MIXED HODGE STRUCTURES

LIVIU I. NICOLAESCU

1. Filtered vector spaces

Let $\Bbbk$ denote one of the fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. A decreasing (resp. increasing) *filtration* of a $\Bbbk$-vector space V is a collection of subspaces

$$\{ F^p = F^p(V) \subset V;\ p \in \mathbb{Z} \} \quad (\text{resp. } \{ F_p = F_p(V) \subset V;\ p \in \mathbb{Z} \}$$

such that $F^p(V) \supset F^{p+1}(V)$ (resp. $F_p(V) \subset F_{p+1}(V)$ for all $p \in \mathbb{Z}$. The filtration is called *finite* if there exist integers $m > n$ such that $F^m(V) = 0$ and $F^n(V) = V$ (resp. $F_m(V) = V$, $F_n(V) = 0$).

Observe that given a decreasing filtration $\{F^p(V)\}$ we can form an increasing filtration $F_p(V) := F^{-p}(V)$. In the remainder of this section we will work exclusively with decreasing filtration so we will drop the attribute decreasing. In this case for $v \in V$ we use the notation

$$F(v) \geq p \Longleftrightarrow v \in F^p(V).$$

To a filtered space $(V, F^\bullet)$ we can associate a graded vector space

$$\mathbf{Gr}^\bullet_F(V) = \bigoplus_{p \in \mathbb{Z}} \mathbf{Gr}^p_F(V), \quad \mathbf{Gr}^p({}_F(V) := F^p(V)/F^{p+1}(V).$$

Suppose we are given a filtration $F^\bullet(V)$ on a vector space V. This induces a filtration $F^\bullet(U)$ on a subspace X by the rule

$$F^p(X) := X \cap F^p(V),$$

and filtration induces a filtration on the quotient V/U

$$F^p(V/X) = F^p(V)/X \cap F^p(V) \cong (F^p(V) + X)/X.$$

For any integer n we defined the shifted filtration $F[n]^\bullet := F^{\bullet + n}$.

Given two vector subspaces $X \subset Y \subset V$ we can regard the quotient Y/X as a subspace of the quotient V/X. We have dual descriptions for Y/X: as a quotient of the subspace Y or as a subspace of the quotient A/X. We obtain in this way two filtrations on Y/X: a quotient filtration induced from the filtration of Y as a subspace of V and the filtration induced from the quotient filtration on V/X. These two filtrations coincide and we will refer to this unique filtration as the induced filtration on Y/X.

Suppose we are given two filtered vector spaces $F^\bullet(U)$ and $F^\bullet(V)$. A *morphism of filtered spaces* is a linear map $L : U \to V$ compatible with the filtrations, i.e.

$$L(F^\bullet(U)) \subset F^\bullet(V).$$

A morphism of filtered spaces $L : U \to V$ is called an *isomorphism* of filtered spaces if it is invertible and the inverse L^{-1} is also compatible with the filtrations.

Note that $\ker L$ and $\mathbf{Im}\, L$ are equipped with natural filtrations.

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Example 1.1. Suppose V is the vector space $\mathbb{k}^2$ equipped with the canonical basis $\{e_1, e_2\}$. Define

$$F^1(V) = \operatorname{span}_{\mathbb{k}}\{e_1, e_2\},\;\; F^2(V) = \operatorname{span}_{\mathbb{k}}\{e_2\},\;\; F^p(V) = 0,\;\; \forall p > 2.$$

and $L : V \to V$ is the linear nilpotent map $e_1 \mapsto e_2 \mapsto 0$. Note that L is compatible with the filtrations. We have

$$\mathbf{Im}\, L = \operatorname{span}\{e_2\},\;\; F^1(\mathbf{Im}\, L) = F^2(\mathbf{Im}\, L) = \mathbf{Im}\, L,\;\; F^p(\mathbf{Im}\, L) = (0),\;\; \forall p > 2.$$

$$\ker L = \operatorname{span}\{e_2\},\;\; F^1(\ker L) = F^2(\ker L) = \ker L,\;\; F^p(\ker L) = 0,\;\; \forall p > 2.$$

