SPECTRAL SEQUENCES and ALL THAT

Talk ①
Fri, Feb 25, '0

Allegra Berliner

Brief Recap:

$F: \mathcal{A} \to \mathcal{B}$ a left-exact functor of abelian cat's

<u>main situation we'll have in mind:</u>

$\mathcal{A} = Sh_{\mathbb{Z}}(X)$ $\mathcal{B} = Ab$

$F = \Gamma(X, -)$ $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$.

"global section functor

Move on to

- Complexes of objects in an abelian cat $C^+(\mathcal{A})$

complex $A^\bullet \in C^+(\mathcal{A})$ is of form $\cdots \to A^n \xrightarrow{d} A^{n+1} \xrightarrow{d} \cdots$

where $A^n = 0$ for all n suff. negativ

- any object $A \in \mathcal{A}$ is in $C^+(\mathcal{A})$ by natural inclusion $\mathcal{A} \hookrightarrow C^+$

ASSOCIATED RIGHT DERIVED FUNCTORS:

$R^k F: \mathcal{A} \to \mathcal{B}$

- replace $A \in \mathcal{A}$ with a complex $(I^\bullet, d)$ of injectives in $C^+(\mathcal{A})$ such that $0 \to A \to I^0 \to I^1 \to \cdots$ exact

"injective resolution"

then $R^k F(A) = \dfrac{\ker(F(I^k \xrightarrow{d} F(I^{k+1}))}{\operatorname{im}(F(I^{k-1}) \xrightarrow{d} F(I^k))}$ = kth cohomology group of the complex $F(I^\bullet, d)$.

A is F-acyclic if $R^k F(A) = 0 \quad \forall\, k > 0$.

Generalized (Grothendieck) Construction

Let $A^\bullet \in C^+(\mathcal{A})$ (eg: $A^\bullet$ is a complex of sheaves)

- replace $A^\bullet$ with a quasi-isomorphic complex $I^\bullet$ (of inject

notat'n: $A^\bullet \xrightarrow{\sim} I^\bullet \Rightarrow$ q-i $\{$ie, $H^k(A^\bullet) \cong H^k(I^\bullet)\}$

Then get

$$\begin{tikzcd} & A^\bullet \arrow[d, "q\text{-}i"] \\ F(I^\bullet) & I^\bullet \arrow[l] \end{tikzcd}$$

$F(I^\bullet) = \mathbb{R}F(A^\bullet)$ "derived functor" Grothendieck.

- the cohomology of this:

$R^k F(A^\bullet) = \mathbb{R}F^k(A^\bullet) = \mathbb{H}^k(F(I^\bullet))$ hypercohomology.

Properties:

$A^\bullet \xrightarrow{\sim} B^\bullet \Longrightarrow \mathbb{R}F^i(A^\bullet) \cong \mathbb{R}F^i(B^\bullet)$

Fact: If $B^\bullet$ is F-acyclic and $A^\bullet \xrightarrow{\sim} B^\bullet$,

$\mathbb{R}F^k(A^\bullet) \cong H^k(F(B^\bullet))$

hypercoh$(A^\bullet) \cong$ actual coh. of $F(B^\bullet)$

Special case of

$\Gamma : Sh_{\mathbb{Z}}(X) \longrightarrow ab$

$\Gamma(X, -)$ acyclic sheaves:

Injective Flabby Soft.

Recall:
$H^k(X; \mathcal{F}) = R^k\Gamma(X, \mathcal{F})$
$H^k(X; \underline{\mathbb{C}}) = H^k(X, \mathbb{C})$

Complexes & filtrations

- We move from $H^\bullet(X;\mathcal{F}) \rightsquigarrow H^\bullet(X;\text{complex of sheaves})$

  such as: Cech complex
  Dolbeault complex

To calculate H^* in coefficients complex of sheaves,

USE: CARTAN - EILENBERG RESOLUTIONS

if: $F: \mathcal{A} \to \mathcal{B}$

Let: $(C^\bullet, D_1)$ cplx in $C^+(\mathcal{A})$

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
I^{0,1} & I^{1,1} & I^{2,1} & \\
I^{0,0} \arrow[u] & I^{1,0} \arrow[u] & I^{2,0} \arrow[u] & I^{3,0} \\
C^0 \arrow[u, "D_2"] \arrow[r, "D_1"'] & C^1 \arrow[u] \arrow[r, "D_1"'] & C^2 \arrow[u] \arrow[r, "D_1"'] & C^3 \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

