Tame flows

Liviu I. Nicolaescu

University of Notre Dame

May 22, 2011
1. Basic constructions

2. Properties of tame flows

3. Morse-like tame flows

4. Combinatorial Morse theory

5. Flip-flops and Morse-like flows
Here we introduce the main character
Here we introduce the main character

Denote by $\mathcal{C}$ the pfaffian closure of $\mathbb{R}_{an}$. 
Here we introduce the main character

Denote by $\mathcal{C}$ the pfaffian closure of $\mathbb{R}_{an}$. In the sequel, we refer to the sets or maps in $\mathcal{C}$ as *tame*, or *definable* or *constructible*.
Here we introduce the main character

Denote by $\mathcal{C}$ the pfaffian closure of $\mathbb{R}_{an}$. In the sequel, we refer to the sets or maps in $\mathcal{C}$ as *tame*, or *definable* or *constructible*.

**Definition**

A **tame flow** is a pair $(X, \Phi)$ such that
Here we introduce the main character

Denote by $\mathcal{C}$ the pfaffian closure of $\mathbb{R}_{an}$. In the sequel, we refer to the sets or maps in $\mathcal{C}$ as *tame*, or *definable* or *constructible*.

**Definition**

A **tame flow** is a pair $(X, \Phi)$ such that

- $X$ is a tame set.
Here we introduce the main character

Denote by $\mathcal{C}$ the pfaffian closure of $\mathbb{R}_{an}$. In the sequel, we refer to the sets or maps in $\mathcal{C}$ as *tame*, or *definable* or *constructible*.

**Definition**

A *tame flow* is a pair $(X, \Phi)$ such that

- $X$ is a tame set.
- $\Phi$ is a continuous tame map $\mathbb{R} \times X \to X$, $(t, x) \mapsto \Phi^t(x)$ such that

\[
\Phi^{t+s}(x) = \Phi^t(\Phi^s(x)) \quad \forall t, s \in \mathbb{R}, \ x \in X.
\]
The canonical flow on \([0, 1]\)

We have the following simple but fundamental example of tame flow.

\[ X = [0, 1]. \]

\[ \Phi_t \] is the flow generated by the differential equation on \(\mathbb{R}\)

\[
\dot{x} = x(x - 1).
\]

The interval \([0, 1]\) is invariant with respect to this flow and

\[
\Phi_t(x_0) = e^{-t}x_0 + 1 - x_0 e^{-t}, \quad \forall x_0 \in [0, 1].
\]

The endpoints 0, 1 are stationary points of this flow and

\[
\lim_{t \to \infty} \Phi_t(x) = 0, \quad \lim_{t \to -\infty} \Phi_t(x) = 1, \quad \forall x \in (0, 1).
\]
The canonical flow on $[0, 1]$

We have the following simple but fundamental example of tame flow.
The canonical flow on $[0, 1]$

We have the following simple but fundamental example of tame flow.

- $X = [0, 1]$. 
The canonical flow on $[0, 1]$

We have the following simple but fundamental example of tame flow.

- $\mathcal{X} = [0, 1]$.
- $\Phi^t$ is the flow generated by the differential equation on $\mathbb{R}$

\[
\frac{dx}{dt} = x(x - 1).
\]
The canonical flow on $[0, 1]$

We have the following simple but fundamental example of tame flow.

- $X = [0, 1]$.
- $\Phi^t$ is the flow generated by the differential equation on $\mathbb{R}$

$$\dot{x} = x(x - 1).$$

The interval $[0, 1]$ is invariant with respect to this flow and
The canonical flow on $[0, 1]$ 

We have the following simple but fundamental example of tame flow.

- $X = [0, 1]$.
- $\Phi^t$ is the flow generated by the differential equation on $\mathbb{R}$
  \[
  \dot{x} = x(x - 1).
  \]

The interval $[0, 1]$ is invariant with respect to this flow and

\[
\Phi^t(x_0) = \frac{e^{-t}x_0}{1 - x_0 + e^{-t}x_0}, \quad \forall x_0 \in [0, 1].
\]
The canonical flow on $[0, 1]$

We have the following simple but fundamental example of tame flow.

- $\mathcal{X} = [0, 1]$.
- $\Phi^t$ is the flow generated by the differential equation on $\mathbb{R}$
  \[ \dot{x} = x(x - 1). \]

The interval $[0, 1]$ is invariant with respect to this flow and

\[ \Phi^t(x_0) = \frac{e^{-t}x_0}{1 - x_0 + e^{-t}x_0}, \quad \forall x_0 \in [0, 1]. \]

- The endpoints 0, 1 are stationary points of this flow and
The canonical flow on $[0, 1]$

We have the following simple but fundamental example of tame flow.

- $X = [0, 1]$.
- $\Phi^t$ is the flow generated by the differential equation on $\mathbb{R}$

\[
\dot{x} = x(x - 1).
\]

The interval $[0, 1]$ is invariant with respect to this flow and

\[
\Phi^t(x_0) = \frac{e^{-t}x_0}{1 - x_0 + e^{-t}x_0}, \quad \forall x_0 \in [0, 1].
\]

The endpoints 0, 1 are stationary points of this flow and

\[
\lim_{t \to \infty} \Phi^t(x) = 0, \quad \lim_{t \to -\infty} \Phi^t(x) = 1, \quad \forall x \in (0, 1).
\]
The cone construction

Suppose $\Phi$ is a tame flow over a tame set $X$. The cone over $X$ is the definable quotient $C(X) := \left[0, 1\right] \times X / \left\{1 \times X\right\}$. The slice $\left\{1\right\} \times X$ is mapped to the vertex of the cone, denoted by $\ast$. Denote by $\pi : \left[0, 1\right] \times X \to C(X)$ the natural projection. Observe that $\pi$ induces a definable homeomorphism $\left[0, 1\right) \times X \to C(X) \ast := C(X) \left\{\ast\right\}$.

We thus have two natural projections the shadow $\sigma : C(X) \ast \to X$ and the altitude $\alpha : C(X) \to \left[0, 1\right]$.
The cone construction

Suppose $\Phi$ is a tame flow over a tame set $X$. 
Suppose \( \Phi \) is a tame flow over a tame set \( X \). The cone over \( X \) is the definable quotient

\[
C(X) := [0, 1] \times X / \{1\} \times X.
\]
The cone construction

Suppose $\Phi$ is a tame flow over a tame set $X$. The cone over $X$ is the definable quotient

$$C(X) := [0, 1] \times X / \{1\} \times X.$$ 

The slice $\{1\} \times X$ is mapped to the vertex of the cone, denoted by $\ast$. 
The cone construction

Suppose $\Phi$ is a tame flow over a tame set $X$. The cone over $X$ is the definable quotient

$$C(X) := [0, 1] \times X / \{1\} \times X.$$  

The slice $\{1\} \times X$ is mapped to the vertex of the cone, denoted by $\ast$. Denote by $\pi : [0, 1] \times X \to C(X)$ the natural projection.
The cone construction

Suppose $\Phi$ is a tame flow over a tame set $X$. The cone over $X$ is the definable quotient

$$C(X) := [0, 1] \times X / \{1\} \times X.$$ 

The slice $\{1\} \times X$ is mapped to the vertex of the cone, denoted by $\ast$. Denote by $\pi : [0, 1] \times X \to C(X)$ the natural projection. Observe that $\pi$ induces a definable homeomorphism

$$[0, 1) \times X \to C(X)^\ast := C(X) \setminus \{\ast\}.$$
The cone construction

Suppose $\Phi$ is a tame flow over a tame set $X$. The cone over $X$ is the definable quotient

$$C(X) := [0, 1] \times X / \{1\} \times X.$$  

The slice $\{1\} \times X$ is mapped to the vertex of the cone, denoted by $\ast$. Denote by $\pi : [0, 1] \times X \to C(X)$ the natural projection. Observe that $\pi$ induces a definable homeomorphism

$$[0, 1) \times X \to C(X)^\ast := C(X) \setminus \{\ast\}.$$  

We thus have two natural projections
The cone construction

Suppose $\Phi$ is a tame flow over a tame set $X$. The cone over $X$ is the definable quotient

$$C(X) := [0, 1] \times X / \{1\} \times X.$$  

The slice $\{1\} \times X$ is mapped to the vertex of the cone, denoted by $\ast$. Denote by $\pi : [0, 1] \times X \to C(X)$ the natural projection. Observe that $\pi$ induces a definable homeomorphism

$$[0, 1) \times X \to C(X)^\ast := C(X) \setminus \{\ast\}.$$  

We thus have two natural projections the shadow $\sigma : C(X)^\ast \to X$ and
The cone construction

Suppose $\Phi$ is a tame flow over a tame set $X$. The cone over $X$ is the definable quotient

$$C(X) := [0, 1] \times X / \{1\} \times X.$$ 

The slice $\{1\} \times X$ is mapped to the vertex of the cone, denoted by $\ast$. Denote by $\pi : [0, 1] \times X \to C(X)$ the natural projection. Observe that $\pi$ induces a definable homeomorphism

$$[0, 1) \times X \to C(X)^\ast := C(X) \setminus \{\ast\}.$$ 

We thus have two natural projections the shadow $\sigma : C(X)^\ast \to X$ and the altitude $\alpha : C(X) \to [0, 1]$. 

Liviu I. Nicolaescu (Notre Dame)
The cone of a flow

The shadow $\sigma(p)$ is governed by the flow $\Phi$ on $X$. The altitude $\alpha(p)$ changes according to the canonical flow on $[0,1]$. The vertex $*$ is a stationary repelling point.
The cone of a flow

The shadow $\sigma(p)$ is governed by the flow $\Phi$ on $X$. 
The shadow $\sigma(p)$ is governed by the flow $\Phi$ on $X$. The altitude $\alpha(p)$ changes according to the canonical flow on $[0, 1]$. 
The shadow $\sigma(p)$ is governed by the flow $\Phi$ on $X$. The altitude $\alpha(p)$ changes according to the canonical flow on $[0, 1]$. The vertex $*$ is a stationary repelling point.
Flows on affine simplices

- Consider points $v_0, v_1, \ldots, v_n \in \mathbb{R}^N$ such that the vectors $\overrightarrow{v_0v_1}, \ldots, \overrightarrow{v_0v_1}$ are linearly independent.
Flows on affine simplices

- Consider points $v_0, v_1, \ldots, v_n \in \mathbb{R}^N$ such that the vectors $\overrightarrow{v_0v_1}, \ldots, \overrightarrow{v_0v_1}$ are linearly independent.
- The affine simplex spanned by $v_0, \ldots, v_n$ is a cone with vertex $v_n$ over the affine simplex spanned by $v_0, \ldots, v_{n-1}$ etc.
Flows on affine simplices

- Consider points $v_0, v_1, \ldots, v_n \in \mathbb{R}^N$ such that the vectors $\overrightarrow{v_0v_1}, \ldots, \overrightarrow{v_0v_1}$ are linearly independent.
- The affine simplex spanned by $v_0, \ldots, v_n$ is a cone with vertex $v_n$ over the affine simplex spanned by $v_0, \ldots, v_{n-1}$ etc.
- Iterating the cone construction starting from the canonical flow on the unit interval we deduce that any linear ordering on the set of vertices of an affine simplex determines a canonical tame flow on the simplex.
Flows on affine simplices

- Consider points $v_0, v_1, \ldots, v_n \in \mathbb{R}^N$ such that the vectors $\overrightarrow{v_0v_1}, \ldots, \overrightarrow{v_0v_1}$ are linearly independent.

- The affine simplex spanned by $v_0, \ldots, v_n$ is a cone with vertex $v_n$ over the affine simplex spanned by $v_0, \ldots, v_{n-1}$ etc.