We have an induced filtration on $V/\ker L$

$$F^1(V/\ker L) = V/\ker L,\;\; F^p(V/\ker L) = 0,\;\; \forall p > 1.$$

We see that the natural map

$$V/\ker L \to \mathbf{Im}\, L$$

is *not an isomorphism of filtered spaces.*

□

Definition 1.2. A morphism of filtered spaces $(U, F^\bullet) \to (V, F^\bullet)$ is said to be *strict* if

$$F^p(V) \cap \mathbf{Im}\, L = L\big(\, F^p(U)\,\big),$$

i.e. for $v \in V$, $F(v) \geq p$, the equation $Lu = v$ has a solution $u \in U$, then it has a solution satisfying the additional condition $F(u) \geq p$.

□

The next result explains the role of the strictness condition in avoiding pathologies of the type illustrated in Example 1.1. Its proof is left to the reader.

Proposition 1.3. *Suppose $L : (U, F^\bullet) \to (V, F^\bullet)$ is a morphism of filtered spaces. Then the following are equivalent.*
(a) L is strict.
(b) The induced map $U/\ker L \to \mathbf{Im}\, L$ is an isomorphism *of filtered spaces.*

□

Observe that if $F^\bullet$ and $G^\bullet$ are two filtrations on the same vector space then we have natural isomorphisms

$$\mathbf{Gr}^m_F\, \mathbf{Gr}^n_G(V) \cong \mathbf{Gr}^n_G\, \mathbf{Gr}^m_F(V),\;\; \forall m, n \in \mathbb{Z}.$$

Definition 1.4. Let $n \in \mathbb{Z}$. Two finite filtrations $F^\bullet$ and $\hat{F}^\bullet$ on the $\mathbb{k}$-vector space V are said to be *n-complementary* if

$$\mathbf{Gr}^p_F\, \mathbf{Gr}^q_{\hat{F}}(V) = 0,\;\; \forall p + q \neq n.$$

□

Proposition 1.5. *Suppose F and $\hat{F}$ are two finite filtrations on the vector space V. We set $V^{p,q} := F^p(V) \cap \hat{F}^q(V)$. The following statements are equivalent.*

*(a) The finite filtrations $F^\bullet$ and $\hat{F}^\bullet$ are n-*complementary.

(b)

$$F^p(V) \cong \bigoplus_{j \geq p} V^{j, n-j},\;\; \hat{F}^q(V) \cong \bigoplus_{k \geq q} V^{n-k, k}.$$

(c)
$$F^p(V)\cap \hat{F}^q(V)=0,\ \ F^p(V)+\hat{F}^q(V)=V,\ \ \forall p+q=n+1.$$

Proof Clearly (b) $\Longrightarrow$ (a). Let us prove that (a) $\Longrightarrow$ (b).