- take injective resol'n of each object C^i in $C^\bullet$.
- this is a really nice cplx w/ injective kernels & coboundaries
- D_1 and D_{11} anticommute

We get: a DOUBLE COMPLEX, $(I^{\bullet,\bullet}, D_1, D_2)$

- can condense double cplx into a regular cplx, $(\mathcal{I}^\bullet, d)$

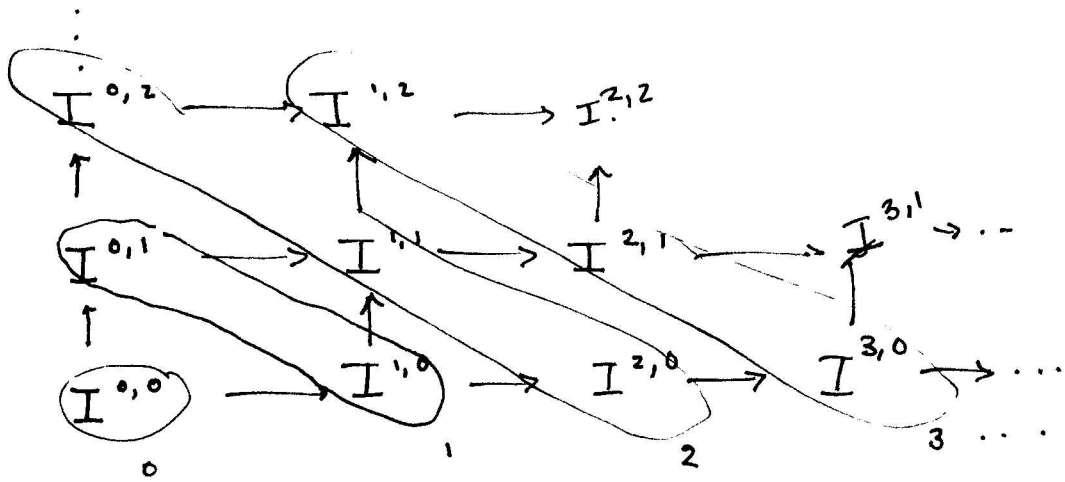
$$(\mathcal{I}^\bullet, d) = \mathrm{Tot}(I^{\bullet,\bullet}) = \begin{cases} \mathcal{I}^k = \bigoplus_{r+s=k} I^{r,s} \\ d = D_1 + D_2 \end{cases}$$

<u>Fact:</u> We get quasi-iso $(C^\bullet, D_1) \simeq (\mathcal{I}^\bullet, D)$.

So in fact, $\mathbb{R}^k F(C^\bullet, d) = H^k(F(\mathrm{Tot}\, I^{\bullet\bullet})) = H^k F(\mathcal{I}, d)$.

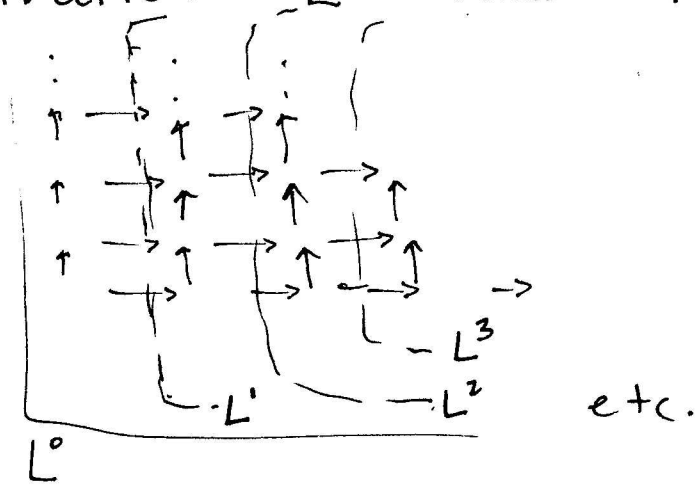
- Would like to distill info of $H^k(F(\mathrm{Tot}\, I^{\cdot,\cdot}))$.