- Iterating the cone construction starting from the canonical flow on the unit interval we deduce that any linear ordering on the set of vertices of an affine simplex determines a canonical tame flow on the simplex.
Suppose $X$ is a compact tame set equipped with a definable triangulation, i.e., a tame homeomorphism $F: K \to X$, where $K$ is a finite affine simplicial complex. Fix a dynamical orientation of the triangulation, i.e., a labeling of the vertices with real numbers so that no two adjacent vertices have the same label. This induces linear orders on the set of vertices of any face and thus tame flows on each of the (closed) faces of $K$. These flows match on the overlaps of faces, and thus define a tame flow on $K$ which can be transported via $F$ to a tame flow on $X$. This is called the simplicial flow determined by the triangulation and the dynamical orientation. Thus, tame flows exist on any compact tame set.
Suppose $X$ is a compact tame set equipped with a definable triangulation, i.e.,
Suppose $X$ is a compact tame set equipped with a definable triangulation, i.e., a tame homeomorphism $F : K \to X$, where $K$ is a finite affine simplicial complex. This induces linear orders on the set of vertices of any face and thus tame flows on each of the (closed) faces of $K$. These flows match on the overlaps of faces, and thus define a tame flow on $K$ which can be transported via $F$ to a tame flow on $X$. This is called the simplicial flow determined by the triangulation and the dynamical orientation. Thus, tame flows exist on any compact tame set.
Simplicial flows

- Suppose $X$ is a compact tame set equipped with a definable triangulation, i.e., a tame homeomorphism $F : K \to X$, where $K$ is a finite affine simplicial complex.
- Fix a *dynamical orientation* of the triangulation, i.e.,
Simplicial flows

Suppose $X$ is a compact tame set equipped with a definable triangulation, i.e., a tame homeomorphism $F : K \to X$, where $K$ is a finite affine simplicial complex.

Fix a *dynamical orientation* of the triangulation, i.e., a labeling of the vertices with real numbers so that no two adjacent vertices have the same label.

This induces linear orders on the set of vertices of any face and thus tame flows on each of the (closed) faces of $K$. These flows match on the overlaps of faces, and thus define a tame flow on $K$ which can be transported via $F$ to a tame flow on $X$.

This is called the simplicial flow determined by the triangulation and the dynamical orientation.

Thus, tame flows exist on any compact tame set.
Suppose $X$ is a compact tame set equipped with a definable triangulation, i.e., a tame homeomorphism $F : K \to X$, where $K$ is a finite affine simplicial complex.

Fix a *dynamical orientation* of the triangulation, i.e., a labeling of the vertices with real numbers so that no two adjacent vertices have the same label.

This induces linear orders on the set of vertices of any face and thus tame flows on each of the (closed) faces of $K$. 
Simplicial flows

- Suppose $X$ is a compact tame set equipped with a definable triangulation, i.e., a tame homeomorphism $F : K \to X$, where $K$ is a finite affine simplicial complex.
- Fix a dynamical orientation of the triangulation, i.e., a labeling of the vertices with real numbers so that no two adjacent vertices have the same label.
- This induces linear orders on the set of vertices of any face and thus tame flows on each of the (closed) faces of $K$.
- These flows match on the overlaps of faces, and thus define a tame flow on $K$ which can be transported via $F$ to a tame flow on $X$. 
Suppose $X$ is a compact tame set equipped with a definable triangulation, i.e., a tame homeomorphism $F : K \to X$, where $K$ is a finite affine simplicial complex.

Fix a \textit{dynamical orientation} of the triangulation, i.e., a labeling of the vertices with real numbers so that no two adjacent vertices have the same label.

This induces linear orders on the set of vertices of any face and thus tame flows on each of the (closed) faces of $K$.

These flows match on the overlaps of faces, and thus define a tame flow on $K$ which can be transported via $F$ to a tame flow on $X$. This is called the \textit{simplicial flow} determined by the triangulation and the dynamical orientation.
Simplicial flows

Suppose $X$ is a compact tame set equipped with a definable triangulation, i.e., a tame homeomorphism $F : K \to X$, where $K$ is a finite affine simplicial complex.

Fix a *dynamical orientation* of the triangulation, i.e., a labeling of the vertices with real numbers so that no two adjacent vertices have the same label.

This induces linear orders on the set of vertices of any face and thus tame flows on each of the (closed) faces of $K$.

These flows match on the overlaps of faces, and thus define a tame flow on $K$ which can be transported via $F$ to a tame flow on $X$. This is called the *simplicial flow* determined by the triangulation and the dynamical orientation.

Thus, tame flows exist on any compact tame set.
A simplicial flow
A simplicial flow

We depict below a special case of the above construction of tame flows on triangulated tame sets.
A simplicial flow

We depict below a special case of the above construction of tame flows on triangulated tame sets.
Barycentric flows

Suppose that $X$ is a triangulated compact tame set. Consider the barycentric subdivision of the triangulation. The vertices of the barycentric subdivision are the barycenters of the faces of the original triangulation. Label the barycenter $b_\sigma$ of a face $\sigma$ with the number $\text{dim} \sigma$. This defines a dynamical orientation of the barycentric subdivision. The tame flow associated to this dynamical orientation is called the barycentric flow.
Suppose that $X$ is a triangulated compact tame set. Consider the barycentric subdivision of the triangulation.

The vertices of the barycentric subdivision are the barycenters of the faces of the original triangulation.

Label the barycenter $b_\sigma$ of a face $\sigma$ with the number $\dim \sigma$. This defines a dynamical orientation of the barycentric subdivision.

The tame flow associated to this dynamical orientation is called the barycentric flow.
Suppose that $X$ is a triangulated compact tame set. Consider the barycentric subdivision of the triangulation.

The vertices of the barycentric subdivision are the barycenters of the faces of the original triangulation.
Suppose that $X$ is a triangulated compact tame set. Consider the barycentric subdivision of the triangulation.

The vertices of the barycentric subdivision are the barycenters of the faces of the original triangulation.

Label the barycenter $b_\sigma$ of a face $\sigma$ with the number $\dim \sigma$. This defines a dynamical orientation of the barycentric subdivision.
Suppose that $X$ is a triangulated compact tame set. Consider the barycentric subdivision of the triangulation.

The vertices of the barycentric subdivision are the barycenters of the faces of the original triangulation.

Label the barycenter $b_\sigma$ of a face $\sigma$ with the number $\dim \sigma$. This defines a dynamical orientation of the barycentric subdivision.

The tame flow associated to this dynamical orientation is called the *barycentric flow*. 
A barycentric flow
A barycentric flow
Theorem (Nico., 2007)

Let $M$ be a compact, real analytic manifold of dimension $m$, equipped with a real analytic metric $g$ and a real analytic Morse function $f$. Suppose that $
abla_g f$, the gradient of $f$ with respect to the metric $g$, satisfies the linearization condition

$$(L)$$

Near every critical point $p$ of $f$ we can find real analytic local coordinates $x_1,...,x_m$ and nonzero real numbers $\lambda_1,...,\lambda_m$ such that $x_i(p) = 0$, $\forall i$ and

$$\nabla_g f = \sum_{i=1}^{m} \lambda_i x_i \frac{\partial}{\partial x_i}.$$ 

Then the flow generated by $\nabla_g f$ is a tame flow.
Theorem (Nico., 2007)

Let $M$ be a compact, real analytic manifold of dimension $m$, equipped with a real analytic metric $g$ and a real analytic Morse function $f$.
Theorem (Nico., 2007)

Let $M$ be a compact, real analytic manifold of dimension $m$, equipped with a real analytic metric $g$ and a real analytic Morse function $f$. Suppose that $\nabla^g f$, the gradient of $f$ with respect to the metric $g$, satisfies the linearization condition

$$\nabla^g f = \sum_{i=1}^{m} \lambda_i x_i \frac{\partial}{\partial x_i}.$$ 

Then the flow generated by $\nabla^g f$ is a tame flow.
Let $M$ be a compact, real analytic manifold of dimension $m$, equipped with a real analytic metric $g$ and a real analytic Morse function $f$. Suppose that $\nabla^g f$, the gradient of $f$ with respect to the metric $g$, satisfies the linearization condition

$$\nabla^g f = \sum_{i=1}^{m} \lambda_i x^i \frac{\partial}{\partial x^i}.$$
Theorem (Nico., 2007)

Let $M$ be a compact, real analytic manifold of dimension $m$, equipped with a real analytic metric $g$ and a real analytic Morse function $f$. Suppose that $\nabla^g f$, the gradient of $f$ with respect to the metric $g$, satisfies the linearization condition

\[(L)\quad \text{Near every critical point } p \text{ of } f \text{ we can find real analytic local coordinates } x^1, \ldots, x^m \text{ and nonzero real numbers } \lambda_1, \ldots, \lambda_m \text{ such that } x^i(p) = 0, \forall i \text{ and}
\]

$$\nabla^g f = \sum_{i=1}^{m} \lambda_i x^i \frac{\partial}{\partial x^i}.$$  

Then the flow generated by $\nabla^g f$ is a tame flow.
About the condition (L)

A theorem of Poincaré and Siegel gives a sufficient condition for (L) in terms of the arithmetics of the vector \( \vec{\mu} = (\mu_1, \ldots, \mu_m) \) consisting of the eigenvalues \( \mu_i \) of the Hessian of \( f \) at the critical point \( p \).

This condition requires that there exist \( C, \nu > 0 \) such that for any \( \vec{k} = (Z \geq 0)^m \) satisfying
\[
|\vec{k}| := \sum k_i \geq 2
\]
we have
\[
|\mu_j - \vec{\mu} \cdot \vec{k}| \geq C |\vec{k}|^{\nu}, \quad \forall j = 1, \ldots, m.
\]

One can show that if \( \nu > m - 2 \), then almost all vectors \( \vec{\mu} \in \mathbb{R}^m \) satisfy the \((C, \nu)\)-condition for some \( C > 0 \).
About the condition (L)

- A theorem of Poincaré and Siegel gives a sufficient condition for (L) in terms of the arithmetics of the vector $\vec{\mu} = (\mu_1, \ldots, \mu_m)$ consisting of the eigenvalues $\mu_j$ of the Hessian of $f$ at the critical point $p$. 

One can show that if $\nu > m - 2$, then almost all vectors $\vec{\mu} \in \mathbb{R}^m$ satisfy the $(C,\nu)$-condition for some $C > 0$. 

Liviu I. Nicolaescu (Notre Dame) Tame flows May 22, 2011 13 / 41
A theorem of Poincaré and Siegel gives a sufficient condition for (L) in terms of the arithmetics of the vector $\vec{\mu} = (\mu_1, \ldots, \mu_m)$ consisting of the eigenvalues $\mu_i$ of the Hessian of $f$ at the critical point $p$.

This condition requires that there exist $C, \nu > 0$ such that for any $\vec{k} = (\mathbb{Z}_{\geq 0})^m$ satisfying

$$|\vec{k}| := \sum_i k_i \geq 2$$

we have
About the condition (L)

- A theorem of Poincaré and Siegel gives a sufficient condition for (L) in terms of the arithmetics of the vector \( \vec{\mu} = (\mu_1, \ldots, \mu_m) \) consisting of the eigenvalues \( \mu_i \) of the Hessian of \( f \) at the critical point \( p \).

- This condition requires that there exist \( C, \nu > 0 \) such that for any \( \vec{k} = (\mathbb{Z}_{\geq 0})^m \) satisfying

\[
|\vec{k}| := \sum_i k_i \geq 2
\]

we have

\[
\left| \mu_j - \vec{\mu} \cdot \vec{k} \right| \geq \frac{C}{|\vec{k}|^\nu}, \quad \forall j = 1, \ldots, m.
\]  

\((C, \nu)\)
About the condition (L)

- A theorem of Poincaré and Siegel gives a sufficient condition for \((L)\) in terms of the arithmetics of the vector \(\bar{\mu} = (\mu_1, \ldots, \mu_m)\) consisting of the eigenvalues \(\mu_i\) of the Hessian of \(f\) at the critical point \(p\).

- This condition requires that there exist \(C, \nu > 0\) such that for any \(\vec{k} = (\mathbb{Z}_{\geq 0})^m\) satisfying 

\[
|\vec{k}| := \sum_i k_i \geq 2
\]

we have 

\[
\left| \mu_j - \bar{\mu} \cdot \vec{k} \right| \geq \frac{C}{|\vec{k}|^{\nu}}, \quad \forall j = 1, \ldots, m.
\]

- One can show that if \(\nu > \frac{m-2}{2}\), then almost all vectors \(\bar{\mu} \in \mathbb{R}^m\) satisfy the \((C, \nu)\)-condition for some \(C > 0\).
Using the Poincaré-Siegel theorem we deduce the following result.

Theorem (Nico., 2007)

Let \( M \) be a compact, real analytic manifold of dimension \( m \) and \( f : M \to \mathbb{R} \) a real analytic Morse function on \( M \).

