CRITICAL SETS OF RANDOM LINEAR COMBINATIONS OF EIGENFUNCTIONS

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ABSTRACT. Given a compact, connected Riemann manifold without boundary $(M, g)$ of dimension $m$ and a large positive constant $L$ we denote by $U_L$ the subspace of $C^\infty (M)\) spanned by eigenfunctions of the Laplacian corresponding to eigenvalues $\leq L$. We equip $U_L$ with the standard Gaussian probability measure induced by the $L^2$-metric on $U_L$, and we denote by $N_L$ the expected number of critical points of a random function in $U_L$. We prove that $N_L \sim C_m \dim U_L$ as $L \to \infty$, where $C_m$ is an explicit positive constant that depends only on the dimension $m$ of $M$ and satisfying the asymptotic estimate $\log C_m \sim \frac{m^2}{2} \log m$ as $m \to \infty$.

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1. INTRODUCTION

Suppose that $(M, g)$ is a smooth, compact Riemann manifold of dimension $m > 1$. We denote by $|dV_g|$ the volume density on $M$ induced by $g$. For any $u, v \in C^\infty (M)$ we denote by $(u, v)_g$ their $L^2$ inner product,

$$(u, v)_g := \int_M u(x)v(x) |dV_g(x)|.$$

The $L^2$-norm of a smooth function $u$ is then

$$\|u\| := \sqrt{(u, u)_g}.$$

Let $\Delta_g : C^\infty (M) \to C^\infty (M)$ denote the scalar Laplacian defined by the metric $g$. For $L > 0$ we set

$$U_L = U_L(M, g) := \bigoplus_{\lambda \in [0, L]} \ker (\lambda - \Delta_g), \quad d(L) := \dim U_L.$$

We equip $U_L$ with the Gaussian probability measure.

$$d\gamma_L(u) := (2\pi)^{-d(L)/2} e^{-\frac{\|u\|^2}{2}} |du|.$$


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For any \( u \in U_L \) we denote by \( N_L(u) \) the number of critical points of \( u \). If \( L \) is sufficiently large then \( N_L(u) \) is finite with probability 1. We obtain in this fashion a random variable \( N_L \), and we denote by \( E(N_L) \) its expectation

\[
E(N_L) := \int_{U_L} N_L(u) d\gamma_L(u).
\]

In this paper we investigate the behavior of \( E(N_L) \) as \( L \to \infty \). More precisely we will prove the following result.

**Theorem 1.1.** There exists a positive constant \( C = C(m) \) that depends only on the dimension of \( M \), such that

\[
E(N_L) \sim C(m) \dim U_L \quad \text{as} \quad L \to \infty.
\]

The constant \( C(m) \) can be expressed in terms of certain statistics on the space \( S_m \), the space of symmetric \( m \times m \) matrices. We denote \( d\gamma_s \) the Gaussian measure\(^1\) on \( S_m \) given by

\[
d\gamma_s(X) = \frac{1}{(2\pi)^{m(m+1)/4}} \mu_m^{-\frac{m}{2}} e^{-\frac{1}{2}(\text{tr} \ X^2 - \frac{1}{m+2} (\text{tr} \ X)^2)} 2^{\frac{1}{2}}(\frac{m}{2}) \prod_{i \leq j} dx_{ij},
\]

\[
\mu_m = 2^{\frac{m}{2}} + (m + 2)^{m-1}.
\]

Then

\[
C(m) = \left( \frac{4\pi}{m+4} \right)^{\frac{m}{2}} \Gamma\left( 1 + \frac{m}{2} \right) \int_{S_m} \left| \det X \right| d\gamma_s(X) =: I_m.
\]

We can say something about the behavior of \( C(m) \) as \( m \to \infty \).

**Theorem 1.2.**

\[
\log C(m) \sim \log I_m \sim \frac{m}{2} \log m \quad \text{as} \quad m \to \infty.
\]

The proof of (1.1) relies on a Kac-Price type integral formula proved by the author in [15] that expresses the expected number of critical points of a function in \( U_L \) as an integral

\[
E(N_L) = \int_M \rho_L(\mathbf{x}) |dV_\mathbf{g}(\mathbf{x})|.
\]

Using some basic ideas from random field theory we reduce the large \( L \) asymptotics of \( \rho_L \) to questions concerning the asymptotics of the spectral function of the Laplacian. Fortunately, these questions were recently settled by X. Bin [4] by refining the wave kernel method of L. Hörmander, [11]. We actually prove a bit more. We show that

\[
\lim_{L \to \infty} L^{-m/2} \rho_L(\mathbf{x}) = \frac{C(m) \omega_m}{(2\pi)^m}, \quad \text{uniformly in} \ \mathbf{x} \in M,
\]

where \( \omega_m \) denotes the volume of the unit ball in \( \mathbb{R}^m \). Using the classical Weyl estimates (3.2) we see that (1.4) implies (1.1).

The equality (1.4) has an interesting interpretation. We can think of \( \rho_L(\mathbf{x}) |dV_\mathbf{g}(\mathbf{x})| \) as the expected number of critical points of a random function in \( U_L \) inside an infinitesimal region of volume \( |dV_\mathbf{g}(\mathbf{x})| \) around the point \( \mathbf{x} \). From this point of view we see that (1.4) states that for large \( L \) we expect the critical points of a random function in \( U_L \) to be uniformly distributed.

\(^1\)We refer to Appendix B for a detailed description of a 3-parameter family Gaussian measures \( d\Gamma_{a,b,c} \) on \( S_m \) that includes \( d\gamma_s \) as \( d\gamma_s = d\Gamma_{3,1,1} \).
This rather vague statement could be made more precise if we had estimates for the variance \( V(N_L) \) of \( N_L \). We are inclined to believe that as \( L \to \infty \) the ratio

\[ q_L = \frac{V(N_L)}{E(N_L)} \]

has a finite limit \( q(M, g) \). Such a result would show that the random variable \( L^{-\frac{m}{2}} N_L \) is highly concentrated near its mean value.

We obtain the asymptotics of \( C(m) \) by relying on a technique used by Y.V. Fyodorov \[9\] in a related context. This reduces the asymptotics of the integral \( I_m \) to known asymptotics of the 1-point correlation function in random matrix theory, more precisely, Wigner’s semi-circle law.

The paper is structured as follows. Section 2 contains a description of the integral formula alluded to above, including several reformulations in the language of random processes. Section 3 contains the proof of the asymptotic estimate (1.1), while section 4 contains the proof of the estimate (1.2). For the reader’s convenience we have included in Appendix A a brief survey of the main facts about Gaussian measures and Gaussian processes used in the proof, while Appendix B contains a detailed description of a family of Gaussian measures on the space \( S_m \) of real, symmetric \( m \times m \) matrices. These measures play a central role in the proof of (1.1) and we could not find an appropriate reference for the mostly elementary facts discussed in this appendix.

2. A SINGLE INTEGRAL FORMULA

A key component in the proof of Theorem 1.1 is an integral formula that we proved in \[15\]. We recall it in this section, and then we formulate it in the language of random fields à la \[1\].

Suppose \( M \) is a compact manifold without boundary. Set \( m := \dim M \). For any nonnegative integer \( k \), any point \( p \in M \) and any \( f \in C^\infty(M) \) we will denote by \( j_k(f, p) \) the \( k \)-th jet of \( f \) at \( p \).

Fix a finite dimensional vector space \( U \subset C^\infty(M) \). Set \( N := \dim U \). We have an evaluation map

\[ \text{ev} = \text{ev}^U : M \to U^\vee := \text{Hom}(U, \mathbb{R}), \quad p \mapsto \text{ev}_p, \]

where for any \( p \in M \) the linear map \( \text{ev}_p : U \to \mathbb{R} \) is given by

\[ \text{ev}_p(u) = u(p), \quad \forall u \in U. \]

If \( k \) is nonnegative integer then we say that \( U \) is \( k \)-ample if for any \( p \in M \) and any \( f \in C^\infty(M) \) there exists \( u \in U \) such that

\[ j_k(u, p) = j_k(f, p). \]

In the remainder of this section we will assume that \( U \) is 1-ample. This implies that the evaluation map \( \text{ev}^U \) is an immersion. Moreover, as explained in \[14, \S 1.2\], the 1-ampleness condition also implies that almost all functions \( u \in U \) are Morse functions and thus have finite critical sets. For any \( u \in U \) we denote by \( N(u) \) the number of critical points of \( u \).

We fix an inner product \( h(-, -) \) on \( U \) and we denote by \( \| - \|_h \) the resulting Euclidean norm. We will refer to the pair \((U, h)\) as the sample space. We set

\[ S_h(U) := \{ u \in U; \quad |u|_h = 1 \}. \]

Using the metric \( h \) we can regard the evaluation map a smooth map

\[ \text{ev} : M \to U. \]

\[ ^2 \text{In } [15] \text{ we proved that } q(S^4) \approx 0.4518. \]
We define the expected number of critical points of a function in \( U \) to be the quantity

\[
\mathcal{N}(U, h) := \frac{1}{\sigma_{n-1}} \int_{S_h(U)} \mathcal{N}(u) |dA_h(u)| = \int_U \mathcal{N}(u) \frac{|u|^2}{(2\pi)^{n-2}} |dV_h(u)|,
\]

(2.1)

where \( \sigma_{n-1} \) denotes the "area" of the unit sphere in \( \mathbb{R}^n \),

\[
\sigma_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.
\]

\(|dA_h|\) denotes the "area" density on \( S_h(U) \), and \(|dV_h(u)|\) denotes the volume density on \( U \) determined by the metric \( h \). A priori, the expected number of critical points could be infinite, but in any case, it is independent of any choice of metric on \( M \).