- 2 ways to <u>filter</u> $I^{\cdot,\cdot}$ to get this:



Total complex = circled parts

Filtration $L^{\cdot}$ takes only some of total complex



etc.

<u>Induced Decreasing Filtration (on columns) of total complex</u>

$$L^p I^k = \bigoplus_{\substack{r+s=k \\ r \geq p}} I^{r,s}$$

(Total complex where all info before p^{th} col. is replaced by 0's)

- Can do analogously along rows - truncate all before q^{th} in q^{th} filtration.

a decreasing filtration on a complex $K^\bullet$

$\parallel$

decreasing family of subcomplexes of $K^\bullet$

- uses differential of $K^\bullet$ restricted toget

$$L^p K^k \xrightarrow{d} L^p K^{k+1}.$$

$(K^\bullet, L)$ = cplx equipped w/decrg filtration L.

Filtrations & Cohomology

We began with: $\left\{\begin{matrix}\text{object}\\ C\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{resol'n of } C\\ \parallel \\ \text{cplx } I^\bullet\end{matrix}\right\} \rightsquigarrow R^iF(C)$

$\left\{\begin{matrix}\text{cplx}\\ C^\bullet\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}q\text{-}i \text{ cplx}\\ I^\bullet\\ (\text{double} \to \text{total}\end{matrix}\right\} \longrightarrow \mathbb{R}F(C^\bullet)$

now $\left\{\begin{matrix}\text{filtered}\\ \text{cplx}\\ (A^\bullet, L)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{filtered}\\ \text{cohomology of } (A^\bullet, L)\end{matrix}\right\}.$

When we filter cplx, $\rightsquigarrow$ natural filtering of cohomology

define pth filtration of $H^i(A^\bullet)$: $L^p H^i(A^\bullet) = \operatorname{Im}(H^i(L^p(A^\bullet)) \to H^i(A^\bullet))$

image of coh. of pth fil into coh of $A^\bullet$.

To understand "filtered cohomology", we want to know:

$$\mathrm{Gr}_p^L H^i(A^\cdot) = \frac{L^p H^i(A^\cdot)}{L^{p+1} H^i(A^\cdot)}$$

the "p^{th} Graded object associated to $H^i(A^\cdot)$

Understanding/Knowing these = "knowing" $H^i(A$

$\Downarrow$

$\boxed{\text{New goal:}}$ Calculate cohomology of a filtered complex.

Technique: Spectral Sequences

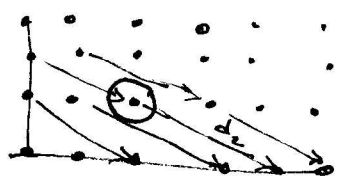
Spec Seq = a series of double complexes $E_r^{\cdot,\cdot}$ (page

with $E_{r+1}^{\cdot,\cdot} = H^\cdot(E_r^{\cdot,\cdot})$ taken w.r.t d_r

r^{th} differe

d_r is of bidegree $\begin{tikzcd} \\ \text{down } r-1 \arrow[d] & \\ & \text{right}_r \end{tikzcd}$ d_r

ie, on $E_2^{\cdot,\cdot}$



so for $E_3^{p,q}$, take $\dfrac{\ker(d_3 \text{ coming out of } E_2^{p,q})}{\mathrm{Im}(d_3 \text{ going into } E_2^{p,q})}$

- Same idea holds in general.

- to be a spectral sequence, we also define specifically what we plug in for $E_0^{\cdot,\cdot}$:

$$E_0^{p,q} = Gr_p^L A^{p+q} = L^p(A^{p+q}) / L^{p+1}(A^{p+q})$$

"p^{th} graded obj. associated to A^{p+q}"

- third condition: "it converges to $H^{\cdot}(A^{\cdot})$"

$$E_\infty^{p,q} = Gr_p^L H^{p+q}(A^{\cdot})$$

meaning, when $E_r^{p,q}$ "stabilizes":

ie, $E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots := \underline{\underline{E_\infty^{p,q}}}$

this happens at some r, possibly a different r for varying (p,q).

