Seiberg-Witten invariants and surface singularities

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(X, p) (germ) of isolated surface singularity (*i.s.s.* for brevity). Assume X is Stein.

1 Topological Invariants

The link. Embed $(X, p) \hookrightarrow (\mathbb{C}^N, 0)$, and set

$$M = X \cap S_{\varepsilon}^{2N-1}(0).$$

M is an oriented 3-manifold independent on the embedding and $\varepsilon \ll 1$.

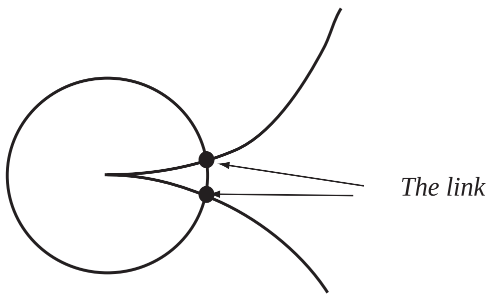


Figure 1: *The link of an isolated singularity*

Good resolutions. A *resolution* of (X, p) is a pair $(\tilde{X}, \pi)$ where

- $\tilde{X}$ is a smooth complex surface;
- $\tilde{X} \xrightarrow{\pi} X$ is holomorphic;
- $\tilde{X} \setminus \pi^{-1}(p) \to X \setminus p$ is biholomorphic;

The resolution is called *good* if the exceptional divisor $E := \pi^{-1}(p)$ is a normal crossing divisor i.e its irreducible components $(E_i)_{1\leq i\leq n}$ are smooth curves intersecting transversally.

FACT. *Good resolutions exist but are not unique. There exists a unique minimal resolution $\hat{X}$, i.e. a resolution containing no -1-spheres. There exists a unique minimal good resolution. (It may have -1 spheres, but when blown down the exceptional divisor will no longer be a normal crossing divisor). Any other resolution is obtained from the minimal one by blowing-up/down -1 spheres.*

Suppose $\tilde{X}$ is a resolution of X. We set

$$\Lambda = \Lambda(\tilde{X}) := \mathrm{span}_{\mathbb{Z}}\{E_i\} \subset H_2(\tilde{X}, \mathbb{Z}),$$

$$\Lambda_+(\tilde{X}) := \Big\{\sum_i m_i E_i \in \Lambda;\ \ m_i \geq 0\Big\}.$$

Theorem. (D. Mumford) *The symmetric matrix $(E_i \cdot E_j)_{i,j}$ is < 0.*

The dual resolution graph. Suppose $(\tilde{X}, \pi)$ is a good resolution of the i.s.s. (X, p) with exceptional divisor $E = \bigcup_i E_i$. The (dual) resolution graph is a decorated graph $\Gamma = \Gamma_{\tilde{X}}$ obtained as follows.

- There is one vertex v_i for each component E_i.
- Two vertices v_i, v_j, $i \neq j$ are connected by $E_i \cdot E_j$ edges.
- Each vertex v_i is decorated by two integers, the genus g_i of E_i, and the self intersection number $e_i := E_i^2$.

We see that $\tilde{X}$ is a plumbing of disk bundles over the Riemann surfaces E_i, with plumbing instructions contained in the graph Γ. The boundary of this plumbing is precisely the link of the singularity.

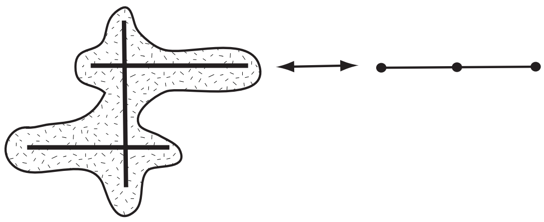


Figure 2: *A plumbing and its associated dual graph*

Theorem. (W.Neumann) *Suppose (X_i, p), $i = 0, 1$ are two i.s.s. Denote by M_i their links, and by $\tilde{X}_i$ their minimal good resolutions. The following statements are equivalent.*

(a) The graphs $\Gamma_{\tilde{X}_i}$ are isomorphic (as weighted graphs).

(b) The links M_i are diffeomorphic as oriented 3-manifolds.

Definition. We say that a property of an i.s.s. is *topological* if it can be described in terms of the combinatorics of the dual graph of the minimal good resolution.

