SHEAVES

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ABSTRACT. We describe the Abelian categories of sheaves over topological spaces, natural functors between them, and several remarkable resolutions of sheaves.

CONTENTS

1. PRESHEAVES

A *presheaf* on X is a correspondence $\mathcal{S}$ which associates to each nonempty open subset $U \subset X$ a nonempty set $\mathcal{S}(U)$ and to every inclusion $U \hookrightarrow V$ a map

$$r_V^U = r(\mathcal{S})_V^U : \mathcal{S}(V) \to \mathcal{S}(U)$$

such that $r_U^U = \mathbb{1}_{\mathcal{S}(U)}$ and for any $U \hookrightarrow V \hookrightarrow W$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{S}(W) \arrow[r, "r_W^V"] \arrow[dr, "r_W^U"'] & \mathcal{S}(V) \arrow[d, "r_V^U"] \\ & \mathcal{S}(U) \end{tikzcd} \iff r_W^U = r_V^U \circ r_W^V.$$

r_V^U is called the restriction map. For simplicity we will often denote it by $|_U$. The set $\mathcal{S}_0(U)$ is referred to as the *space of sections of* $\mathcal{S}_0$ *over* U and it is often denoted by $\Gamma(U, \mathcal{S}_0)$.

A *morphism of pre-sheaves* $\mathcal{S}_0$ and $\mathcal{S}_1$ is a collection of maps

$$\phi = \big\{ \phi_U : \mathcal{S}_0(U) \to \mathcal{S}_1(U);\ \ U \text{ open subset of } X \big\}$$

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such that for any inclusion $U\subset V$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_0(V) & \xrightarrow{\phi_V} & \mathcal{S}_1(V)\\ {\scriptstyle r^U_V}\downarrow & & \downarrow{\scriptstyle r^U_V}\\ \mathcal{S}_0(U) & \xrightarrow[\phi_U]{} & \mathcal{S}_1(U)\end{array} \Longleftrightarrow r^U_V\phi_V=\phi_U r^U_V. \tag{1.1}$$

We obtain in this fashion a category $\mathbf{pSh}(X)$, of presheaves on X. If $\mathcal{C}$ is a subcategory of **Set** then a pre-sheaf of $\mathcal{C}$-objects is a presheaf such that every set $\mathcal{S}(U)$ is an object in $\mathcal{C}$ and the restriction maps are morphisms in $\mathcal{C}$. A morphism of presheaves of $\mathcal{C}$-objects is a morphisms of presheaves of sets ϕ such that for any open set U the map ϕ_U is a morphism in $\mathcal{C}$. We denote by $\mathbf{pSh}_{\mathcal{C}}(X)$ the resulting subcategory of presheaves of $\mathcal{C}$-objects over X.

If R is a commutative ring with 1, we denote by ${}_R\mathbf{Mod}$ the category of R-modules and we set $\mathbf{pSh}_R(X)=\mathbf{pSh}_{{}_R\mathbf{Mod}}$. We set

$$\Gamma(\emptyset,\mathcal{F})=0,\;\;\forall\mathcal{F}\in\mathbf{pSh}_R(X).$$

Note that for any two presheaves of R-modules $\mathcal{S}_0,\mathcal{S}_1$ the space of morphisms $\mathrm{Hom}_{\mathbf{pSh}_R}(\mathcal{S}_0,\mathcal{S}_1)$ is naturally an R-module. We denote by $\underline{R}={}_X\underline{R}$ the constant presheaf on X determined by the requirement that $\Gamma(U,\underline{R})$ consists of the continuous maps $U\to R$, where R is equipped with the *discrete* topology. Equivalently, the sections in $\Gamma(U,{}_X\underline{R})$ are the locally constant maps $U\to R$.

For any morphism $\phi\in\mathrm{Hom}_{\mathbf{pSh}_R}(\mathcal{S}_0,\mathcal{S}_1)$, any inclusion of open sets $U\hookrightarrow V$ and any $s\in\mathcal{S}_0(V)$ we deduce from the commutative diagram (1.1) that

$$\phi_V(s)=0\Longrightarrow\phi_U(s\,|_U)=0.$$

Thus the restriction maps of $\mathcal{S}_0$ induce morphisms

$$|_U\colon\ker\big(\,\mathcal{S}_0(V)\xrightarrow{\phi_V}\mathcal{S}_1(V)\,\big)\to\ker\big(\,\mathcal{S}_0(U)\xrightarrow{\phi_U}\mathcal{S}_1(U)\,\big).$$

Hence the correspondence

$$U\longmapsto(\ker\phi)(U):=\ker\big(\,\mathcal{S}_0(U)\xrightarrow{\phi_U}\mathcal{S}_1(U)\,\big)\subset\mathcal{S}_0(U)$$

defines a presheaf on X called the *kernel* of ϕ and denoted by $\ker\phi$.

Arguing in a similar fashion we deduce that the correspondence

$$U\longmapsto(\mathrm{im}\,\phi)(U):=\mathrm{im}\big(\,\mathcal{S}_0(Y)\xrightarrow{\phi_U}\mathcal{S}_1(U)\,\big)\subset\mathcal{S}_1(U).$$

is a presheaf called the *image* of ϕ and denoted by $\mathrm{im}\,\phi$. Similarly we obtain presheaves

$$\mathrm{coker}\,\phi=\mathcal{S}_1/\,\mathrm{im}\,\phi,\;\;\Gamma(U,\mathrm{coker}\,\phi)=\mathcal{S}_1(U)/\phi_U(\,\mathcal{S}_0(U)\,),$$

$$\mathrm{coim}\,\phi=\mathcal{S}_1/\ker\phi,\;\;\Gamma(U,\mathrm{coim}\,\phi)=\mathcal{S}_0(U)/\ker\phi_U.$$

Note that we have a natural isomorphism of presheaves

$$\mathrm{coim}\,\phi\to\mathrm{im}\,\phi.$$

This shows that $\mathbf{pSh}_R$ is naturally an Abelian category[1].

[1]Recall that a category is called *additive* if $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ has a natural structure of Abelian group, there exists a zero object (i.e. simultaneously initial and terminal) and there exist finite direct sums. A category is called *Abelian* if it is additive, every morphism ϕ has a kernel and cokernel, and the canonical map $\mathrm{coim}\,\phi\to\mathrm{im}\,\phi$ is an isomorphism. For more details we refer to [1, 4, 5].