Example of resolution $\rightsquigarrow$ spectral seq:

Let $(C^{\cdot}, d)$ = complex

- then the Cartan-Eilenberg resol'n is injective.
- columns will be exact because that's the property of a resolution.

$$\begin{array}{ccccc} I^{0,\cdot} & & I^{1,\cdot} & & \\ \uparrow & & \uparrow & & \\ C^0 & \to & C^1 & \to & \cdots \end{array}$$

$\Rightarrow$ we put in $L^p(I^{\cdot,k}) = \bigoplus_{\substack{r+s=k \\ r \geq p}} I^{r,s}$

for filtration.

So E_0 page has input

$$E_0^{p,q} = Gr_p^L I^{p+q} = L^p(I^{p+q}) / L^{p+1}(I^{p+q})$$

$$= \frac{\bigoplus_{\substack{r+s=p+q \\ r \geq p}} I^{r,s}}{\bigoplus_{\substack{r+s=p+q \\ r \geq p+1}} I^{r,s}} = \frac{\text{objects of } (\mathrm{Tot}\, I)^{p+q} \text{ from } p \text{ on}}{\text{Obj. of } (\mathrm{Tot}\, I)^{p+q} \text{ from } (p+1) \text{ on}}$$

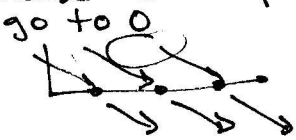
$$= \text{objects of } (\mathrm{Tot}\, I)^{p+q} \text{ in col } p = I^{p,q}$$

$$d_0 = (-1)^p D_2.$$

<u>ON</u> $E_1^{p,q} = H^q_{D_2}(I^{p,\cdot})$ $\quad d_1 = (D_1)_*$ (induced cobdry on cohomology): $H^q(I^{p,\cdot}) \to H^q(I^{p+1,\cdot})$

But exactness $\Rightarrow H^q(I^{p,\cdot}) = 0$ except at $q=0$.

Thus $E_2^{p,q} = H^p_{D_1} H^q_{D_2}(I^\bullet)$ differentials are 0 (must be) because all maps come from 0, go to 0



$$\Downarrow$$

$$E_\infty^{p,q} = Gr_p^L H^{p+q}(I^\cdot) \cong Gr_p^L H^{p+q}(C^\bullet)$$

$$\|$$

$$E_2^{p,q} = H^p_{D_1} H^q_{D_2}(I^\cdot) = H^{p+q}(I^\cdot)$$

Filtered Complexes & cohomology: adding derived functors

general cplxes w/ filtrations

$\rightsquigarrow$

filtration of cohomology

Now, consider derived functors with this:

$\mathcal{A} \xrightarrow{F} \mathcal{B}$

Let $(A^\cdot, L) =$ filtered cplx $\in C^+(\mathcal{A})$.

- induces filtration of the derived functors $RF(A^\cdot)$

$$L^p R^i F(A^\cdot) = \mathrm{Im}\left(R^i F(L^p A^\cdot) \longrightarrow R^i F(A^\cdot)\right)$$

Thus if we begin w/ $A^\cdot \in C^+(\mathcal{A})$ w/ filtrat'n $L^\cdot$, then can get

$$A^\cdot \xrightarrow[q\text{-}i]{} I^\cdot_{\text{injective}}$$ with a filtration s.t.

$L^p I^\cdot$ which are q-i to $L^p A^\cdot$

This gives:
$$H^i(F(L^p I^\cdot)) := R^i F(L^p A^\cdot)$$

So since $L^p(F(I^\cdot)) = F(L^p I^\cdot)$, this gives us a filtration

$$L^p R^i F(A^\cdot) \text{ of } R^i F(A^\cdot).$$

- can input this into a spectral sequence.