□

The arithmetic genus. $\tilde{X}$ resolution of (X,p), $E=\cup_i E_i$, the exceptional divisor. Note that every $Z=\sum_i n_i E_i \in \Lambda_+$ can be identified with a compact complex curve on $\tilde{X}$. The *arithmetic genus* of Z is defined by

$$p_a(Z)=1+\frac{1}{2}\big(Z\cdot Z+\langle K_{\tilde{X}},Z\rangle\big),$$

where $K_{\tilde{X}}\in H^2(\tilde{X},\mathbb{Z})$ is the canonical line bundle of $\tilde{X}$. When Z is a smooth curve $p_a(Z)$ is the usual genus of Z. Set

$$p_a(\tilde{X}):=\sup\big\{p_a(Z);\;\; Z\in\Lambda_+\setminus 0\big\}.$$

This nonnegative integer is independent of the resolution and thus it is a *topological* invariant of (X,p). We will denote it by $p_a(X,p)$, and we will refer to it as the arithmetic genus of the singularity.

The canonical cycle. (X,p) - i.s.s. and $(\tilde{X},\pi)$ is a resolution. The canonical cycle is the cycle $Z_K=Z_K(\tilde{X})\in\Lambda\otimes\mathbb{Q}$ defined by

$$Z_K\cdot E_j=-\langle K_{\tilde{X}},E_j\rangle=2-p_a(E_j)+E_j^2,\forall i.$$

Set

$$\gamma(\tilde{X})=Z_{K_{\tilde{X}}}^2+b_2(\tilde{X})\in\mathbb{Q}.$$

This number is independent of the resolution $\tilde{X}$, and thus it is a topological invariant of (X,p). We will denote it by $\gamma(X,p)$. Note that if $\tilde{X}$ is the minimal good resolution then $Z_{K_{\tilde{X}}}$ is a topological invariant of M.

Observe that

$$\gamma(X,p)=\Big(K_{\tilde{X}}^2-\big(2\chi(\tilde{X})+3\mathrm{sign}\,(\tilde{X})\big)\Big)+2-2b_1(\tilde{X}). \tag{γ}$$

Definition. *Suppose (X,p) is an i.s.s., and $(\tilde{X},\pi,E)$ is a good resolution. The singularity is called* **Gorenstein** *if $K_{\tilde{X}}\mid_{\tilde{X}\setminus E}$ is* **holomorphically** *trivial. The singularity is called* **numerically Gorenstein** *if $K_{\tilde{X}}\mid_{\tilde{X}\setminus E}$ is* **topologically** *trivial.*

Observe that (X,p) is numerically Gorenstein iff

$$K_{\tilde{X}}\in H^2(\tilde{X},\partial\tilde{X};\mathbb{Z})\Longleftrightarrow Z_K\in\Lambda.$$

Example. All local complete intersection singularities are Gorenstein. Recall that the i.s.s. (X,p) is a local complete intersection singularity if near p it can is described as the zero set of a holomorphic map $F:\mathbb{C}^N\to\mathbb{C}^{N-2}$. □

2 Analytic invariants

The geometric genus. (X,p) i.s.s., X Stein, $\tilde{X}$ resolution.

$$\tilde{X}\text{ Levi pseudoconvex}\Longrightarrow\dim H^k(\tilde{X},\mathcal{O}_{\tilde{X}})<\infty,\;\;\forall k\geq 1.$$

The integer $\dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is independent of the resolution, and thus it is an *analytic* invariant of (X,p). It is called the *geometric genus* and is denoted by $p_g(X,p)$. It is known that

$$p_g(X,p) \geq p_a(X,p).$$

Example. Suppose $L \to \Sigma$ is a degree $d < 0$ holomorphic line bundle over the Riemann surface Σ of genus g. By a theorem of Grauert there exists a natural Stein space X with an isolated singularity at $p \in X$, and a holomorphic map

$$(L, \Sigma) \xrightarrow{\pi} (X, p)$$

which makes L a good resolution of (X,p) with exceptional divisor $\Sigma \hookrightarrow L$. Then

$$p_a(X,p) = g, \;\; Z_K = \Big(1 + \frac{2-2g}{d}\Big)\Sigma, \;\; \gamma(X,p) = d\Big(1 + \frac{2-2g}{d}\Big)^2 + 1,$$

$$p_g(X,p) = \sum_{n\geq 0} \dim H^1\big(\Sigma, \mathcal{O}(-nL)\big) \stackrel{Serre}{=} \sum_{n\geq 0} \dim H^0\big(\Sigma, \mathcal{O}(K_\Sigma + nL)\big). \qquad (p_g)$$

We deduce that $p_g(X,p)$ depends on the complex structure on Σ, and on the complex structure on L, i.e. on the holomorphic embedding $\Sigma \hookrightarrow L$. These dependencies on analytic data become irrelevant under appropriate topological constraints.