Example 1.1. (a) Suppose X is a smooth manifold. For every open set $U \subset X$ we denote by $\mathcal{A}^p_X(U)$ (or $\mathcal{A}^p(U)$ if there is no confusion concerning X) the space of degree p, complex valued, smooth, differential forms on U. In local coordinates $(x^1, \cdots, x^n)$ on U such a differential form has the description

$$\omega = \sum_\alpha \omega_\alpha dx^\alpha, \tag{1.2}$$

where the coefficients ω_α are smooth, complex valued functions, the summation is carried over all the ordered multi-indices $\alpha = (\alpha_1, \cdots, \alpha_p)$ and

$$dx^\alpha = dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}.$$

The exterior differential induces a natural morphism of presheaves

$$d : \mathcal{A}^p_X \to \mathcal{A}^{p+1}_X.$$

If ω is as in (1.2) then

$$d\omega = \sum_{k=1}^n dx^k \wedge \Big(\sum_\alpha \frac{\partial \omega_\alpha}{\partial x^k} dx^\alpha \Big).$$

We obtain in this fashion a complex of presheaves

$$(\mathcal{A}^\bullet_X, d) : \quad \mathcal{A}^0_X \xrightarrow{d} \cdots \to \mathcal{A}^p_X \xrightarrow{d} \mathcal{A}^{p+1}_X \to \cdots$$

called the *DeRham complex.* As the case $X = S^1$ shows this complex may not be acyclic. Note that we can identify the constant presheaf $\underline{\mathbb{C}}$ with the kernel of the morphism $d : \mathcal{A}^0_X \to \mathcal{A}^1_X$.

(b) Suppose X is a *complex* manifold. For every open set $U \subset X$ we denote by $\mathcal{A}^{p,q}_X(U)$ the vector space of complex valued, smooth forms of bi-degree (p, q) on U. If we use holomorphic coordinates $(z^1, \cdots, z^n)$ on U then $\omega \in \mathcal{A}^{p,q}_X(U)$ has the local description

$$\omega = \sum_{\alpha,\beta} \omega_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta, \tag{1.3}$$

where the summation is carried over all ordered multi-indices $\alpha = (\alpha_1, \cdots, \alpha_p)$, $\beta = (\beta_1, \cdots, \beta_q)$ and

$$dz^\alpha = dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p}, \;\; d\bar{z}^\beta = d\bar{z}^{\alpha_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}.$$

We have natural morphisms of sheaves

$$\partial : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1,q}_X, \;\; \bar{\partial} : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p,q+1}_X.$$

If ω is as in (1.3) then

$$\partial\omega = \sum_x dz^k \wedge \Big(\sum_{\alpha,\beta} \frac{\partial \omega_{\alpha\beta}}{\partial z^k} dz^\alpha \wedge d\bar{z}^\beta \Big)$$

$$\bar{\partial}\omega = \sum_x d\bar{z}^k \wedge \Big(\sum_{\alpha,\beta} \frac{\partial \omega_{\alpha\beta}}{\partial \bar{z}^k} dz^\alpha \wedge d\bar{z}^\beta \Big)$$

Note that

$$\mathcal{A}^m_X = \bigoplus_{p+q=m} \mathcal{A}^{p,q}_X \;\text{ and }\; d = \partial + \bar{\partial}.$$

From the equalities $d^2 = \partial^2 = \bar{\partial}^2 = 0$ we deduce that we can form a *double complex*

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r, "\partial"] & \mathcal{A}^{p,q+1} \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"] & \mathcal{A}^{p+1,q+1} \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"] & \cdots \\
\cdots \arrow[r, "\partial"] & \mathcal{A}^{p,q} \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"] & \mathcal{A}^{p+1,q} \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"] & \cdots \\
 & \vdots \arrow[u, "\bar{\partial}"] & \vdots \arrow[u, "\bar{\partial}"] &
\end{tikzcd} \qquad (\mathcal{A}^{\bullet,\bullet}, \partial, \bar{\partial}))$$

The rows and the columns of this complex are simple complexes called the *Dolbeault complexes*. The total complex of this double complex is precisely the DeRham complex.

We denote by $\Omega_X^{p,q}$ the kernel of $\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$. $\Omega_X^{p,q}$ is called the (pre)sheaf of holomorphic (p,q)-forms on X. Note that the sections of $\Omega^{0,0}$ are precisely the (locally defined) holomorphic functions on X. This (pre)sheaf is usually denoted by $\mathcal{O}_X$ and it is called the *structural (pre)sheaf* of the complex manifold X. Note that the sequence

$$\Omega_X^{0,0} \xrightarrow{\partial} \Omega_X^{1,0} \to \cdots \to \Omega_X^{p,0} \xrightarrow{\partial} \Omega_X^{p+1,0} \to \cdots$$

is a complex of (pre)sheaves called the *holomorphic DeRham complex*. Note that on Ω_X^m we have $d = \partial$.

□

2. Sheaves

Suppose that $\mathcal{S} \in \mathbf{pSh}_R(X)$. Given any family $\mathcal{U} = \big(U_i\big)_{i\in I}$ of open subsets of X we set

$$|\mathcal{U}| := \bigcup_{i\in I} U_i,\;\; U_{ij} = U_i \cap U_j,\;\; \Gamma(\mathcal{U}, \mathcal{S}) = \Big\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{S}(U_i);\;\; \forall i,j :\;\; s_i\,|_{U_{ij}} = s_j\,|_{U_{ij}} \Big\}.$$

Note that we have a natural map

$$|_{\mathcal{U}} \colon \Gamma(|\mathcal{U}|, \mathcal{S}) \to \Gamma(\mathcal{U}, \mathcal{S}),\;\; s \longmapsto \big(s\,|_{U_i} \big)_{i\in U_i}.$$

In general this map is neither injective, nor surjective. A presheaf $\mathcal{S}$ is called a *sheaf* if for any family $\mathcal{U}$ of open subsets of X the map $|_{\mathcal{U}}$ is a bijection.

Example 2.1. Suppose X is a set equipped with the discrete topology (i.e. every subset is open). For example X could be the set of integers equipped with the topology induced from the Euclidean topology of the real axis. We consider the presheaves $\mathcal{F}$ and $\hat{\mathcal{F}}$ of $\mathbb{C}$-vector spaces on X defined by

$$\Gamma(S, \mathcal{F}) = \bigoplus_{s\in S} \mathbb{C},\;\; \Gamma(S, \hat{\mathcal{F}}) = \prod_{s\in S} \mathbb{C}.$$

The restriction maps are given by the natural projections. $\hat{\mathcal{F}}$ is a sheaf, while $\mathcal{F}$ is not.

□

To any presheaf $\mathcal{S} \in \mathbf{pSh}_R(X)$ we can associate a sheaf $\mathcal{F}^+$ in the following canonical way. First, for $x \in X$ define the stalk of $\mathcal{S}$ in x to be the R-module

$$\mathcal{S}_x = \varinjlim_{U \ni x} \Gamma(U, \mathcal{S}),$$

By definition, for any open neighborhood U of x we have a map

$$[-]_x : \Gamma(U, \mathcal{S}) \to \mathcal{S}_x, \;\; s \mapsto [s]_x.$$

$[s]_x$ is called the *germ* of s at x. We now define $\Gamma(U, \mathcal{S}^+)$ to be the subset of $\prod_{u \in U} \mathcal{S}_u$ consisting of families of germs $(s_u)_{u \in U}$ such that every $u \in U$ there exist an open neighborhood V in U and a section $\tilde{s} \in \Gamma(V, \mathcal{S})$ such that

$$[\tilde{s}]_v = s_v, \;\; \forall v \in V.$$

It is not hard to check that $\mathcal{F}^+$ is indeed a sheaf. It is also called the *sheafification*[2] of $\mathcal{F}$. Note that by construction

$$\Gamma(U, \mathcal{F}) \subset \Gamma(U, \mathcal{F}^+).$$

In Example 2.1 $\hat{\mathcal{F}}$ is the sheafification of $\mathcal{F}$.