Note that for $p+q\neq n$ we have
$$\mathbf{Gr}^p_F\,\mathbf{Gr}^q_{\hat{F}}(V)=0 \Longleftrightarrow F^p\,\mathbf{Gr}^q_{\hat{F}}(V)=F^{p+1}\,\mathbf{Gr}^q_{\hat{F}}(V)$$
$$\Longleftrightarrow F^p(V)\cap\hat{F}^q(V)/F^p(V)\cap\hat{F}^{q+1}(V)=F^{p+1}(V)\cap\hat{F}^q(V)/F^{p+1}(V)\cap\hat{F}^{q+1}(V)$$
$$\Longleftrightarrow F^p(V)\cap\hat{F}^q(V)=F^p(V)\cap\hat{F}^{q+1}(V)+F^{p+1}(V)\cap\hat{F}^q(V).$$
If $p'+q'\gg n$ then $F^{p'}(V)\cap\hat{F}^{q'}(V)=0$ and by descending induction over $p'+q'$ we deduce
$$F^p(V)\cap\hat{F}^q(V)=0,\ \ \forall p+q>n.$$
The equality
$$F^p\,\mathbf{Gr}^q_{\hat{F}}(V)=F^{p+1}\,\mathbf{Gr}^q_{\hat{F}}(V)$$
is also equivalent to
$$\big(\,F^p(V)\cap\hat{F}^q(V)+\hat{F}^{q+1}(V)\,\big)/\hat{F}^{q+1}(V)=\big(\,F^{p+1}(V)\cap\hat{F}^q(V)+\hat{F}^{q+1}(V)\,\big)/\hat{F}^{q+1}(V)$$
so that
$$F^p(V)\cap\hat{F}^q(V)+\hat{F}^{q+1}(V)=F^{p+1}(V)\cap\hat{F}^q(V)+\hat{F}^{q+1}(V),\ \ \forall p+q<n.$$
If we make the change in variables $p\to p+1$ we deduce that for every $p+q<n+1$ and every $p'<p$ we have
$$F^{p'}(V)\cap\hat{F}^q(V)+\hat{F}^{q+1}(V)=F^p(V)\cap\hat{F}^q(V)+\hat{F}^{q+1}(V)$$
If p' is sufficiently small we have $F^{p'}(V)=V$ and we deduce
$$\hat{F}^q(V)=F^p(V)\cap\hat{F}^q(V)+\hat{F}^{q+1}(V),\ \ \forall p+q<n+1.$$
Now, if we choose $p+q=n$ then $F^p(V)\cap\hat{F}^q(V)\cap\hat{F}^{q+1}(V)=0$ and we deduce
$$\hat{F}^q(V)=F^p(V)\cap\hat{F}^q(V)\oplus\hat{F}^{q+1}(V),\ \ \forall p+q<n+1.$$
Since for large q we have $\hat{F}^q(V)=0$ we deduce by descending induction over q that
$$\hat{F}^q(V)=\bigoplus_{k\geq q}F^{n-k}(V)\cap\hat{F}^k(V).$$
This finishes the proof of (a) $\Longrightarrow$ (b) since the roles of F and $\hat{F}$ are symmetric.

Clearly (b) $\Longrightarrow$ (c). To prove the opposite implication note that the equality
$$V=F^p(V)\oplus\hat{F}^{q+1}(V),\ \ p+q=n$$
implies
$$\hat{F}^q(V)=V^{p,q}\oplus\hat{F}^{q+1}(V).$$
We conclude again by descending induction on q.

□

2. Complementary triple filtrations

Suppose $F^\bullet$, $\hat{F}^\bullet$ and $W_\bullet$ are three finite filtrations on the $\Bbbk$-vector space V, $F^\bullet$ and $\hat{F}^\bullet$ decreasing, $W_\bullet$ increasing. They are called *complementary* if

$$\mathbf{Gr}_F^p \mathbf{Gr}_{\hat{F}}^q \mathbf{Gr}_n^W n(V) = 0, \;\; \forall p+q \neq n.$$

Equivalently, this means that for every integer m the filtrations induced by F and $\hat{F}$ on $\mathbf{Gr}_W^{-n}$ are n-complementary. For simplicity we write

$$V^p := F^p(V), \;\; \hat{V}^q := \hat{F}^q(V), \;\; V_n = W_n(V),$$

$$V_n^p = V^p \cap V_n, \;\; \hat{V}_n^q = \hat{V}^q \cap V_n,$$

$$I^{p,q} = V_{p+q}^p \cap \Big(\hat{V}_{p+q}^q + \hat{V}_{p+q-1}^q + \hat{V}_{p+q-2}^{q-1} + \cdots \Big) = V_{p+q}^p \cap \Big(\hat{V}_{p+q}^q + \hat{V}_{p+q-2}^{q-1} + \hat{V}_{p+q-3}^{q-2} + \cdots \Big),$$

$$\hat{I}^{p,q} = \hat{V}_{p+q}^q \cap \Big(V_{p+q}^p + V_{p+q-1}^p + V_{p+q-2}^{p-1} + \cdots \Big) = \hat{V}_{p+q}^q \cap \Big(V_{p+q}^p + V_{p+q-2}^{p-1} + V_{p+q-3}^{p-2} + \cdots \Big).$$

We have the following key structural result.

