Sheaf cohomology

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Abstract

We introduce the right derived functors of the global section functor and describe its' main properties. We will present the basic methods of computing sheaf cohomology and the relations with other cohomology theories such as Cech and singular cohomology. Proofs (more or less detailed) of various important results e.g. Cech-de Rham and Dolbeault theorems should form the core of this presentation. The main reference is Voisin[2002](ch. 4)

Contents

1 The global section functor

Let X be a topological space. Denote by Sh_X the category of sheaves of abelian groups defined on X and by Ab_{gr} the category of abelian groups. Sh_X is an abelian category and has enough injectives since every sheaf $\mathcal{F} \in Sh_X$ can be mapped

$$\mathcal{F} \hookrightarrow \prod_{x \in X} \mathcal{F}_x \hookrightarrow \prod_{x \in X} \mathcal{I}_x \quad (*)$$

where the second map is the direct product of stalkwise injections $\mathcal{F}_x \hookrightarrow \mathcal{I}_x$ into injective abelian groups, for which the sheaf of discontinuous sections $\prod_{x \in X} \mathcal{I}_x$ is easily seen to be injective. Examples :

Recall that for a left exact functor $F : \underline{A} \to \underline{B}$ between two abelian categories with A having the extra property of possessing enough injectives we can construct the right derived functors $R^i F : \underline{A} \to \underline{B}$ which takes an object $a \in \underline{A}$ to the i-th cohomology object of the complex obtained after applying F to an injective resolution of a. We will be concerned exclusively with the global section functor

$$\Gamma : Sh_X \to Ab_{gr}$$

$$\Gamma(\mathcal{F}) := \Gamma(X, \mathcal{F}) := \mathcal{F}(X)$$

for which the derived functors have the classical notation:

$$H^i(X, \mathcal{F}) := R^i \Gamma(\mathcal{F})$$

Properties:

a) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$
b) $H^n(X, \mathcal{F}) = 0 \qquad \forall n < 0$
c) $H^n(X, \mathcal{F}) = 0 \qquad \forall i > 0 \;\; \& \;\; \mathcal{F} \;\; injective \; (acyclic)$
d) every short exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

gives rise to a long exact sequence:

$$\to H^i(X, \mathcal{F}_2) \to H^{i+1}(X, \mathcal{F}_1) \to H^{i+1}(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}_2) \to$$

We recall the following basic property of the right derived functors of which we will make extensive use:

Proposition 1.1. *The right derived functors of a left exact functor $F : \underline{A} \to \underline{B}$ can be computed from any acyclic resolution, i.e. if*

$$0 \to a \to a_0 \to a_1 \to a_2 \to$$

is a resolution of $a \in \underline{A}$ such that $R^i F(a_j) = 0, \;\; \forall j, \forall i \geqslant 1$ then

$$R^i F(a) = H^i(F(a_\bullet)) \;\; \forall i \geqslant 1$$

The reason for which such a property is useful is that, in practice, it is pretty hard to come up with concrete injective resolutions so one will try to use some other kind of acyclic objects. In fact this idea can be very nicely formalized as follows. For a left exact functor F as above we call a subcategory $\underline{C} \subseteq \underline{A}$, F-injective if it satisfies the following properties:
i) $\forall a \in \underline{A}$, $\exists\ c \in \underline{C}$ and $i : a \hookrightarrow c$ monomorphism
ii) every exact sequence

$$0 \to c \to a \to a_1 \to 0$$

such that $c \in \underline{C}$ induces an exact sequence

$$0 \to F(c) \to F(a) \to F(a_1) \to 0$$

iii) for every exact sequence

$$0 \to c_1 \to c_2 \to c_3 \to 0$$

such that $c_1, c_2 \in \underline{C}$ implies $c_3 \in \underline{C}$

Every F-injective object is acyclic and the full subcategory of injective objects is F-injective for any left exact functor F. Hence the whole construction of right derived functors can be carried over with suitable chosen F-injective subcategories. (details in [Kashiwara 1990]). It is indeed the case for the global section functor and the Γ-injective subcategory we'll be concerned in what follows is the category of flabby sheaves.