We can organize the collection of sheaves as a category $\mathbf{Sh}_R(X)$. The correspondence $\mathcal{F} \to \mathcal{F}^+$ is a functor $\mathbf{pSh}_R(X) \to \mathbf{Sh}_R(X)$. Note that $\mathbf{Sh}_R(X)$ is by definition a full subcategory of $\mathbf{pSh}_R(X)$ and thus we have an inclusion functor $i : \mathbf{Sh}_R(X) \to \mathbf{pSh}_R(X)$.

For any $\mathcal{F} \in \mathbf{pSh}_R(X)$, $\mathcal{G} \in \mathbf{Sh}_R(X)$ and $\phi \in \mathrm{Hom}_{\mathbf{pSh}_R}(\mathcal{F}, \mathcal{G})$ there exists an induced morphism $\phi^+ \in \mathrm{Hom}_{\mathbf{Sh}_R}(\mathcal{F}^+, \mathcal{G})$. One can show that resulting map

$$\mathrm{Hom}_{\mathbf{pSh}_R}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathbf{Sh}_R}(\mathcal{F}^+, \mathcal{G})$$

is an isomorphism of R-module. We can write this as

$$\mathrm{Hom}_{\mathbf{Sh}_R}(\mathcal{F}^+, \mathcal{G}) \cong_n \mathrm{Hom}_{\mathbf{pSh}_R}\big(\mathcal{F}, i(\mathcal{G})\big), \;\; \forall \mathcal{F} \in \mathbf{pSh}_R(X), \;\; \mathcal{G} \in \mathbf{Sh}_R(X),$$

where $\cong_n$ means the isomorphism in natural with respect to the "variables" $\mathcal{F}$ and $\mathcal{G}$. In modern terms it says that $^+$ is the *left adjoint* of i.

Given $\mathcal{F}, \mathcal{G} \in \mathbf{Sh}_R$ and $\phi \in \mathrm{Hom}_{\mathbf{Sh}_R}(\mathcal{F}, \mathcal{G})$ we can define $\ker \phi, \operatorname{coker} \phi, \operatorname{im} \phi, \operatorname{coim} \phi \in \mathbf{Sh}_R(X)$ by sheafifying the corresponding constructions for presheaves. One can show that $\mathbf{Sh}_R(X)$ is an Abelian category.

Any $\phi \in \mathrm{Hom}_{\mathbf{Sh}_R}(\mathcal{F}, \mathcal{G})$ induces morphisms between stalks

$$\phi_x \in \mathrm{Hom}_R(\mathcal{F}_x, \mathcal{G}_x), \;\; \forall x \in X.$$

A short sequence $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ of sheaves and morphism of sheaves is exact in $\mathbf{Sh}_R(X)$ if and only if for every $x \in X$ the sequence

$$\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

is exact in ${}_R\mathbf{Mod}$. More explicitly this means that for every $x \in X$, any neighborhood U of x and any section $\mathcal{G} \in \Gamma(U, \mathcal{G})$ such that $\psi_U(g) = 0$ there exists a neighborhood $V \subset U$ containing x and a section $f \in \Gamma(V, \mathcal{F})$ such that

$$\phi_V(f) = g\,|_V \,.$$

If $\mathcal{R}$ is a sheaf of rings on X then we can define in an obvious fashion the category $\mathbf{Sh}_{\mathcal{R}}(X)$ of sheaves of $\mathcal{R}$-modules.

[2]We refer to [1, II.4.12] or [7] for a more conceptual construction of the sheafification.

For any sheaf $\mathcal{S} \in \mathbf{Sh}_R(X)$, any open set U and any $s \in \Gamma(U, \mathcal{S})$ we define the support of s to be

$$\operatorname{supp} s = \{u \in U;\ \ [s]_0 \neq 0\}.$$

Note that $\operatorname{supp} s$ is closed in U with respect to the subspace topology on U.

Example 2.2. All the presheaves defined in Example 1.1 are sheaves of complex vector spaces. If We denote by $\mathcal{A} = \mathcal{A}^0$ the sheaf smooth complex valued functions on X then $\mathcal{A}$ is naturally a sheaf of $\mathbb{C}$ algebras and $\mathcal{A}^m$ is a sheaf of $\mathcal{A}$-modules. □

Theorem 2.3. *(a)* **(Smooth Poincaré Lemma)** *Suppose X is a smooth manifold. Then the smooth DeRham sequence*

$$\mathcal{A}_X^0 \to \cdots \to \mathcal{A}_X^m \xrightarrow{d} \mathcal{A}_X^{m+1} \to \cdots$$

is exact $\mathbf{Sh}_{\mathbb{C}}(X)$.
(b) **(Dolbeault Lemma)** *Suppose X is a complex manifold. Then the Dolbeault sequences*

$$\mathcal{A}_X^{\bullet,0} \to \cdots \to \mathcal{A}_X^{\bullet,q} \xrightarrow{\bar\partial} \mathcal{A}_X^{\bullet,q+1} \to \cdots$$

and

$$\mathcal{A}_X^{0,\bullet} \to \cdots \to \mathcal{A}_X^{p,\bullet} \xrightarrow{\partial} \mathcal{A}_X^{p+1,\bullet} \to \cdots$$

are exact.
(c) **(Holomorphic Poincaré Lemma)** *Suppose X is a complex manifold. Then the holomorphic DeRham complex*

$$\Omega_X^0 \xrightarrow{\partial} \cdots \to \Omega_X^m \xrightarrow{\partial} \Omega_X^{m+1} \to \cdots$$

is exact.

□

For a proof of this theorem we refer to [3].

Let us rephrase the above facts in the language of homological algebra. Suppose $\mathcal{A}$ is an Abelian category and

$$(A^\bullet, d) := \cdots A^n \xrightarrow{d} A^{n+1} \to \cdots,\ \ n \in \mathbb{Z}$$

is a co-chain complex. We denote its homology objects by $H^\bullet(A, d_A)$. We denote by $\mathbf{Com}(\mathcal{A})$ the additive category of complexes of $\mathcal{A}$ objects. Then a morphism of cochain complexes $\phi : (A^\bullet, d_A) \to (B^\bullet, d_B)$ is called a *quasi-isomorphism*, or a *resolution* if it induces an isomorphism in homology. Every object $A \in \mathcal{A}$ defines a complex $[A]$

$$[A]^n = \begin{cases} A & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \neq 0. \end{cases}$$

A complex $(C^\bullet, d)$ is called a *resolution* of the object A if it is quasi-isomorphic to $[A]$.