Proposition 2.1.

$$V_n = \bigoplus_{p+q\leq n} I^{p,q}, \tag{2.1}$$

$$V_n = \bigoplus_{p+q\leq n} \hat{I}^{p,q} \tag{2.2}$$

$$V^p = \bigoplus_{k\geq p} \bigoplus_q I^{k,q}, \tag{2.3}$$

$$\hat{V}^q = \bigoplus_{k\geq q} \bigoplus_p \hat{I}^{p,k}. \tag{2.4}$$

Proof The fact that the triple filtration is complementary is equivalent to the fact that for every p and every n we have we have

$$V_n/V_{n-1} = F^p(V_n/V_{n-1}) \oplus \hat{F}^{n+1-p}(V_n/V_{n-1})$$

$$= (V_n^p + V_{n-1})/V_{n-1} \oplus (\hat{V}^{n+1-p} + V_{n-1})/V_{n-1} \Longleftrightarrow$$

$$V_n = V_n^p + \hat{V}_n^{n+1-p} + V_{n-1}, \;\; (V_n^p + V_{n-1}) \cap (\hat{V}_n^{n+1-p} + V_{n-1}) = V_{n-1}. \tag{2.5}$$

For every n we also have the equality

$$V_n/V_{n-1} = \bigoplus_{p+q=n} \mathbf{Gr}_F^p \mathbf{Gr}_{\hat{F}}^q (V_n/V_n - 1) = \bigoplus_{p+q=n} F^p(V_n/V_{n-1}) \cap \hat{F}^q(V_n/V_{n-1}).$$

Let $\alpha \in \mathbf{Gr}_F^p \mathbf{Gr}_{\hat{F}}^q (V_n/V_{n-1})$, $p+q=n$. Then we can represent it by an element $v \in V_n^p$ and by an element in $u \in \hat{V}_n^q$. Moreover

$$v - u \in V_{n-1}.$$

Lemma 2.2. *For every $k \geq 0$ there exist vectors $u_k, v_k \in V_n$, unique modulo V_{n-k-2}, which represent α, such that*

$$v_k \in V_n^p, \;\; u_k \in \hat{V}_n^q + \hat{V}_{n-1}^q + \hat{V}_{n-2}^{q-1} + \cdots + \hat{V}_{n-k-1}^{q-k}, \;\; w_k = v_k - u_k \in V_{n-k-2}.$$

Proof We will use induction over $k \geq 0$. Let us first prove the $k = 0$ statement. Choose $v \in V_n^p$ and $u \in \hat{V}_n^q$ both representing α. Then

$$v = u + w, \quad w \in V_{n-1}.$$

Using the equality

$$V_{n-1} = V_{n-1}^p + \hat{V}_{n-1}^q + V_{n-2}$$

we can write

$$v - u = w = x_0 + y_0 + w_0, \quad x_0 \in V_{n-1}^p, \quad y_0 \in \hat{V}_{n-1}^q, \quad w_0 \in V_{n-2}.$$

Hence

$$\underbrace{(v - x_0)}_{v_0} = \underbrace{(u + y_0)}_{u_0} + w_0.$$

Suppose $\alpha = 0$ and $v_0 \in V_n^p$ and $u_0 \in \hat{V}_n^q + \hat{V}_{n-1}^q = \hat{V}_n^q$ represent α. Then

$$v_0 \in V_{n-1}^p, \quad u_0 \in \hat{V}_{n-1}^q.$$

In particular we deduce that

$$v_0, u_0 \in (V_{n-1}^p + V_{n-2}) \cap (\hat{V}_{n-1}^q + V_{n-2}) \overset{(2.5)}{\subset} V_{n-2}.$$