Then for a generic real analytic metric \( g \) on \( M \) the flow generated by the gradient \( \nabla_g f \) is tame.
Using the Poincaré-Siegel theorem we deduce the following result.
Using the Poincaré-Siegel theorem we deduce the following result.

**Theorem (Nico., 2007)**

Let $M$ be a compact, real analytic manifold of dimension $m$ and $f : M \to \mathbb{R}$ a real analytic Morse function on $M$. Then for a generic real analytic metric $g$ on $M$ the flow generated by the gradient $\nabla g f$ is tame.
Using the Poincaré-Siegel theorem we deduce the following result.

**Theorem (Nico., 2007)**

Let $M$ be a compact, real analytic manifold of dimension $m$ and $f : M \to \mathbb{R}$ a real analytic Morse function on $M$. Then for a generic real analytic metric $g$ on $M$ the flow generated by the gradient $\nabla^g f$ is tame.
Asymptotic properties of tame flows

Suppose that $\Phi : \mathbb{R} \times X \to X$ is a tame flow on a compact tame set. For any $x \in X$ the limits $\Phi^{\pm \infty}(x) := \lim_{t \to \pm \infty} \Phi^t(x)$ exist and are stationary points of the flow. The maps $\Phi^{\pm \infty} : X \to X$ are tame surjections onto the set $\text{Cr}_\Phi$ of stationary points of $\Phi$.

For a stationary point $x_0 \in X$ we define the stable variety of $x_0$ to be the set $W^+_\Phi(x_0) := \{x \in X; \Phi^{\infty}(x) = x_0\}$.

The unstable variety of $x_0$ is the set $W^-\Phi(x_0) := \{x \in X; \Phi^{-\infty}(x) = x_0\}$.

The (un)stable varieties are tame sets.
Asymptotic properties of tame flows

Suppose that $\Phi : \mathbb{R} \times X \rightarrow X$ is a tame flow on a compact tame set.

For any $x \in X$ the limits

$$\Phi_{\pm\infty}(x) := \lim_{t \to \pm\infty} \Phi_t(x)$$

exist and are stationary points of the flow.

The maps $\Phi_{\pm\infty} : X \rightarrow X$ are tame surjections onto the set $C_{\Phi}$ of stationary points of $\Phi$.

For a stationary point $x_0 \in X$ we define the

stable variety of $x_0$ to be the set

$$W^+_{\Phi}(x_0) := \{x \in X; \Phi_{\infty}(x) = x_0\}.$$  

The unstable variety of $x_0$ is the set

$$W^-_{\Phi}(x_0) := \{x \in X; \Phi_{-\infty}(x) = x_0\}.$$
Asymptotic properties of tame flows

Suppose that $\Phi : \mathbb{R} \times X \to X$ is a tame flow on a compact tame set.

For any $x \in X$ the limits

$$\Phi^{\pm \infty}(x) := \lim_{t \to \pm \infty} \Phi^t(x)$$

exist and are stationary points of the flow.
Asymptotic properties of tame flows

Suppose that $\Phi : \mathbb{R} \times X \to X$ is a tame flow on a compact tame set.

- For any $x \in X$ the limits
  \[ \Phi^{\pm \infty}(x) := \lim_{t \to \pm \infty} \Phi^t(x) \]
  exist and are stationary points of the flow.

- The maps $\Phi^{\pm \infty} : X \to X$ are tame surjections onto the set $\text{Cr}_\Phi$ of stationary points of $\Phi$. 
Asymptotic properties of tame flows

Suppose that $\Phi : \mathbb{R} \times X \to X$ is a tame flow on a compact tame set.

- For any $x \in X$ the limits
  \[ \Phi^{\pm \infty}(x) := \lim_{t \to \pm \infty} \Phi^t(x) \]
  exist and are stationary points of the flow.
- The maps $\Phi^{\pm \infty} : X \to X$ are tame surjections onto the set $\text{Cr}_\Phi$ of stationary points of $\Phi$.
- For a stationary point $x_0 \in X$ we define the *stable variety* of $x_0$ to be the set
  \[ W^+_\Phi(x_0) := \{ x \in X ; \Phi^\infty(x) = x_0 \} . \]
Asymptotic properties of tame flows

Suppose that $\Phi : \mathbb{R} \times X \to X$ is a tame flow on a compact tame set.

- For any $x \in X$ the limits
  $\Phi^{\pm\infty}(x) := \lim_{t \to \pm\infty} \Phi^t(x)$
  exist and are stationary points of the flow.

- The maps $\Phi^{\pm\infty} : X \to X$ are tame surjections onto the set $\text{Cr}_\Phi$ of stationary points of $\Phi$.

- For a stationary point $x_0 \in X$ we define the *stable variety* of $x_0$ to be the set
  $$W^+_\Phi(x_0) := \{ x \in X ; \Phi^\infty(x) = x_0 \}.$$ 
  The *unstable variety* of $x_0$ is the set
  $$W^-_\Phi(x_0) := \{ x \in X ; \Phi^{-\infty}(x) = x_0 \}.$$
Asymptotic properties of tame flows

Suppose that $\Phi : \mathbb{R} \times X \to X$ is a tame flow on a compact tame set.

- For any $x \in X$ the limits

$$
\Phi^{\pm \infty}(x) := \lim_{t \to \pm \infty} \Phi^t(x)
$$

exist and are stationary points of the flow.

- The maps $\Phi^{\pm \infty} : X \to X$ are tame surjections onto the set $\text{Cr}_\Phi$ of stationary points of $\Phi$.

- For a stationary point $x_0 \in X$ we define the \textit{stable variety} of $x_0$ to be the set

$$
W^+_\Phi(x_0) := \left\{ x \in X; \ \Phi^\infty(x) = x_0 \right\}.
$$

The \textit{unstable variety} of $x_0$ is the set

$$
W^-_\Phi(x_0) := \left\{ x \in X; \ \Phi^{-\infty}(x) = x_0 \right\}.
$$

The (un)stable varieties are tame sets.
The above flow has two stationary points, \( a \) and \( b \), and the overlap \( W^+ \cap W^- \) is the interior of the disk. This overlap is filled by homoclinic trajectories.
The above flow has two stationary points, $a, b$, and the overlap $W^+ (b) \cap W^- (b)$ is the interior of the disk. This overlap is filled by homoclinic trajectories.
Suppose that $X$ is a compact tame space equipped with a triangulation $S$. Denote by $\Phi$ the associated barycentric flow.

The stationary points of $\Phi$ are the barycenters $b_\sigma$ of the faces $\sigma \in S$.

The unstable variety of the stationary point $b_\sigma$ is the relative interior of the face $\sigma \in S$. 

Liviu I. Nicolaescu (Notre Dame)

Tame flows

May 22, 2011
Suppose that $X$ is a compact tame space equipped with a triangulation $S$. Denote by $\Phi$ the associated barycentric flow.
Suppose that $X$ is a compact tame space equipped with a triangulation $\mathcal{S}$. Denote by $\Phi$ the associated barycentric flow.

The stationary points of $\Phi$ are the barycenters $b_{\sigma}$ of the faces $\sigma \in \mathcal{S}$. 
Suppose that $X$ is a compact tame space equipped with a triagulation $\mathcal{S}$. Denote by $\Phi$ the associated barycentric flow.

The stationary points of $\Phi$ are the barycenters $b_\sigma$ of the faces $\sigma \in \mathcal{S}$.

The unstable variety of the stationary point $b_\sigma$ is the \textit{relative interior} of the face $\sigma \in \mathcal{S}$. 
If $X$ is an oriented smooth, tame manifold, of dimension $m$ equipped with a piecewise linear triangulation $S$. Denote by $\Phi$ the associated barycentric flow. For any $\sigma \in S$, the stable variety of $b_\sigma$ is tamely homeomorphic to an open disk of dimension $m - \dim \sigma$ and intersects the face $\sigma$ "transversally" at one point, the barycenter $b_\sigma$. The resulting cell decomposition $X = \bigcup_{\sigma \in S} W^+(b_\sigma)$ is called the cell decomposition dual to the triangulation $S$. It plays a central role in the classical proof of the Poincaré duality theorem. If we fix an orientation $o_\sigma$ on each face $\sigma \in S$, then we have an induced orientation $o_\perp \sigma$ on $W^+(b_\sigma)$ uniquely determined by the equality $o_\perp \sigma \wedge o_\sigma =$ orientation of $M$. Liviu I. Nicolaescu (Notre Dame)
Barycentric flows on manifolds

If $X$ is an oriented smooth, tame manifold, of dimension $m$ equipped with a \textit{piecewise linear} triangulation $S$. Denote by $\Phi$ the associated barycentric flow.
Barycentric flows on manifolds

- If $X$ is an oriented smooth, tame manifold, of dimension $m$ equipped with a *piecewise linear* triangulation $S$. Denote by $\Phi$ the associated barycentric flow.
- For any $\sigma \in S$, the stable variety of $b_\sigma$ is tamely homeomorphic to an open disk of dimension $m - \dim \sigma$ and intersects the face $\sigma$ “transversally” at one point, the barycenter $b_\sigma$. 
Barycentric flows on manifolds

- If $X$ is an oriented smooth, tame manifold, of dimension $m$ equipped with a *piecewise linear* triangulation $S$. Denote by $\Phi$ the associated barycentric flow.

- For any $\sigma \in S$, the stable variety of $b_\sigma$ is tamely homeomorphic to an open disk of dimension $m - \dim \sigma$ and intersects the face $\sigma$ “transversally” at one point, the barycenter $b_\sigma$.

- The resulting cell decomposition

$$X = \bigcup_{\sigma \in S} W^+(b_\sigma)$$

is called the cell decomposition dual to the triangulation $S$. 
Barycentric flows on manifolds

- If $X$ is an oriented smooth, tame manifold, of dimension $m$ equipped with a *piecewise linear* triangulation $S$. Denote by $\Phi$ the associated barycentric flow.
- For any $\sigma \in S$, the stable variety of $b_\sigma$ is tamely homeomorphic to an open disk of dimension $m - \dim \sigma$ and intersects the face $\sigma$ “transversally” at one point, the barycenter $b_\sigma$.
- The resulting cell decomposition

$$X = \bigcup_{\sigma \in S} W^+(b_\sigma)$$

is called the cell decomposition dual to the triangulation $S$. It plays a central role in the classical proof of the Poincaré duality theorem.
If $X$ is an oriented smooth, tame manifold, of dimension $m$ equipped with a *piecewise linear* triangulation $S$. Denote by $\Phi$ the associated barycentric flow.

For any $\sigma \in S$, the stable variety of $b_\sigma$ is tamely homeomorphic to an open disk of dimension $m - \dim \sigma$ and intersects the face $\sigma$ “transversally” at one point, the barycenter $b_\sigma$.

The resulting cell decomposition

$$X = \bigcup_{\sigma \in S} W^+(b_\sigma)$$

is called the cell decomposition *dual* to the triangulation $S$. It plays a central role in the classical proof of the Poincaré duality theorem.