Definition 1.2. *$\mathcal{F} \in Sh_X$ is called flabby if for $\forall V \subseteq U$ inclusions of open sets the restriction $\mathcal{F}(U) \twoheadrightarrow \mathcal{F}(V)$ is surjective.*

Remark 1) It's enough to ask that $\mathcal{F}(X) \to \mathcal{F}(V)$ be surjective.
2) The sheaf of discontinuous sections $\prod_{x \in X} \mathcal{F}_x$ is flabby.

Proposition 1.3. *A flabby sheaf is acyclic.*

Proof: Let $\mathcal{F}$ be the flabby sheaf into consideration and let $\mathcal{F} \hookrightarrow \mathcal{I}$ be an inclusion into a flabby injective sheaf (see $*$). We have a corresponding short exact sequence:

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$$

Claim: $\mathcal{I}(U) \to \mathcal{G}(U)$ is surjective for every open set U.
Consequences: i) $\mathcal{G}$ is flabby;
ii) $H^1(X, \mathcal{F}) = 0$ since in the induced long exact sequence we have $H^1(X, \mathcal{I}) = 0$ (in fact $H^i(X, \mathcal{I}) = 0,\ \forall i \geqslant 1$) and $\Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G})$ surjective;
iii) from i) and ii) conclude that $H^1(X, \mathcal{G}) = 0$ and use induction in the long exact sequence

$$\to H^{i-1}(X, \mathcal{I}) \to H^{i-1}(X, \mathcal{G}) \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{I}) \to$$

to infer that $H^k(X, \mathcal{F}) \simeq H^{k-1}(X, \mathcal{G}) = 0\ \ \forall k \geqslant 1$.
For the proof of the claim one uses a Zorn lemma argument (which won't be given in full detail) in the following way. Let $\alpha \in \mathcal{G}(U)$ be a section and let $A_\alpha = \{(\beta, V) \mid \beta \in \mathcal{I}(V),\ \ V \subseteq U \ \ \text{and } \alpha|_V = \beta \text{ via } \mathcal{I}(V) \to \mathcal{G}(V)\}$. Then A_α is endowed with the obvious order and one shows that a maximal element of A_α is indeed a pair (β, U) using the fact

that $\forall(\beta_1, V_1), (\beta_2, V_2) \in A_\alpha,\ \ \exists(\beta_3, V_1 \cup V_2) \in A_\alpha$ such that $\beta_3|_{V_i} = \beta_i,\ \ i = 1, 2.$ ∎

The "canonical" flabby resolution is usually called the Godement resolution and is defined as follows:

$$0 \to \mathcal{F} \overset{\varphi}{\hookrightarrow} \prod_{x \in X} \mathcal{F}_x \overset{\psi}{\hookrightarrow} \prod_{x \in X} (\ Coker\ \varphi_x) \hookrightarrow \prod_{x \in X} (\ Coker\ \psi_x) \hookrightarrow$$

2 Sheaf cohomology and integral cohomology

For certain "nice" spaces sheaf cohomology is really a natural generalization of the singular cohomology. Let $\mathbb{Z}_X$ denote the sheaf associated to the constant presheaf $U \to \mathbb{Z}$ (the sheaf of locally constant functions) on the topological space X and $H^i_{sing}(X, \mathbb{Z})$ the singular cohomology groups with integer coefficients. Recall that a locally contractible topological space X has the property that each point $x \in X$ has a basis consisting of contractible neighborhoods. This section is dedicated to proving the following:

Proposition 2.1. *If X is locally contractible then*

$$H^i_{sing}(X, \mathbb{Z}) \simeq H^i(X, \mathbb{Z}_X)$$

Proof: Let $\mathcal{C}^q$ be the sheaf associated to the presheaf of q-th singular cochains $U \to C^q_{sing}(U, \mathbb{Z})$.
Step 1 We prove first that:

$$0 \to \mathbb{Z}_X \to \mathcal{C}^0 \to \mathcal{C}^1 \to$$

is a flabby resolution of $\mathbb{Z}_X$. The exactness in degree > 0 is straightforward if we think that for U contractible the following sequence is exact:

$$C^{q-1}_{sing}(U, \mathbb{Z}) \overset{\partial}{\to} C^q_{sing}(U, \mathbb{Z}) \overset{\partial}{\to} C^{q+1}_{sing}(U, \mathbb{Z})$$