Suppose X is a smooth manifold. The smooth Poincaré lemma is equivalent to the statement that the natural inclusion of $\underline{{}_X\mathbb{C}}$ in the DeRham complex $(\mathcal{A}_X^\bullet, d)$ is a resolution. If X is a smooth manifold then the Dolbeault lemma shows that the inclusion

$$\Omega_X^p \to (\mathcal{A}_X^{p,\bullet}, \bar\partial)$$

is a resolution. Similarly, the holomorphic Poincaré lemma shows that the inclusion

$$\underline{{}_X\mathbb{C}} \hookrightarrow (\Omega_X^\bullet, \partial)$$

is a resolution.

These resolutions of the constant sheaf use special features of the topology of X, such as a smooth structure or a complex structure. We describe below a construction of a resolution of an arbitrary sheaf on an arbitrary space.

Example 2.4 (The Godement resolution)**.** Let X be a topological space. For every sheaf $\mathcal{F} \in \mathbf{Sh}_R(X)$ we denote by $D(\mathcal{F})$ the sheaf defined by

$$\Gamma(U, D(\mathcal{F})) = \prod_{u \in U} \mathcal{F}_u.$$

In other words, the sections of $D(f)$ are possibly discontinuous sections of $\mathcal{F}$. Note that we have an inclusion $\mathcal{F} \to D(\mathcal{F})$. We can now describe inductively the Godement resolution $(\mathcal{F}^\bullet, d)$. We set $\mathcal{F}^n = 0$ for $n < 0$, $\mathcal{F}^0 = D(f)$. Suppose we have already constructed $d_{n-1} : \mathcal{F}^{n-1} \to \mathcal{F}^n$. Then we set

$$d_n : \mathcal{F}^n \to \operatorname{coker} d_{n-1} \hookrightarrow D(\operatorname{coker} d_{n-1}) =: \mathcal{F}^{n+1}.$$

□

3. Direct and inverse images of sheaves

Suppose $f : X \to Y$ is a continuous map. We want to construct covariant functors

$$f_* : \mathbf{Sh}_R(X) \to \mathbf{Sh}_R(Y), \;\; f^{-1} : \mathbf{Sh}_R(Y) \to \mathbf{Sh}_R(X),$$

called respectively the *direct* and *inverse image* functors. Given $\mathcal{F} \in \mathbf{Sh}_R(X)$ we observe that the presheaf $f_*\mathcal{F}$ given by

$$Y \supset V \longmapsto \Gamma(V, f_*\mathcal{F}) = \Gamma(f^{-1}V, \mathcal{F})$$

is already a sheaf. Next, for $g \in \mathbf{Sh}_R(Y)$ and any open set $U \subset X$ we set

$$\Gamma(U, f^{-1}\mathcal{G}) = \varinjlim_{V \supset f(U)} \Gamma(V, \mathcal{G}),$$

where in the above limit V ranges over the open neighborhoods of $f(U)$ in Y. Then we define $f^{-1}\mathcal{G}$ to be the sheaf associated to the presheaf

$$X \supset U \longmapsto \Gamma(U, f^{-1}\mathcal{G}).$$

Note that

$$(f^{-1}\mathcal{G})_x \cong \mathcal{G}_x, \;\; \Gamma(V, f_*f^{-1}\mathcal{G}) = \Gamma(f^{-1}V, f^{-1}\mathcal{G}).$$

From the definition

$$\Gamma(f^{-1}V, f^{-1}\mathcal{G}) = \varinjlim_{W \supset f(f^{-1}V)} \Gamma(W, \mathcal{G})$$

we deduce the existence of a natural morphism

$$\Gamma(V, \mathcal{G}) \to \Gamma(f^{-1}V, f^{-1}\mathcal{G}) = \Gamma(V, f_*f^{-1}\mathcal{G})$$

This defines a natural morphism of sheaves $\alpha : \mathcal{G} \to f_*f^{-1}\mathcal{G}$ called the *adjunction morphism.* Given a sheaf $\mathcal{F} \in \mathbf{Sh}_R(X)$ and a morphism $\phi \in \operatorname{Hom}_{\mathbf{Sh}_R(X)}(f^{-1}\mathcal{G}, f)$ we obtain a morphism $\phi^\vee \in \operatorname{Hom}_{\mathbf{Sh}_R(Y)}(\mathcal{G}, f_*\mathcal{F})$ as the composition

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\alpha"] \arrow[dr, "\phi^\vee"'] & f_*f^{-1}\mathcal{G} \arrow[d, "f_*(\phi)"] \\
 & f_*\mathcal{F}
\end{tikzcd}$$

Proposition 3.1. *The correspondence $\phi \longmapsto \phi^{\vee}$ defines a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{Sh}_R(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \cong_n \mathrm{Hom}_{\mathbf{Sh}_R(Y)}(\mathcal{G}, f_*\mathcal{F}), \ \ \forall \mathcal{F}, \mathcal{G}. \tag{3.1}$$

□

For a proof see [4, Thm. 4.8]. The above proposition states that f_* is the right adjoint of f^{-1}. The next result follows easily from the definition.

Proposition 3.2. *(a) f^{-1} is an exact functor while f_* is only left exact, i.e. for every exact sequence $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ in $\mathbf{Sh}_R(X)$ the sequence*

$$0 \to f_*\mathcal{S}' \to f_*\mathcal{S} \to f_*\mathcal{S}''$$

is exact in $\mathbf{Sh}_R(Y)$.
(b) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous maps then we have equivalences of functors

$$(g \circ f)_* \cong g_* \circ f_*, \ \ (g \circ f)^{-1} \cong f^{-1} \circ g^{-1}.$$

□

Example 3.3. (a) Suppose $c = c_X \to \{pt\}$ is the collapse map. Note that $\mathbf{Sh}_R(pt) = {}_R\mathbf{Mod}$ so the direct image gives a functor $c_* : \mathbf{Sh}_R(X) \to \mathcal{R}\,\mathbf{Mod}$. This is precisely the *global sections functor*

$$\mathbf{Sh}_R(X) \ni \mathcal{F} \longmapsto \Gamma(X, \mathcal{F}).$$

Note that

$$c^{-1}R \cong {}_X\underline{R}, \ \ c_*c^{-1}R = c_*(\underline{R}) = \Gamma(X, \underline{R}) \in {}_R\mathbf{Mod}\,.$$