The integral formula needed in the proof of Theorem 1.1 expresses \( \mathcal{N}(U, h) \) as the integral of an explicit density on \( M \). To describe this formula it is convenient to fix a metric \( g \) on \( M \). We will express \( \mathcal{N}(U, h) \) as an integral

\[
\int_M \rho_g(p) |dV_g(p)|.
\]

The function \( \rho_g \) does depend on \( g \) but the density \( \rho_g(p) |dV_g(p)| \) is independent of \( g \). The concrete description of \( \rho_g(p) \) relies on several fundamental objects naturally associated to the triplet \( (U, h, g) \).

For any \( p \in M \) we set

\[
U_p^0 := \{ u \in U; \ dA(u) = 0 \}.
\]

The 1-ampleness assumption on \( U \) implies that for any \( x \in M \) the subspace \( U_p^0 \) has codimension \( m \) in \( U \) so that

\[
\dim U_p^0 = N - m.
\]

The differential of the evaluation map at \( p \) is a linear map \( A_p : T_p M \to U \), and we denote by \( J_g(p) \) its Jacobian, i.e., the norm of the induced map \( \Lambda^m : \Lambda^m T_p M \to \Lambda^m U \). Equivalently, if \( (e_1, \ldots, e_m) \) is \( g \)-orthonormal basis of \( T_p M \), then

\[
J_g(p)^2 = \det \left[ h(A_p e_i, A_p e_j) \right]_{1 \leq i, j \leq m}.
\]

Since \( ev^U \) is an immersion we have \( J_g(p) \neq 0, \forall x \in M \).

For any \( p \in M \) and any \( u \in U_p^0 \), the Hessian of \( u \) at \( p \) is a well defined symmetric bilinear form on \( T_p M \) that can be identified via the metric \( g \) with a symmetric endomorphism of \( T_p M \). We denote this symmetric endomorphism by \( Hess_p(u, g) \). In [15] we proved that

\[
\mathcal{N}(U, h) = (2\pi)^{-\frac{n}{2}} \int_M \frac{1}{J_g(p)} \left( \int_{U_p^0} \left| \det Hess_p(u, g) \right| \frac{|u|^2}{(2\pi)^{n-m}} |dV_h(u)| \right) |dV_g(p)|.
\]

(2.2)

This formula looks hopeless in a general context for two immediately visible reasons.

- The Jacobian \( J_g(p) \) seems difficult to compute.
- The integral \( I_p \) in (2.2) may be difficult to compute since the domain of integration \( U_p^0 \) may be impossible to pin down.
We will deal with these difficulties simultaneously by relying on some probabilistic principles inspired from [1]. For the reader’s convenience we have gathered in Appendix A the basic probabilistic notions and facts need in the sequel.

We denote by $\sigma = \sigma_U$ the pullback of the metric $h$ on $U$ via the evaluation map. We will refer to it as the stochastic metric associated to the sample space $(U, h)$. It is convenient to have a local description of the stochastic metric.

Fix an orthonormal basis $\psi_1, \ldots, \psi_N$ of $U$. The evaluation map $eva^U : M \to U$ is then given by

$$M \ni x \mapsto \sum_n \psi_n(x) : \psi_n \in U.$$  

If $p \in M$ and $U$ is an open coordinate neighborhood of $p$ with coordinates $x = (x^1, \ldots, x^m)$ then

$$\sigma_p(\partial_{x^i}, \partial_{x^j}) = \sum_n \frac{\partial \psi_n}{\partial x^i}(p) \frac{\partial \psi_n}{\partial x^j}(p), \quad \forall 1 \leq i, j m. \quad (2.3)$$

Note that if the collection $(\partial_{x^i})_{1 \leq i \leq m}$ forms an $g$-orthonormal frame of $T_p M$ then

$$J_g(p)^2 = \det \left[ \sigma_p(\partial_{x^i}, \partial_{x^j}) \right]_{1 \leq i, j \leq m}. \quad (2.4)$$

To the sample space $(U, h)$ we associate in a tautological fashion a Gaussian random field on $M$ as follows. The measure $d\gamma_h$ in (2.1) is a probability measure and thus $(U, d\gamma_h)$ is naturally a probability space. We have a natural map

$$\xi : M \times U \to \mathbb{R}, \quad M \times U \ni (p, u) \mapsto \xi_p(u) := u(p)$$

The collection of random variables $(\xi_p)_{p \in M}$ is a random field on $M$. As explained in Appendix A this is in fact a Gaussian random field.

Using the orthonormal basis $(\psi_k)$ of $U$ we obtain a linear isometry

$$\mathbb{R}^N \ni t = (t_1, \ldots, t_n) \mapsto u_t = \sum_k t_k \psi_k \in U,$$

with inverse $u \mapsto t_k(u) = h(u, \psi_k)$. For any $p \in M$ and any $t \in \mathbb{R}^N$ we have

$$\xi_p(u_t) = \sum_k t_k \psi_k(p).$$

The covariance kernel of this field is the function $\mathcal{E} = \mathcal{E}_U : M \times M \to \mathbb{R}$ given by

$$\mathcal{E}(p, q) = \mathcal{E}(\xi_p, \xi_q) = \sum_{j, k=1}^N \left( \int_{\mathbb{R}^N} t_j t_k d\gamma_N(t) \right) \psi_j(p) \psi_k(q)$$

$$\quad = \sum_{k=1}^M \psi_k(p) \psi_k(q), \quad (2.5)$$

where $d\gamma_N$ is the canonical Gaussian measure on $\mathbb{R}^N$.

If $p \in M$ and $U$ is an open coordinate neighborhood of $p$ with coordinates $x = (x^1, \ldots, x^m)$ such that $x(p) = 0$, then we can rewrite (2.3) in terms of the covariance kernel alone

$$\sigma_p(\partial_{x^i}, \partial_{x^j}) = \frac{\partial^2 \mathcal{E}(x, y)}{\partial x^i \partial y^j}|_{x=y=0}. \quad (2.6)$$

Note that for any vector field $X$ determines a new Gaussian random field on $M$, the derivative of $u$ along $X$. We obtain Gaussian random variables

$$u \mapsto Xu(p), \quad u \mapsto Yu(p),$$
The last equality justifies the attribute stochastic attached to the metric $\sigma$.

We denote by $\nabla$ the Levi-Civita connection of the metric $g$. The Hessian of a smooth function $f : \to \mathbb{R}$ with respect to the metric $g$ is the symmetric $(0,2)$-tensor $\nabla^2 f$ on $M$ defined by the equality
\[
\nabla^2 f(X,Y) := XY f - (\nabla_X Y)f, \quad \forall X,Y \in \text{Vect}(M).
\]

If $p$ is a critical point of $f$ then $\nabla^2_pf$ is the usual Hessian of $f$ at $p$. More generally, if $(x^1, \ldots, x^m)$ are $g$-normal coordinates at $p$ then
\[
\nabla^2_pf(\partial_{x^i}, \partial_{x^j}) = \partial^2_{x^ix^j}f(p), \quad \forall 1 \leq i, j \leq m.
\]

For any $p \in M$ and any $f \in C^\infty(M)$ we identify using the metric $g_p$ the bilinear form $\nabla^2_pf$ on $T_pM$ with an element of $S(T_pM)$, the vector space of symmetric endomorphisms of the Euclidean space $(T_pM, g_p)$. For any $p \in M$ we have two random Gaussian vectors
\[
U \ni u \mapsto \nabla^2_pu \in S(T_pM), \quad U \ni u \mapsto du(p) \in T^*_xM.
\]

Note that the expectation of both random vectors are trivial while (2.6) shows that the covariance form of $du(p)$ is the metric $\sigma_p$.

To proceed further we need to make an additional assumption on the sample space $U$. Namely, in the remainder of this section we will assume that it is 2-ample. In this case the map
\[
U \ni u \mapsto \nabla^2_pu \in S(T_pM)
\]
is surjective so the Gaussian random vector $\nabla^2_pu$ is nondegenerate. A simple application of the co-area formula shows that the integral $I_p$ in (2.2) can be expressed as a conditional expectation
\[
I_p = E\left( |\det \nabla^2_pu| \mid du(p) = 0 \right).
\]

The covariance form of the pair of random variables $\nabla^2_pu$ and $du(p)$ is the bilinear map
\[
\Omega : S(T_pM)^\vee \times T_pM \to \mathbb{R},
\]
\[
\Omega(\xi, \eta) = E\left( \langle \xi, \nabla^2_pu \rangle \cdot \langle du, \eta \rangle \right), \quad \forall \xi \in S^\vee_m, \quad \eta \in T_pM.
\]

Using the natural inner products on $S(T_pM)$ and $T_pM$ defined by $g_p$ we can regard the covariance form as a linear operator
\[
\Omega_p : T_pM \to S(T_pM).
\]

Similarly, we can identify the covariance forms of $\nabla^2_pu$ and $du$ with symmetric positive definite operators
\[
S\nabla^2_pu : S(T_pM) \to S(T_pM)
\]
and respectively
\[
S_{du(p)} : T_pM \to T_pM.
\]

Using the regression formula (A.4) we deduce that
\[
E\left( |\det \nabla^2_pu| \mid du(p) = 0 \right) = E( |\det Y_p|),
\]

where $Y_p : U \to S(T_pM)$ is a Gaussian random vector with mean value zero and covariance operator
\[
\Xi_p = \Xi_{Y_p} := S\nabla^2_pu - \Omega_p^{-1} \Omega_p^{\dagger} : S(T_pM) \to S(T_pM).
\]

When $\Xi_p$ is invertible we have
\[
E( |\det Y_p|) = (2\pi)^{-\frac{\dim S(T_pM)}{2}} (\det \Xi_p)^{-\frac{1}{2}} \int_{S(T_pM)} |\det Y| e^{-\frac{\langle Y, \Xi_p^{-1} Y \rangle}{2}} dV_g(Y). \quad \text{(2.10)}
\]
Observing that 
\[ J_g(p) = (\det S_{dU(p)})^{\frac{1}{2}}, \]
we deduce that when \( U \) is 2-ample we have
\[ N(U, h) = \frac{1}{(2\pi)^m} \int_M (\det S_{dU(p)})^{-\frac{1}{2}} E(|\det Y_p|) |dV_g(p)|, \]
where \( Y_p \) is a Gaussian random symmetric endomorphism of \( T_p M \) with expectation 0 and covariance operator \( \Xi_p \) described by (2.9).

