SPECTRAL SEQUENCES

and ALL THAT

Algebra Berliner

Brief Recap:

\[ F : A \to B \] a left-exact functor of abelian cat's

main situation we'll have in mind:

\[ A = S^h_{X} \] \[ B = \text{Ab} \]

\[ F = \Gamma (X, -) \to \Gamma (X, Y) \]

"global section functor"

Move on to

- Complexes of objects in an abelian cat \( C^+(A) \)

complex \( A^\bullet \in C^+(A) \) is of form \( \cdots \to A^n \to A^{n+1} \to \cdots \)

where \( A^n = 0 \) for all \( n \) suf\( f \) negative

- any object \( A \in A \) is in \( C^+(A) \) by natural inclusion \( A \to C^+(A) \)

ASSOCIATED RIGHT DERIVED FUNCTORS:

\( R^k F : A \to B \)

- replace \( A \in A \) with a complex \( (I^\bullet, d) \) of injectives in \( C^+(A) \)

such that \( 0 \to A \to I^\cdot \to I^\cdot \to \cdots \) \# exact

"injective resolution"

then \( R^k F(A) = \frac{\ker (F(I^k) \to F(I^{k+1}))}{\text{im} (F(I^{k-1}) \to F(I^k))} \) = \( k \)-th cohomology group of the complex \( F(I^\cdot, d) \).
A is \( F \)-acyclic if \( R^k F(A) = 0 \) \( \forall k > 0 \).

**Generalized (Grothendieck) Construction**

Let \( A^* \in \mathcal{C}^+(\mathcal{A}) \) (reg. \( A^* \) is a complex of sheaves)

- replace \( A^* \) with a quasi-isomorphic complex \( I^* \) (st-injective)

notation: \( A^* \cong I^* \implies q_i [\text{i.e., } H^k(A^*) \cong H^k(I^*)] \)

Then get

\[
\begin{array}{c}
A^* \\
\downarrow q_i \\
F(I^*) \\
\cong \mathbb{R}F(A^*) \text{ "derived functor" Grothendieck.}
\end{array}
\]

-the cohomology of this:

\[
R^k F(A^*) = \mathbb{R}F^k(A^*) = H^k(F(I^*)) \text{ hypercohomology.}
\]

**Properties:**

\( A^* \cong B^* \implies \mathbb{R}F^i(A^*) \cong \mathbb{R}F^i(B^*) \)

**Fact:** If \( B^* \) is \( F \)-acyclic and \( A^* \cong B^* \),

\[
\mathbb{R}F^k(A^*) \cong H^k(F(B^*))
\]

hypercoh. \( (A^*) \cong \) actual coh. of \( F(B^*) \)

**Special case of**

\[ \Pi: \text{Sh}_\mathbb{Z}(X) \to \text{Ab} \]

\( \Pi \) (\( X \)-acyclic sheaves):

- **Injective**
- **Flabby**
- **Soft**
Complexes & filtrations
- We move from \( H^*(X; F) \) to \( H^*(X; \text{complex of sheaves}) \)
  such as: Čech complex
  Dolbeault complex

To calculate \( H^* \) in coefficients \( \boxtimes \) complex of sheaves,

USE: CARTAN - EILENBERG RESOLUTIONS

\[ F: A \rightarrow B \]

Let: \((C^*, D_1, D_2)\) cplx in \( C^*(A) \)

\[
\begin{array}{ccc}
C^0 & \rightarrow & C^1 & \rightarrow & C^2 & \rightarrow & \cdots \\
\rightarrow & D_1 & \rightarrow & D_1 & \rightarrow & \cdots \\
\end{array}
\]

- take injective resoln of each object \( C_i \) in \( C^* \).
- this is a really nice cplx w/ injective kernels & coboundaries
- \( D_1 \) and \( D_2 \) anticommute

We get: a DOUBLE COMPLEX, \((I^{*,*}, D_1, D_2)\)

- can condense double cplx into a regular cplx, \((E^*, d)\)

\[(E^*, d) = \text{Tot}(I^{*,*}) = \bigoplus_{r+s=k} I^{r,s} \]

\[ d = D_1 + D_2 \]

\textbf{Fact:} we get quasi-iso \((C^*, D_1) \cong (E^*, D)\).