If we fix an orientation $o_\sigma$ on each face $\sigma \in S$, then we have an induced orientation $o_\sigma^\perp$ on $W^+(b_\sigma)$ uniquely determined by the equality $o_\sigma^\perp \wedge o_\sigma = \text{orientation of } M$. 
Theorem (Nico., 2007)

Suppose that $M$ is a compact, oriented tame smooth manifold and $\Phi$ is a tame flow on $M$. Denote by $\Gamma_t$ the graph of $\Phi_t$. It is a submanifold of $M \times M$ tamely diffeomorphic to $M$ and as such it defines a current of integration $[\Gamma_t]$. Then the volume of $\Gamma_t$ is bounded from above by a constant independent of $t$ and as $t \to \infty$ the current $[\Gamma_t]$ converges weakly (and in the flat distance) to an integral subanalytic current $[\Gamma_\infty]$. The current $[\Gamma_\infty]$ is a deformation of the diagonal, i.e., it is homologous in the sense of integral subanalytic currents with the current supported by the diagonal $[\Gamma_0]$. 
Theorem (Nico., 2007)

Suppose that $M$ is a compact, oriented tame smooth manifold and $\Phi$ is a tame flow on $M$. Denote by $\Gamma_t$ the graph of $\Phi_t$. It is a submanifold of $M \times M$ tamely diffeomorphic to $M$ and as such it defines a current of integration $[\Gamma_t]$. Then the volume of $\Gamma_t$ is bounded from above by a constant independent of $t$ and as $t \to \infty$ the current $[\Gamma_t]$ converges weakly (and in the flat distance) to an integral subanalytic current $[\Gamma_\infty]$. The current $[\Gamma_\infty]$ is deformation of the diagonal, i.e., it is homologous in the sense of integral subanalytic currents with the current supported by the diagonal $[\Gamma_0]$. 
### Theorem (Nico., 2007)

Suppose that $M$ is a compact, oriented tame smooth manifold and $\Phi$ is a tame flow on $M$. Denote by $\Gamma_t$ the graph of $\Phi^t$. Then the volume of $\Gamma_t$ is bounded from above by a constant independent of $t$ and as $t \to \infty$ the current $[\Gamma_t]$ converges weakly (and in the flat distance) to an integral subanalytic current $[\Gamma_\infty]$. The current $[\Gamma_\infty]$ is deformation of the diagonal, i.e., it is homologous in the sense of integral subanalytic currents with the current supported by the diagonal $[\Gamma_0]$.
Theorem (Nico., 2007)

Suppose that $M$ is a compact, oriented tame smooth manifold and $\Phi$ is a tame flow on $M$. Denote by $\Gamma_t$ the graph of $\Phi^t$. It is a submanifold of $M \times M$ tamely diffeomorphic to $M$ and as such it defines a current of integration $[\Gamma_t]$.
Theorem (Nico., 2007)

Suppose that $M$ is a compact, oriented tame smooth manifold and $\Phi$ is a tame flow on $M$. Denote by $\Gamma_t$ the graph of $\Phi^t$. It is a submanifold of $M \times M$ tamely diffeomorphic to $M$ and as such it defines a current of integration $[\Gamma_t]$.

Then the volume of $\Gamma_t$ is bounded from above by a constant independent of $t$ and as $t \to \infty$ the current $[\Gamma_t]$ converges weakly (and in the flat distance) to an integral subanalytic current $[\Gamma_\infty]$. 

Theorem (Nico., 2007)

Suppose that $M$ is a compact, oriented tame smooth manifold and $\Phi$ is a tame flow on $M$. Denote by $\Gamma_t$ the graph of $\Phi^t$. It is a submanifold of $M \times M$ tamely diffeomorphic to $M$ and as such it defines a current of integration $[\Gamma_t]$.

Then the volume of $\Gamma_t$ is bounded from above by a constant independent of $t$ and as $t \to \infty$ the current $[\Gamma_t]$ converges weakly (and in the flat distance) to an integral subanalytic current $[\Gamma_\infty]$. The current $[\Gamma_\infty]$ is deformation of the diagonal, i.e., it is homologous in the sense of integral subanalytic currents with the current supported by the diagonal $[\Gamma_0]$. 
Suppose that $M$ is a compact, oriented tame smooth manifold and $\Phi$ is a tame flow on $M$. Then for any smooth form $\alpha$, the pullbacks $(\Phi^t)^* \alpha$ have a limit in the sense of currents as $t \to \infty$. 

Liviu I. Nicolaescu (Notre Dame)
Corollary

Suppose that $M$ is a compact, oriented tame smooth manifold and $\Phi$ is a tame flow on $M$. 

Corollary

Suppose that $M$ is a compact, oriented tame smooth manifold and $\Phi$ is a tame flow on $M$. Then for any smooth form $\alpha$ on $M$ the pullbacks $(\Phi^t)^*\alpha$ have a limit in the sense of currents as $t \to \infty$. 
Corollary

Suppose that $M$ is a tame, compact oriented smooth manifold of dimension $m$ equipped with a piecewise linear triangulation $S$. Fix orientations $o_\sigma$ on each of the faces $\sigma \in S$ and denote by $\Phi$ the barycentric flow on $M$ determined by $S$. Then the following hold.

(a) For any smooth, $k$-form $\alpha$ on $M$, the pullback $(\Phi^t)^* \alpha$ converges in the sense of currents as $t \to -\infty$ to the $(m-k)$-dimensional current $\left(\Phi^{\infty}\right)^* \alpha = \sum_{\dim \sigma = m-k} (-1)^k ((m-k) \int (W + (b_\sigma)^*, o_\sigma)) \alpha$ where $[(W - (b_\sigma)^*, o_\sigma)]$ is the current of integration over the interior of $\sigma$ equipped with the orientation $o_\sigma$.

(b) If $\alpha$ is a closed $k$-form on $M$, then the current $\left(\Phi^{\infty}\right)^* \alpha$ is a $(m-k)$-dimensional simplicial cycle whose homology class is Poincaré dual to the cohomology class of $\alpha$. 
Suppose that $M$ is a tame, compact oriented smooth manifold of dimension $m$ equipped with a piecewise linear triangulation $S$. Fix orientations $o_\sigma$ on each of the faces $\sigma \in S$ and denote by $\Phi$ the barycentric flow on $M$ determined by $S$. Then the following hold.

(a) For any smooth, $k$-form $\alpha$ on $M$, the pullback $(\Phi^t)^* \alpha$ converges in the sense of currents as $t \to -\infty$ to the $(m - k)$-dimensional current

$$(\Phi^{-\infty})^* \alpha := \sum_{\dim \sigma = m-k} (-1)^{k(m-k)} \left( \int (W^+(b_\sigma), o_\sigma^\perp) \alpha \right) [W^-(b_\sigma), o_\sigma],$$

where $[W^-(b_\sigma), o_\sigma]$ is the current of integration over the interior of $\sigma$ equipped with the orientation $o_\sigma$. 

(b) If $\alpha$ is a closed $k$-form on $M$, then the current $(\Phi^{-\infty})^* \alpha$ is a $(m - k)$-dimensional simplicial cycle whose homology class is Poincaré dual to the cohomology class of $\alpha$. 

Liviu I. Nicolaescu (Notre Dame) Tame flows May 22, 2011
Corollary

Suppose that $M$ is a tame, compact oriented smooth manifold of dimension $m$ equipped with a piecewise linear triangulation $S$. Fix orientations $o_\sigma$ on each of the faces $\sigma \in S$ and denote by $\Phi$ the barycentric flow on $M$ determined by $S$. Then the following hold.

(a) For any smooth, $k$-form $\alpha$ on $M$, the pullback $(\Phi^t)^*\alpha$ converges in the sense of currents as $t \to -\infty$ to the $(m - k)$-dimensional current

\[
(\Phi^{-\infty})^*\alpha := \sum_{\dim \sigma = m - k} (-1)^{k(m-k)} \left( \int_{(W^+(b_\sigma), o_\sigma)} \alpha \right) [W^-(b_\sigma), o_\sigma],
\]

where $[W^-(b_\sigma), o_\sigma]$ is the current of integration over the interior of $\sigma$ equipped with the orientation $o_\sigma$.

(b) If $\alpha$ is a closed $k$-form on $M$, then the current $(\Phi^{-\infty})^*\alpha$ is a $(m - k)$-dimensional simplicial cycle whose homology class is Poincaré dual to the cohomology class of $\alpha$. 

Morse-like tame flows

Definition

A tame flow $\Phi$ on a compact tame set $X$ is said to be Morse-like if it satisfies the following conditions.

The set of stationary points is finite.

The flow admits a tame Lyapunov function, i.e., a tame continuous function $f: X \to \mathbb{R}$ that decreases strictly along the nontrivial orbits of $\Phi$.

In other words, for any nonstationary point $x \in X$ the function $t \mapsto f(\Phi_t(x))$ is strictly decreasing.

A triplet $(X, \Phi, f)$ with the above properties is called a Morse triplet.
Morse-like tame flows

**Definition**

A tame flow $\Phi$ on a compact tame set $X$ is said to be **Morse-like** if it satisfies the following conditions.

1. The set of stationary points is finite.
2. The flow admits a tame Lyapunov function, i.e., a tame continuous function $f: X \to \mathbb{R}$ that decreases strictly along the nontrivial orbits of $\Phi$.

In other words, for any nonstationary point $x \in X$ the function $t \mapsto f(\Phi_t(x))$ is strictly decreasing.

A triplet $(X, \Phi, f)$ with the above properties is called a **Morse triplet**.
A tame flow $\Phi$ on a compact tame set $X$ is said to be Morse-like if it satisfies the following conditions.

- The set of stationary points is finite.
Morse-like tame flows

Definition

A tame flow $\Phi$ on a compact tame set $X$ is said to be Morse-like if it satisfies the following conditions.

- The set of stationary points is finite.
- The flow admits a tame Lyapunov function, i.e., a tame continuous function $f : X \to \mathbb{R}$ that decreases strictly along the nontrivial orbits of $\Phi$. 
A tame flow $\Phi$ on a compact tame set $X$ is said to be Morse-like if it satisfies the following conditions.

- The set of stationary points is finite.
- The flow admits a tame Lyapunov function, i.e., a tame continuous function $f : X \to \mathbb{R}$ that decreases strictly along the nontrivial orbits of $\Phi$. In other words, for any nonstationary point $x \in X$ the function

$$ t \mapsto f\left( \Phi^t(x) \right) $$

is strictly decreasing.

In other words, for any nonstationary point $x \in X$ the function

$$ t \mapsto f\left( \Phi^t(x) \right) $$

is strictly decreasing.
Morse-like tame flows

Definition

A tame flow $\Phi$ on a compact tame set $X$ is said to be \textbf{Morse-like} if it satisfies the following conditions.

- The set of stationary points is finite.
- The flow admits a tame Lyapunov function, i.e., a tame continuous function $f : X \to \mathbb{R}$ that decreases strictly along the nontrivial orbits of $\Phi$. In other words, for any nonstationary point $x \in X$ the function

$$
  t \mapsto f\left(\Phi^t(x)\right)
$$

is strictly decreasing.

A triplet $(X, \Phi, f)$ with the above properties is called a \textit{Morse triplet}. 

Liviu I. Nicolaescu (Notre Dame)
Examples of Morse-like tame flows

If $X$ is a compact, real analytic manifold and $f: X \to \mathbb{R}$ is a real analytic Morse function, then for a generic real analytic metric $g$ the flow generated by $-\nabla g f$ is a Morse-like tame flow on $M$. The function $f$ is a Lyapunov function for this flow.

Any simplicial flow is Morse-like. The cone of a Morse-like tame flow is a Morse-like tame flow.
Examples of Morse-like tame flows

- If $X$ is a compact, real analytic manifold and $f : X \rightarrow \mathbb{R}$ is a real analytic Morse function, then for a generic real analytic metric $g$ the flow generated by $-\nabla^g f$ is a Morse-like tame flow on $M$. 

  The function $f$ is a Lyapunov function for this flow.

  Any simplicial flow is Morse-like.

  The cone of a Morse-like tame flow is a Morse-like tame flow.
Examples of Morse-like tame flows

- If $X$ is a compact, real analytic manifold and $f : X \to \mathbb{R}$ is a real analytic Morse function, then for a generic real analytic metric $g$ the flow generated by $-\nabla^g f$ is a Morse-like tame flow on $M$. The function $f$ is a Lyapunov function for this flow.
Examples of Morse-like tame flows

- If $X$ is a compact, real analytic manifold and $f : X \rightarrow \mathbb{R}$ is a real analytic Morse function, then for a generic real analytic metric $g$ the flow generated by $-\nabla^g f$ is a Morse-like tame flow on $M$. The function $f$ is a Lyapunov function for this flow.
- Any simplicial flow is Morse-like.
Examples of Morse-like tame flows

1. If $X$ is a compact, real analytic manifold and $f : X \to \mathbb{R}$ is a real analytic Morse function, then for a generic real analytic metric $g$ the flow generated by $-\nabla^g f$ is a Morse-like tame flow on $M$. The function $f$ is a Lyapunov function for this flow.

2. Any simplicial flow is Morse-like.

3. The cone of a Morse-like tame flow is a Morse-like tame flow.
The Conley index

Suppose $\Phi : \mathbb{R} \times X \to X$ is a flow on the compact metric space $X$. For $t \in \mathbb{R}$ and $x \in X$ we set $t \cdot x := \Phi_t(x)$. For any $W \subset X$ we set $I(W) := \{ w \in W ; t \cdot w \in W, \forall t \in \mathbb{R} \}$. In other words, $I(W)$ is the largest flow invariant subset of $W$. A compact subset $S \subset X$ is called an isolated invariant set if it admits a compact neighborhood $W$ such that $S = I(W)$. We say that $W$ is an isolating neighborhood of $S$. 