The adjunction morphism $\alpha : R \to \Gamma(X, \underline{R})$ associates to each $r \in R$ the constant section of $\underline{R}$ equal to r.
(b) Consider the unit circle $S^1 = \{z \in \mathbb{C};\ \ |z| = 1\}$ and the double cover

$$p : S^1 \to S^1, \ \ z \mapsto z^2.$$

We would like to understand $p_*\underline{\mathbb{C}}$. Note that for any connected subset $U \subsetneq S^1$ the preimage $p^{-1}(U)$ consists of two connected components $V_{\pm}$ and thus

$$\Gamma(U, p_*\underline{\mathbb{C}}) \cong \Gamma(p^{-1}(U), \underline{\mathbb{C}}) \cong \Gamma(V_-, \mathbb{C}) \oplus \Gamma(V_+, \mathbb{C}) \cong \mathbb{C}^2.$$

On the other hand $p^{-1}(S^1) = S^1$ and we deduce

$$\Gamma(S^1, p_*\underline{\mathbb{C}}) = \Gamma(S^1, \underline{\mathbb{C}}) \cong \mathbb{C}.$$

The sheaf $p_*\underline{\mathbb{C}}$ is only locally constant. Its stalks are all isomorphic to $\mathbb{C}^2$ and for this reason we say it is locally constant sheaf of complex vector spaces of rank 2. For the classification of locally constant sheaves of vector spaces we refer to [4, §IV.9].
(c) Suppose $x \in X$ and j_x denotes the inclusion $\{x\} \hookrightarrow X$. Then $j_x^{-1}\mathcal{F} = \mathcal{F}_x$, $\forall \mathcal{F} \in \mathbf{Sh}_R(X)$. More generally, suppose $S \subset X$ is a *closed* subset and $j : S \hookrightarrow X$ denotes the natural embedding. For any sheaf $\mathcal{F}$ on X we set

$$\mathcal{F}_S := j_*j^{-1}\mathcal{F} \in \mathbf{Sh}_R(X).$$

Note that for every open set $V \subset X$ we have

$$\Gamma(V, \mathcal{F}_S) = \Gamma(V \cap S, j^{-1}\mathcal{F}) = \varinjlim_{W \supset V \cap S} \Gamma(W, \mathcal{F}).$$

Hence we conclude that

$$(\mathcal{F}_S)_x \cong \begin{cases} \mathcal{F}_x & \text{if} \ \ x \in S \\ 0 & \text{if} \ \ x \notin S \end{cases}.$$

The adjunction morphism $\alpha : \mathcal{F} \to j_*j^{-1}\mathcal{F}$ is called in this case the *attachment map*. We will denoted it by $\mathbf{a}^S$. The induced morphism $\mathbf{a}^S_x : \mathcal{F}_x \to (\mathcal{F}_S)_x$ satisfies

$$\mathbf{a}^S_x = \begin{cases} \mathbb{1}_{\mathcal{F}_x} & \text{if} \quad x \in S \\ 0 & \text{if} \quad x \notin S \end{cases}. \tag{3.2}$$

We set $U = X \setminus S$, $\mathcal{F}_U := \ker \alpha \in \mathbf{Sh}_R(X)$. Note that

$$(\mathcal{F}_U)_x \cong \begin{cases} \mathcal{F}_x & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \notin U \end{cases}.$$

We have a fundamental exact sequence of sheaves

$$0 \to \mathcal{F}_{X\setminus S} \to \mathcal{F} \to \mathcal{F}_S \to 0. \tag{3.3}$$

Note that if $Z \subset X$ is either closed, or open, and $\mathcal{F}$ is a sheaf on X we have defined a sheaf $\mathcal{F}_Z$ whose stalks are trivial outside Z, and coincide with the stalks of $\mathcal{F}$ on Z. The correspondence $\mathcal{F} \to \mathcal{F}_Z$ is an *exact* covariant functor $\mathbf{Sh}_R(X) \to \mathbf{Sh}_R(X)$.

In general if X is a topological space and Z a subset of X then the following are equivalent.
(i) The subset Z is locally closed, i.e. it is the intersection of a closed set in X with an open subset of X.
(ii) For every sheaf $\mathcal{F}$ on X there exists a sheaf $\mathcal{F}_Z$ on X such that

$$(\mathcal{F}_Z)_x = \begin{cases} 0 & \text{if} \quad x \in X \setminus Z \\ \mathcal{F}_x & \text{if} \quad x \in Z. \end{cases}$$

For a proof of this fact we refer to [2, II.2.9]. For example, the set

$$\{z \in \mathbb{C};\ \ \mathbf{Im}\, z > 0\} \cup \{0\}$$

is not locally closed.
(d) Suppose $U \subset X$ is an *open* set and $i : U \hookrightarrow X$ denotes the natural inclusion. For any sheaf $\mathcal{F} \in \mathbf{Sh}_R(X)$ the inverse image $i^{-1}\mathcal{F}$ is the restriction of $\mathcal{F}$ to U and it is often denoted by $\mathcal{F}\mid_U$. We set

$$\Gamma_U\mathcal{F} := i_*i^{-1}\mathcal{F} \in \mathbf{Sh}_R(X).$$

For any open set $V \subset X$ we have

$$\Gamma\big(V, \Gamma_U\mathcal{F}\big) = \Gamma(V \cap U, \mathcal{F}).$$

This implies

$$(\Gamma_U\mathcal{F})_x \cong \begin{cases} \mathcal{F}_x & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \in X \setminus \bar{U} \end{cases}$$

If $x \in \bar{U} \setminus U$ then it is possible that $(\Gamma_U\mathcal{F})_x \neq 0$. Take for example

$$X = \mathbb{R}^2,\ \ U = \{(x, y);\ xy \neq 0\},\ \ x = (0, 0),\ \ \mathcal{F} = \mathcal{A}_{\mathbb{R}^2}. \tag{3.4}$$

The function $\frac{1}{xy}$ is a section of $\Gamma_U\mathcal{A}$ on $\mathbb{R}^2$ but it is not equal to a section of $\mathcal{A}$ in any neighborhood of zero.

If U' is an open subset of U then we have a natural morphism $\Gamma_U\mathcal{F} \to \Gamma_{U'}\mathcal{F}$ defined by

$$\Gamma(V, \Gamma_U\mathcal{F}) = \Gamma(U \cap V, \mathcal{F}) \ni f \mapsto f\mid_{U'\cap V} \in \Gamma(U' \cap V, \mathcal{F}) = \Gamma(V, \Gamma'_U\mathcal{F}).$$

We will denote this morphism as a restriction map, $|^U_{U'}$ or $|_{U'}$.

We set $S = X \setminus U$ and we denote by $\Gamma_S\mathcal{F} \in \mathbf{Sh}_S(X)$ the kernel of the adjunction morphism $\alpha : \mathcal{F} \to i_*i^{-1}\mathcal{F}$. The example (3.4) shows that α *need not be onto.* Since the functor i_* is left exact and i^{-1} is exact we deduce that for any open set U the functor $\Gamma_U : \mathbf{Sh}_R(X) \to \mathbf{Sh}_R(X)$ is left exact.