Spectral Sequences for $\mathcal{A} \xrightarrow{F} \mathcal{B}$:

sseq I: ${}_{I}E_1^{p,q} = R^q F(C^p)$ so, ${}_{I}E_2^{p,q} = H^p(R^q F(C), R^q F($

abuts to $\mathbb{R}^{p+q}F(C^\cdot, d)$.

this one degenerates at the E_2-page (when we use a C-E resolutio

sseq II: ${}_{II}E_2^{p,q} = R^p F(H^q(C^\cdot, d))$

also abuts to $\mathbb{R}^{p+q}F(C^\cdot, d)$.

in a C-E resolution, all coboundaries, cocycles are also injective throughout the s.seq.

$\Rightarrow H^*(F(C^\cdot)) \cong F(H^*(C^\cdot))$.

So we get that $C^\cdot \longrightarrow I^\cdot$ injective resol'n.

$H^p(F(I^{\cdot,q}), F(d_I)) \cong F(H^p(I^{\cdot,q}, d_I)) \quad \forall p,q$

$R^p F(H^q(C^\cdot, d)) = H^p F(\quad \mathcal{J}^2 \quad)$

this is the fundamental property we use to understand why the 2nd seq. also abuts to $\mathbb{R}^{p+q}F(C^\cdot, d)$

$\Rightarrow E_\infty^{p,q} = Gr_L^p \, \mathbb{R}^{p+q}F(C^\cdot, d)$

Spectral Sequences for composed functors –

$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$

F. left exact, maps injectives in $\mathcal{A}$ to G-acyclics in $\mathcal{B}$

G- left exact.

Note: in this case $R(G \circ F) \cong RG \circ RF$.

- Let $M \in \mathcal{A}$.

- resolve $M \longrightarrow (I^\cdot, d)$

Then $R^i GF(M) = H^i(GF(I^\cdot), GF(d))$

THEOREM $\exists$ a canonical filtration L on the objects $R^i(GF)(M)$ and a spectral sequence

$$E_r^{p,q} \underset{\text{abuts to}}{\Longrightarrow} R^*(GF)(M)$$

with E_2 input term

$$E_2^{p,q} = R^p G(R^q F(M))$$

$$E_\infty^{p,q} = Gr_L^p R^{p+q} GF(M)$$

- will do this by using $K^\cdot = F(I^\cdot)$ in our regular "derived functor s.seq" from previous page.

Special case: Leray spectral sequence.

$f: X \to Y$ conts map of spaces.

$$\begin{tikzcd} \underset{\mathcal{A}}{Sh_{\mathbb{Z}} X} \arrow[r, "f_*"] \arrow[rr, bend right, "\Gamma(X,-)"'] & \underset{\mathcal{B}}{Sh_{\mathbb{Z}}(Y)} \arrow[r, "\Gamma(Y,-)"] & \underset{\mathcal{C}}{Ab} \end{tikzcd}$$

Let $f_* = F$. $\Gamma(Y, -) = G$.

- input into Grothendieck sseq.

Theorem (Leray)

For every sheaf $\mathcal{F}$ on X, there exists a canonical filtration L on $H^q(X, \mathcal{F})$ and a spectral sequence

$$E_v^{p,q} \Rightarrow H^{p+q}(X, \mathcal{F})$$

that is canonical starting from E_2, and satisfies

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F})), \quad E_\infty^{p,q} = Gr_p^L H^{p+q}(X, \mathcal{F})$$

(Leray Spectral Sequence)

Using Leray SS to calculate $H^*(\mathbb{C}P^2)$

$$\begin{array}{c} S^5 \\ \downarrow f \\ \mathbb{C}P^2 \end{array}$$ Hopf fibration - $\forall$ pt $x \in \mathbb{C}P^2$, $f^{-1}(x) \simeq S^1$.

our functors:

$$Sh(S^5) \xrightarrow{f_*} Sh(\mathbb{C}P^2) \xrightarrow{\Gamma(\mathbb{C}P^2, -)} ab$$

- Do for constant sheaf $\underline{\mathbb{C}}$.

$$E_2^{p,q} = H^p(\mathbb{C}P^2; \underbrace{R^q f_* \underline{\mathbb{C}}}_{\text{a sheaf}}) \Rightarrow Gr_p^L H^{p+q}(S^5; \underline{\mathbb{C}})$$
$$= Gr_p^L H^{p+q}(S^5; \mathbb{C}).$$

What is $R^q f_* \underline{\mathbb{C}}$?