To compute the above integral we choose normal coordinates \((x_1, \ldots, x^n)\) near \( p \) and thus we can orthogonally identify \( T_p M \) with \( \mathbb{R}^m \). We can view the random variable \( \nabla^2_{\xi} u \) as a random variable
\[ H^P: U \rightarrow S_m := \mathbb{S}(\mathbb{R}^m), \quad U \ni u \mapsto H^P(u) \in S_m, \quad H^P_i(u) = \partial^2_{x_i x_j} u(p), \]
and the random variable \( dU(p) \) as a random variable
\[ D^P: U \rightarrow \mathbb{R}^m, \quad u \mapsto D^P u \in \mathbb{R}^m, \quad D^P_i u = \partial_{x_i} u(p). \]
The covariance operator \( S_{dU(p)} \) of the random variable \( D^P \) is given by the symmetric \( m \times m \) matrix with entries
\[ \sigma_p(\partial_{x_i}, \partial_{x_j}) = \frac{\partial^2 \mathcal{E}(x, y)}{\partial x^i \partial y^j}|x=y=0. \]
To compute the covariance form \( \Sigma_{H^P} \) of the random matrix \( H^P \) we observe first that we have a canonical basis \( (\xi_{ij})_{1 \leq i \leq j \leq m} \) of \( S^\vee_m \) so that \( \xi_{ij} \) associates to a symmetric matrix \( A \) the entry \( a_{ij} \) located in the position \((i, j)\). Then
\[ (\Sigma_{H^P})_{ij, kl} = E(\ H^P_{ij}(u), H^P_{kl}(u) \ ) = E(\ \partial^2_{x_i x_j} u(x) \partial^2_{x_k x_l} u(x) \ ) \]
\[ = \sum\limits_{n=1}^{N} \partial^2_{x_i x_j} \psi_n(x) \partial^2_{x_k x_l} \psi_n(x) = \frac{\partial^4 \mathcal{E}(x, y)}{\partial x^i \partial x^j \partial y^k \partial y^l}|x=y=0. \]
Similarly we have
\[ \Omega(\xi_{ij}, \partial_{x_k}) = E(\partial^2_{x_i x_j} u(p), \partial_{x_k} u(p) \ ) = \frac{\partial^3 \mathcal{E}(x, y)}{\partial x^i \partial x^j \partial y^k}|x=y=0. \]
To identify \( \Omega \) with an operator it suffices to observe that \( (\partial_{x_k}) \) is an orthonormal basis of \( T_p M \), while the collection \( \{\xi_{ij}\}_{1 \leq i, j \leq m} \subset S^\vee_m \)
\[ \xi_{ij} = \begin{cases} \xi_{ij}, & i = j \\ \sqrt{2} \xi_{ij}, & i < j \end{cases} \]
is an orthonormal basis of \( S^\vee_m \). If we denote by \( \hat{\xi}_{ij} \) the dual orthonormal basis of \( S_m \), then
\[ \Omega \partial_{x_k} = \sum_{1 \leq i, j} \Omega(\xi_{ij}, \partial_{x_k}) \hat{\xi}_{ij}. \]

3. The proof of Theorem 1.1

We fix an orthonormal basis of \( L^2(M, g) \) consisting of eigenfunctions \( \Psi_n \) of \( \Delta_g \),
\[ \Delta_g \Psi_n = \lambda_n \Psi_n, \quad n = 0, 1, \ldots, \quad \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots. \]
The collection \( \{\Psi_n\}_{n \leq L} \) is therefore an orthonormal basis of \( U_L \) so that the covariance kernel of the Gaussian field determined by \( U_L \) is
\[ \mathcal{E}_L(p, q) = \sum_{\lambda_n \leq L} \Psi_n(p) \Psi_n(q). \]
This function is also known as the spectral function associated to the Laplacian. Observe that

\[ \int_M \mathcal{E}_L(p, p) |dV_g(p)| = \dim U_L. \]

In the groundbreaking work [11], L. Hörmander used the kernel of the wave group \( e^{it\sqrt{-\Delta}} \) to produce refined asymptotic estimates for the spectral function. More precisely he showed (see [11] or [12, §17.5])

\[ \mathcal{E}_L(p, p) = \frac{\omega_m}{(2\pi)^m} L^\frac{m}{2} + O(L^{\frac{m-1}{2}}) \quad \text{as} \quad L \to \infty, \]  

(3.1)

uniformly with respect to \( p \in M \). Above, \( \omega_m \) denotes the volume of the unit ball in \( \mathbb{R}^m \). This implies immediately the classical Weyl estimates

\[ \dim U_L \approx \frac{\omega_m}{(2\pi)^m} \text{vol}_g(M) L^\frac{m}{2}. \]  

(3.2)

Recently (2004), X. Bin [4] used Hörmander’s approach to produce substantially refined asymptotic estimates of the behavior of the spectral function in a infinitesimal neighborhood of the diagonal.

To formulate these estimates we set \( \lambda := L^\frac{1}{2} \). Fix a point \( p \) and normal coordinates \( x = (x^1, \ldots, x^m) \) at \( p \). Note that \( x(p) = 0 \). For any multi-indices \( \alpha, \beta \in \mathbb{Z}_{\geq 0}^m \) we set

\[ \mathcal{E}_L^{\alpha, \beta}(p) := \frac{\partial^{\alpha+\beta} \mathcal{E}(x, y)}{\partial x^\alpha \partial y^\beta} \bigg|_{x=y=0} \]

X. Bin proved that for any multi-indices \( \alpha, \beta \in \mathbb{Z}_{\geq 0}^m \) we have

\[ \mathcal{E}_L^{\alpha, \beta}(p) = C_m(\alpha, \beta) \lambda^{m+|\alpha+|\beta|} + O\left( \lambda^{m+|\alpha+|\beta|} \right) \]

(3.3)

where

\[ C_m(\alpha, \beta) = \begin{cases} 
0, & \alpha - \beta \notin (2\mathbb{Z})^m \\
\left(\frac{-1}{(2\pi)^m}\right)^{|\alpha+|\beta|} \int_{B^m} x^{\alpha+\beta} |dx|, & \alpha - \beta \in (2\mathbb{Z})^m,
\end{cases} \]

(3.4)

and \( B^m \) denotes the unit ball

\[ B^m = \{ x \in \mathbb{R}^m; \ |x| = 1 \}. \]

The estimates (3.3) are uniform in \( p \in M \). Using (A.6) we deduce (compare with (B.13))

\[ \frac{1}{(2\pi)^m} \int_{B^m} x^{2\alpha+2\beta} |dx| = \frac{1}{(4\pi)^\frac{m}{2} \Gamma(1 + \frac{m}{2})} \int_{\mathbb{R}^m} x^{\alpha+\beta} e^{-|x|^2 / \pi} |dx|. \]

We set

\[ K_m = C_m(\alpha, \alpha), \quad |\alpha| = 1, \]

so that

\[ K_m = \frac{1}{(4\pi)^\frac{m}{2} \Gamma(2 + \frac{m}{2})} \int_{\mathbb{R}^m} x^1 e^{-|x|^2 / \pi} |dx| = \frac{1}{2(4\pi)^\frac{m}{2} \Gamma(2 + \frac{m}{2})}. \]

For any \( i \leq j \) define \( \alpha_{ij} \in \mathbb{Z}^m \) so that

\[ x^{\alpha_{ij}} = x_i x_j. \]

For \( i \leq j \) and \( k \leq j \) we set

\[ C_m(i, j; k, \ell) = C_m(\alpha_{ij}, \alpha_{k\ell}) = \frac{1}{(4\pi)^\frac{m}{2} \Gamma(3 + \frac{m}{2})} \int_{\mathbb{R}^m} x_i x_j x_k x_\ell e^{-|x|^2 / \pi} |dx|. \]
For \( i < j \) we have
\[
C_m(i, i; j, j) = \frac{1}{(4\pi)^{m/2} \Gamma(3 + \frac{m}{2})} \int_{\mathbb{R}^m} x_i^2 x_j^2 e^{-|x|^2} |dx| = \frac{1}{4(4\pi)^{m/2} \Gamma(3 + \frac{m}{2})} =: c_m.
\]
Finally
\[
C_m(i, i; i, i) = \frac{1}{(4\pi)^{m/2} \Gamma(3 + \frac{m}{2})} \int_{\mathbb{R}^m} x_i^4 e^{-|x|^2} |dx| = \frac{3}{4(4\pi)^{m/2} \Gamma(3 + \frac{m}{2})} = 3c_m,
\]
and
\[
C_m(i, j; i, j) = C_m(i, i; j, j),
\]
We denote by \( \sigma^L \) the stochastic metric on \( M \) determined by the sample space \( U_L, L \gg 0 \). As explained in the previous section the covariance form of the random vector \( U_L \ni \mathbf{u} \mapsto d\mathbf{u}(p) \in T_p^*M = \sigma^L_p \), and from (3.3) we deduce
\[
\sigma^L_p(\partial_{x^i}, \partial_{x^j}) = \frac{\partial^2 \mathcal{E}_L(x, y)}{\partial x^i \partial y^j} \bigg|_{x=y=0} = K_m \lambda^{m+2} \delta_{ij} + O(\lambda^{m+1})
\]
\[
= K_m \lambda^{m+2} g_p(\partial_{x^i}, \partial_{x^j}) + O(\lambda^{m+1}) \text{ as } L \to \infty, \text{ uniformly in } p. \tag{3.5}
\]
In particular, if \( S^L_{du(p)} \) denotes the covariance operator of the random vector \( d\mathbf{u}(p) \), then we deduce from the above equality that
\[
S^L_{du(p)} = K_m \lambda^{m+2} \mathbb{1}_m + O(\lambda^{m+1}), \text{ uniformly in } p, \tag{3.6}
\]
and invoking (2.11) we deduce
\[
J_g^L(p) = \left( \det S^L_{du(p)} \right)^{\frac{1}{2}} = K_m^\frac{m}{2} \lambda^{\frac{m(m+2)}{2}} + O(\lambda^{\frac{m(m+2)}{2} - 1}), \text{ uniformly in } p. \tag{3.7}
\]
Denote by \( \Sigma^L_{HP} \) the covariance form of the random matrix
\[
U_L \ni \mathbf{u} \mapsto \nabla^2_{\mathbf{u}} \mathbf{u} \in S(T_p^*M) = S_m.
\]
Using (2.13) and (3.3) we deduce
\[
\Sigma^L_{HP} = c_m \lambda^{m+4} \Sigma_{3,1,1} + O(\lambda^{m+3}), \text{ uniformly in } p, \tag{3.8}
\]
where the positive definite, symmetric bilinear form \( \Sigma_{3,1,1} : S^\vee_m \times S^\vee_m \to \mathbb{R} \) is described by the equalities (B.2a) and (B.2b). We denote by \( \Gamma_{3,1,1} \) the centered Gaussian measure on \( S_m \) with covariance form \( \Sigma_{3,1,1} \).