The sheaf $\Gamma_S\mathcal{F}$ can be alternatively characterized by

$$\Gamma(U,\Gamma_S\mathcal{F}) = \big\{ f\in\Gamma(U,\mathcal{F});\ \ \operatorname{supp} f\subset S\,\big\}.$$

We leave it as an exercise to prove that the functor $\Gamma_S : \mathbf{Sh}_R(X)\to\mathbf{Sh}_R(X)$ is left exact.

The functors $\Gamma_S,\Gamma_U : \mathbf{Sh}_R(X)\to\mathbf{Sh}_R(X)$ defined above play an important role in local cohomology.

□

4. The Čech resolutions

Given a family $(\mathcal{F}_j)_{j\in J}$ of sheaves of R-modules we denote by $\prod_{j\in J}\mathcal{F}_j$ the sheaf defined by

$$\Gamma(U,\prod_{j\in J}\mathcal{F}_j) := \prod_{j\in J}\Gamma(U,\mathcal{F}_J).$$

We want to point out that the natural morphism $\Big(\prod_{j\in J}\mathcal{F}_j\Big)_x\to\prod_{j\in J}(\mathcal{F}_j)_x$ *need not be an isomorphism.*

Suppose $\mathcal{U}=(U_i)_{i\in I}$ is an open cover of the topological space X. We fix a *total order*[3] on I. For every subset $\alpha\in I$ we define

$$U_\alpha = \bigcap_{i\in\alpha}U_i.$$

The *nerve* of the cover is the collection of all finite subset $\alpha\subset I$ such that $U_\alpha\neq\emptyset$. For $\alpha=\{\alpha_0,\cdots,\alpha_p\}\in N(\mathcal{U})$ we set $\dim\alpha := p$. We will refer to such a set as a *p-simplex.* The 0-simplices are called *vertices.* We will write $\alpha=(\alpha_0,\cdots,\alpha_p)$ when we want to indicate that the vertices of α are arranged in increasing order. We set

$$N_p(\mathcal{U}) = \big\{\alpha\in N(\mathcal{U});\ \ \dim\alpha = p\big\}.$$

The nerve is a simplicial complex in the sense that if $\beta\in N(\mathcal{U})$ and $\alpha\subset\beta$ then $\alpha\in N(\mathcal{U})$. For every $\alpha=(\alpha_0,\cdots,\alpha_p)\in N_p(\mathcal{U})$ and $0\le i\le p$ we define its *i-th face*, or the face opposite to the vertex α_i to be the $(p-1)$-simplex

$$F_i\alpha = \alpha\setminus\{\alpha_i\}\in N_{p-1}(\mathcal{U}).$$

Now define[4]

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) = \prod_{\alpha\in N_p(\mathcal{U})}\Gamma_{U_\alpha}\mathcal{F}\in\mathbf{Sh}_R(X).$$

and

$$\delta : \mathcal{C}^p(\mathcal{U},\mathcal{F})\to\mathcal{C}^{p+1}(\mathcal{U},\mathcal{F}),\quad \delta = \prod_{\beta\in N_{p+1}(\mathcal{U})}\delta_\beta$$

where

$$\delta_\beta = \sum_{i=0}^{p+1}(-1)^i\delta_\beta^{F_i\beta},$$

---

[3]We can even choose a well ordering on a set, that is a total order such that every nonempty subset has minimal element. The existence of such orderings on an arbitrary subset is equivalent to the axiom of choice.

[4]Our definition differs from that in [8, vol. I, §4.1.3]. That definition is incorrect since it uses direct sums as opposed to direct products.

and $\delta_\beta^{F_i\beta}$ is the composition

$$\begin{tikzcd}
\prod_{\alpha\in N_p(\mathcal{U})} \Gamma_{U_\alpha}\mathcal{F} \arrow[r, "\pi_{F_i\beta}"] \arrow[dr, dashed, "\delta_\beta^{F_i\beta}"'] & \Gamma_{U_{F_i\beta}}\mathcal{F} \arrow[d, "|^{U_\beta}_{U_{F_i\beta}}"] \\
& \Gamma_{U_\beta}\mathcal{F}
\end{tikzcd}.$$

If V is an open subset of X then

$$\Gamma\big(V, \mathcal{C}^p(\mathcal{U},\mathcal{F})\big) = \prod_{\alpha\in N_p(\mathcal{U})} \Gamma(V\cap U_\alpha,\mathcal{F}).$$

We can represent $s \in \Gamma\big(V, \mathcal{C}^p(\mathcal{U},\mathcal{F})\big)$ as a collection

$$\Big\{\, s(\alpha);\ \ \alpha \in N_p(\mathcal{U}),\ \ s(\alpha)\in\Gamma(V\cap U_\alpha,\mathcal{F})\,\Big\}.$$

Then we can represent δs as a collection $\{\,\delta s\,(\beta);\ \ \beta\in N_{p+1}(\mathcal{U})\,\}$ where

$$\delta s\,(\beta) = \sum_{i=0}^{p+1}(-1)^i s\,(F_i\beta)\,|_{U_\beta} = \sum_{i=0}^{p+1}(-1)^i s\,(\beta_0,\cdots,\hat{\beta}_i,\cdots,\beta_{p+1})\,|_{U\cap U_\beta}\in\Gamma(V\cap U_\beta,\mathcal{F}).$$

Using the total ordering $<$ on I we define for any vertex $v\in\beta$ the integers

$$\nu(v,\beta) = \#\big\{\, j\in\beta;\ \ j<v\,\big\},\ \ \epsilon(v,\beta) = (-1)^{\nu(v,\beta)}.$$

The integer $\nu(v,\beta)$ describes the position of the vertex v in the set β arranged in increasing order. The definition of δ can be rewritten

$$\delta s\,(\beta) = \sum_{v\in\beta}\epsilon(v,\beta)\cdot s\,(F_v\beta)\,|_{U_\beta}\ .$$

Note that we have a natural morphism of complexes

$$r : [\mathcal{F}] \to \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}),\ \ \Gamma(V,\mathcal{F}) \mapsto \big(f\,|_{V\cap U_i})_{i\in I}\in\mathcal{C}^0(\mathcal{U},\mathcal{F}).$$

Example 4.1. Suppose $\mathcal{U} = \{U_0, U_1\}$. Then

$$\mathcal{C}^0(\mathcal{U},\mathcal{F}) = \Gamma_{U_0}\mathcal{F}\times\Gamma_{U_1}\mathcal{F},\ \ \mathcal{C}^1(\mathcal{U},\mathcal{F}) = \Gamma_{U_{01}}\mathcal{F}.$$