So in fact, \( R^k F(C^*, d) = H^k (F(\text{Tot} I^{*,*})) = H^k F(E^*, d) \).
- Would like to distill info of $H^k(F(Tot I'))$

- 2 ways to filter $I'$ to get this:

- Total complex = circled parts

Filtration $L^0$ takes only some of total complex

Induced Decreasing Filtration (on columns) of total complex

$L^p I^K = \bigoplus_{rs = k, r \geq p} I^{r,s}$

(Total complex where all info before $p^{th}$ col. is replaced by 0's)

- Can do analogously along rows - truncate all before $q^{th}$ in $q^{th}$ filtration.
A decreasing filtration on a complex $K^\bullet$.

- Decreasing family of subcomplexes of $K^\bullet$.
- Uses differential of $K^\bullet$ restricted together.

\[ L^p K^k \rightarrow L^p K^{k+1} \]

$(K^\bullet, L) = \text{cplx equipped w/ decr\y filtration } L$.

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**Filtrations & Cohomology**

We begin with:

- Object $C$  \( \rightarrow \)  \{ resolution of $C$ \}

- Complex $C^\bullet$  \( \rightarrow \)  \{ $q-i$ complex $I^\bullet$ \}

Now:

- Filtered complex $(A^\bullet, L)$  \( \rightarrow \)  \{ filtered cohomology of $(A^\bullet, L)$ \}

When we filter cplx, use natural filtering of cohomology.

Define $p$th filtration of $H^\bullet(A^\bullet)$:

\[ L^p H^i(A^\bullet) = \text{Im}(H^i(L^p(A^\bullet)) \rightarrow H^i(A^\bullet)) \]

(Im of cohom of $p$th filt into cohom of $A^\bullet$).
To understand "filtered cohomology," we want to know:

\[ \text{Gr}_p^L H^i(A) = \frac{L^p H^i(A)}{L^{p+1} H^i(A)} \]

the \( p \)-th graded object associated to \( H^i(A) \)

Understanding/knowing these = "knowing" \( H^i(A) \)

**New goal:** Calculate cohomology of a filtered complex.

**Technique:** Spectral Sequences

\[ \text{Spec Seq} = \text{a series of double complexes } E_r^{i,j} \text{' (page up)} \]

with \( E_r^{i,j} = H^i(E_r^{i,j-1}) \) taken w.r.t \( dr \)

\( dr \) is of bi-degree \( \downarrow \text{down left} \uparrow \text{right up} \)

i.e. on \( E_r^{i,j} \):

so for \( E_3^{p,q} \), take \( \ker (d_3 \text{ coming out of } E_2^{p,q}) \)

\[ \text{Im (} d_3 \text{ going into } E_2^{p,q} \text{)} \]

- Same idea holds in general.
to be a spectral sequence, we also define specifically what we plug in for \( E_0 \):

\[
E_0^{p,q} = \text{Gr}_p^L A^{p+q} = \frac{L^p(A^{p+q})}{L^{p+1}(A^{p+q})}
\]

"\( \text{Gr}_p \) graded obj. associated to \( A^{p+q} \)

- third condition: "it converges to \( H^-(A^\cdot) \)"

\[
E_\infty^{p,q} = \text{Gr}_p^L H^{p+q}(A^\cdot)
\]

meaning, when \( E_r^{p,q} \) "stabilizes":

\[
E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots = E_\infty^{p,q}
\]

this happens at some \( r \), possibly a different \( r \) for varying \((p, q)\).

Example of resolution \( \rightarrow \) spectral seq:

let \((C^\cdot, d)\) = complex

- then the Cartan-Eilenberg resol'n is injective.