Liviu I. Nicolaescu (Notre Dame)
Suppose $\Phi : \mathbb{R} \times X \to X$ is a flow on the compact metric space $X$. For any $W \subset X$ we set $I(W) := \{ w \in W; t \cdot w \in W, \forall t \in \mathbb{R} \}$. In other words, $I(W)$ is the largest flow invariant subset of $W$. A compact subset $S \subset X$ is called an isolated invariant set if it admits a compact neighborhood $W$ such that $S = I(W)$. We say that $W$ is an isolating neighborhood of $S$. 

Liviu I. Nicolaescu (Notre Dame)
The Conley index

- Suppose $\Phi : \mathbb{R} \times X \to X$ is a flow on the compact metric space $X$.
- For $t \in \mathbb{R}$ and $x \in X$ we set $t \cdot x := \Phi^t(x)$. 
Suppose $\Phi : \mathbb{R} \times X \rightarrow X$ is a flow on the compact metric space $X$.
For $t \in \mathbb{R}$ and $x \in X$ we set $t \cdot x := \Phi^t(x)$.
For any $W \subset X$ we set
\[ I(W) := \{ w \in W; \ t \cdot w \in W, \ \forall t \in \mathbb{R} \} . \]
Suppose $\Phi : \mathbb{R} \times X \to X$ is a flow on the compact metric space $X$.

For $t \in \mathbb{R}$ and $x \in X$ we set $t \cdot x := \Phi^t(x)$.

For any $W \subset X$ we set

$$I(W) := \{ w \in W ; \ t \cdot w \in W , \ \forall t \in \mathbb{R} \}.$$ 

In other words, $I(W)$ is the largest flow invariant subset of $W$. 
The Conley index

Suppose $\Phi : \mathbb{R} \times X \to X$ is a flow on the compact metric space $X$.

For $t \in \mathbb{R}$ and $x \in X$ we set $t \cdot x := \Phi^t(x)$.

For any $W \subset X$ we set

$$I(W) := \{w \in W; \ t \cdot w \in W, \ \forall t \in \mathbb{R}\}.$$ 

In other words, $I(W)$ is the largest flow invariant subset of $W$.

A compact subset $S \subset X$ is called a isolated invariant set if it admits a compact neighborhood $W$ such that $S = I(W)$. 
The Conley index

Suppose \( \Phi : \mathbb{R} \times X \to X \) is a flow on the compact metric space \( X \).

For \( t \in \mathbb{R} \) and \( x \in X \) we set \( t \cdot x := \Phi^t(x) \).

For any \( W \subset X \) we set
\[
I(W) := \{ w \in W; \ t \cdot w \in W, \ \forall t \in \mathbb{R} \}.
\]

In other words, \( I(W) \) is the largest flow invariant subset of \( W \).

A compact subset \( S \subset X \) is called a isolated invariant set if it admits a compact neighborhood \( W \) such that \( S = I(W) \). We say that \( W \) is an isolating neighborhood of \( S \).
The Conley index

Suppose that $S$ is an isolated invariant set. An index pair of $S$ is a pair of compact sets $(N, N^-)$, $N^- \subset N$, satisfying the following conditions.

$(I_1)$ $N \setminus N^-$ is a neighborhood of $S$, and $S = \text{cl}(N \setminus N^-)$.

$(I_2)$ $N^-$ is positively invariant in $N$, i.e., $x \in N^-, t \geq 0, [0, t] \cdot x \subset N \implies [0, t] \cdot x \subset N^-$.

$(I_3)$ $N^-$ is an exit set for $N$, i.e., if $x \in N$ and $[0, \infty) \cdot x \not\subset N$, then there exists $t \geq 0$ such that $t \cdot x \in N^-$ and $[0, t] \cdot x \subset N^-$. 

Liviu I. Nicolaescu (Notre Dame) Tame flows May 22, 2011 25 / 41
The Conley index

- Suppose that $S$ is an isolated invariant set. An index pair of $S$ is a pair of compact sets $(N, N^-)$, $N^- \subset N$, satisfying the following conditions.

1. $N \setminus N^-$ is a neighborhood of $S$, and $S = \text{cl}(N \setminus N^-)$.
2. $N^-$ is positively invariant in $N$, i.e., $x \in N^-, t \geq 0, [0, t] \cdot x \subset N^-$.
3. $N^-$ is an exit set for $N$, i.e., if $x \in N$ and $[0, \infty) \cdot x \not\subset N$, then there exists $t \geq 0$ such that $t \cdot x \in N^-$ and $[0, t] \cdot x \subset N^-$. 

Liviu I. Nicolaescu (Notre Dame)
Suppose that $S$ is an isolated invariant set. An **index pair of $S$** is a pair of compact sets $(N, N^-)$, $N^- \subset N$, satisfying the following conditions.

\[ (I_1) \quad N \setminus N^- \text{ is a neighborhood of } S, \text{ and } S = I(\text{cl}(N \setminus N^-)), \]

\[ (I_2) \quad N^- \text{ is positively invariant in } N, \text{ i.e., } x \in N^-, t \geq 0, [0, t] \cdot x \subset N \Rightarrow [0, t] \cdot x \subset N^-. \]
The Conley index

Suppose that $S$ is an isolated invariant set. An index pair of $S$ is a pair of compact sets $(N, N^-)$, $N^- \subset N$, satisfying the following conditions.

$(I_1)$ $N \setminus N^-$ is a neighborhood of $S$, and $S = I(\text{cl}(N \setminus N^-))$, $\text{cl} := \text{closure}$
The Conley index

Suppose that $S$ is an isolated invariant set. An index pair of $S$ is a pair of compact sets $(N, N^{-})$, $N^{-} \subset N$, satisfying the following conditions.

$(I_1)$ \( N \setminus N^{-} \) is a neighborhood of $S$, and $S = l(\text{cl}(N \setminus N^{-}))$, \( \text{cl} := \text{closure} \)

$(I_2)$ $N^{-}$ is positively invariant in $N$, i.e.,

\[
x \in N^{-}, \quad t \geq 0, \quad [0, t] \cdot x \subset N \Rightarrow [0, t] \cdot x \subset N^{-}.
\]
The Conley index

Suppose that $S$ is an isolated invariant set. An index pair of $S$ is a pair of compact sets $(N, N^-)$, $N^- \subset N$, satisfying the following conditions.

(I$_1$) $N \setminus N^-$ is a neighborhood of $S$, and $S = I(\text{cl}(N \setminus N^-))$, \text{cl} := \text{closure}

(I$_2$) $N^-$ is positively invariant in $N$, i.e.,

$$ x \in N^-, \quad t \geq 0, \quad [0, t] \cdot x \subset N \Rightarrow [0, t] \cdot x \subset N^-.$$

(I$_3$) $N^-$ is an exit set for $N$, i.e.,
The Conley index

- Suppose that $S$ is an isolated invariant set. An index pair of $S$ is a pair of compact sets $(N, N^-)$, $N^- \subset N$, satisfying the following conditions.

  \begin{enumerate}[(I_1)]
    \item $N \setminus N^-$ is a neighborhood of $S$, and $S = I(\text{cl}(N \setminus N^-))$, \text{cl} := \text{closure}$
    \item $N^-$ is positively invariant in $N$, i.e.,
      \[ x \in N^-, \quad t \geq 0, \quad [0, t] \cdot x \subset N \Rightarrow [0, t] \cdot x \subset N^-.
    \]
    \item $N^-$ is an exit set for $N$, i.e.,
      \[ \text{if } x \in N \text{ and } [0, \infty) \cdot x \not\subset N, \]
  \end{enumerate}
The Conley index

Suppose that $S$ is an isolated invariant set. An index pair of $S$ is a pair of compact sets $(N, N^-)$, $N^- \subset N$, satisfying the following conditions.

$I_1$ $N \setminus N^-$ is a neighborhood of $S$, and $S = I(\text{cl}(N \setminus N^-))$, $\text{cl} :=$ closure

$I_2$ $N^-$ is positively invariant in $N$, i.e.,

$$x \in N^-, \ t \geq 0, \ [0, t] \cdot x \subset N \Rightarrow [0, t] \cdot x \subset N^-.$$  

$I_3$ $N^-$ is an exit set for $N$, i.e.,

If $x \in N$ and $[0, \infty) \cdot x \not\subset N$,

then there exists $t \geq 0$ such that

$$t \cdot x \in N^- \ \text{and} \ \ [0, t] \cdot x \subset N.$$
The Conley index

Theorem (C. Conley)

Suppose that $S$ is an isolated invariant set. Then the following hold.

(a) For any neighborhood $U$ of $S$ there exists an index pair $(N, N - )$ of $S$ such that $\text{cl}(N \setminus N - ) \subset U$.

(b) If $(N_0, N_0 - )$ and $(N_1, N_1 - )$ are index pairs of $S$, then the topological spaces $N_0 / N_0 -$ and $N_1 / N_1 -$ are homotopy equivalent. Their common homotopy type is called the Conley index of the invariant set $S$. We denote it by $I_\Phi(S)$.
Suppose that $S$ is an isolated invariant set. Then the following hold.

(a) For any neighborhood $U$ of $S$ there exists an index pair $(N, N^-)$ of $S$ such that $\text{cl}(N \setminus N^-) \subset U$.

(b) If $(N_0, N_0^-)$ and $(N_1, N_1^-)$ are index pairs of $S$, then the topological spaces $N_0 / N_0^-$ and $N_1 / N_1^-$ are homotopy equivalent. Their common homotopy type is called the Conley index of the invariant set $S$. We denote it by $I_\Phi(S)$.
The Conley index

Theorem (C. Conley)

Suppose that $S$ is an isolated invariant set. Then the following hold.

(a) For any neighborhood $U$ of $S$ there exists an index pair $(N, N^-)$ of $S$ such that

$$\text{cl}(N \setminus N^-) \subset U.$$

(b) If $(N_0, N_0^-)$ and $(N_1, N_1^-)$ are index pairs of $S$, then the topological spaces $N_0/N_0^-$ and $N_1/N_1^-$ are homotopy equivalent. Their common homotopy type is called the Conley index of the invariant set $S$. We denote it by $I\Phi(S)$. 
The Conley index

Theorem (C. Conley)

Suppose that $S$ is an isolated invariant set. Then the following hold.

(a) For any neighborhood $U$ of $S$ there exists an index pair $(N, N^-)$ of $S$ such that

$$\text{cl}(N \setminus N^-) \subset U.$$ 

(b) If $(N_0, N_0^-)$ and $(N_1, N_1^-)$ are index pairs of $S$, 

Their common homotopy type is called the Conley index of the invariant set $S$. We denote it by $I(\Phi)(S)$. 

Liviu I. Nicolaescu (Notre Dame)
The Conley index

Theorem (C. Conley)

Suppose that $S$ is an isolated invariant set. Then the following hold.

(a) For any neighborhood $U$ of $S$ there exists an index pair $(N, N^-)$ of $S$ such that

$$\text{cl}(N \setminus N^-) \subset U.$$ 

(b) If $(N_0, N_0^-)$ and $(N_1, N_1^-)$ are index pairs of $S$, then the topological spaces $N_0/N_0^-$ and $N_1/N_1^-$ are homotopy equivalent.
The Conley index

Theorem (C. Conley)

Suppose that $S$ is an isolated invariant set. Then the following hold.

(a) For any neighborhood $U$ of $S$ there exists an index pair $(N, N^-)$ of $S$ such that

$$\text{cl}(N \setminus N^-) \subset U.$$ 

(b) If $(N_0, N_0^-)$ and $(N_1, N_1^-)$ are index pairs of $S$, then the topological spaces $N_0/N_0^-$ and $N_1/N_1^-$ are homotopy equivalent. Their common homotopy type is called the Conley index of the invariant set $S$. We denote it by $\mathcal{I}_\Phi(S)$.
Example

Let \((M, g)\) be a smooth compact Riemannian manifold and \(f : M \to \mathbb{R}\) be a smooth Morse function. Denote by \(\Phi\) the flow generated by \(-\nabla g f\).