- Σ is rational, i.e. $g = 0$.
- Σ is elliptic, i.e. $g = 1$.
- The degree of L is sufficiently negative, $\deg L \leq -g$.

In all these cases $p_g(X,p) = p_a(X,p) = g$.

□

Smoothings. A smoothing of an i.s.s. (X,p) is a proper flat map $(\mathcal{X}, q) \xrightarrow{F} (\mathbb{C}, 0)$ together with an embedding $\imath : (X,p) \hookrightarrow (\mathcal{X}, q)$ which induces an isomorphism $(X,p) \cong (F^{-1}(0), q)$. For $t \in \mathbb{C}^*$ sufficiently small the fiber $X_t := f^{-1}(t)$ is smooth. Its topology is independent of t. X_t is called the *Milnor fiber* of the smoothing. The Milnor number μ of the smoothing is $b_2(X_t)$.

Example. Suppose (X,p) is the complete intersection, described by the zero set of a map $F : \mathbb{C}^N \to \mathbb{C}^{N-2}$. To construct smoothings of (X,p) its suffices to pick a line through the origin $L \subset \mathbb{C}^{N-2}$ and set $\mathcal{X} := F^{-1}(L)$.

□

Theorem. (Durfee-Laufer-Steenbrink-Wahl) *Suppose (X,p) is a smoothable Gorenstein i.s.s. We denote by F its Milnor fiber. Then*

$$p_g(X,p) = -\frac{1}{8}\mathrm{sign}\,(F) - \frac{1}{8}\gamma(X,p).$$

Motivated by the above result, we define the *virtual signature* of an i.s.s. by

$$\sigma_{virt}(X,p) := -8p_g(X,p) - \gamma(X,p).$$

For smoothable Gorenstein singularities, the virtual signature is the signature of the Milnor fiber.

3 The Main Problem and a Bit of History

The Main Problem. *How much information about the analytic structure of the i.s.s. (X, p) is encoded in the topology of its link M. In particular, can we determine $p_g(X, p)$ from combinatorial data contained in the dual resolution graph of the minimal good resolution?*

History. Work of the past four decades indicates that the link often contains nontrivial information about the analytic structure.

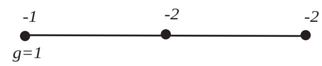
❶ (D. Mumford, 1961) (X, p) is smooth at p if and only if the link is $\cong S^3$.

❷ (M. Artin, 1962-66) $p_g(X, p) = 0 \Longleftrightarrow p_a(X, p) = 0$. In this case, the link M is a rational homology sphere ($\mathbb{Q}HS$ for brevity).

❸ (H. Laufer, 1977) Assume that (X, p) is elliptic, i.e. $p_a(X, p) = 1$, and Gorenstein. Then the condition $p_g(X, p) = 1$ is topological.

❹ (A. Nemethi, 1999) Assume that (X, p) is elliptic, Gorenstein, and the link M is a $\mathbb{Q}HS$. Then $p_g(X, p)$ is equal to a certain topological invariant of M, the length of the elliptic sequence defined by S.S.-T. Yau.

Remark. (a) M is a $\mathbb{Q}HS$ iff Γ is a tree and all the components E_i are rational curves.
(b) The condition that M is a $\mathbb{Q}HS$ cannot be removed from ❹. To see this consider the singularities $(X_1, 0) = \{x^2 + y^3 + z^{18} = 0\}$, $(X_2, 0) = \{z^2 + y(x^4 + y^6) = 0\}$. They have isomorphic resolution graphs (see below), but $p_g(X_1, 0) = 3$, $p_g(X_{,2}\,, 0) = 2$. □



