1. Prove that if \( S \) is a subset of 10 numbers from \( \{1, \ldots, 100\} \) then there are two nonempty subsets \( A, B \) with \( A \cap B = \emptyset \) such that the sum of the numbers in \( A \) is equal to the sum of the numbers in \( B \).

2. Prove that if \( S \) is any subset of 55 numbers chosen from \( \{1, 2, \ldots, 100\} \) then there are two elements of \( S \) differing by exactly 9.

3. If \( a, b \) are integers, then \( \gcd(a, b) \) means the greatest common divisor of \( a \) and \( b \). Prove that for any integers \( a \) and \( b \) there are integers \( s \) and \( t \) such that \( sa + tb = \gcd(a, b) \).

4. Let \( a, b, c, d \) be positive integers such that \( ad - be = 1 \). Show that the fraction \( (a + b)/(c + d) \) is in lowest terms.

5. Prove that some positive multiple of 21 has 241 as its last 3 digits.

6. Prove that for any set of \( n \) integers, there is a subset of them whose sum is divisible by \( n \).

7. Prove that if \( 2n + 1 \) and \( 3n + 1 \) are both perfect squares, then \( n \) is divisible by 40.

8. Prove that there are no integers \( x \) and \( y \) for which \( x^2 + 3xy - 2y^2 = 122 \).

9. Do there exist 1,000,000 consecutive integers such that each one contains a repeated prime factor?

10. Prove that for any integer \( n \) there is a multiple of \( n \) whose base 10 representation contains only 1’s and 0’s.

11. The Fibonacci sequence is the sequence \( F_1, F_2, F_3, \ldots \) where \( F_1 = F_2 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 3 \). Prove that if \( k \) is a divisor of some Fibonacci number then it is a divisor of infinitely many Fibonacci numbers.

12. How many 0’s does 100! end with?

13. Determine, as a function of the integer \( n \), the number of ordered pairs \((x, y)\) such that \( 1/x + 1/y = 1/n \).

14. Prove that not both integers \( 2^n - 1 \) and \( 2^n + 1 \) can be prime.

15. Find all pairs \((m, n)\) of positive integers such that \(|3^n - 2^m| = 1\)

16. (a) Find all pairs \((m, n)\) of positive integers such that \(|3^n - 2^m| = 1\)
   (b) Find all 4-tuples \((p, q, m, n)\) where \( p, q \) are prime and \( m, n \) are positive integers that satisfy \(|p^n - q^m| = 1\).

17. Prove that there are infinitely many natural numbers \( a \) with the property that \( n^4 + a \) is not prime for any natural number \( n \).