For \(r \in \mathbb{R}\) we set \(M_r := \{x \in M ; f(x) \leq r\}\).

Assume that \(c\) is a critical value of \(f\) and the level set \(\{f = c\}\) contains a unique critical point \(p\) of \(f\) of index \(k\).

The set \(\{p\}\) is an isolated invariant set of \(\Phi\).

For \(\varepsilon > 0\) sufficiently small the pair \((M_c + \varepsilon, M_c - \varepsilon)\) is an index pair of \(\{p\}\).

The central theorem of Morse theory implies that \(M_c + \varepsilon\) is homotopic to a space obtained from \(M_c - \varepsilon\) by attaching a disk of dimension \(k\).

Thus \(I_{\Phi}(p) \simeq M_c + \varepsilon / M_c - \varepsilon \simeq S^k\).
Example

- Let \((M, g)\) be a smooth compact Riemannian manifold and \(f : M \to \mathbb{R}\) be a smooth Morse function.
Example

Let \((M, g)\) be a smooth compact Riemannian manifold and \(f : M \to \mathbb{R}\) be a smooth Morse function. Denote by \(\Phi\) the flow generated by \(-\nabla^g f\).
Example

Let \((M, g)\) be a smooth compact Riemannian manifold and \(f : M \rightarrow \mathbb{R}\) be a smooth Morse function. Denote by \(\Phi\) the flow generated by \(-\nabla^g f\).

For \(r \in \mathbb{R}\) we set \(M^r := \{ x \in M ; \; f(x) \leq r \} \).
Example

- Let $(M, g)$ be a smooth compact Riemannian manifold and $f : M \to \mathbb{R}$ be a smooth Morse function. Denote by $\Phi$ the flow generated by $-\nabla^g f$.
- For $r \in \mathbb{R}$ we set $M^r := \{ x \in M ; \ f(x) \leq r \}$.
- Assume that $c$ is a critical value of $f$ and the level set $\{ f = c \}$ contains a unique critical point $p$ of $f$ of index $k$. 

Liviu I. Nicolaescu (Notre Dame)
Example

Let \((M, g)\) be a smooth compact Riemannian manifold and \(f : M \to \mathbb{R}\) be a smooth Morse function. Denote by \(\Phi\) the flow generated by \(-\nabla^g f\).

For \(r \in \mathbb{R}\) we set \(M^r := \{ x \in M; \ f(x) \leq r \}\).

Assume that \(c\) is a critical value of \(f\) and the level set \(\{ f = c \}\) contains a unique critical point \(p\) of \(f\) of index \(k\).

The set \(\{p\}\) is an isolated invariant set of \(\Phi\).
Example

Let \((M, g)\) be a smooth compact Riemannian manifold and \(f : M \rightarrow \mathbb{R}\) be a smooth Morse function. Denote by \(\Phi\) the flow generated by \(-\nabla^g f\).

For \(r \in \mathbb{R}\) we set \(M^r := \{ x \in M; \ f(x) \leq r \}\).

Assume that \(c\) is a critical value of \(f\) and the level set \(\{ f = c \}\) contains a unique critical point \(p\) of \(f\) of index \(k\).

The set \(\{p\}\) is an isolated invariant set of \(\Phi\).

For \(\varepsilon > 0\) sufficiently small the pair \((M^{c+\varepsilon}, M^{c-\varepsilon})\) is an index pair of \(\{p\}\).
Example

- Let \((M, g)\) be a smooth compact Riemannian manifold and \(f : M \to \mathbb{R}\) be a smooth Morse function. Denote by \(\Phi\) the flow generated by \(-\nabla^g f\).
- For \(r \in \mathbb{R}\) we set \(M^r := \{ x \in M; \ f(x) \leq r \}\).
- Assume that \(c\) is a critical value of \(f\) and the level set \(\{ f = c \}\) contains a unique critical point \(p\) of \(f\) of index \(k\).
- The set \(\{p\}\) is an isolated invariant set of \(\Phi\).
- For \(\varepsilon > 0\) sufficiently small the pair \((M^{c+\varepsilon}, M^{c-\varepsilon})\) is an index pair of \(\{p\}\).
- The central theorem of Morse theory implies that \(M^{c+\varepsilon}\) is homotopic to a space obtained from \(M^{c-\varepsilon}\) by attaching a disk of dimension \(k\).
Example

- Let \((M, g)\) be a smooth compact Riemannian manifold and \(f : M \to \mathbb{R}\) be a smooth Morse function. Denote by \(\Phi\) the flow generated by \(-\nabla^g f\).
- For \(r \in \mathbb{R}\) we set \(M^r := \{x \in M; \ f(x) \leq r\}\).
- Assume that \(c\) is a critical value of \(f\) and the level set \(\{f = c\}\) contains a unique critical point \(p\) of \(f\) of index \(k\).
- The set \(\{p\}\) is an isolated invariant set of \(\Phi\).
- For \(\varepsilon > 0\) sufficiently small the pair \((M^{c+\varepsilon}, M^{c-\varepsilon})\) is an index pair of \(\{p\}\).
- The central theorem of Morse theory implies that \(M^{c+\varepsilon}\) is homotopic to a space obtained from \(M^{c-\varepsilon}\) by attaching a disk of dimension \(k\). Thus

\[ J_\Phi(p) \cong M^{c+\varepsilon} / M^{c-\varepsilon} \cong S^k. \]
Example
Example

\[ \dot{x} = x, \quad \dot{y} = -y \]

with Lyapunov function

\[ f = -x^2 + y^2. \]

\[ I(p) = W_{-\varepsilon(p)}/\partial W_{-\varepsilon(p)} \]

\[ W_{-\varepsilon(p)} = W(p) \cap \{ f \geq f(p) - \varepsilon \} \]

\[ \partial W_{-\varepsilon(p)} = W(p) \cap \{ f = f(p) - \varepsilon \} \]
Flow \( \dot{x} = x, \dot{y} = -y \) with Lyapunov function \( f = -x^2 + y^2 \).
Flow $\dot{x} = x$, $\dot{y} = -y$ with Lyapunov function $f = -x^2 + y^2$.

$I(p) = W_\varepsilon^-(p) / \partial W_\varepsilon^-(p)$, $W_\varepsilon^-(p) = W^-(p) \cap \{ f \geq f(p) - \varepsilon \}$, $\partial W_\varepsilon^-(p) := W^-(p) \cap \{ f = f(p) - \varepsilon \}$
Theorem (Nico., 2007)

Suppose that $(X, \Phi, f)$ is a Morse triplet. Let $p$ be a stationary point of $\Phi$. For any $\varepsilon > 0$ we set
\[
W^-\varepsilon(p) := W^-(p) \cap \{ f \geq f(p) - \varepsilon \},
\]
\[
\partial W^-\varepsilon(p) := W^-(p) \cap \{ f = f(p) - \varepsilon \}.
\]
Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ we have
\[
I_{\Phi}(p) \simeq W^-\varepsilon(p) / \partial W^-\varepsilon(p).
\]
Theorem (Nico., 2007)

Suppose that \((X, \Phi, f)\) is a Morse triplet.
Theorem (Nico., 2007)

Suppose that \((X, \Phi, f)\) is a Morse triplet. Let \(p\) be a stationary point of \(\Phi\).

\[
\begin{align*}
W_{-\varepsilon}(p) & := W_{-\varepsilon}(p) \cap \{f \geq f(p) - \varepsilon\}, \\
\partial W_{-\varepsilon}(p) & := W_{-\varepsilon}(p) \cap \{f = f(p) - \varepsilon\}.
\end{align*}
\]

Then there exists \(\varepsilon_0 > 0\) such that, for any \(\varepsilon \in (0, \varepsilon_0)\) we have

\[
I_{\Phi}(p) \cong W_{-\varepsilon}(p) / \partial W_{-\varepsilon}(p).
\]
Theorem (Nico., 2007)

Suppose that \((X, \Phi, f)\) is a Morse triplet. Let \(p\) be a stationary point of \(\Phi\). For any \(\varepsilon > 0\) we set

\[
W^-_\varepsilon(p) := W^-(p) \cap \{ f \geq f(p) - \varepsilon \},
\]

\[
\partial W^-_\varepsilon(p) := W^-(p) \cap \{ f = f(p) - \varepsilon \}.
\]
Computing the Conley index

**Theorem (Nico., 2007)**

Suppose that \((X, \Phi, f)\) is a Morse triplet. Let \(p\) be a stationary point of \(\Phi\). For any \(\varepsilon > 0\) we set

\[
W_{\varepsilon}^{-}(p) := W^{-}(p) \cap \{ f \geq f(p) - \varepsilon \},
\]

\[
\partial W_{\varepsilon}^{-}(p) := W^{-}(p) \cap \{ f = f(p) - \varepsilon \}.
\]

Then there exists \(\varepsilon_0 > 0\) such that, for any \(\varepsilon \in (0, \varepsilon_0)\) we have

\[
I_\Phi(p) \simeq W_{\varepsilon}^{-}(p)/\partial W_{\varepsilon}^{-}(p).
\]
Suppose that $X$ is a compact tame set equipped with a triangulation $S$. Denote by $S'$ the barycentric subdivision of $S$. The vertices of $S'$ are the barycenters $b_{\sigma}$ of the faces $\sigma \in S$. Two vertices $b_{\sigma}$ and $b_{\tau}$ of $S'$ are adjacent iff $\sigma \subset \tau$ or $\tau \subset \sigma$.

A dynamical orientation on $S'$ is then a map $\sigma \mapsto \omega(b_{\sigma}) \in \mathbb{R}$ such that $\omega(b_{\sigma}) \neq \omega(b_{\tau})$ if $\sigma \subset \tau$ or $\tau \subset \sigma$.

A dynamical violation at $\sigma \in S$ is a face $\tau \in S$ such that either $\tau \subset \sigma$ and $\omega(b_{\tau}) > \omega(b_{\sigma})$, or $\sigma \subset \tau$ and $\omega(b_{\sigma}) > \omega(b_{\tau})$. 

Liviu I. Nicolaescu (Notre Dame)
Combinatorial Morse functions

Suppose that $X$ is a compact tame set equipped with a triangulation $S$. Denote by $S'$ the barycentric subdivision of $S$.
Combinatorial Morse functions

- Suppose that $X$ is a compact tame set equipped with a triangulation $S$. Denote by $S'$ the barycentric subdivision of $S$.
- The vertices of $S'$ are the barycenters $b_\sigma$ of the faces $\sigma \in S$. 
Combinatorial Morse functions

- Suppose that $X$ is a compact tame set equipped with a triangulation $S$. Denote by $S'$ the barycentric subdivision of $S$.
- The vertices of $S'$ are the barycenters $b_\sigma$ of the faces $\sigma \in S$. Two vertices $b_\sigma$ and $b_\tau$ of $S'$ are adjacent iff $\sigma \subset \tau$ or $\tau \subset \sigma$. 

Combinatorial Morse functions

- Suppose that $X$ is a compact tame set equipped with a triangulation $S$. Denote by $S'$ the barycentric subdivision of $S$.
- The vertices of $S'$ are the barycenters $b_\sigma$ of the faces $\sigma \in S$. Two vertices $b_\sigma$ and $b_\tau$ of $S'$ are adjacent iff $\sigma \subset \tau$ or $\tau \subset \sigma$.
- A dynamical orientation on $S'$ is then a map
  
  $$S \ni \sigma \mapsto \omega(b_\sigma) \in \mathbb{R}$$
  
  such that $\omega(b_\sigma) \neq \omega(b_\tau)$ if $\sigma \subset \tau$ or $\tau \subset \sigma$. 
Combinatorial Morse functions

- Suppose that $X$ is a compact tame set equipped with a triangulation $S$. Denote by $S'$ the barycentric subdivision of $S$.
- The vertices of $S'$ are the barycenters $b_{\sigma}$ of the faces $\sigma \in S$. Two vertices $b_{\sigma}$ and $b_{\tau}$ of $S'$ are adjacent iff $\sigma \subset \tau$ or $\tau \subset \sigma$.
- A dynamical orientation on $S'$ is then a map
  \[ S \ni \sigma \mapsto \omega(b_{\sigma}) \in \mathbb{R} \]
  such that $\omega(b_{\sigma}) \neq \omega(b_{\tau})$ if $\sigma \subset \tau$ or $\tau \subset \sigma$.
- A **dynamical violation** at $\sigma \in S$ is a face $\tau \in S$ such that either
Combinatorial Morse functions

Suppose that $X$ is a compact tame set equipped with a triangulation $S$. Denote by $S'$ the barycentric subdivision of $S$.