❺ (Fintushel-Stern, Neumann-Wahl, 1990) Suppose (X, p) is a Brieskorn complete intersection singularity and the link M is an *integral* homology sphere ($\mathbb{Z}HS$). Then

$$\text{Casson}\,(M) = -\frac{1}{8}\sigma_{virt}(X, p).$$

Here we recall that a Brieskorn complete intersection singularity is a complete intersection singularity of the form

$$\left\{\begin{array}{ccccccc} a_{11}z_1^{p_1} & + & \cdots & + & a_{1n}z_n^{p_n} & = & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(n-2)1}z_1^{p_1} & + & \cdots & + & a_{(n-2)n}z_n^{p_n} & = & 0 \end{array}\right.$$

Remark. If in ❺ we assume only that M is a $\mathbb{Q}HS$ then the obvious generalization

$$\text{Casson-Walker}\,(M) = -\frac{1}{8}\sigma_{virt}(X, p).$$

is no longer true.

□

4 The Main Conjecture and Evidence in its Favor

The Main Conjecture. Suppose (X, p) is a rational, or Gorenstein singularity such that its link is a $\mathbb{Q}HS$. Then M is equipped with a canonical $spin^c$ structure σ_{can}, *which depends only on the resolution graph* Γ *of the minimal good resolution*, and

$$\boldsymbol{sw}_M(\sigma_{can}) = -\frac{1}{8}\sigma_{virt}(X, p),$$

where $\boldsymbol{sw}_M(\sigma_{can})$ denotes the Seiberg-Witten invariant of the canonical $spin^c$ structure. In particular, if M is a $\mathbb{Z}HS$ then there is an unique $spin^c$ structure on M whose Seiberg-Witten invariant equals the Casson invariant of M so that

$$\text{Casson}\,(M) = -\frac{1}{8}\sigma_{virt}(X, p).$$

$\square$

Evidence. We need to describe the various terms in the Main Conjecture.

Denote by $\tilde{X}$ the minimal good resolution of (X, p). Then

$$\Lambda = H_2(\tilde{X}, \mathbb{Z}), \;\; H^2(\tilde{X}, \mathbb{Z}) \cong \check{\Lambda} := \mathrm{Hom}(\Lambda, \mathbb{Z})$$

Set $H := H_1(M, \mathbb{Z})$, and denote the group operation on H multiplicatively. The intersection form on Λ defines an embedding $\Lambda \hookrightarrow \check{\Lambda}$, and we have

$$H \cong \check{\Lambda}/\Lambda.$$

H acts freely and transitively on the set $Spin^c(M)$ of $spin^c$ structures on M

$$H \times Spin^c(M) \ni (h, \sigma) \mapsto h \cdot \sigma \in Spi^c(M).$$

To define the canonical $spin^c$ structure σ_{can} let us recall that a choice of a $spin^c$ structure on M is equivalent to a choice of an almost complex structure on the stable tangent bundle $\underline{\mathbb{R}} \oplus TM$ of M. The stable tangent bundle of M is equipped with a natural complex structure induced by the complex structure on $\tilde{X}$. σ_{can} is the $spin^c$ structure associated to this complex structure.

$*$ **Proposition.** σ_{can} can be described only in terms of the combinatorics of $\Gamma_{\tilde{X}}$.

Proof. Denote by $\boldsymbol{lk}_M : H \times H \to \mathbb{Q}/\mathbb{Z}$ the linking form of M. An enhancement of $\boldsymbol{lk}_M$ is a function

$$q : H \to \mathbb{Q}/\mathbb{Z}$$

such that

$$q(h_1 h_2) - q(h_1) - q(h_2) = \boldsymbol{lk}_M(h_1, h_2), \;\; \forall h_1, h_2 \in H.$$

There is a natural bijection between $Spin^c(M)$ and the set of enhancements, $\sigma \mapsto q_\sigma$. Recalling that $H \cong \check{\Lambda}/\Lambda$ we define

$$q_{can} : \check{\Lambda}/\Lambda \to \mathbb{Q}, \;\; q_{can}(h) = -\frac{1}{2}\big(K_{\tilde{X}} \cdot \check{h} + \check{h} \cdot \check{h}\big)\mathrm{mod}\mathbb{Z}$$

for every $h \in \check{\Lambda}/\Lambda$, and every $\check{h} \in \check{\Lambda}$ which projects onto h. The expression in the right hand side depends only on Γ. σ_{can} is the *spin*^{c} structure corresponding to q_{can}.

$\square$

Remark. (a) q_{can} first appeared in work of Looijenga-Wahl.
(b) From the above description of σ_{can} and the equality (γ) we deduce that $\gamma(X,p) - 2$ equals the Gompf invariant of the *spin*^{c} structure σ_{can}.