hence the same is true for the sequence of stalks:

$$C^{q-1}_{sing,\ x} \overset{\partial_x}{\to} C^q_{sing,\ x} \overset{\partial_x}{\to} C^{q+1}_{sing,\ x}$$

because X is locally contractible and so we can take the direct limit with respect to a basis of contractible neighborhoods. In conclusion, the corresponding sequence for $\mathcal{C}^q$ is exact. $Ker(\mathcal{C}^0 \to \mathcal{C}^1) = \mathbb{Z}_X$ follows again from the fact that X is locally contractible since for every cocycle $\varphi \in C^0_{sing}(U, \mathbb{Z})$ with U contractible and for every two points $x, y \in U$ which can be connected by a path α we get

$$0 = \partial\varphi(\alpha) = \varphi(\partial\alpha) = \varphi(y) - \varphi(x)$$

Therefore φ is locally constant and hence the same is true for every cocycle $\phi \in \mathcal{C}^0,\ \partial\varphi = 0$ for which we know that locally is a φ as above. The fact that $\mathcal{C}^q$ is flabby follows from the fact that C^q_{sing} is flabby $\forall q$.
Step 2 There exists a natural (with respect to the restriction morphisms) isomorphism

$$\frac{C^q_{sing}(V, \mathbb{Z})}{C^q_{sing}(V, \mathbb{Z})_0} \overset{\sim}{\longrightarrow} \mathcal{C}^q(V, \mathbb{Z})$$

where $C^q_{sing}(V,\mathbb{Z})_0 := \{\varphi \in C^q_{sing}(V,\mathbb{Z})| \ \exists\ \mathcal{V} = (V_i)_{i\in I}$ open covering of V such that $\varphi|_{C_q(V_i,\mathbb{Z})} = 0\}$
To see this consider the natural morphism:

$$\varphi \in C^q_{sing}(V,\mathbb{Z}) \xrightarrow{+} \mathcal{C}^q(V,\mathbb{Z}) \ni \varphi^+$$

Then $\varphi^+ = 0 \Leftrightarrow \varphi|_{V_i} = 0$ for some open covering $(V_i)_{i\in I}$ of V, $\Leftrightarrow \varphi|_{C_q(V_i,\mathbb{Z})} = 0$. So the kernel of the map is just $C^q_{sing}(V,\mathbb{Z})_0$. To see surjectivity take $\beta \in \mathcal{C}^q(V,\mathbb{Z})$ and let $\tilde{\beta}_i \in C^q_{sing}(V_i,\mathbb{Z})$ such that $\beta|_{V_i} = \tilde{\beta}_i$, for some open covering $(V_i)_{i\in I}$ of V. Define $\tilde{\beta} : C_q(V,\mathbb{Z}) \to \mathbb{Z}$ by

$$\tilde{\beta}(\alpha) = \begin{cases} \tilde{\beta}_i(\alpha) & \text{if } Im\ \alpha \subseteq U_i \text{ for some } U_i; \\ 0 & \text{otherwise.} \end{cases}$$

It is well defined since $\tilde{\beta}_i(\alpha) = \tilde{\beta}_j(\alpha)$ if $Im\ (\alpha) \subseteq V_i \cap V_j$. Moreover $\beta = \tilde{\beta}^+$.
Step 3 The complexes $C^*_{sing}(X,\mathbb{Z})$ and $C^*_{sing}(X,\mathbb{Z})/C^*_{sing}(X,\mathbb{Z})_0$ are quasi-isomorphic. This is essentially the theorem of small chains. Let us fix one covering $\mathcal{V} = (V_i)_{i\in I}$ of X. If we denote by $C^{\mathcal{V}}_*(X,\mathbb{Z})$ the chain complex which has as generators singular p-simplexes $\sigma : \triangle^p \to X$ with the image contained in some $V_i \in \mathcal{V}$ then a well known result (see for example Spanier[1966], pag 178) says that the inclusion $C^{\mathcal{V}}_*(X,\mathbb{Z}) \to C_*(X,\mathbb{Z})$ is a chain equivalence (i.e. equivalence in the homotopy category). Hence the same is true about the dual map $\pi_{\mathcal{V}} : C^*_{sing}(X,\mathbb{Z}) \to C^{\mathcal{V}\,*}_{sing}(X,\mathbb{Z})$. Now $Ker\ \pi_{\mathcal{V}} = C^{\mathcal{V}\,*}_{sing}(X,\mathbb{Z})_0$ is the set of cochains which are zero when restricted to $C^*(V_i,\mathbb{Z})$ for every $i \in I$. The chain equivalence says that the complex $Ker\ \pi_{\mathcal{V}}$ is acyclic. We can now take the direct limit after $\mathcal{V}$ in the exact sequence:

$$0 \to Ker\ \pi_{\mathcal{V}} \longrightarrow C^*_{sing}(X,\mathbb{Z}) \xrightarrow{\pi_{\mathcal{V}}} C^{\mathcal{V}\,*}_{sing}(X,\mathbb{Z}) \to 0$$

and use the fact that $Ker\ \pi = C^*_{sing}(X)_0$ is still acyclic ($\pi = \varinjlim \pi_{\mathcal{V}}$) to finish the proof.
Step 4

$$H^q(X,\mathbb{Z}_X) \simeq H^q(\mathcal{C}^*(X)) \simeq H^q(C^*_{sing}(X,\mathbb{Z})/_{C^*_{sing}(X,\mathbb{Z})_0}) \simeq H^q(C^*_{sing}(X,\mathbb{Z})) = H^q_{sing}(X,\mathbb{Z})\ \blacksquare$$

The same theorem is valid for any constant coefficients.

3 Soft and fine sheaves

Until now we have worked in full generality. In this section we restrict our attention to paracompact topological spaces. We make this restriction because in our applications the space X will always be paracompact.

Definition 3.1. *A sheaf $\mathcal{F}$ on a paracompact topological space X is called soft if $\mathcal{F}(X) \to \mathcal{F}(K)$ is surjective for every closed set K, where $\mathcal{F}(K) = \Gamma(K, \mathcal{F}|_K)$.*

A sheaf $\mathcal{F}$ is called fine if the sheaf $\mathcal{H}om(\mathcal{F},\mathcal{F})$ is soft.

We make some important remarks. The first is that every fine sheaf is soft. If $\mathcal{A}$ is a soft sheaf of rings with unit (we'll take here as a definition for the unit to be a global section

which restricted to any open subset is also a unit) and $\mathcal{F}$ is a sheaf of $\mathcal{A}$-modules then $\mathcal{F}$ is fine. Moreover we have the following equivalent definitions:
soft: for every K closed and for every $s \in \mathcal{F}$ and every locally finite covering $(U_\alpha)_\alpha$ of K in X there exists $s_\alpha \in \mathcal{F}(X)$ with *supp* $s_\alpha \subseteq U_\alpha$ such that $s = \sum s_\alpha|_K$
fine: in the case $\mathcal{A}$ is a sheaf of rings with unit: there exists a partition of unity subordinate to any locally finite covering $(U_\alpha)_\alpha$ of X, i.e. there exists $s_\alpha \in \mathcal{A}(X)$ with *supp* $s_\alpha \subseteq U_\alpha$ such that $\sum s_\alpha = 1$. In what follows $\mathcal{F}$ will always be supposed to be a sheaf of $\mathcal{A}$-modules.

Details can be found in (Bredon [1967])

Proposition 3.2. *If $\mathcal{F}$ is fine sheaf then $H^i(X, \mathcal{F}) = 0$, i.e. $\mathcal{F}$ is acyclic.*

Proof: We take the Godement resolution of $\mathcal{F}$, complex which we denote by $\mathcal{I}_\bullet$. We have :

$$H^i(X, \mathcal{F}) = Ker\ (\Gamma(X, \mathcal{I}^i) \to \Gamma(X, \mathcal{I}^{i+1}))/Im\ (\Gamma(X, \mathcal{I}^{i-1}) \to \Gamma(X, \mathcal{I}^i))$$