The equality (2.14) coupled with (3.3) imply that the covariance operator \( \Omega^L_p \) satisfies
\[
\Omega^L_p = O(\lambda^{m+2}), \text{ uniformly in } p. \tag{3.9}
\]
Using (3.6), (3.8) and (3.9) we deduce that the covariance operator \( \Xi^L_p \) defined as in (2.9) satisfies the estimate
\[
\Xi^L_p = c_m \lambda^{m+4} \hat{Q}_{3,1,1} + O(\lambda^{m+2}), \text{ as } L \to \infty, \text{ uniformly in } p, \tag{3.10}
\]
where \( \hat{Q}_{3,1,1} \) is the covariance operator associated to the covariance form \( \Sigma_{3,1,1} \) and it is described explicitly in (B.3). If we denote by \( d\Gamma_L \) the Gaussian measure on \( S_m \) with covariance operator \( \Xi^L_p \), we deduce that
\[
d\Gamma_L(Y) = \frac{1}{(2\pi)^{Nm/2} (\det \Xi^L_p)^{\frac{1}{2}}} e^{-\frac{\langle Y, Y \rangle}{2}} \cdot 2^{\frac{m}{2}} \prod_{i \leq j} dy_{ij},
\]
where
\[ N_m = \dim S_m = \frac{m(m + 1)}{2}. \]

Let us observe that \(|dY|\) is the Euclidean volume element on \(S_m\) defined by the natural inner product on \(S_m\), \((X, Y) = \text{tr}(XY)\). We set
\[ c_L := c_m \lambda^{m+4}, \quad Q_p^L = \frac{1}{c_L} \Xi_p^L. \]

Using (A.7) we deduce that
\[
\frac{1}{(2\pi)^{\frac{N_m}{2}} (\det \Xi_p^L)^{\frac{1}{2}}} \int_{S_m} |\det Y| e^{-\frac{(\Xi_p^L Y, Y)}{2}} |dY| = \frac{(c_L)^m}{(2\pi)^{\frac{N_m}{2}} (\det Q_p^L)^{\frac{1}{2}}} \int_{S_m} |\det Y| e^{-\frac{(Q_p^L Y, Y)}{2}} |dY|.
\]

From the estimate (3.10) we deduce that
\[ Q_p^L \to \hat{Q}_{3,1,1} \text{ as } L \to \infty, \text{ uniformly in } p. \]

We conclude that
\[ E(\det Y_p) = \int_{S_m} |\det Y| d\Gamma_L(Y) \sim c_m^\frac{m}{2} \lambda^{m(m+4)/2} \int_{S_m} |\det Y| d\Gamma_{3,1,1}(Y). \quad (3.11) \]

The measure \(d\Gamma_{3,1,1}\) is described explicitly in (B.11), more precisely
\[ d\Gamma_{3,1,1}(Y) = \frac{1}{(2\pi)^{\frac{N_m}{2}} \sqrt{\mu_m}} \cdot e^{-\frac{1}{4} \left( \text{tr} Y^2 - \frac{1}{m+2} (\text{tr} Y)^2 \right)} |dY|,
\]
where \(\mu_m\) is given by (B.12). Using (2.12), (3.7) and (3.11) we deduce that
\[ E(N_L) \sim \left( \frac{c_m}{K_m} \right)^{\frac{m}{2}} \lambda^{\frac{m(m+4)}{2} - \frac{m(m+2)}{2}} \text{vol}_p(M) \int_{S_m} |\det Y| d\Gamma_{3,1,1}(Y)
\]
\[ \sim \left( \frac{c_m}{K_m} \right)^{\frac{m}{2}} \left( \frac{2\pi}{\omega_m} \right)^m \dim U_L. \quad (3.2)
\]

Observe that
\[ \frac{c_m}{K_m} = \frac{\Gamma(2 + \frac{m}{2})}{2\Gamma(3 + \frac{m}{2})} = \frac{1}{m+4}, \quad \omega_m = \frac{\pi^m}{\Gamma(1 + \frac{m}{2})}, \quad \frac{2\pi^m}{\omega_m} = (4\pi)^{\frac{m}{2}} \Gamma \left( 1 + \frac{m}{2} \right).
\]

This completes the proof of (1.1) and (1.4). \(\Box\)

4. THE PROOF OF THEOREM 1.2

We begin by describing the large \(m\) behavior of the integral
\[ I_m := \frac{1}{(2\pi)^{\frac{m(m+1)}{4}} \sqrt{\mu_m}} \int_{S_m} |\det X| e^{-\frac{1}{4} \left( \text{tr} X^2 - \frac{1}{m+4} (\text{tr} X)^2 \right)} |dX|. \]

We follow the strategy in [9]; see also [8, §1.5]. Recall first the classical equality
\[ \int_{\mathbb{R}} e^{-\left( at^2 + bt + c \right)} |dt| = \left( \frac{\pi}{a} \right)^{\frac{1}{2}} e^{\frac{b^2}{4a}} \Delta = b^2 - 4ac, \quad a > 0,
\]
which follows from the well known identity
\[ at^2 + bt + c = a \left( t - \frac{b}{2a} \right)^2 - \frac{\Delta}{4a}. \]
For any real numbers $u, v, w$, we have
\[
(u^2 + v \text{tr}(X + wt1_m))^2 = (u + mw^2)^2 + 2vw \text{tr} X + v \text{tr} X^2
\]
\[= a(u, v, w)^2 + b(u, v, w)t + c(u, v, w).\]

We seek $u, v, w$ such that
\[
b^2 - 4ac = -\frac{1}{4} \left( \text{tr} X^2 - \frac{1}{m + 2} (\text{tr} X)^2 \right).
\]

We have
\[
b^2 - 4ac = \frac{v^2 w^2}{u + mw^2} (\text{tr} X)^2 - \frac{v}{u + mw^2} \text{tr} X^2
\]
This implies
\[
v = \frac{1}{4}, \quad \frac{v^2 w^2}{u + mw^2} = \frac{1}{4(m + 2)}.
\]

We deduce
\[
v w^2 = \frac{1}{(m + 2)}, \quad v = \frac{1}{4} (u + mw^2) \iff u = 4v - mw^2.
\]

Hence
\[
w^2 = \frac{1}{v(m + 2)}, \quad u = 4v - \frac{m}{v(m + 2)}.
\]

We choose $v = \frac{1}{2}$ so that
\[
w^2 = \frac{2}{m + 2}, \quad u = 2 - \frac{2m}{m + 2} = \frac{4}{m + 2}, \quad a(u, v, w) = 4v = 2,
\]
\[e^{-\frac{1}{4} (\text{tr} X^2 - \frac{1}{m + 2} (\text{tr} X)^2)} = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{ut^2}{m + 2}} e^{-\frac{1}{2} \text{tr} (X + t \sqrt{\frac{2}{m + 2} 1_m})^2} dt
\]
\[= \left( \frac{2(m + 2)}{\pi} \right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{1}{2} \text{tr} (X + \sqrt{2} 1_m)^2} e^{-s^2}ds = \left( \frac{m + 2}{2} \right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{1}{2} \text{tr} (X - s 1_m)^2} \cdot \frac{e^{-2s^2}}{\sqrt{\frac{\pi}{2}} ds}.
\]

Hence
\[
I_m = \left( \frac{m + 2}{2\pi} \right)^{\frac{1}{2}} \left( m + \frac{1}{4} \right) \sqrt{\mu_m} \cdot \left( \int_{\mathbb{S}_m} \left| \det X | e^{-\frac{1}{2} \text{tr} (X - s 1_m)^2} |dX| \right) d\gamma(s)
\]
\[= A_m \int_{\mathbb{S}_m} \left( \int_{\mathbb{S}_m} \left| \det (x 1_m - Y) | e^{-\frac{1}{2} \text{tr} Y^2} |dY| \right) d\gamma(x).
\]

For any $O(n)$ invariant function $f : \mathbb{S}_n \to \mathbb{R}$ we have a Weyl integration formula (see [2, 8, 13]).
\[
\int_{\mathbb{S}_n} f(X)|dX| = \frac{1}{2^n} \int_{\mathbb{R}^n} f(\lambda)|\Delta_m(\lambda)| |d\lambda|,
\]
where
\[
\Delta_n(\lambda) := \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i),
\]
and the constant $Z_n$ is defined by the equality

$$Z_n = \frac{\int_{\mathbb{R}^n} e^{-\frac{1}{2}|\lambda|^2} |\Delta_m(\lambda)| \ |d\lambda|}{\int_{\mathbb{R}^n} e^{-\frac{1}{2}trX^2} |dX|}.$$  