$\square$

The Seiberg-Witten invariant is a function

$$\boldsymbol{sw}_M : Spin^c(M) \to \mathbb{Q}, \;\; \sigma \mapsto \boldsymbol{sw}_M(\sigma).$$

$\boldsymbol{sw}_M(\sigma)$= # of Seiberg-Witten σ-monopoles + the Kreck-Stolz invariant of σ (a certain combination of eta invariants). For each $\sigma \in Spin^c(M)$ define

$$\boldsymbol{SW}_{M,\sigma} : H \to \mathbb{Q}, \;\; \boldsymbol{SW}_{M,\sigma}(h) = \boldsymbol{sw}_M(h^{-1} \cdot \sigma).$$

One can give a combinatorial description of this invariant. For each *spin*^{c} structure σ, the Reidemeister-Turaev torsion of (M, σ) is a function

$$\mathfrak{T}_{M,\sigma} : H \to \mathbb{Q}.$$

Denote by CW_M the Casson-Walker invariant of M, and define the modified Reidemeister-Turaev torsion of M by

$$\mathfrak{T}^0_{M,\sigma} : H \to \mathbb{Q}, \;\; \mathfrak{T}^0_{M,\sigma}(h) := \frac{1}{|H|} CW_M + \mathfrak{T}_{M,\sigma}(h), \;\; \forall h \in H.$$

Theorem. (L.I. Nicolaescu) For every $\sigma \in Spin^c(M)$ we have

$$\boldsymbol{SW}_{M,\sigma} \equiv \mathfrak{T}^0_{M,\sigma}.$$

Denote by $\hat{H}$ the Pontryagin dual of H, $\hat{H} := \mathrm{Hom}\big(H, U(1)\big)$. The Fourier transform of $\mathfrak{T}_{M,\sigma_{can}}$ is the function

$$\hat{\mathfrak{T}}_{M,\sigma_{can}} : \hat{H} \to \mathbb{C}, \;\; F(\chi) = \sum_{h \in H} \mathfrak{T}_{M,\sigma_{can}}(h) \bar{\chi}(h).$$

The Fourier inversion formula implies

$$\boldsymbol{sw}_M(\sigma_{can}) = \boldsymbol{SW}_{M,\sigma_{can}}(1) = \frac{1}{|H|} CW_M + \frac{1}{|H|} \sum_{\chi \in \hat{H}} \hat{\mathfrak{T}}_{M,\sigma_{can}}(\chi). \tag{$*$}$$

Theorem.(Lescop-Raţiu) *The Casson-Walker invariant can be described* **explicitly** *in terms of the combinatorics of* Γ.

Main Technical Result. (Nemethi-Nicolaescu) $\hat{\mathfrak{T}}_{M,\sigma_{can}}$ *can be described* **explicitly** *in terms of the combinatorics of* $\Gamma_{\tilde{X}}$.

Idea of Proof. Using surgery formulæ we produce an *explicit* holomorphic regularization R_M of $\hat{\mathfrak{T}}_{M,\sigma_{can}}$. This is an element in the group algebra $\mathbb{C}(t)[\hat{H}]$,

$$R_M = \sum_{\chi \in \hat{H}} R_\chi(t)\chi, \quad R_\chi \in \mathbf{C}(t) = \text{the field of rational functions in one variable} \qquad (**)$$

such that for every character χ

$$\lim_{t \to 1} R_\chi(t) = \hat{\mathfrak{T}}_{M,\sigma_{can}}(\chi).$$

□

In applications the sum in the right-hand side of $(*)$ is difficult to compute if the combinatorics of the graph is very involved.

Theorem. (Nemethi-Nicolaescu) *The Main Conjecture is true for* **all** *the quasihomogeneous singularities whose links are $\mathbb{Q}HS$'s.*

Idea of Proof. For a quasihomogeneous singularity (X, p) the resolution graph is star-shaped and the sum in $(*)$ simplifies somewhat. Our expression for the holomorphic regularization R_M in $(**)$ is formally identical to the Poincaré series associated to the Universal Abelian cover of (X, p) introduced by W.Neumann. The proof of the Main Conjecture in this case relies on a formula for $p_g(X, p)$ of Dolgachev-Pinkham, and on some ideas of Neumann and Zagier.

□