For any open set $V\subset X$ we have

$$\Gamma\big(\,V, \mathcal{C}^0(\mathcal{U},\mathcal{F})\,\big) = \Gamma(V\cap U_0,\mathcal{F})\times\Gamma(V\cap U_1,\mathcal{F}),\ \ \Gamma\big(\,V,\mathcal{C}^1(\mathcal{U},\mathcal{F})\,\big) = \Gamma(V\cap U_{01},\mathcal{F})$$

and $\delta : \Gamma\big(\,V,\mathcal{C}^0(\mathcal{U},\mathcal{F})\,\big)\to\Gamma\big(\,V,\mathcal{C}^1(\mathcal{U},\mathcal{F})\,\big)$ is given by $\delta(s_0,s_1) = s_0\,|_{V\cap U_{01}} - s_1\,|_{V\cap U_{01}}$. □

Theorem 4.2. *Suppose $\mathcal{U}$ is an open cover of the topological space X. $\mathcal{F}\in\mathbf{Sh}_R(X)$ then the natural morphism of complexes $[\mathcal{F}]\to\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$ is a resolution called the* Čech resolution determined by the open cover $\mathcal{U}$. *In other words the long sequence*

$$0\to\mathcal{F}\xrightarrow{r}\mathcal{C}^0(\mathcal{U},\mathcal{F})\to\cdots\to\mathcal{C}^p(\mathcal{U},\mathcal{F})\xrightarrow{\delta}\mathcal{C}^{p+1}(\mathcal{U},\mathcal{F})\to\cdots$$

is exact in $\mathbf{Sh}_R(X)$.

Proof The exactness at $\mathcal{F}$ and $\mathcal{C}^0$ follows from the properties of a sheaf, namely that for any open set the map

$$\Gamma(V, \mathcal{F}) \to \Gamma(V \cap \mathcal{U}, \mathcal{F})$$

is bijective. Let us prove the exactness at $\mathcal{C}^p$, $p \geq 1$.

It suffices to show that for every $x \in X$ the sequence

$$\mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})_x \xrightarrow{\delta^{p-1}} \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x \xrightarrow{\delta^p} \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})_x$$

is exact.

Let $x \in X$ and $s_x \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x$ such that $\delta^p s_x = 0$. We can represent the germ s_x by a section $s \in \Gamma(V, \mathcal{C}^p(\mathcal{U}, \mathcal{F}),)$, where V is an open neighborhood of x. By shrinking V if needed we can assume that V is contained in some open set $U_\ell \in \mathcal{U}$. Set

$$St(x, \mathcal{U}) := \big\{ \alpha \in N(\mathcal{U});\ \ V \cap U_\alpha \neq \emptyset \big\}.$$

$St(x, \mathcal{U})$ is a simplicial subcomplex of $N(\mathcal{U})$. Note also that

$$\alpha \in St(x, \mathcal{U}) \Longrightarrow U_\ell \cap U_\alpha \neq \emptyset \Longrightarrow \{\ell\} \cup \alpha \in N(\mathcal{U}).$$

This subcomplex is the *star* of the vertex ℓ in $N(\mathcal{U})$ and in particular it is *contractible*. This is essentially the reason for the acyclicity of the Čech complex.

We denote by $St_m(x, \mathcal{U})$ the set of m-simplices in $St(x, \mathcal{U})$ and by $St_m(x, \mathcal{U})'$ the set of m-simplices which do not contain ℓ as a vertex. Note that we have a natural map

$$St_{p-1}(x, \mathcal{U})' \ni \alpha \longmapsto \ell * \alpha := \{\ell\} \cup \alpha \in St_p(x, \mathcal{U}).$$

Geometrically, $\ell * \alpha$ is the cone on α with vertex ℓ. The germ s_x is represented by a collection of sections

$$s\,(\alpha) \in \Gamma(V \cap U_\alpha, \mathcal{F}),\ \ \alpha \in St_p(X, \mathcal{U}).$$

The condition $\delta^p s_x = 0$ is equivalent to

$$\sum_{i=1}^{p+1} (-1)^i s\,(F_i \beta)\,|_{V \cap U_\beta} = 0,\ \ \forall \beta \in St(x, \mathcal{U})_{p+1}.$$

Now define

$$\sigma \in \prod_{\beta \in St_{p-1}(x, \mathcal{U})} \Gamma(V, \Gamma_{U_\beta} \mathcal{F}) = \prod_{\beta \in St_{p-1}(x, \mathcal{U})} \Gamma(V \cap U_\beta, \mathcal{F})$$

$$\sigma\,(\beta) = \begin{cases} 0 & \text{if} \quad \ell \in \beta \\ \epsilon(\ell, \ell * \beta) \cdot s\,(\ell * \beta) & \text{if} \quad \ell \notin \beta \end{cases}$$

We want to prove that

$$\delta \sigma\,(\alpha) = s\,(\alpha),\ \ \forall \alpha \in St(x, \mathcal{U}).$$

We distinguish two cases.

- $\ell \in \alpha$ so that $\alpha = \{\ell, \beta_1, \cdots, \beta_p\} = \ell * \beta$. Then

$$\delta \sigma\,(\alpha) = \epsilon(\ell, \alpha) \sigma\,(\beta) = s\,(\ell * \beta) = s\,(\alpha).$$

- $\ell \notin \alpha = (\alpha_0, \cdots, \alpha_p)$. Then

$$\delta \sigma\,(\alpha) = \sum_{j \in \alpha}^{p} \epsilon(j, \alpha) \sigma\,(F_j \alpha)\,|_{V \cap U_\alpha} = \sum_{i=0}^{p} \epsilon(j, \alpha) \cdot \epsilon(\ell, \ell * F_i \alpha) \cdot s\,(\ell * F_i \alpha)\,|_{V \cap U_\alpha}$$

$$= s\,(\alpha) - \delta s\,(\ell * \alpha) = s\,(\alpha).$$

Consider for example the situation depicted in Figure 1 where $p = 2$, the simplex α is

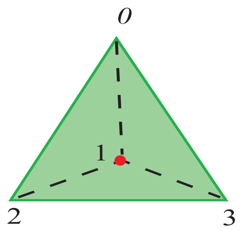

FIGURE 1. *A three-dimensional simplex.*

$(\alpha_0, \alpha_1, \alpha_2) = (0, 2, 3)$ (marked in green), and the vertex ℓ is $\ell = 1$ (marked in red). The condition that s is a cocycle translates into

$$0 = \delta s\,(0, 1, 2, 3) = s, (1, 2, 3) - s\,(0, 2, 3) + s\,(0, 1, 3) - s\,(0, 1, 2).$$

Note that for $j > 1$

$$\sigma, (0, j) = -s\,(0, 1, j),$$

while for $j > i > 1$ we have

$$\sigma\,(i, j) = s\,(1, i, j).$$

Then

$$\begin{aligned} \delta\sigma\,(0, 2, 3)\rangle &= \sigma\,(2, 3) - \sigma\,(0, 3) + \sigma\,(0, 2) \\ &= s\,(1, 2, 3) + s\,(0, 1, 3) - s\,(0, 1, 2). \end{aligned}$$

In view of the cocycle condition the last expression is equal to $s\,(1, 2, 3)$.