The denominator is equal to $(2\pi)^{\frac{\dim S_n}{2}}$ while the numerator is given by ([13, §17.6])

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}|\lambda|^2} |\Delta_m(\lambda)| \ |d\lambda| = 2^{\frac{3n}{2}} \prod_{j=1}^{n} \Gamma\left(1 + \frac{j}{2}\right),$$

so that

$$Z_n = \frac{2^{\frac{3n}{2}} \prod_{j=1}^{n} \Gamma\left(1 + \frac{j}{2}\right)}{(2\pi)^{\frac{\dim S_n}{2}}}. \quad (4.1)$$

Now observe that for any $\lambda_0 \in \mathbb{R}$ we have

$$f_m(\lambda_0) = \frac{1}{Z_m} \int_{\mathbb{R}^m} e^{-\frac{1}{2}|\lambda|^2} \prod_{j=1}^{n} |\lambda_j - \lambda_0||\Delta_m(\lambda)| \ |d\lambda|$$

$$= e^{\frac{1}{2}x_3 \lambda_3} \int_{\mathbb{R}^m} e^{-\frac{1}{2}\sum_{i=1}^{m} \lambda_i^2} |\Delta_{m+1}(\lambda_0, \lambda_1, \ldots, \lambda_m)| \ |d\lambda_1 \cdots d\lambda_m|$$

$$= e^{\frac{1}{2}x_3 \lambda_3} \frac{1}{Z_{m+1}} \int_{\mathbb{R}^m} e^{-\frac{1}{2} \sum_{i=1}^{m+1} \lambda_i^2} |\Delta_{m+1}(\lambda_0, \lambda_1, \ldots, \lambda_m)| \ |d\lambda_1 \cdots d\lambda_m|.$$  

The function $\rho_m(x) = nR_m(x)$ is known in random matrix theory as the 1-point correlation function of the Gaussian orthogonal ensemble of symmetric $n \times n$ matrices, [6, §4.4.1], [13, §4.2]. We conclude that

$$I_m = \frac{A_m Z_{m+1}}{Z_m} \int_{\mathbb{R}} R_{m+1}(x) e^{-\frac{x^2}{2}} d\gamma(x) = \frac{A_m Z_{m+1}}{Z_m} \int_{\mathbb{R}} R_{m+1}(x) \sqrt{\frac{2}{\pi}} e^{-\frac{3x^2}{2}} dx.$$  

We have

$$Z_{m+1} \over Z_m = \frac{2^{\frac{3}{2}} \Gamma\left(m + \frac{3}{2}\right)}{2^{m+1} \cdot 2^{m+1}} ,$$

$$\sqrt{\frac{2}{\pi}} A_m Z_{m+1} = \frac{2^{\frac{3}{2}} \Gamma\left(m + \frac{3}{2}\right)}{2^{2m+1} \cdot 2^{m+1}} \times \sqrt{\frac{2}{\pi}} \times \frac{(m + 2)^{\frac{3}{2}}}{2^{\frac{1}{2}}(2\pi)^{\frac{m(m+1)}{4}}} \sqrt{\mu_m}$$

$$= \frac{\Gamma\left(m + \frac{3}{2}\right)}{2^{m-1} \sqrt{\pi}} \times \frac{1}{2^{m(m+1)} \cdot 2^{m+1} \cdot \frac{1}{4} (m + 2)^{\frac{m-2}{2}}} = \frac{\Gamma\left(m + \frac{3}{2}\right)}{\sqrt{\pi} 2^{m^2 + \frac{1}{2} (m + 2)^{\frac{m-2}{2}}}}.$$  

We deduce

$$I_m = \frac{\Gamma\left(m + \frac{3}{2}\right)}{\sqrt{\pi} 2^{m^2 + \frac{1}{2} (m + 2)^{\frac{m-2}{2}}} \int_{\mathbb{R}} R_{m+1}(x) e^{-\frac{3x^2}{2}} dx$$

$$= \frac{\Gamma\left(m + \frac{3}{2}\right)}{\sqrt{\pi} 2^{m^2 + \frac{1}{2} (m + 2)^{\frac{m-2}{2}}}} \int_{\mathbb{R}} \rho_{m+1}(x) e^{-\frac{3x^2}{2}} dx.$$  

(4.2)
To proceed further we use as guide Wigner’s theorem, [2, Thm. 2.1.1] stating that the sequences of probability measures $\bar{\rho}_n(x)dx$,

$$\bar{\rho}_n(x) := \frac{1}{\sqrt{n}} \rho_n(\sqrt{n}x),$$

converges weakly to the semi-circle probability measure $^3\rho(x)dx$,

$$\rho(x) = \frac{1}{\pi} \begin{cases} \sqrt{2 - x^2}, & |x| \leq \sqrt{2} \\ 0, & |x| > \sqrt{2}. \end{cases}$$

(4.3)

We deduce

$$\int_\mathbb{R} \rho_n(x)e^{-\frac{x^2}{2}}dx = n \int_\mathbb{R} \rho_n(\sqrt{n}s)e^{-\frac{3ns^2}{2}}|ds| = n^\frac{3}{2} \int_\mathbb{R} e^{-\frac{3ns^2}{2}} \bar{\rho}_n(s)ds$$

$$= \left( \frac{2\pi}{3} \right)^{\frac{1}{2}} n\int_\mathbb{R} \frac{(3n)^{\frac{1}{2}}e^{-\frac{3ns^2}{2}}}{(2\pi)^{\frac{3}{2}}} \cdot \bar{\rho}_n(s)ds. \tag{4.4}$$

We observe that the Gaussian measures $w_n(s)ds$ converge to the Dirac delta measure concentrated at the origin. The correlation $\rho_n(x)$ in terms of Hermite polynomials, [13, Eq. (7.2.32) and §A.9],

$$\rho_n(x) = \sum_{k=0}^{n-1} \psi_k(x)^2 + \frac{n}{2} \int_\mathbb{R} \varepsilon(x - t)\psi_n(t)dt + \alpha_n(x), \tag{4.5}$$

where

$$\psi_n(x) = \frac{1}{(2^n n! \sqrt{\pi})^{\frac{1}{2}}} e^{-\frac{x^2}{2}} H_n(x), \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}),$$

$$\alpha_n(x) = \begin{cases} 0, & n \in 2\mathbb{Z} + 1, \\ \frac{\psi_n(x)}{\int_k \psi_n(x)dx}, & n \in \mathbb{Z}, \end{cases}$$

and

$$\varepsilon(x) = \begin{cases} \frac{1}{2}, & x > 0 \\ 0, & x = 0, \\ -\frac{1}{2}, & x < 0. \end{cases}$$

From the Christoffel-Darboux formula [16, Eq. (5.5.9)] we deduce

$$\pi^{\frac{1}{2}} e^{x^2} \sum_{k=0}^{n-1} \psi_k(x)^2 = \sum_{k=1}^{n-1} \frac{1}{2^k k!} H_k(x)^2 = \frac{1}{2^n (n-1)!} \left( H'_n(x)H_{n-1}(x) - H_n(x)H'_n(x) \right)$$

Using the recurrence formula $H'_n(x) = 2xH_n(x) - H_{n+1}(x)$ we deduce

$$H'_n(x)H_{n-1}(x) - H_n(x)H'_n(x) = H^2_n(x) - H_{n-1}(x)H_{n+1}(x)$$

and

$$k_n(x) = \frac{e^{-x^2}}{2^n (n-1)! \pi^{\frac{1}{2}}} \left( H^2_n(x) - H_{n-1}(x)H_{n+1}(x) \right).$$

---

3The difference between our definition (4.3) of the semicircle measure and the one in [2, Thm. 2.1.1] is due to the fact that the covariances of our random matrices differ by a factor from those in [2]. Our conventions agree with those in [13].
We set
\[ \bar{k}_n(x) := \frac{k_n(\sqrt{n}x)}{\sqrt{n}}. \]
The integrals entering into the definition of \( \ell_n \) in (4.4) can be given a more explicit description using the generating function, [16, Eq. (5.5.6)],
\[ \sum_{n=0}^{\infty} H_n(x) \frac{w^n}{n!} = e^{2wx-w^2}. \]
Using the refined asymptotic estimates for Hermite polynomials [7] and [16, Thm.8.22.9, Thm.8.91.3] we deduce that
\[ \lim_{n \to \infty} \int_{\mathbb{R}} (\bar{\rho}_n(s) - \rho(x)) w_n(s) ds = 0. \]
On the other hand,
\[ \lim_{n \to \infty} \int_{\mathbb{R}} \rho(s) w_n(s) ds = \rho(0) = \frac{\sqrt{2}}{\pi}. \]
Using the above equality in (4.2) and (4.4) we deduce that
\[ I_m \sim \frac{2\Gamma(m + \frac{3}{2})}{\pi 2^{m^2+m-1}(m + 2)^{\frac{m^2}{2}}} \quad \text{as} \quad m \to \infty. \]
We now invoke Stirling’s formula to conclude that
\[ \Gamma(m + \frac{3}{2}) \sim \sqrt{2\pi} \left( m + \frac{3}{2} \right)^{m + \frac{1}{2}} e^{-m - \frac{3}{2}} \]
\[ \sim \sqrt{2\pi} m^{m+\frac{1}{2}} \left( m + \frac{3}{2m} \right)^{\frac{m}{2}} e^{-m - \frac{3}{2}} \sim \sqrt{2\pi} \left( \frac{m}{e} \right)^m. \]
We have
\[ (m + 2)^{\frac{m^2}{2}} = m^{m^2} \left( m + \frac{2}{m} \right)^{\frac{m^2}{2}} \sim m^{\frac{m^2}{2}} e. \]
Thus
\[ \log I_m \sim \frac{m - 3}{2} \log m \sim \frac{m}{2} \log m, \quad \text{as} \quad m \to \infty. \quad \text{(4.6)} \]
From (1.2) we deduce that
\[ \log C(m) = \log I_m + \frac{m}{2} \log 4\pi + \log \Gamma \left( 1 + \frac{m}{2} \right) - \frac{m}{2} \log(m + 4). \]
Stirling’s formula and (4.6) imply that
\[ \log C(m) \sim \log I_m \quad \text{as} \quad m \to \infty. \]
This proves (1.3). □