Suppose we found (u_k, v_k, w_k). Using the identity

$$V_{n-k-1} = V_{n-k-1}^p + \hat{V}_{n-k-1}^{q-k} + V_{n-k-2}$$

we can write

$$w_k = w_k' + w_{k+1}'' + w_{k+1}, \quad w_k' \in V_{n-k-1}^p, \quad w_k'' \in \hat{V}_{n-1-k}^{q-k}, \quad w_k \in V_{n-k-2}.$$

Then

$$\underbrace{(v_k - w_k')}_{v_{k+1}} = \underbrace{(u_k + w_k')}_{u_{k+1}} + w_{k+1}$$

Suppose we have a pair $v_k \in V_p^n$, $u_k \in V_n$ of elements representing $0 \in \mathbf{Gr}_F^p \mathbf{Gr}_{\hat{F}}^q (V_n/V_{n-1})$ and

$$v_k \in V_n^p, \quad u_k \in \hat{V}_n^q + \hat{V}_{n-1}^q + \hat{V}_{n-2}^{q-1} + \cdots + \hat{V}_{n-1-k}^{q-k}, \quad v_k - u_k \in V_{n-k-2}$$

Since v_k represents 0 we conclude by induction over k that

$$v_k \in V_{n-k-1}^p.$$

Let us write

$$u_k = \underbrace{\left(\hat{v}_n^q + \hat{v}_{n-2}^{q-1} + \cdots + \hat{v}_{n-k}^{q-k+1}\right)}_{:= u_{k-1}' \in \hat{V}^{q-k+1}} + \hat{v}_{n-k-1}^{q-k} = v_k - w_k \in V_{n-1-k}^p + V_{n-k-2}.$$

Using again the induction hypothesis we deduce

$$u_k' \in V_{n-k-1},$$

so that

$$u_k = u_k' + \hat{v}_{n-k-1}^{q-k} \in \hat{V}_{n-k-1}^{q-k}$$

Hence

$$u_k \in \hat{V}_{n-k+1}^{q-k} \cap (V_{n-k-1}^p + V_{n-k-2}) \overset{(2.5)}{\subset} V_{n-k-2}.$$

This completes the proof of Lemma 2.2.

$\square$

Lemma 2.2 shows that the projection $V_{p+q} \to V_{p+q}/V_{p+q-1}$ maps $I^{p,q} \subset V_{p+q}$ isomorphically onto $\mathbf{Gr}_F^p \mathbf{Gr}_{\hat{F}}^q V_{p+q}/V_{p+q-1}$. In particular $I^{p,q} \cap V_{p+q-1} = 0$. The equality (2.1) follows by an induction over n.

To prove (2.3) fix an integer p and $v \in V^p$. Denote by m the least integer m such that $v \in V_m$. We can then write

$$v = \sum_{p+q \leq m} v^{p,q}, \;\; v^{p,q} \in I^{p,q}.$$

Then the image of v in $\mathbf{Gr}_m V$ belongs to $F^p \mathbf{Gr}_m^W V$ so that $v^{i,m-i} = 0$, $\forall i < p$. Hence

$$v = \sum_{i \geq m} v^{i,m-i} + \underbrace{\sum_{p+q \leq m-1} v^{p,q}}_{\in V_{m-1}}.$$

The equality (2.3) is obtained by iterating the above procedure.

The equalities (2.2) and (2.4) are obtained in a similar fashion.