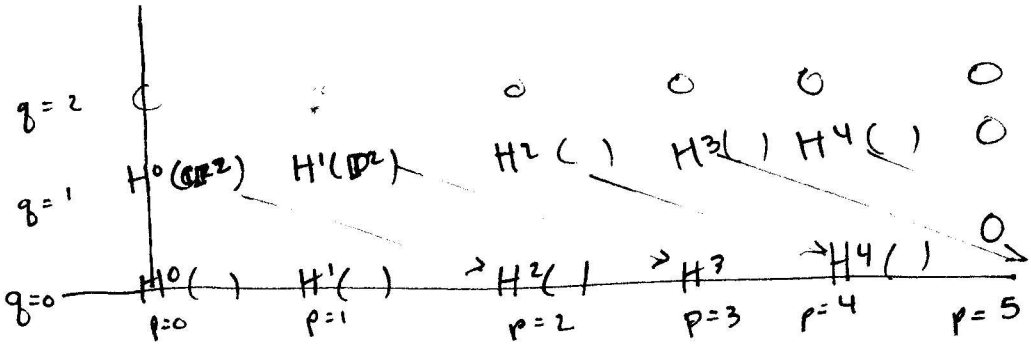
- stalk over any $x \in \mathbb{C}P^2 = H^q(f^{-1}(x); \mathbb{C})$ since that's the sheaf cohom.

- the Hopf fibration is a bundle, so any $y \in \mathbb{C}P^2$ has $f^{-1}(y) \simeq f^{-1}(x) \Rightarrow$
$H^q(f^{-1}(x), \mathbb{C}) \cong H^q(f^{-1}(y); \mathbb{C}) = H^q(S^1, \mathbb{C}$

<u>So</u> $R^q f_* \underline{\mathbb{C}}$ is a LOCALLY CONSTANT SHEAF <u>over a simply connected space</u>, $\mathbb{C}P^2$.

Thus $R^q f_* \underline{\mathbb{C}} = H^q(S^1; \mathbb{C}) = \begin{cases} \mathbb{C} & q = 0, 1 \\ 0 & \text{else} \end{cases}$

E_2 page is

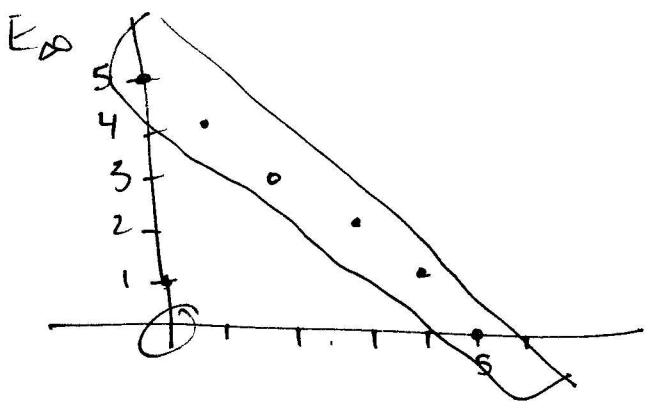


dim of $\mathbb{P}^2 = 4$ so no cohom above here

$E_2^{p,0} = H^p(\mathbb{CP}^2; \mathbb{C})$

$E_2^{p,1} = H^p(\mathbb{CP}^2; \mathbb{C})$.

$E_\infty^{p,q} = Gr_p^L H^{p+q}(S^5; \mathbb{C})$ - only on $(0,0)$ and diagonal $p+q=5$



filtration of cohom. of S^5

* This SS degenerates after E (differentials after E_3 & beyond are

- using Poincaré duality and where differentials map to 0, can deduce cohomology of $\mathbb{CP}^2 = \begin{cases} \mathbb{C} & 0,2,4 \\ 0 & 1,3 \end{cases}$