**APPENDIX A. GAUSSIAN MEASURES AND GAUSSIAN RANDOM FIELDS**

For the reader’s convenience we survey here a few basic facts about Gaussian measures. For more details we refer to [5]. A **Gaussian measure** on \( \mathbb{R} \) is a Borel measure \( \gamma_{m,\sigma} \) of the form
\[ \gamma_{m,\sigma}(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx. \]
The scalar \( m \) is called the **mean** while \( \sigma \) is called the **standard deviation**. We allow \( \sigma \) to be zero in which case
\[ \gamma_{m,0} = \delta_m = \text{The Dirac measure on} \mathbb{R} \text{ concentrated at} m. \]
Suppose that \( V \) is a finite dimensional vector space. A **Gaussian measure** on \( V \) is a Borel measure \( \gamma \) on \( V \) such that, for any \( \xi \in V^\vee \), the pushforward \( \xi_*(\gamma) \) is a Gaussian measure on \( \mathbb{R} \),
\[ \xi_*(\gamma) = \gamma_{m(\xi),\sigma(\xi)}. \]
One can show that the map \( V^\vee \ni \xi \mapsto m(\xi) \in \mathbb{R} \) is linear, and thus can be identified with a vector \( m_\gamma \in V \) called the **barycenter** or **expectation** of \( \gamma \) that can be alternatively defined by the equality
\[ m_\gamma = \int_V v d\gamma(v). \]
The Gaussian measure \( \gamma \) is uniquely determined by its Fourier transform \( \hat{\gamma} : V^\vee \to \mathbb{R} \),
\[ \hat{\gamma}(\xi) = e^{im(\xi)-\frac{1}{2}\sigma(\xi)^2}. \]
Moreover, there exists a nonnegative definite, symmetric bilinear map
\[ \Sigma : V^\vee \times V^\vee \to \mathbb{R} \]
such that
\[ \sigma(\xi)^2 = \Sigma(\xi,\xi), \quad \forall \xi \in V^\vee. \]
The form \( \Sigma \) is called the **covariance form** and can be identified with a linear operator \( S : V^\vee \to V \) such that
\[ \Sigma(\xi,\eta) = \langle \xi, S\eta \rangle, \quad \forall \xi,\eta \in V^\vee, \]
where \( \langle -, - \rangle : V^\vee \times V \to \mathbb{R} \) denotes the natural bilinear pairing between a vector space and its dual. The operator \( S \) is called the **covariance operator** and it is explicitly described by the integral formula
\[ \langle \xi, S\eta \rangle = \Lambda(\xi,\eta) = \int_V \langle \xi, v - m_\gamma \rangle \langle \eta, v - m_\gamma \rangle d\gamma(v). \]
The Gaussian measure is said to be **nondegenerate** if \( \Sigma \) is nondegenerate, and it is called **centered** if \( m = 0 \). A nondegenerate Gaussian measure on \( V \) is uniquely determined by its covariance form and its barycenter.

**Example A.1.** Suppose that \( U \) is an \( n \)-dimensional Euclidean space with inner product \( \langle -, - \rangle \). We use the inner product to identify \( U \) with its dual \( U^\vee \). If \( A : U \to U \) is a symmetric, positive definite operator, then
\[ d\gamma_A(x) = \frac{1}{(2\pi)^{\frac{1}{2}} \sqrt{\det A}} e^{-\frac{1}{2}(A^{-1}u,u)} |du| \]
(A.1)
is a centered Gaussian measure on \( U \) with covariance form described by the operator \( A \). □
If $V$ is a finite dimensional vector space equipped with a Gaussian measure $\gamma$ and $L : V \to U$ is a linear map then the pushforward $L_{\ast} \gamma$ is a Gaussian measure on $U$ with barycenter

$$m_{L_{\ast} \gamma} = L(m_\gamma)$$

and covariance form

$$\Sigma_{L_{\ast} \gamma} : U^\vee \times U^\vee \to \mathbb{R}$$

given by

$$\Sigma_{L_{\ast} \gamma}(\eta, \eta) = \Sigma_\gamma(L^\vee \eta, L^\vee \eta), \quad \forall \eta \in U^\vee,$$

where $L^\vee : U^\vee \to V^\vee$ is the dual (transpose) of the linear map $L$. Observe that if $\gamma$ is nondegenerate and $L$ is surjective, then $L_{\ast} \gamma$ is also nondegenerate.

Suppose $(S, \mu)$ is a probability space. A Gaussian random vector on $(S, \mu)$ is a (Borel) measurable map

$$X : S \to V, \quad V \text{ finite dimensional vector space}$$

such that $X_{\ast} \mu$ is a Gaussian measure on $V$. We denote this measure by $\gamma_X$ and by $\Sigma_X$ (respectively $S_X$) its covariance form (respectively operator),

$$\Sigma_X(\xi_1, \xi_2) = E(\langle \xi_1, X - E(X) \rangle \langle \xi_2, X - E(X) \rangle).$$

Note that the expectation of $\gamma_X$ is precisely the expectation of $X$. The random vector is called nondegenerate if the Gaussian measure $\gamma_X$ is such.

Suppose that $X_j : S \to V_1$, $j = 1, 2$, are two Gaussian random vectors such that the direct sum

$$X_1 \oplus X_2 : S \to V_1 \oplus V_2$$

is also a Gaussian random vector with associated Gaussian measure

$$\gamma_{X_1 \oplus X_2} = p_{X_1 \oplus X_2}(x_1, x_2) |dx_1| |dx_2|.$$

We obtain a bilinear form

$$\text{cov}(X_1, X_2) : V_1^\vee \times V_2^\vee \to \mathbb{R}, \quad \text{cov}(X_1, X_2)(\xi_1, \xi_2) = \Sigma(\xi_1, \xi_2),$$

called the covariance form. The random vectors $X_1$ and $X_2$ are independent if and only if they are uncorrelated, i.e.,

$$\text{cov}(X_1, X_2) = 0.$$

We can form the random vector $E(X_1 | X_2)$, the conditional expectation of $X_1$ given $X_2$. If $X_1$ and $X_2$ are independent then

$$E(X_1 | X_2) = E(X_1),$$

while at the other end, we have

$$E(X_1 | X_1) = X_1.$$

To find the formula in general we fix Euclidean metrics $(-, -)_V$ on $V$. We can then identify $\text{cov}(X_1, X_2)$ with a linear operator $\text{Cov}(X_1, X_2) : V_2 \to V_1$, via the equality

$$E(\langle \xi_1, X_1 \rangle \langle \xi_2, X_2 \rangle) = \text{cov}(X_1, X_2)(\xi_1, \xi_2)
= \langle \xi_1, \text{Cov}(X_1, X_2)\xi_2 \rangle,
\quad \forall \xi_1 \in V_1^\vee, \quad \xi_2 \in V_2^\vee,$$

where $\xi_2^\bot \in V_2$ denotes the vector metric dual to $\xi_2$. The operator $\text{cov}(X_1, X_2)$ is called the covariance operator of $X_1, X_2$. If $X_2$ is nondegenerate, then we have the regression formula

$$E(X_1 | X_2) = \text{Cov}(X_1, X_2)S^{-1}_{X_2}(X_2 - E(X_2)) + E(X_1). \quad (A.2)$$

To prove this we follow the elegant argument in [3, Prop. 1.2]. Observe first that it suffices to prove that it suffices to prove the equality in the special case when $E(X_1) = 0$ and $E(X_2) = 0$, which is
what we will assume in the sequel. We seek a linear operator \( C : V_2 \to V_1 \) such that the random vector \( Y = X_1 - CX_2 \) is independent of \( X_2 \). If such an operator exists then

\[
E(X_1|X_2) = E(Y|X_2) + E(CX_2|X_2) = E(Y) + CX_2 = CX_2.
\]

Since the random vector \( X_1 - CX_2 \) is Gaussian the operator \( C \) must satisfy the constraint

\[
cov(X_1 - CX_2, X_2) = 0 \iff Cov(X_1 - CX_2, X_2) = Cov(X_1, X_2) - Cov(CX_2, X_2)
\]

To find \( C \) we note that

\[
\langle \xi_1, Cov(CX_2, X_2)\xi_2 \rangle = E\left( \langle \xi_1, CX_2 \rangle \langle \xi_2, X_2 \rangle \right) = E\left( \langle C^\vee \xi_1, X_2 \rangle \langle \xi_2, X_2 \rangle \right) = \Sigma_{X_2}(C^\vee \xi_1, \xi_2) = \langle \xi_1, C\Sigma_{X_2} \xi_2 \rangle
\]

Hence identifying \( V_2 \) with \( V_2^\vee \) via the Euclidean metric \((-,-)_{V_2}\), we can regard \( \Sigma_{X_2} \) as a linear, symmetric nonnegative operator \( V_2 \to V_2 \), and we deduce

\[
Cov(CX_2, X_2) = C\Sigma_{X_2} = Cov(X_1, X_2),
\]

which shows that

\[
C = Cov(X_1, X_2)S_{X_2}^{-1}.
\]

The conditional probability density of \( X_1 \) given that \( X_2 = x_2 \) is the function

\[
p_{X_1|X_2=x_2}(x_1) = \frac{p_{X_1\oplus X_2}(x_1, x_2)}{\int_{V_1} p_{X_1\oplus X_2}(x_1, x_2)|dx_1|}.
\]

For a measurable function \( f : V_1 \to \mathbb{R} \) the conditional expectation \( E(f(X_1)|X_2 = x_2) \) is the (deterministic) scalar

\[
E(f(X_1)|X_2 = x_2) = \int_{V_1} f(x_1)p_{X_1|X_2=x_2}(x_1)|dx_1|.
\]

Again, if \( X_2 \) is nondegenerate, then we have the regression formula

\[
E(f(X_1)|X_2 = x_2) = E\left( f(Y + Cx_2) \right)
\]

where \( Y : \mathcal{S} \to V_1 \) is a Gaussian vector with

\[
E(Y) = E(X_1) - CE(X_2), \quad S_Y = S_{X_1} - Cov(X_1, X_2)S_{X_2}^{-1} Cov(X_2, X_1),
\]

and \( C \) is given by (A.3).