□

Definition 2.3. Suppose ${}_0V$, ${}_1V$ are $\Bbbk$-vectors spaces and $\big({}_iF^\bullet, {}_i\hat{F}^\bullet, {}_iW_\bullet\big)$ is a complementary triple filtration on ${}_iV$, $i = 0, 1$. A linear map $L : {}_0V \to {}_1V$ is said to be a *morphism of complementary triple filtrations of bidegree* $(r, s) \in \mathbb{Z} \times \mathbb{Z}$ if

$$L({}_0F^\bullet) \subset {}_1F^{\bullet+r}, \;\; L({}_0\hat{F}^\bullet) \subset {}_1\hat{F}^{\bullet+s},$$

$$L({}_0W_\bullet) \subset {}_1W_{\bullet+r+s}.$$

□

Proposition 2.4. *Suppose* $L : ({}_0V, {}_0F^\bullet, {}_0\hat{F}^\bullet, {}_0W^\bullet) \to ({}_1V, {}_1F^\bullet, {}_1\hat{F}^\bullet, {}_1W^\bullet)$ *is a morphism of bidegree* (r, s) *of complementary triple filtrations. Then the following hold.*

(a) L *is strict with respect to each of the filtrations, i.e.*

$${}_1F^{p+r} \cap L({}_0V) = L({}_0F^p), \;\; {}_1\hat{F}^{q+s} \cap L({}_0V) = L({}_0\hat{F}^q)$$

$${}_1W_{n+r+s} \cap L({}_0V) = L({}_0W_n).$$

(b) The triple filtrations induced on $\ker L$ *and* $\operatorname{coker} L$ *are complementary.*

Proof (a) For simplicity assume $(r, s) = (0, 0)$. Define ${}_jI^{p,q} \subset {}_jV$, $j = 0, 1$ as in Proposition 2.1. The strictness follows from the inclusions

$$L({}_0I^{p,q}) \subset {}_1I^{p,q}$$

(b) Let us first prove the statement about $\ker L$. Assume $(r, s) = (0, 0)$. We define ${}_0V^p$, ${}_0\hat{V}^q$ etc. as in the proof of Proposition 2.1. Set

$$K = \ker L, \;\; K^p = K \cap {}_0V^p, \;\; \hat{K}^q = K \cap {}_0\hat{V}^q, \;\; K_n^p = K \cap {}_0V_n^p \;\text{ etc.}$$

We need to prove that for every n, p we have

$$K_n = (K_n^p + K_{n-1}) + (\hat{K}_n^{n+1-p} + K_{n-1}), \;\; (K_n^p + K_{n-1}) \cap (\hat{K}_n^{n+1-p} + K_{n-1}) = K_{n-1}.$$

The equality $(K_n^p + K_{n-1}) \cap (\hat{K}_n^{n+1-p} + K_{n-1}) = K_{n-1}$ follows from the equality

$$({}_0V_n^p + {}_0V_{n-1}) \cap ({}_0\hat{V}_n^{n+1-p} + {}_0V_{n-1}) = {}_0V_{n-1}.$$

Every $x \in {}_0V$ has a unique decomposition

$$x = \sum_{p,q} x^{p,q}, \quad x^{p,q} \in {}_0I^{p,q}.$$

Then

$$x \in {}_0V_n \Longleftrightarrow x^{p,q} = 0, \quad \forall p+q > n,$$

and

$$x \in K \Longleftrightarrow L(x^{p,q}) = 0, \quad forall p,q.$$

Let $x \in K_n$. Then

$$x = \sum_{p+q \le n} x^{p,q}, \quad Lx^{p,q} = \sum_{p+q=n} x^{p,q} + \underbrace{\sum_{p+q<n} x^{p,q}}_{:=w}$$

$$= \underbrace{\sum_{k \ge p} x^{k,m-k}}_{:=x'} + \underbrace{\sum_{k \le p-1} x^{k,n-k}}_{:=x''} + w$$

Clearly $x' \in K_n^p$, $w \in K_{n-1}$. Now observe that for $k \le p-1$ we have

$$x^{k,m-k} \in {}_0V_n^k \cap ({}_0\hat{V}_n^{n-k} + \hat{V}_{n-1}^{n-k} + {}_0\hat{V}_{n-k-2}^{n-k-1} + \cdots) \in {}_0\hat{V}^{n+1-p} + {}_0V_{n-1}$$