We want to show that if $\alpha \in \Gamma(X, \mathcal{I}^i)$ is an i-cocycle then it is also a coboundary. To begin with, we note that, for some local covering $(U_j)_{j \in J}$, α can be written

$$\alpha|_{U_j} = d\alpha_j$$

where $\alpha_j \in \mathcal{I}^{i-1}(U_j)$. This is just the exactness of the resolution $\mathcal{I}_\bullet$. Now we can suppose without loss of generality that $(U_j)_{j \in J}$ is locally finite. Take a partition of unity $(f_j)_{j \in J}$ subordinate to this covering with $f_j \in \mathcal{A}(X)$ and denote by

$$\tilde{\alpha} := \sum f_j \alpha_j$$

where by $f_j \alpha_j$ we mean a global section which equals 0 outside U_j and $\alpha_j f_j|_{U_j}$ on U_j (stalk-wise, of course). It's easy to check that writing $\alpha = \sum f_j \alpha|_{U_j}$ we get $d\tilde{\alpha} = \alpha$. ■

Examples:
1) The sheaf of continuous functions on a paracompact space is soft. This follows from the fact that every paracompact space is normal and from Tietze's extension theorem.
2) Let X be a smooth manifold. Then the sheaves of smooth differential forms Ω^p, $p = 0, 1...$ are fine since they are sheaves of modules over $\Omega^p := \mathcal{C}^\infty$ which has a partition of unity property.
3) Let X be a complex manifold, E a holomorphic vector bundle and $\mathcal{A}^{p,q}(E) := \mathcal{A}^{p,q} \otimes \mathcal{E}$: the sheaves of (p, q) differential forms with values in E. All these are fine sheaves.

We have two important corollaries of what has been just said.

Theorem 3.3. *(de Rham) Let X be a smooth manifold. Then*

$$\frac{Ker\ d : \Omega^k(X) \to \Omega^{k+1}(X)}{Im\ d : \Omega^{k-1}(X) \to \Omega^k(X)} \simeq H^k(X, \mathbb{R}) \simeq H^k_{sing}(X, \mathbb{R})$$

Theorem 3.4. *Let X be a complex manifold and E a holomorphic vector bundle. Then*

$$H^q(X, \mathcal{E}) = \frac{Ker\ \bar{\partial} : \mathcal{A}^{0,q} \to \mathcal{A}^{0,q+1}}{Im\ \bar{\partial}\mathcal{A}^{0,q-1} \to \mathcal{A}^{0,q}}$$

We state separately the particular case when $E = \bigwedge^p T^*X$, T^*X being the *holomorphic* cotangent bundle :

Theorem 3.5. *(Dolbeault)*

$$H^q(X, \Omega^p) = H^{p,q}_{\bar{\partial}}(X)$$

4 Čech cohomology

The subject of this section is what was first called sheaf cohomology. It is still the way that some authors prefer to introduce the topic rather than via the derived functors approach. The connection between the two is given in the Leray theorem. We'll see in a moment what it says.

Let X be a topological space and $\mathcal{F} \in Sh_X$. Let $\mathcal{U} = (U_i)_{i\in I}$ be a covering with open sets of X such that the index set I is well ordered. For every $J \subset I$ subset with $q+1$ elements we denote by $U_J := \bigcap_{i\in J} U_i$ and by $\mathcal{F}_{U_J}$ the sheaves on X:

$$\mathcal{F}_{U_J}(V) := \mathcal{F}(U_J \cap V), \quad \forall V = \overset{\circ}{V} \subseteq X$$

These sheaves can alternatively be defined by taking the pull-back followed by the direct image of $\mathcal{F}$ under the inclusion $U_J \hookrightarrow X$ Consider the sequence:

$$0 \to \mathcal{F} \to \prod_{i\in\ I} \mathcal{F}_{U_i} \rightrightarrows \prod_{|J|=2} \mathcal{F}_{U_J} \rightrightarrows$$

where the boundary maps are defined as follows. Denote by $\mathcal{F}^q$ the sheaf $\prod_{|J|=q+1} \mathcal{F}_{U_J}$. Let $\sigma = (\sigma_J)(\sigma_J) \in \mathcal{F}(V \cap U_J)$ for some $J \subseteq I$ with $|J| = q+1$. Then define $(\partial\sigma)_{\tilde{J}} \in \mathcal{F}(V \cap U_{\tilde{J}})$, where $\tilde{J} = \{j_0, ..., j_{q+1}\}$, $j_0 < ... < j_{q+1}$:

$$(\partial\sigma)_{\tilde{J}} = \sum_i (-1)^i \sigma_{\tilde{J}-\{j_i\}}|_{V\cap U_{\tilde{J}}}$$