Let us point out that if \( X : \mathcal{S} \to U \) is a Gaussian random vector and \( L : U \to V \) is a linear map, then the random vector \( LX : \mathcal{S} \to V \) is also Gaussian. Moreover

\[
E(LX) = LE(X), \quad \Sigma_{LX}(\xi, \xi) = \Sigma_X(L^\vee \xi, L^\vee \xi), \quad \forall \xi \in V^\vee,
\]

where \( L^\vee : V^\vee \to U^\vee \) is the linear map dual to \( L \). Equivalently,

\[
S_{LX} = LS_XL^\vee.
\]

A random field (or function) on a set \( T \) is a map

\[
\xi : T \times (\mathcal{S}, \mu) \to \mathbb{R}, \quad (t, s) \mapsto \xi_t(s)
\]

such that

- \((\mathcal{S}, \mu)\) is a probability space, and
- for any \( t \in T \) the function \( \xi_t : \mathcal{S} \to \mathbb{R} \) is measurable, i.e., it is a random variable.
Thus, a random field on $T$ is a family of random variables $\xi_t$ parameterized by the set $T$. For simplicity we will assume that all these random variables have finite second moments. For any $t \in T$ we denote by $\mu_t$, the expectation of $\xi_t$. The covariance function or kernel of the field is the function $C_\xi : T \times T \to \mathbb{R}$ defined by

$$C_\xi(t_1, t_2) = E((\xi_{t_1} - \mu_{t_1})(\xi_{t_2} - \mu_{t_2})) = \int_S (\xi_{t_1}(s) - \mu_{t_1})(\xi_{t_2}(s) - \mu_{t_2}) \, d\mu(s).$$

The field is called Gaussian if for any finite subset $F \subset T$ the random vector

$$S \in s \mapsto (\xi_t(s))_{t \in F} \in \mathbb{R}^F$$

is a Gaussian random vector. Almost all the important information concerning a Gaussian random field can be extracted from its covariance kernel.

Here is a simple method of producing Gaussian random fields on a set $T$. Choose a finite dimensional space $U$ of real valued functions on $T$. Once we fix a Gaussian measure $d\gamma$ on $U$ we obtain tautologically a random field

$$\xi : T \times U \to \mathbb{R}, \ (t, u) \mapsto \xi_t(u) = u(t).$$

This is a Gaussian field since for any finite subset $F \subset T$ the random vector

$$\Xi : U \to \mathbb{R}^F, \ u \mapsto (u(t))_{t \in F}$$

is Gaussian because the map $\Xi$ is linear and thus the pushforward $\Xi_*d\gamma$ is a Gaussian measure on $\mathbb{R}^F$. For more information about random fields we refer to [1, 3, 10].

In the conclusion of this section we want to describe a few simple integral formulas.

**Proposition A.2.** Suppose $V$ is an Euclidean space of dimension $N$, $f : U \to \mathbb{R}$ is a locally integrable, positively homogeneous function of degree $k \geq 0$, and $A : U \to U$ is a positive definite symmetric operator. Denote by $B(U)$ the unit ball of $V$ centered at the origin, and by $S(U)$ its boundary. Then the following hold

$$\frac{1}{\pi^\frac{N}{2}} \int_{B(U)} |f(u)| \, du = \frac{1}{\Gamma\left(1 + \frac{k+N}{2}\right)} \int_U f(u) e^{-|u|^2/\pi} \, du. \quad (A.6)$$

$$\int_U f(u) d\gamma_{t,A}(u) = t^\frac{k}{2} \int_U f(u) d\gamma_A(u) \ \forall \ t > 0, \quad (A.7)$$

where $d\gamma_A$ is the Gaussian measure defined by (A.1).

**Proof.** We have

$$\int_{B(U)} |f(u)| \, du = \int_0^1 t^{k+N-1} \left( \int_{S(U)} f(u) |dA(u)| \right) = \frac{1}{k+N} \int_{S(U)} f(u) |dA(u)|.$$

On the other hand

$$\frac{1}{\pi^\frac{N}{2}} \int_U f(u) e^{-|u|^2/\pi} \, du = \frac{1}{\pi^\frac{N}{2}} \left( \int_0^\infty t^{k+N-1} e^{-t^2/2} \, dt \right) \int_{S(U)} f(u) |dA(u)|$$

$$= \frac{1}{\pi^\frac{N}{2}} \Gamma\left( \frac{k+N}{2} \right) \left( \frac{k+N}{2} \right)^\frac{k+N}{2} \int_{B(U)} f(u) |du|$$

$$= \frac{1}{\pi^\frac{N}{2}} \Gamma\left( \frac{1 + k+N}{2} \right) \int_{B(U)} f(u) |du|.$$

This proves (A.6). The equality (A.7) follows by using the change in variables $u = t^\frac{1}{2} v$. \qed
APPENDIX B. GAUSSIAN RANDOM SYMMETRIC MATRICES

We want to describe in some detail a 3-parameter family of centered Gaussian measures on $S_m$, the vector space of real symmetric $m \times m$ matrices, $m > 1$.

For any $1 \leq i \leq j$ define $\xi_{ij} \in S_m^\vee$ so that for any $A \in S_m$

$$\xi_{ij}(A) = a_{ij} = \text{the } (i, j)\text{-entry of the matrix } A.$$ 

The collection $(\xi_{ij})_{1 \leq i \leq j \leq m}$ is a basis of the dual space $S_m^\vee$. We denote by $(E_{ij})_{1 \leq i \leq j}$ the dual basis of $S_m$. More precisely, $E_{ij}$ is the symmetric matrix whose $(i, j)$ and $(j, i)$ entries are 1 while all the other entries are equal to zero. For any $A \in S_m$, we have

$$A = \sum_{i \leq j} \xi_{ij}(A) E_{ij}.$$ 

The space $S_m$ is equipped with an inner product

$$( -, - ) : S_m \times S_m \to \mathbb{R}, \quad (A, B) = \text{tr}(AB), \quad \forall A, B \in S_m.$$ 

This inner product is invariant with respect to the action of $SO(m)$ on $S_m$. We set

$$\hat{E}_{ij} := \begin{cases} E_{ij}, & i = j \\ \frac{1}{\sqrt{2}} E_{ij}, & i < j. \end{cases}$$ 

The collection $(\hat{E}_{ij})_{i \leq j}$ is a basis of $S_m$ orthonormal with respect to the above inner product. We set

$$\hat{\xi}_{ij} := \begin{cases} \xi_{ij}, & i = j \\ \sqrt{2} \xi_{ij}, & i < j. \end{cases}$$ 

The collection $(\hat{\xi}_{ij})_{i \leq j}$ the orthonormal basis of $S_m^\vee$ dual to $(\hat{E}_{ij})$.

To any numbers $a, b, c$ satisfying the inequalities

$$a - b, c, a + (m - 1)b > 0. \quad (B.1)$$ 

we will associate a centered Gaussian measure $\Gamma_{a,b,c}$ on $S_m$ uniquely determined by its covariance form

$$\Sigma = \Sigma_{a,b,c} : S_m^\vee \times S_m^\vee \to \mathbb{R}$$

defined as follows:

$$\Sigma(\xi_{ii}, \xi_{ii}) = a, \quad \Sigma(\xi_{ii}, \xi_{jj}) = b, \quad \forall i \neq j, \quad (B.2a)$$

$$\Sigma(\xi_{ij}, \xi_{ij}) = c, \quad \Sigma(\xi_{ij}, \xi_{k\ell}) = 0, \quad \forall i < j, k \leq \ell, (i, j) \neq (k, \ell). \quad (B.2b)$$

To see that $\Sigma_{a,b,c}$ is positive definite if $a, b, c$ satisfy (B.1) we decompose $S_m^\vee$ as a direct sum of subspaces

$$S_m^\vee = D_m \oplus \emptyset_m,$$

$$D_m = \text{span } \{ \xi_{ii}; 1 \leq i \leq m \}, \quad \emptyset_m = \text{span } \{ \xi_{ij}; 1 \leq i < j \leq m \}, \quad \dim \emptyset_m = \binom{m}{2}.$$ 

With respect to this decomposition, and the corresponding bases of these subspaces the matrix $Q_{a,b,c}$ describing $\Sigma_{a,b,c}$ with respect to the basis $(\xi_{ij})$ has a direct sum decomposition

$$Q_{a,b,c} = G_m(a, b) \oplus c I_m(\frac{m}{2}),$$

where $G_m(a, b)$ is the $m \times m$ symmetric matrix whose diagonal entries are equal to $a$ while all the off diagonal entries are all equal to $b$.