- columns will be exact because that's the property of a resolution.

\[
\Rightarrow \text{we put in } L^r(I^{\cdot, k}) = \bigoplus_{r+s = k} L^s p
\]

for filtration.
So $E_0$ page has input

$$E_0^{p,q} = \text{Gr}_p^{L} (I^{p+q}) = L^p (I^{p+q}) / L^{p+1} (I^{p+q})$$

= \underbrace{\bigoplus_{r + s = p + q, r \geq p} I^{r,s}}_{\text{obj. of } (\text{Tot } I)^{p+q} \text{ from } p \text{ on}}

= \underbrace{\bigoplus_{r + s = p + q, r \geq p+1} I^{r,s}}_{\text{obj. of } (\text{Tot } I)^{p+q} \text{ from } (p+1) \text{ on}}

= \text{objects of } (\text{Tot } I)^{p+q} \text{ in col } p = I^{p,q}

$$d_0 = (-1)^p D_2.$$  

$E_1^{p,q} = H^q_D(I^{p,*})$  

$$d_1 = (D^*_1) \phi : H^q_D(I^{p,*}) \to H^q_D(I^{p+1,*})$$

**But exactness** $\Rightarrow H^q_D(I^{p,*}) = 0$ except at $q = 0$.

Thus $E_2^{p,q} = H^p_{D^*_1} H^q_D(I^{*,*})$  

**differentials are 0** (must be) because all maps come from $\partial_0 / \partial_0 + 0$.

$$E^{p,q}_\infty = \text{Gr}_p^{L} H^{p+q}(C^*) = \text{Gr}_p^{L} H^{p+q}(C^*)$$

$$E^{p,q}_2 = H^p_{D^*_1} H^q_D(I^{*,*}) = H^{p+q}(C^*)$$
Filtered Complexes & cohomology: adding derived functors

General cplx with filtrations

\[ \mathcal{F} \]

Filtration of cohomology

Now, consider derived functors with this:
\[ A \longrightarrow B \]

Let \( (A', L) \) be filtered cplx in \( C^+(A) \).

- induces filtration of the derived functors \( R^iF(A') \)

\[ L^p R^iF(A') = \text{Im} \left( R^iF(L^p A') \longrightarrow R^iF(A') \right) \]

Thus if we begin with \( A' \in C^+(A) \) w/ filtration \( L' \),

then can get
\[ A' \longrightarrow I' \]

with a filtration s.t.

\[ L^p I' \text{ which are q-i to } L^p A' \]

This gives:
\[ H^i(F(L^p I')) = R^iF(L^p A') \]

So since \( L^p(F(I')) = F(L^p I') \), this gives us a filtration

\[ L^p R^iF(A') \text{ of } R^iF(A') \]

- can input this into a spectral sequence.
Spectral Sequences for $A \xrightarrow{F} B$:

**seq I:** $\pi E_{1}^{p,q} = R^{p} F(C^{p})$ so, $E_{2}^{p,q} = H^{p}(R^{q} F C^{p}, R^{q} F C^{p})$

abuts to $IR^{p+q} F(C^{p}, d)$.

This one degenerates at the $E_{2}$-page when we use a $C-E$ resolution.

**seq II:** $\pi E_{2}^{p,q} = R^{p} F(H^{q} C^{p}, d)$

also abuts to $IR^{p+q} F(C^{p}, d)$.

In a $C-E$ resolution, all coboundaries, cocycles are also injective throughout the s.seq.

$$\Rightarrow H^{p}(F C^{p}, d)) \cong F(H^{p}(C^{p}, d))$$

so we get that $C^{p} \rightarrow I^{p}$ injective resol'n.

$$H^{p}(F(I^{p}, d_{I}^{p}), F(H^{p}(I^{p}, d_{I}^{p}))) \cong F(H^{p}(I^{p}, d_{I}^{p}), d_{I}^{p})$$

$$R^{p} F(H^{q} C^{p}, d)) = H^{p} F(C^{p}, d)$$

This is the fundamental property we use to understand why the 2nd seq. also abuts to $IR^{p+q} F(C^{p}, d)$.

$$\Rightarrow E_{\infty}^{p,q} = Gr_{L}^{p}(R^{p+q} F(C^{p}, d))$$
Spectral sequences for composed functors

\[ \begin{align*}
A & \xrightarrow{F} B \xrightarrow{G} C \\
\end{align*} \]

- \( F \) is left exact, maps injectives in \( A \) to \( G \)-acyclics in \( B \)
- \( G \) is left exact.

