Epilogue

*A whole is that which has a beginning, a middle and an end.*

Aristotle *Poetics*

We can now take a step back and enjoy the view. Think of the places we’ve been and of the surprises we’ve uncovered! I hope this long and winding road we took has strengthened the idea that Mathematics is One Huge Question, albeit that it appears in different shapes, colours and flavors in the minds of the eccentric group of people we call mathematicians.

I think the sights you’ve seen are so breathtaking that even the clumsiest guide cannot ruin the pleasure of the mathematical tourist. I also have some good news for the thrill seeker. There is a lot more out there, and hereafter, you are on your own. Still, I cannot help but mention some of the trails that have been opened and are now advancing into the Unknown. (This is obviously a biased selection.)

We’ve learned that counting the monopoles on a 4-manifold can often be an extremely rewarding endeavour. The example of Kähler surfaces suggests that individual monopoles are carriers of interesting geometric information. As explained in [64], even the knowledge that monopoles exist can lead to nontrivial conclusions. What is then the true nature of a monopole? The experience with the Seiberg-Witten invariants strongly suggests that the answers to this vaguely stated question will have a strong geometric flavour.

In dimension 4, the remarkable efforts of C.H. Taubes [128, 129, 130, 131], have produced incredibly detailed answers and raised more refined questions.

One subject we have not mentioned in this book but which naturally arises when dealing with more sophisticated gluing problems is that of the
gauge theoretic invariants of 3-manifolds. There is a large collection of such invariants (see [24, 40, 41, 64, 71, 72, 81, 82, 84] and the references therein) which, with few exceptions, are very difficult to compute. The nature of 3-dimensional monopoles is a very intriguing subject and there have been some advances [64, 66, 93, 101], suggesting that these monopoles reflect many shades of the underlying geometry. These studies also seem to indicate that 3-dimensional contact topology ought to have an important role in elucidating the nature of monopoles.

One important event unfolding as we are writing these lines (December 1999) is the incredible tour de force of Paul Feehan and Thomas Leness, who in a long sequence of very difficult papers ([30]) are establishing the original prediction of Seiberg and Witten that the “old” Yang-Mills theory is topologically equivalent to the new Seiberg-Witten theory. While on this subject, we have to mention the equally impressive work in progress of Andrei Teleman [132] directed towards the same goal but adopting a different tactic. Both these efforts are loosely based on an idea of Pidstrigach and Tyurin.

Gauge theory has told us that the low-dimensional world can be quite exotic and unruly. At this point there is no one generally accepted suggestion about how one could classify the smooth 4-manifolds, but there is a growing body of counterexamples to most common sense guesses. Certain trends have developed, and there is a growing acceptance of the fact that geometry ought to play a role in any classification scheme. In any case, the world is ready for the next Big Idea.
Bibliography


http://www.math.harvard.edu/~kronheim/


http://www.math.harvard.edu/~kronheim/


[71] Y. Lim: Seiberg-Witten invariants for 3-manifolds in the case $b_1 = 0$ or 1, preprint, 1998.


http://abel.math.harvard.edu/HTML/Individuals/Curtis_T_McMullen.html


http://www.maths.warwick.ac.uk/gt/


http://math.polytechnique.fr/cmat/vidussi/


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