Hence $x'' \in ({}_0\hat{V}_n^{n+1-p} + V_{n-1}) \cap K$. Using the decomposition of ${}_0$ as a direct sum of ${}_0\hat{I}^{p,q}$ we conclude that

$$x'' \in (\hat{K}_n^{n+1-p} + K_{n-1}).$$

Now let $C = \operatorname{coker} L$, $U = \mathbf{Im}\, L$. Since L is strict with respect to all the filtrations it induces an isomorphism of filtered spaces

$$(C, F^\bullet, \hat{F}^\bullet, W_\bullet) \to (U, F^\bullet, \hat{F}^\bullet, W_\bullet)$$

so it suffices to show that the induced filtrations on U are complementary. This is achieved via an argument similar to the one used in the case of $\ker L$.

□

Remark 2.5. Suppose $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of $\Bbbk$-vector spaces such that V' and V'' are equipped with complementary triple filtration. Then one can equip V is a natural complementary triple filtration such that the linear maps in the exact sequence are morphisms of bidegree $(0,0)$ of triple filtrations. □

3. Mixed Hodge structures

A *mixed Hodge structure* is a triplet $H = (H_\mathbb{Z}, F^\bullet, W_\bullet)$ where $H_\mathbb{Z}$ is a free Abelian group, $W_\bullet$ is an increasing filtration of $H_\mathbb{Q} = H_\mathbb{Z} \otimes \mathbb{Q}$, $F^\bullet$ is a decreasing filtration on $H_\mathbb{C}$ such that if $\bar{F}^\bullet$ denotes the filtration obtained from $F^\bullet$ by complex conjugation then $(F^\bullet, \bar{F}^\bullet, W_\bullet)$ is a complementary triple filtration on $H_\mathbb{C}$. The mixed Hodge structure is called *pure of weight* m if $W_m = H_\mathbb{Q}$ and $W_n = 0$ for all $n < m$.

A morphism of bidegree (r,r) between two mixed Hodge structure $L : {}_0H \to {}_1H$ is a morphism of Abelian groups $L : {}_0H_\mathbb{Z} \to {}_1H_\mathbb{Z}$ such that the induced map ${}_0H_\mathbb{C} \to {}_1H_\mathbb{C}$ is a morphism of bidegree (r,r) of complementary triple filtrations. A morphism of mixed Hodge structures will be by definition a morphism of bidegree $(0,0)$.

Example 3.1. Suppose X is a compact Kähler manifold of complex dimension m. We set $H^k(X,\mathbb{Z})_0 = H^k(X,\mathbb{Z})/\mathrm{Tors}$. Then $H^k(X,\mathbb{Z})_0$ is equipped with a pure Hodge structure of weight k where

$$F^pH^k(X,\mathbb{C}) = \bigoplus_{j=p}^{k} H^{j,k-j}(X).$$

Suppose now $f : X \to Y$ is a holomorphic map between two compact Káhler manifolds of complex dimensions m and respectively n. We set $r = n - m$. Then the pullback

$$f^* : H^k(Y,\mathbb{Z})_0 \to H^k(X,\mathbb{Z})_0$$

induces a morphism of pure Hodge structures of bidegree $(0,0)$. Denote by $(-,-)_X$ the Poincaré duality pairing on X

$$(-,-)_X : H^\bullet(X,\mathbb{Z})_0 \times H^{2m-\bullet}(X,\mathbb{Z})_0 \to \mathbb{Z},\ \ (\alpha,\beta)_X = \langle \alpha\cup\beta, [X]\rangle,$$

where $\langle -,-\rangle$ denotes the Kronecker pairing between homology and cohomology. We have a Gysin map

$$f_! : H^k(X,\mathbb{Z})_0 \to H^{k+2r}(Y,\mathbb{Z})_0$$

defined by

$$(f_!\alpha,\beta)_Y = (\alpha, f^*\beta)_X,\ \ \forall \alpha\in H^k(X,\mathbb{Z})_0,\ \ \beta\in H^{2m-k}(Y,\mathbb{Z})_) = H^{2n-(k+2r)}(Y,\mathbb{Z})_0.$$