Note: in this case \( R(G \circ F) \cong RG \circ RF \).

- Let \( M \in A \).
- Resolve \( M \to (I^\cdot, d) \)

Then \( R^i GF(M) = H^i(GF(I^\cdot), GF(d)) \)

**Theorem**

\( \exists \) a canonical filtration \( L \) on the objects \( R^i(GF)(M) \) and a spectral sequence

\[ \begin{align*}
E^{p,q}_r \Rightarrow R^*(GF)(M) \\
\end{align*} \]

with \( E_2 \) input term

\[ \begin{align*}
E^{p,q}_2 & = R^p G(R^q F(M)) \\
E^{p,q}_\infty & = Gr^p_L R^{p+q} GF(M) \\
\end{align*} \]

- will do this by using \( K^* = F(I^\cdot) \)
- in our regular "derived functor s. seq" from previous page.
Special case: Leray spectral sequence.

\[ f: X \to Y \text{ conts map of spaces.} \]

\[ \begin{array}{c}
\mathcal{S}_X \xrightarrow{f_*} \mathcal{S}_Y \xrightarrow{\Pi(Y,-)} \operatorname{Ab} \\
\Lambda \quad \text{B} \\
\Pi(X,-)
\end{array} \]

Let \( f_* = F \). \( \Pi(Y, -) = G \).

- input into Grothendieck sseq.

**Theorem (Leray)**

For every sheaf \( F \) on \( X \), there exists a canonical filtration \( L \) on \( H^\bullet(X, F) \) and a spectral sequence

\[ E_\infty^{p,q} \Rightarrow H^{p+q}(X, F) \]

that is canonical starting from \( E_2 \), and satisfies

\[ E_2^{p,q} = H^p(Y, R^q f_*(F)), \quad E_\infty^{p,q} = \operatorname{Gr}_p H^{p+q}(X, F) \]

(Leray Spectral Sequence)
Using Leray SS to calculate \( H^*(\mathbb{CP}^2) \)

\[ S^5 \xrightarrow{f} \text{Hopf Fibration} \quad \forall \text{pt } x \in \mathbb{CP}^2, \quad f^{-1}(x) \approx S^1 \]

our functors:

\[ \text{Sh}(S^5) \xrightarrow{f_*} \text{Sh}(\mathbb{CP}^2) \xrightarrow{\mathcal{F}(\mathbb{CP}^2, -)} \text{Ab} \]

Do for constant sheaf \( \mathbb{C} \).

\[ E_{i}^{p,q} = H^p(\mathbb{CP}^2; R^q f_* \mathbb{C}) \Rightarrow \text{Gr}_p^L H^q(S^5; \mathbb{C}) \]

\[ = \text{Gr}_p^L H^q(S^1; \mathbb{C}) \]

What is \( R^q f_* \mathbb{C} \)?

- Stalk over any \( x \in \mathbb{CP}^2 = H^q(f^{-1}(x); \mathbb{C}) \)
  
  Since that's the sheaf cohom.

- The Hopf Fibration is a bundle, so any \( y \in \mathbb{CP}^2 \) has

\[ f^{-1}(y) \approx f^{-1}(x) \Rightarrow H^q(f^{-1}(x); \mathbb{C}) = H^q(f^{-1}(y); \mathbb{C}) = H^q(S^1; \mathbb{C}) \]

So \( R^q f_* \mathbb{C} \) is a locally constant sheaf over a simply connected space, \( \mathbb{CP}^2 \).

Thus \( R^q f_* \mathbb{C} = H^q(S^1; \mathbb{C}) = \begin{cases} \mathbb{C} & q = 0, 1 \\ 0 & \text{else} \end{cases} \)
$E_2$ page is

$E_2^{p, q} = \text{Gr}_p^L H^{p+q}(S^5; \mathbb{C})$ - only on $(0, 0)$ and diagonal $p+q = 5$

This $SS$ degenerates after $E_3$ and beyond.

Using Poincaré duality and where differentials map to 0, we can deduce cohomology of $CP^2$: $\{0, 1, 3, 4\}$.