□

Remark 4.3. As explained in [5, §2.8], the Čech complex of sheaves associated to an open cover and a sheaf is locally the cone complex of a morphisms between complexes. This explains its acyclicity. □

Suppose now $\mathcal{S} = (S_i)_{i \in I}$ is a *locally finite closed cover* of X. This means that the sets S_i are closed, their union is X and any point $x \in X$ has a neighborhood which intersects only finitely many of the sets S_i. Fix a total order on I. The nerve $N(\mathcal{S})$ of this cover is defined as before. For every simplex $\alpha \in N(\mathcal{S})$ we set as before

$$S_\alpha = \bigcap_{v \in \alpha} S_v.$$

Note that if $\alpha \subset \beta \in N(\mathcal{S})$ then $S_\alpha \supset S_\beta$ and we have a natural isomorphism $\mathcal{F}_{S_\beta} \cong (\mathcal{F}_{S_\alpha})_{S_\beta}$. In particular we have a natural attachment map $\mathbf{a}_\alpha^\beta : \mathcal{F}_{S_\alpha} \to \mathcal{F}_{S_\beta}$. Now define

$$\mathcal{C}^p(\mathcal{S}, \mathcal{F}) := \prod_{\alpha \in N_p(\mathcal{S})} \mathcal{F}_{S_\alpha} \;\text{ and }\; \delta^p : \mathcal{C}^p(\mathcal{S}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{S}, \mathcal{F}),\;\; \delta = \prod_{\beta \in N_{p+1}(\mathcal{S})} \delta_\beta,$$

where

$$\delta_\beta = \sum_{v \in \beta} \epsilon(v, \beta) \mathbf{a}_{F_v \beta}^\beta.$$

Note that we have a natural attachment map

$$\mathbf{a}^{\mathcal{S}} : \mathcal{F} \to \mathcal{C}^0(\mathcal{S}, \mathcal{F}), \;\; \mathbf{a}^{\mathcal{S}} = \prod_{S \in \mathcal{S}} \mathbf{a}^S.$$

Theorem 4.4. *Suppose X is a $\mathbf{T}_3$-space.*[5] *If $\mathcal{S}=(S_i)_{i\in I}$ is a locally finite closed cover of X then the long exact sequence of sheaves*

$$0\to\mathcal{F}\xrightarrow{\mathbf{a}^{\mathcal{S}}}\mathcal{C}^0(\mathcal{S},\mathcal{F})\xrightarrow{\delta^0}\cdots\xrightarrow{\delta^{p-1}}\mathcal{C}^p(\mathcal{S},\mathcal{F})\xrightarrow{\delta^p}\cdots$$

is exact. In other words, the morphism of complexes $\mathbf{a}^{\mathcal{S}}:[\mathcal{F}]\to C^\bullet(\mathcal{S},\mathcal{F})$ is a resolution of $\mathcal{F}$ called the Čech resolution determined by the closed cover $\mathcal{S}$.

Proof The injectivity of $\mathbf{a}^{\mathcal{S}}$ follows from the injectivity of the explicit description (3.2) of the attachment maps. The fact that $\operatorname{im}\mathbf{a}^{\mathcal{S}}=\ker\delta^0$ follows from the surjectivity of the attaching maps $\mathbf{a}^S$. We need to check the exactness of

$$\mathcal{C}^{p-1}(\mathcal{S},\mathcal{F})_x\xrightarrow{\delta}\mathcal{C}^p(\mathcal{S},\mathcal{F})_x\xrightarrow{\delta}\mathcal{C}^{p+1}(\mathcal{S},\mathcal{F})_x,$$

for any $x\in X$. Let $s_x\in\mathcal{C}^p(\mathcal{S},\mathcal{F})_x$ such that $\delta s_x=0$. We can represent s_x as the germ of a section $s\in\Gamma\big(U,\mathcal{C}^p(\mathcal{S},\mathcal{F})\big)$, where U is a very small neighborhood of x which intersects finitely many S_i's. Since X is a $\mathbf{T}_3$ we can choose U small enough so that the set S_i intersects U if and only if it contains the point x. We let

$$I_x:=\big\{i\in I;\;\; x\in S_i\big\},\;\;\mathcal{S}_x=(S_i)_{i\in I_x},\;\;\ell=\min I_x.$$

Note that the simplicial complex $N(\mathcal{S}_x)$ is an elementary simplex. In particular it is contractible. The section s is described by a family

$$s(\alpha)\in\Gamma(U,\mathcal{F}_{S_\alpha}),\;\;\alpha\in N_p(\mathcal{S}_x).$$

Now define

$$\sigma\in\prod_{\beta\in N_{p-1}(\mathcal{S}_x)}\Gamma(U,\mathcal{F}_{S_\beta})\subset\Gamma(U,\mathcal{C}^{p-1}(\mathcal{S},\mathcal{F}))$$

by setting

$$\sigma(\beta)=\begin{cases}0 & \text{if}\;\;\ell\in\beta\\ s(\ell*\beta) & \text{if}\;\;\ell\notin\beta\end{cases}$$

A computation identical to the one in the proof of Theorem 4.2 shows that $\delta\sigma=s$.

□

Suppose $\mathcal{U}$ is an open cover of the topological space X. The Čech resolution construction associates to each sheaf $\mathcal{F}\in\mathbf{Sh}_R(X)$ a complex $C^\bullet(\mathcal{U},\mathcal{F})\in\mathbf{Com}(\mathbf{Sh}_R(X)\,)$. This construction is functorial in the sense that any morphism of sheaves $\Phi:\mathcal{F}_0\to\mathcal{F}_1$ induces a morphism of complexes of sheaves

$$\Phi_{\mathcal{U}}:C^\bullet(\mathcal{U},\mathcal{F}_0)\to C^\bullet(\mathcal{U},\mathcal{F}_1)$$

satisfying all the functorial requirements. In particular, to every complex $(\mathcal{F}^\bullet,d)$ we can associate a double complex

$$\Big(C^\bullet(\mathcal{U},\mathcal{F}^\bullet),d_I,d_{II}\Big),\;\;d_I^p=\delta^p:C^p(\mathcal{U},\mathcal{F}^\bullet)\to C^{p+1}(\mathcal{U},\mathcal{F}^\bullet)$$

$$d_{II}^q=(-1)^pd:C^p(\mathcal{U},\mathcal{F}^q)\to C^p(\mathcal{U},\mathcal{F}^{q+1}).$$

[5]We use the conventions in [6, Chap. 4] so X is $\mathbf{T}_3$ if it is Hausdorff and for every point x and any closed set S not containing x there exist disjoint neighborhoods of x and C in X. For example, all paracompact spaces ar regular.

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