It's not hard to see, although a bit messy, that the above sequence is a resolution of the sheaf $\mathcal{F}$. It is called the Čech resolution with respect to the covering $\mathcal{U}$. The groups:

$$\check{H}^q_{\mathcal{U}}(X, \mathcal{F}) := H^q(\mathcal{F}^*(X))$$

are called the *Čech* cohomology groups relative to the covering $\mathcal{U}$ (no surprise about that).

We have the following:

Theorem 4.1. *If the covering $\mathcal{U}$ has the property that $H^q(U_J, \mathcal{F}) = 0$ for every $q \geqslant 1$ and for every $U_J = \cap_{i\in J} U_i$, J finite, then*

$$H^q(X, \mathcal{F}) = \check{H}^q_{\mathcal{U}}(X, \mathcal{F}), \quad \forall q \geqslant 0$$

Proof: We look at the following double complex:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{I}^1 & \rightarrow & \mathcal{I}^{1,0} & \rightarrow & \mathcal{I}^{1,1} \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & \mathcal{I}^0 & \rightarrow & \mathcal{I}^{0,0} & \rightarrow & \mathcal{I}^{0,1} \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{F}^0 & \rightarrow & \mathcal{F}^1 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}$$

where on the first column we have a flabby resolution of $\mathcal{F}$ and each row represents the Cech resolution of the sheaf $\mathcal{I}^q$ with respect to the covering $\mathcal{U}$. The up-going arrows are obtained from the functoriality of the Cech resolution. Let $\mathcal{K}^q = \bigoplus_{p+l=q} \mathcal{I}^{p,l}$ be the simple complex associated to $\mathcal{I}^{\bullet,\bullet}$. The first thing to notice is that $\mathcal{K}$ is a flabby resolution of $\mathcal{F}$. On one side, we get that the property of being flabby is preserved under the following operations with sheaves: sum, product, restriction(pull-back) to an open set, direct image with respect to a continuous function. Hence we infer that each $\mathcal{I}^{p,l}$ is flabby and therefore each $\mathcal{K}^q$ is flabby. On the other side the complex has exact rows and a well-known fact about double complexes says that in this situation the cohomology of the first column is the cohomology of the simple complex. The first column is a resolution of $\mathcal{F}$ so the same must be true about the simple complex. It follows that we can compute the cohomology of $\mathcal{F}$ from this resolution.

$$H^q(X, \mathcal{F}) = H^q(\mathcal{K}^\bullet(X))$$

In order to achieve that we look at the double complex of global sections:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{I}^1(X) & \rightarrow & \mathcal{I}^{1,0}(X) & \rightarrow & \mathcal{I}^{1,1}(X) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & \mathcal{I}^0(X) & \rightarrow & \mathcal{I}^{0,0}(X) & \rightarrow & \mathcal{I}^{0,1}(X) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & \mathcal{F}(X) & \rightarrow & \mathcal{F}^0(X) & \rightarrow & \mathcal{F}^1(X) \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}$$

We notice that the columns are exact. This is a consequence of the fact that the cohomology groups of each "column complex" is a product of groups of the type $H^q(U_J, \mathcal{F})$, which are assumed to be zero for our covering $\mathcal{U}$. Hence the cohomology of the first row equals the cohomology of the simple complex. In our case this means:

$$\check{H}^q_{\mathcal{U}}(X, \mathcal{F}) \simeq H^q(\mathcal{K}^\bullet(X))$$

Together with what has been just said we get the desired isomorphism. ■

We state without proof the following proposition:

Proposition 4.2. *If X is Hausdorff then*

$$\varinjlim \check{H}^q_{\mathcal{U}}(X, \mathcal{F}) \simeq H^q(X, \mathcal{F}).$$

the direct limit being taken "in" the set of all coverings $\mathcal{U}$ which can be seen as a directed set with respect to the operation of refinement.

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