ i^*(e^{k+1}), \ldots, i^*(e^m) \} \).

We see that a pair \( (S, \text{or} \perp) \) consisting of a \( C^1 \), locally subanalytic submanifold \( S \hookrightarrow M \), and a co-orientation \( \text{or} \perp \) defines a subanalytic chain \( [S, \text{or} \perp] \in \mathcal{C}_k(M) \). Observe that

\[ \text{supp} \partial [S, \text{or} \perp] \subset cl(S) \setminus S. \]

Thus, if \( \dim(cl(S) \setminus S) < \dim S - 1 \) then \( \partial [S, \text{or} \perp] = 0 \).

**Definition B.3.** An *elementary cycle* of \( M \) is a co-oriented locally subanalytic submanifold \( (S, \text{or} \perp) \) such that \( \partial [S, \text{or} \perp] = 0 \).

We say that two elementary cycles \( (S_i, \text{or}_i \perp), i = 0, 1 \) intersect quasi-transversally if the following hold.

- The submanifolds \( S_0, S_1 \) intersect transversally.
- \( cl(S_0) \cap cl(S_1) = cl(S_0 \cap S_1) \).

\[ \square \]

Observe that if two elementary cycles \( (S_i, \text{or}_i \perp), i = 0, 1 \), intersect transversally, then the associated subanalytic chains \( [S_i, \text{or}_i \perp] \) are quasi-transversal. The conormal bundle of \( S_0 \cap S_1 \) is the direct sum of the restrictions of the conormal bundles of \( S_0 \) and \( S_1 \),

\[ T^* _{S_0 \cap S_1} M = (T^* _{S_0} M |_{S_0 \cap S_1}) \oplus (T^* _{S_1} M |_{S_0 \cap S_1}) \]

There is natural induced co-orientation \( \text{or}^{-1}_0 \wedge \text{or}^{-1}_1 \) on \( S_0 \cap S_1 \) given by the above ordered direct sum.

**Proposition B.4.** Suppose \( (S_i, \text{or}_i \perp), i = 0, 1 \) are elementary cycles intersecting quasi-transversally. Then

\[ [S_0, \text{or}_0^{-1}] \cdot [S_1, \text{or}_1^{-1}] = [S_0 \cap S_1, \text{or}_0^{-1} \wedge \text{or}_1^{-1}]. \]

\[ ^\dagger \]

I cannot help but remark that it is hard to find a book written during the past three decades which presents a complete proof of the Poincaré duality the “old fashion way”, via triangulations and their dual cell decompositions.
Proof. From (B.3) we deduce that
\[ \partial [S_0, or_0^\perp] \cdot [S_1, or_1^\perp] = 0. \]
On the other hand,
\[ \text{supp}[S_0, or_0^\perp] \cdot [S_1, or_1^\perp] \subset \text{cl}(S_0) \cap \text{cl}(S_1) = \text{cl}(S_0 \cap S_1). \]
Thus, to find the intersection current \([S_0, or_0^\perp] \cdot [S_1, or_1^\perp]\) it suffices to test it with differential forms \(\alpha \in \Omega^{c_0+c_1}(M)\) such that
\[ \text{supp} \alpha \cap \text{cl}(S_0) \cap \text{cl}(S_1) \subset S_0 \cap S_1. \]
Via local coordinates this reduces the problem to the special case when \(S_0, S_1\) are co-oriented subspaces of \(\mathbb{R}^n\) intersecting transversally in which case the result follows by direct computation from the definition. We leave the details to the reader. \(\square\)

References

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