The vertices of $S'$ are the barycenters $b_\sigma$ of the faces $\sigma \in S$. Two vertices $b_\sigma$ and $b_\tau$ of $S'$ are adjacent iff $\sigma \subset \tau$ or $\tau \subset \sigma$.

A dynamical orientation on $S'$ is then a map

$$S \ni \sigma \mapsto \omega(b_\sigma) \in \mathbb{R}$$

such that $\omega(b_\sigma) \neq \omega(b_\tau)$ if $\sigma \subset \tau$ or $\tau \subset \sigma$.

A dynamical violation at $\sigma \in S$ is a face $\tau \in S$ such that either

$$\tau \subset \sigma \quad \text{and} \quad \omega(b_\tau) > \omega(b_\sigma),$$

or
Combinatorial Morse functions

Suppose that $X$ is a compact tame set equipped with a triangulation $S$. Denote by $S'$ the barycentric subdivision of $S$.

The vertices of $S'$ are the barycenters $b_{\sigma}$ of the faces $\sigma \in S$. Two vertices $b_{\sigma}$ and $b_{\tau}$ of $S'$ are adjacent iff $\sigma \subset \tau$ or $\tau \subset \sigma$.

A dynamical orientation on $S'$ is then a map

$$S \ni \sigma \mapsto \omega(b_{\sigma}) \in \mathbb{R}$$

such that $\omega(b_{\sigma}) \neq \omega(b_{\tau})$ if $\sigma \subset \tau$ or $\tau \subset \sigma$.

A dynamical violation at $\sigma \in S$ is a face $\tau \in S$ such that either

$$\tau \subset \sigma \text{ and } \omega(b_{\tau}) > \omega(b_{\sigma}),$$

or

$$\sigma \subset \tau \text{ and } \omega(b_{\sigma}) > \omega(b_{\tau}).$$
Definition (R. Forman)

Suppose that $X$ is a compact tame set equipped with a triangulation $S$.

(a) A combinatorial Morse function on $S$ is a dynamical orientation $\omega$ on the barycentric subdivision $S'$ such that for any $\sigma \in S$ the number of dynamical violation at $\sigma$ is $\leq 1$.

(b) If $\omega$ is a combinatorial Morse function on $S$, then a face $\sigma$ that has no dynamical violations is called critical. All the other faces are called regular.

(c) For a combinatorial Morse function $\omega$ on $S$ we denote by $\Phi_\omega$ the associated simplicial flow. The stationary points of this flow are the barycenters $b_\sigma$ of the faces $\sigma \in S$. We denote by $I_\omega(\sigma)$ the Conley index of $b_\sigma$ with respect to the flow $\Phi_\omega$. 

Liviu I. Nicolaescu (Notre Dame)
Combinatorial Morse functions

Definition (R. Forman)
Suppose that \( X \) is a compact tame set equipped with a triangulation \( S \).

(a) A combinatorial Morse function on \( S \) is a dynamical orientation \( \omega \) on the barycentric subdivision \( S' \) such that for any \( \sigma \in S \) the number of dynamical violations at \( \sigma \) is \( \leq 1 \).

(b) If \( \omega \) is a combinatorial Morse function on \( S \), then a face \( \sigma \) that has no dynamical violations is called critical. All the other faces are called regular.

(c) For a combinatorial Morse function \( \omega \) on \( S \) we denote by \( \Phi_\omega \) the associated simplicial flow. The stationary points of this flow are the barycenters \( b_\sigma \) of the faces \( \sigma \in S \). We denote by \( I_\omega(\sigma) \) the Conley index of \( b_\sigma \) with respect to the flow \( \Phi_\omega \).
### Definition (R. Forman)

Suppose that $X$ is a compact tame set equipped with a triangulation $S$.

(a) A **combinatorial Morse function** on $S$ is a dynamical orientation $\omega$ on the barycentric subdivision $S'$ such that for any $\sigma \in S$ the number of dynamical violation at $\sigma$ is $\leq 1$. 
Combinatorial Morse functions

Definition (R. Forman)

Suppose that $X$ is a compact tame set equipped with a triangulation $S$.

(a) A **combinatorial Morse function** on $S$ is a dynamical orientation $\omega$ on the barycentric subdivision $S'$ such that for any $\sigma \in S$ the number of dynamical violation at $\sigma$ is $\leq 1$.

(b) If $\omega$ is a combinatorial Morse function on $S$, then a face $\sigma$ that has no dynamical violations is called **critical**. All the other faces are called **regular**.
Definition (R. Forman)

Suppose that $X$ is a compact tame set equipped with a triangulation $S$.

(a) A **combinatorial Morse function** on $S$ is a dynamical orientation $\omega$ on the barycentric subdivision $S'$ such that for any $\sigma \in S$ the number of dynamical violation at $\sigma$ is $\leq 1$.

(b) If $\omega$ is a combinatorial Morse function on $S$, then a face $\sigma$ that has no dynamical violations is called **critical**. All the other faces are called **regular**.

(c) For a combinatorial Morse function $\omega$ on $S$ we denote by $\Phi_\omega$ the associated simplicial flow. The stationary points of this flow are the barycenters $b_\sigma$ of the faces $\sigma \in S$. We denote by $I_\omega(\sigma)$ the Conley index of $b_\sigma$ with respect to the flow $\Phi_\omega$. 
Examples of combinatorial Morse functions
Suppose that $X$ is a compact tame set equipped with a triangulation $S$. 
Examples of combinatorial Morse functions

Suppose that $X$ is a compact tame set equipped with a triangulation $S$. Then the map

$$S \ni \sigma \mapsto \dim \sigma$$

is a combinatorial Morse function on $S$. 

Examples of combinatorial Morse functions

Green points correspond to regular faces while red points correspond to critical faces. The red arrow indicates the dynamical violations.
Examples of combinatorial Morse functions

Green points correspond to regular faces while red points correspond to critical faces. The red arrow indicate the dynamical violations.
Suppose $X$ is a compact tame space equipped with a triangulation $\mathcal{S}$ and a combinatorial Morse function $\omega : \mathcal{S} \to \mathbb{R}$.

Theorem (Nico., 2007)
Theorem (Nico., 2007)

Suppose $X$ is a compact tame space equipped with a triangulation $\mathcal{S}$ and a combinatorial Morse function $\omega : \mathcal{S} \to \mathbb{R}$.

(a) If $\sigma$ is a regular face, then the Conley index $I_\omega(b_\sigma)$ is trivial.
The Conley indices of combinatorial Morse functions

Theorem (Nico., 2007)

Suppose $X$ is a compact tame space equipped with a triangulation $S$ and a combinatorial Morse function $\omega : S \to \mathbb{R}$.

(a) If $\sigma$ is a regular face, then the Conley index $\mathcal{J}_\omega(b_\sigma)$ is trivial.
(b) If $\sigma$ is a critical face, then the Conley index $\mathcal{J}_\omega(b_\sigma)$ is homotopic to a sphere of dimension $\dim \sigma$. 
A **tame blowdown map** is a continuous tame map \( \beta : X \rightarrow Y \) such that

1. \( X \) and \( Y \) are compact.
2. There exists a finite subset \( L_\beta \subset Y \) such that the induced map \( \beta : X \setminus \beta^{-1}(L_\beta) \rightarrow Y \setminus L_\beta \) is a homeomorphism.
3. The set \( L_\beta \) is called the **blowup locus**.
4. The set \( E_\beta := \beta^{-1}(L_\beta) \) is called the **exceptional locus**.
5. A **weight** for a tame blowdown map \( \beta : X \rightarrow Y \) is a tame continuous function \( w : Y \rightarrow [0, \infty) \) such that \( L_\beta = w^{-1}(0) \).
Tame blowdowns

- A *tame blowdown map* is a continuous tame map $\beta : X \to Y$ such that
  - $X$ and $Y$ are compact.
A \textit{tame blowdown map} is a continuous tame map $\beta : X \to Y$ such that

\begin{itemize}
  \item $X$ and $Y$ are compact.
  \item There exists a \textit{finite} subset $L_\beta \subset Y$ such that the induced map

  \[ \beta : X \setminus \beta^{-1}(L_\beta) \to Y \setminus L_\beta \]

  is a \textit{homeomorphism}.
\end{itemize}
Tame blowdowns

A *tame blowdown map* is a continuous tame map $\beta : X \to Y$ such that

- $X$ and $Y$ are compact.
- There exists a *finite* subset $L_\beta \subset Y$ such that the induced map
  \[
  \beta : X \setminus \beta^{-1}(L_\beta) \to Y \setminus L_\beta
  \]
  is a *homeomorphism*.

The set $L_\beta$ is called the *blowup locus*. 
A **tame blowdown map** is a continuous tame map $\beta : X \to Y$ such that

- $X$ and $Y$ are compact.
- There exists a *finite* subset $L_\beta \subset Y$ such that the induced map

$$\beta : X \setminus \beta^{-1}(L_\beta) \to Y \setminus L_\beta$$

is a *homeomorphism*.

The set $L_\beta$ is called the **blowup locus**. The set $E_\beta := \beta^{-1}(L_\beta)$ is called the **exceptional locus**.
A \textit{tame blowdown map} is a continuous tame map $\beta : X \rightarrow Y$ such that

- $X$ and $Y$ are compact.
- There exists a \textit{finite} subset $L_\beta \subset Y$ such that the induced map
  \begin{equation*}
  \beta : X \setminus \beta^{-1}(L_\beta) \rightarrow Y \setminus L_\beta
  \end{equation*}
  is a \textit{homeomorphism}.

The set $L_\beta$ is called the \textit{blowup locus}. The set $E_\beta := \beta^{-1}(L_\beta)$ is called the \textit{exceptional locus}.

A \textit{weight} for a tame blowdown map $\beta : X \rightarrow Y$ is a tame continuous function $w : Y \rightarrow [0, \infty)$ such that

\begin{equation*}
L_\beta = w^{-1}(0).
\end{equation*}
Fundamental example
Fundamental example

- Suppose \((X, \Phi, f)\) is a Morse triplet.
Fundamental example

- Suppose \((X, \Phi, f)\) is a Morse triplet.
- For \(r \in \mathbb{R}\) we set \(X_r := f^{-1}(r)\).
Fundamental example

- Suppose \((X, \Phi, f)\) is a Morse triplet.
- For \(r \in \mathbb{R}\) we set \(X_r := f^{-1}(r)\).
- Suppose that
  \[
  f(\text{Cr}_\Phi) = \{c_0 < c_1 < \cdots < c_n\} \subset \mathbb{R}.
  \]
- Set \(\text{Cr}_i = \text{Cr}_\Phi \cap X_{c_i}\).
Fundamental example

- Suppose \((X, \Phi, f)\) is a Morse triplet.
- For \(r \in \mathbb{R}\) we set \(X_r := f^{-1}(r)\).
- Suppose that
  \[
  f(\mathbf{Cr}_\Phi) = \{c_0 < c_1 < \cdots < c_n\} \subset \mathbb{R}.
  \]

Set \(\mathbf{Cr}_i = \mathbf{Cr}_\Phi \cap X_{c_i}\). Note that \(X_{c_0}\) and \(X_{c_n}\) are finite sets.
Fundamental example

- Suppose \((X, \Phi, f)\) is a Morse triplet.
- For \(r \in \mathbb{R}\) we set \(X_r := f^{-1}(r)\).
- Suppose that

\[
f(\text{Cr}_\Phi) = \{ c_0 < c_1 < \cdots < c_n \} \subset \mathbb{R}.
\]

Set \(\text{Cr}_i = \text{Cr}_\Phi \cap X_{c_i}\). Note that \(X_{c_0}\) and \(X_{c_n}\) are finite sets.
- Fix \(r_i \in (c_{i-1}, c_i), \ i = 1, \ldots, n\).
Fundamental example