The the spectrum of $G_m(a, b)$ consists of two eigenvalues: $(a - b)$ with multiplicity $(m - 1)$ and the simple eigenvalue $a - b + mb$. Indeed, if $C_m$ denotes the $m \times m$ matrix with all entries equal to 1, then $G_m(a, b) = (a - b) I_m + bC_m$. The matrix $C_m$ has rank 1 and a single nonzero eigenvalue equal
to \( m \) with multiplicity 1. This proves that \( Q_{a,b,c} \) is positive definite since its spectrum is positive. We denote by \( d\Gamma_{a,b,c} \) the centered Gaussian measure on \( S_m \) with covariance form \( \Sigma_{a,b,c} \).

Since \( S_m \) is equipped with an inner product we can identify \( \Sigma_{a,b,c} \) with a symmetric, positive definite bilinear form on \( S_m \). We would like to compute the matrix \( \tilde{Q} = \tilde{Q}_{a,b,c} \) that describes \( \Sigma_{a,b,c} \) with respect to the orthonormal basis \( \{ \tilde{E}_{ij} \}_{1 \leq i \leq j} \). We have

\[
\tilde{Q}(\tilde{E}_{ii}, \tilde{E}_{ii}) = Q(\tilde{\xi}_{ii}, \tilde{\xi}_{ii}) = a, \quad \tilde{Q}(\tilde{E}_{ii}, \tilde{E}_{jj}) = b, \quad \forall i \neq j,
\]

\[
\tilde{Q}(\tilde{E}_{ij}, \tilde{E}_{ij}) = Q(\tilde{\xi}_{ij}, \tilde{\xi}_{ij}) = 2Q(\tilde{\xi}_{ii}, \tilde{\xi}_{ii}) = 2c, \quad \forall i < j,
\]

Thus

\[
\tilde{Q}_{a,b,c} = G_m(a, b) \oplus 2c \mathbb{I}_{\binom{m}{2}}. \tag{B.3}
\]

If \( | - |_{a,b,c} \) denotes the Euclidean norm on \( S_m \) determined by \( \Sigma_{a,b,c} \) then for

\[
A = \sum_{i \leq j} a_{ij} E_{ij} = \sum_i a_{ii} \tilde{E}_{ii} + \sqrt{2} \sum_{i < j} a_{ij} \tilde{E}_{ij},
\]

we have

\[
|A|_{a,b,c}^2 = a \sum_i a_{ii}^2 + 2b \sum_{i < j} a_{ii} a_{jj} + 4c \sum_{i < j} a_{ij}^2
\]

\[
= (a - b - 2c) \sum_i a_{ii}^2 + b \left( \sum_i a_{ii} \right)^2 + 2c \left( \sum_i a_{ii}^2 + 2 \sum_{i < j} a_{ij}^2 \right)
\]

\[
= (a - b - 2c) \sum_i a_{ii}^2 + b (\text{tr} A)^2 + 2c \text{ tr} A^2.
\]

Observe that when

\[
a - b = 2c \tag{B.4}
\]

we have

\[
|A|_{a,b,c}^2 = b (\text{tr} A)^2 + 2c \text{ tr} A^2 \tag{B.5}
\]

so that the norm \( | - |_{a,b,c} \) and the Gaussian measure \( d\Gamma_{a,b,c} \) are \( \text{SO}(m) \)-invariant. Let us point out that the space \( S_m \) equipped with the Gaussian measure \( d\Gamma_{2,0,1} \) is the well known \( \text{GOE} \), the Gaussian orthogonal ensemble.

To obtain a more concrete description of \( \Gamma_{a,b,c} \) we first identify \( \Sigma_{a,b,c} \) with a symmetric operator

\[
\tilde{Q}_{a,b,c} : S_m \to S_m.
\]

Using (B.3) we deduce that

\[
\tilde{Q}_{a,b,c} = G(a, b) \oplus 2c \mathbb{I}_{\binom{m}{2}}.
\]

Observe that

\[
\det \tilde{Q}_{a,b,c} = (a - b) (a + (m - 1)b)^{m-1} \binom{m}{2}, \tag{B.6}
\]

and

\[
\tilde{Q}_{a,b,c}^{-1} = \tilde{Q}_{a', b', c'} = G_m(a', b') \oplus 2c' \mathbb{I}_{\binom{m}{2}}, \tag{B.7}
\]

where \( 2c' = \frac{1}{2c} \) and the real numbers \( a', b' \) are determined from the linear system

\[
\begin{cases}
  a' - b' = \frac{1}{a-b} \\
  a' + (m-1)b' = \frac{1}{a+(m-1)b}.
\end{cases} \tag{B.8}
\]
We then have
\[
d\Gamma_{a,b,c}(X) = \frac{1}{(2\pi)^{m(m+1)/4}} \left( \det \hat{Q}_{a,b,c} \right)^{\frac{3}{2}} e^{-\frac{1}{2} \hat{(Q}_{a,b,c}^{-1} X, X)} 2^{\frac{m}{2}} \prod_{i<j} dx_{ij}, \tag{B.9}
\]
where
\[
(\hat{Q}_{a,b,c}^{-1} X, X) = \left( a' - b' - \frac{1}{2c} \right) \sum_i x_i^2 + b'(\text{tr} X)^2 + \frac{1}{2c} \text{tr} X^2. \tag{B.10}
\]
The special case \( b = c > 0, a = 3c \) is particularly important for our considerations. We denote by \((-,-)_c\) and respectively \(d\Gamma_c\) the inner product and respectively the Gaussian measure on \( S_m \) corresponding to the covariance form \( \Sigma_{3c,c,c} \).

If we set \( \hat{Q}_c := \hat{Q}_{3c,c,c} \) then we deduce from (B.7) that
\[
\hat{Q}_c^{-1} = \hat{Q}_{a',b',c'} = G_m(a', b') \oplus \mathbb{I} \left( \begin{smallmatrix} \frac{1}{2c} \\ 2 \end{smallmatrix} \right)
\]
where
\[
\begin{cases}
a' - b' = \frac{1}{2c} = 2c' \\
a' + (m-1)b' = \frac{1}{(m+2)c}.
\end{cases}
\]
We deduce
\[
m b' = \frac{1}{m+2} - \frac{1}{2c} = - \frac{m}{2c(m+2)} \Rightarrow b' = - \frac{1}{2c(m+2)}.
\]
Note that the invariance condition (B.4) \( a' - b' = 2c' \) is automatically satisfied so that
\[
(\hat{Q}_c^{-1} X, X) = \frac{1}{2c} \text{tr} X^2 - \frac{1}{2c(m+2)} (\text{tr} X)^2.
\]
Using (B.6) and (B.9) we deduce
\[
d\Gamma_c(X) = \frac{1}{(2\pi c)^{m(m+1)/4} \sqrt{\mu_m}} \cdot e^{-\frac{1}{2} \left( \text{tr} X^2 - \frac{1}{m+2} (\text{tr} X)^2 \right)} 2^{\frac{m}{2}} \prod_{i<j} |dx_{ij}|, \tag{B.11}
\]
where
\[
\mu_m := 2^{m+1} (m+2)^{-m+1}. \tag{B.12}
\]
The inner product \((-,-)_c\) has the alternate description
\[
(A, B)_c = I_c(A, B) := 4c \int_{\mathbb{R}^m} (Ax, x)(Bx, x) \frac{e^{-|x|^2}}{\pi^m} |dx|, \quad V, A, B \in S_m. \tag{B.13}
\]
To verify (B.13) it suffices to show that
\[
I_c(E_{ii}, E_{ii}) = 3c, \quad I_c(E_{ij}, E_{jj}) = c, \quad I_c(E_{ij}, E_{ij}) = 4c, \quad \forall 1 \leq i < j \leq m,
\]
\[
I_c(E_{ij}, E_{k\ell}) = 0, \quad \forall 1 \leq i < j \leq m, \quad k \leq \ell, \quad (i,j) \neq (k,\ell).
\]
To achieve this we need to use the classical identity
\[
\int_{\mathbb{R}^m} x_1^{2p_1} \cdots x_m^{2p_m} e^{-|x|^2} |dx| = \prod_{k=1}^{m} \int_{\mathbb{R}} x_k^{2p_k} e^{-t^2} dt_k = \prod_{k=1}^{m} \Gamma \left( p_k + \frac{1}{2} \right).
\]
Observe that
\[
\int_{\mathbb{R}^m} (E_{ii} x, x)(E_{jj} x, x) \frac{e^{-|x|^2}}{\pi^m} |dx| = \int_{\mathbb{R}^m} x_i^2 x_j^2 \frac{e^{-|x|^2}}{\pi^2} |dx|
\]
\[ \pi^{-\frac{n-m}{2}} \Gamma \left( \frac{1}{2} \right)^{m-2} \begin{cases} \Gamma \left( \frac{5}{2} \right) \Gamma \left( \frac{1}{2} \right), & i = j, \\ \Gamma \left( \frac{3}{2} \right)^2, & i \neq j \end{cases} \begin{align*} = & \frac{1}{4} \times \begin{cases} 3, & i = j \\ 1, & i \neq j \end{cases} \end{align*} \]

Next, if \( i < j \) we have
\[
\int_{\mathbb{R}^m} (E_{ij}(x,x))(E_{ij}(x,x)) e^{-|x|^2} \frac{1}{\pi^{\frac{n}{2}}} |dx| = 4 \int_{\mathbb{R}^m} x_i^2 x_j^2 e^{-|x|^2} \frac{1}{\pi^{\frac{n}{2}}} |dx| = 1.
\]

Finally, if \( i < j, k \leq \ell \) and \( (i,j) \neq (k,\ell) \), then the quartic polynomial
\[
(E_{ij}(x,x))(E_{k\ell}(x,x))
\]
is odd with respect to a reflection \( x_p \mapsto -x_p \) for some \( p = \{i,j,k,\ell\} \) and thus
\[
\int_{\mathbb{R}^m} (E_{ij}(x,x))(E_{k\ell}(x,x)) e^{-|x|^2} \frac{1}{\pi^{\frac{n}{2}}} |dx| = 0.
\]

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