We claim that $f_!$ is a morphism of pure Hodge structures of bidegree (r,r). To see this assume

$$\alpha\in H^{p,q}(X),\ \ p+q=k,\ \ \beta\in H^{p',q'}(Y),\ \ p'+q' = 2m-k.$$

Then

$$(f_!\alpha,\beta)_Y = \int_Y (f_!\alpha)\cup\beta = \int_X \alpha\cup f^*\beta.$$

Note that

$$\int_X \alpha\cup f^*\beta \neq 0 \Longrightarrow (p,q)+(p',q') = (m,m)$$

We deduce in a similar fashion that $f_!\alpha$ should belong to a single component $H^{p'',q''}(Y)$ such that

$$(n,n) = (p'',q'') + (p',q')$$

We conclude that

$$(p'',q'') = (p,q)+(r,r).$$

□

We deduce from the general results in the previous section that if $L : H \to H'$ is a morphism of mixed Hodge structures of bi-degree (r,r) then both $\ker L$ and $\operatorname{coker} L$ with the induced filtrations define mixed Hodge structures.

The results proved so far shows that the category of mixed Hodge structures is Abelian.

Example 3.2. Let us describe another method of constructing a (pure) Hodge structure on the cohomology of a compact Kähler manifold X. Start with the Dolbeault resolution of the constant sheaf $\underline{\mathbb{C}}$

$$0\to\underline{\mathbb{C}}\to\Omega^0_X \xrightarrow{\partial} \Omega^1_X \xrightarrow{\partial} \cdots$$

For each p, the sheaf Ω^p_X of holomorphic $(p,0)$-forms admits a soft resolution

$$\Omega^{p,0}\to(\mathcal{A}^{p;\bullet}_X,\bar\partial)$$

and thus we obtain a double complex $(\mathcal{A}^{\bullet,\bullet}, \partial, \bar{\partial})$ whose associated total complex

$$(\mathcal{A}^{\bullet}_X, d),\ \ \mathcal{A}^m = \bigoplus_{p+q=m} \mathcal{A}^{p,q},\ \ d = \partial + \bar{\partial}$$

is the DeRham complex and it is quasi-isomorphic to $\underline{\mathbb{C}}$. Its hypercohomology is isomorphic to the cohomology of $\underline{\mathbb{C}}$. The complex $K^{\bullet} = \Gamma(X, \mathcal{A}^{\bullet}_X)$ is equipped with a natural decreasing filtration

$$F^p K^m_= \bigoplus_{k\geq p} \Gamma(X, \mathcal{A}^{k,m-k}_X),$$

which defines a spectral sequence converging to $H^{\bullet}(X, \mathbb{C})$. The E_1 term is

$$E_1^{p,q} = H^q(X, \Omega^p_X) = H^{p,q}(X)$$

The differential d_1 on E_1 is ∂ and according to classical Hodge theory every element in $H^q(X, \Omega^p_X)$ can be represented by a (p,q)-form α satisfying

$$\Delta_{\partial}\alpha = \Delta_{\bar{\partial}}\alpha = 0 \Longrightarrow \partial\alpha = 0.$$

Hence the differential d_1 is trivial and thus the spectral sequence stops at the E_1 term. We deduce

$$H^m(X, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X),\ \ H^{p,q} = \overline{H^{q,p}}$$

and thus $H^m(X, \mathbb{C})$ is equipped with a pure Hodge structure of weight m. □

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-4618.
E-mail address: nicolaescu.1@nd.edu