- Suppose \((X, \Phi, f)\) is a Morse triplet.
- For \(r \in \mathbb{R}\) we set \(X_r := f^{-1}(r)\).
- Suppose that
  \[
  f(Cr_\Phi) = \{c_0 < c_1 < \cdots < c_n\} \subset \mathbb{R}.
  \]
  Set \(Cr_i = Cr_\Phi \cap X_{c_i}\). Note that \(X_{c_0}\) and \(X_{c_n}\) are finite sets.
- Fix \(r_i \in (c_{i-1}, c_i), i = 1, \ldots, n\).
- Define \(\beta_i^+ : X_{r_i} \to X_{c_{i-1}},\)
  \[
  \beta_i^+(x) = \Phi\left([0, \infty] \times \{x\}\right) \cap X_{c_{i-1}}.
  \]
Fundamental example

- Suppose \((X, \Phi, f)\) is a Morse triplet.
- For \(r \in \mathbb{R}\) we set \(X_r := f^{-1}(r)\).
- Suppose that

\[
f(\text{Cr}_\Phi) = \{c_0 < c_1 < \cdots < c_n\} \subset \mathbb{R}.
\]

Set \(\text{Cr}_i = \text{Cr}_\Phi \cap X_{c_i}\). Note that \(X_{c_0}\) and \(X_{c_n}\) are finite sets.
- Fix \(r_i \in (c_i-1, c_i)\), \(i = 1, \ldots, n\).
- Define \(\beta_i^+ : X_{r_i} \to X_{c_{i-1}}\),

\[
\beta_i^+(x) = \Phi\left([-\infty, \infty] \times \{x\}\right) \cap X_{c_{i-1}}.
\]

- Define \(\beta_i^- : X_{r_i} \to X_{c_i}\),

\[
\beta_i^-(x) = \Phi\left([-\infty, \infty] \times \{x\}\right) \cap X_{c_i}.
\]
Theorem (Nico., 2007)

- The maps \( \beta_i^+ : X_{r_i} \to X_{c_{i-1}}, \ i = 1, \ldots, n \) are blowdown maps with blowup loci \( Cr^{i-1} \).
**Theorem (Nico., 2007)**

- **The maps** \( \beta_i^+ : X_{r_i} \rightarrow X_{c_{i-1}} , i = 1, \ldots, n \) **are blowdown maps with blowup loci** \( \text{Cr}^{i-1} \).

- **Additionally, the set** \( \{ c_{i-1} \leq f \leq r_i \} \) **is tamely homeomorphic to the mapping cylinder** \( M_{\beta_i^+} \) **of** \( \beta_i^+ \).
Theorem (Nico., 2007)

- The maps $\beta_i^+ : X_{r_i} \rightarrow X_{c_i-1}, i = 1, \ldots, n$ are blowdown maps with blowup loci $C_{r_i}^{i-1}$.
- Additionally, the set $\{c_{i-1} \leq f \leq r_i\}$ is tamely homeomorphic to the mapping cylinder $M_{\beta_i^+}$ of $\beta_i^+$.
- If we define $T_i^+ : X_{c_{i-1}} \rightarrow [0, \infty]$ by the condition $T_i^+(x) = \infty$ if $x \in C_{r_i}^{i-1}$ and
Theorem (Nico., 2007)

- The maps $\beta_i^+ : X_{r_i} \rightarrow X_{c_{i-1}}$, $i = 1, \ldots, n$ are blowdown maps with blowup loci $C_{r_i}^{i-1}$.

- Additionally, the set $\{c_{i-1} \leq f \leq r_i\}$ is tamely homeomorphic to the mapping cylinder $M_{\beta_i^+}$ of $\beta_i^+$.

- If we define $T_i^+ : X_{c_{i-1}} \rightarrow [0, \infty]$ by the condition
  - $T_i^+(x) = \infty$ if $x \in C_{r_i}^{i-1}$ and
  - $\Phi^{-T_i^+}(x)(x) \in X_{r_i}$, if $x \in X_{c_{i-1}} \setminus C_{r_i}^{i-1}$,
Theorem (Nico., 2007)

The maps $\beta_i^+ : X_{r_i} \to X_{c_{i-1}}$, $i = 1, \ldots, n$ are blowdown maps with blowup loci $\text{Cr}^{i-1}$.

Additionally, the set $\{c_{i-1} \leq f \leq r_i\}$ is tamely homeomorphic to the mapping cylinder $M_{\beta_i^+}$ of $\beta_i^+$.

If we define $T_i^+ : X_{c_{i-1}} \to [0, \infty]$ by the condition

- $T_i^+(x) = \infty$ if $x \in \text{Cr}^{i-1}$ and
- $\Phi^{-T_i^+}(x)(x) \in X_{r_i}$, if $x \in X_{c_{i-1}} \setminus \text{Cr}^{i-1}$,

then the map $w_i^+ = \frac{1}{T_i^+}$ is a weight for the blowdown map $\beta_i^+$. Similar results are true for the maps $\beta_i^-$. 

Liviu I. Nicolaescu (Notre Dame) Tame flows May 22, 2011 38 / 41
Theorem (Nico., 2007)

The maps $\beta_i^+ : X_{r_i} \to X_{c_{i-1}}$, $i = 1, \ldots, n$ are blowdown maps with blowup loci $C r_{i-1}$.

Additionally, the set $\{c_{i-1} \leq f \leq r_i\}$ is tamely homeomorphic to the mapping cylinder $M_{\beta_i^+}$ of $\beta_i^+$.

If we define $T_i^+ : X_{c_{i-1}} \to [0, \infty]$ by the condition

- $T_i^+(x) = \infty$ if $x \in C r_{i-1}$ and
- $\Phi^{-T_i^+(x)}(x) \in X_{r_i}$, if $x \in X_{c_{i-1}} \setminus C r_{i-1}$,

then the map $w_i^+ = \frac{1}{T_i^+}$ is a weight for the blowdown map $\beta_i^+$.

Similar results are true for the maps $\beta_i^-$. 

Liviu I. Nicolaescu (Notre Dame)

May 22, 2011 38 / 41
Fundamental Example
We obtain a string of weighted blowdown maps

\[ X_{c_n} \xleftarrow{\beta_n} X_{r_n} \xrightarrow{\beta_n^+} X_{c_{n-1}} \xleftarrow{\beta_{n-1}^-} \cdots \xrightarrow{\beta_2^+} X_{c_1} \xleftarrow{\beta_1^-} X_{r_1} \xrightarrow{\beta_1^+} X_{c_0}, \]

where \( X_{c_n} \) and \( X_{c_0} \) are finite.
We obtain a string of weighted blowdown maps

\[ X_{c_n} \xleftarrow{\beta_n^-} X_{r_n} \xrightarrow{\beta_n^+} X_{c_{n-1}} \xleftarrow{\beta_{n-1}^-} \cdots \xrightarrow{\beta_2^+} X_{c_1} \xleftarrow{\beta_1^-} X_{r_1} \xrightarrow{\beta_n^+} X_{c_0}, \]

where \( X_{c_n} \) and \( X_{c_0} \) are finite.

Moreover, \( X \) is tamely homeomorphic to the space

\[ M_{\beta_n^-} \cup X_{r_n} \cup M_{\beta_n^+} \cup X_{c_{n-1}} \cdots \cup X_{r_1} \cup M_{\beta_1^+}. \]
A flip-flop is a sequence \( C_n \) of blowdown maps

\[
C_n \leftarrow R_n \rightarrow C_{n-1} \beta - n - 1 \leftarrow \cdots \beta + 2 \rightarrow C_1 \beta - 1 \leftarrow R_1 \rightarrow C_0,
\]

where \( C_0 \) and \( C_n \) are finite sets.

A flip-flop is called weighted if all the corresponding blowdown maps are weighted.

The total space of the flip-flop is the space

\[
M(F) := M(\beta - n) \cup R_n M(\beta + n) \cup C_n \cdots \cup R_1 M(\beta + 1).
\]

We see that to any Morse triplet \((X, \Phi, f)\) we have associated a canonical weighted flip-flop \( F(X, \Phi, f) \) with total space \( X \).
A flip-flop is a sequence $\mathcal{F}$ of blowdown maps

$$
C_n \xleftarrow{\beta_n^-} R_n \xrightarrow{\beta_n^+} C_{n-1} \xleftarrow{\beta_{n-1}^-} \cdots \xrightarrow{\beta_2^+} C_1 \xleftarrow{\beta_1^-} R_1 \xrightarrow{\beta_1^+} C_0,
$$

where $C_0$ and $C_n$ are finite sets.
A flip-flop is a sequence $\mathcal{F}$ of blowdown maps

$$
\begin{align*}
C_n & \xleftarrow{\beta_n^-} R_n \xrightarrow{\beta_n^+} C_{n-1} \xleftarrow{\beta_{n-1}^-} \cdots \xrightarrow{\beta_2^+} C_1 \xleftarrow{\beta_1^-} R_1 \xrightarrow{\beta_1^+} C_0,
\end{align*}
$$

where $C_0$ and $C_n$ are finite sets.

A flip-flop is called weighted if all the corresponding blowdown maps are weighted.
Flip-flops

- A **flip-flop** is a sequence $\mathcal{F}$ of blowdown maps

  $$\begin{align*}
  C_n & \xleftarrow{\beta_n^-} R_n \xrightarrow{\beta_n^+} C_{n-1} \xleftarrow{\beta_{n-1}^-} \cdots \xrightarrow{\beta_2^+} C_1 \xleftarrow{\beta_1^-} R_1 \xrightarrow{\beta_1^+} C_0,
  \end{align*}$$

  where $C_0$ and $C_n$ are finite sets.

- A flip-flop is called **weighted** if all the corresponding blowdown maps are weighted.

- The **total space** of the flip-flop is the space

  $$M(\mathcal{F}) := M_{\beta_n^-} \cup R_n M_{\beta_n^+} \cup C_{n-1} \cdots \cup R_1 M_{\beta_1^+}.$$
Flip-flops

- A **flip-flop** is a sequence $\mathcal{F}$ of blowdown maps

  $$C_n \xleftarrow{\beta_n^-} R_n \xrightarrow{\beta_n^+} C_{n-1} \xleftarrow{\beta_{n-1}^-} \cdots \xrightarrow{\beta_2^+} C_1 \xleftarrow{\beta_1^-} R_1 \xrightarrow{\beta_1^+} C_0,$$

  where $C_0$ and $C_n$ are finite sets.

- A flip-flop is called **weighted** if all the corresponding blowdown maps are weighted.

- The **total space** of the flip-flop is the space

  $$M(\mathcal{F}) := M_{\beta_n^-} \cup R_n M_{\beta_n^+} \cup C_{n-1} \cdots \cup R_1 M_{\beta_1^+}.$$

- We see that to any Morse triplet $(X, \Phi, f)$ we have associated a canonical weighted flip-flop $\mathcal{F}(X, \Phi, f)$ with total space $X$. 

Liviu I. Nicolaescu (Notre Dame)
Characterization of Morse-like tame flows
Theorem (Nico., 2007)

For every weighted flip-flop $\mathcal{F}$ there exists a canonical tame Morse-like flow $\Phi = \Phi_{\mathcal{F}}$ on the total space $X = M(\mathcal{F})$ and a tame Lyapunov function $f_{\mathcal{F}}$ such that the flip-flop associated to the Morse triplet $(M(\mathcal{F}), \Phi_{\mathcal{F}}, f_{\mathcal{F}})$ coincides with $\mathcal{F}$.

Conversely, if $\mathcal{F}$ is the flip-flop associated to a Morse-triplet $(X, \Phi, f)$, then $\Phi = \Phi_{\mathcal{F}}$. 
Characterization of Morse-like tame flows

Theorem (Nico., 2007)

- For every weighted flip-flop $\mathcal{F}$ there exists a canonical tame Morse-like flow $\Phi = \Phi_{\mathcal{F}}$ on the total space $X = M(\mathcal{F})$ and a tame Lyapunov function $f_{\mathcal{F}}$ such that the flip-flop associated to the Morse triplet $(M(\mathcal{F}), \Phi_{\mathcal{F}}, f_{\mathcal{F}})$ coincides with $\mathcal{F}$.

- Conversely, if $\mathcal{F}$ is the flip-flop associated to a Morse-triplet $(X, \Phi, f)$, then $\Phi = \Phi_{\mathcal{F}}$. 